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A sharp rearrangement inequality for the fractional maximal operator

by

A. CIANCHI (Firenze), R. KERMAN (St. Catharines, ON),
B. OPIC (Praha) and L. PICK (Praha)

Abstract. We prove a sharp pointwise estimate of the nonincreasing rearrangement of the fractional maximal function of f, $M_\gamma f$, by an expression involving the nonincreasing rearrangement of f. This estimate is used to obtain necessary and sufficient conditions for the boundedness of M_γ between classical Lorentz spaces.

1. Introduction and statement of main results. For $n \in \mathbb{N}$ and $\gamma \in [0, n)$, the *fractional maximal operator* M_γ is defined at $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$(M_\gamma f)(x) = \sup_{Q \ni x} |Q|^{\gamma/n-1} \int_Q |f(y)|\,dy, \qquad x \in \mathbb{R}^n,$$

where the supremum is extended over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes and $|E|$ denotes the n-dimensional Lebesgue measure of a measurable subset E of $\mathbb{R}^n$. For the classical Hardy–Littlewood maximal operator $M := M_0$, the rearrangement inequality

$$cf^{**}(t) \le (Mf)^*(t) \le Cf^{**}(t), \qquad t \in (0, \infty), \tag{1.1}$$

holds, where $f^*(t) = \inf\{\lambda > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| \le t\}$ is the *nonincreasing rearrangement* of f, $f^{**}(t) = t^{-1}\int_0^t f^*(y)\,dy$, and c, C are positive constants depending only on n (cf. [BS, Chapter 3, Theorem 3.8]). Similar sharp rearrangement estimates are known for other classical operators of harmonic analysis such as the Riesz potential

$$(I_\gamma f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}}\,dy, \qquad x \in \mathbb{R}^n,\ \gamma \in (0, n),$$

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and the Hilbert transform

$$(Hf)(x) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)}{x-t}\,dt, \qquad x \in \mathbb{R}.$$

These estimates are of great importance in the study of operators on rearrangement-invariant function spaces as well as in interpolation theory. A sharp rearrangement inequality for M_γ, $\gamma \in (0,n)$, is not available in the existing literature. The aim of this paper is to fill this gap and to present an application of our estimate.

Throughout the paper, we denote by $\mathfrak{M}^+(0,\infty)$ the set of all nonnegative measurable functions on $(0,\infty)$, and by $\mathfrak{M}^+(0,\infty;\downarrow)$ the set of all nonincreasing functions from $\mathfrak{M}^+(0,\infty)$. The symbol $\chi_{(a,b)}$ stands for the characteristic function of an interval $(a,b) \subset (0,\infty)$. The quantity $\omega_n = \pi^{n/2}\Gamma(n/2+1)^{-1}$ is the volume of the unit ball in $\mathbb{R}^n$. We use the letter C for a positive constant, independent of appropriate parameters and not necessarily the same at each occurrence.

THEOREM 1.1. *Let $n \in \mathbb{N}$ and $\gamma \in [0,n)$. Then there exists a positive constant C, depending only on n and γ, such that*

$$(M_\gamma f)^*(t) \le C \sup_{t<\tau<\infty} \tau^{\gamma/n} f^{**}(\tau), \qquad t \in (0,\infty), \tag{1.2}$$

for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Inequality (1.2) is sharp in the sense that for every $\varphi \in \mathfrak{M}^+(0,\infty;\downarrow)$ there exists a function f on $\mathbb{R}^n$ such that $f^* = \varphi$ a.e. on $(0,\infty)$ and

$$(M_\gamma f)^*(t) \ge c \sup_{t<\tau<\infty} \tau^{\gamma/n} f^{**}(\tau), \qquad t \in (0,\infty), \tag{1.3}$$

where, again, c is a positive constant which depends only on n and γ. Moreover, the expression $\sup_{t<\tau<\infty} \tau^{\gamma/n} f^{**}(\tau)$ can be replaced by $(t^{\gamma/n} f^{**}(t) + \sup_{t<\tau<\infty} \tau^{\gamma/n} f^*(\tau))$ in both (1.2) and (1.3).

Observe that, since

$$\sup_{t<\tau<\infty} \tau^{\gamma/n} f^{**}(\tau) = f^{**}(t) \qquad \text{if } \gamma = 0,$$

(1.2) and (1.3) are consistent with (1.1). Further, (1.3) is of the same nature as an analogous estimate for the Riesz potential (cf. [S]), though essentially smaller, as we show below in the remark preceding the proof of Theorem 1.1.

Theorem 1.1 will be used to characterize the classical Lorentz spaces between which M_γ is bounded. Given $p \in (1,\infty)$ and a nonnegative measurable function v on $(0,\infty)$, the *classical Lorentz space* $\Lambda^p(v)$ is the set of all measurable functions f on $\mathbb{R}^n$ such that the quantity

$$\|f\|_{\Lambda^p(v)} = \left(\int_0^\infty (f^*(t))^p v(t)\,dt\right)^{1/p}$$

is finite. Let us recall that classical Lorentz spaces include many familiar function spaces such as Lebesgue, Lorentz or Lorentz–Zygmund spaces.

The boundedness of the Hardy–Littlewood maximal operator M between $\Lambda^p(v)$ and $\Lambda^q(w)$ was characterized in [AM] for $1 < p = q < \infty$ and $v = w$, and in [S] for arbitrary $p, q \in (1,\infty)$ and v, w. For the operator M_γ, we have the following result.

THEOREM 1.2. *Let $n \in \mathbb{N}$, $\gamma \in [0,n)$, $1 < p \le q < \infty$, and let w, v be nonnegative and measurable functions on $(0,\infty)$ with v satisfying $\int_0^x v(t)\,dt < \infty$ for every $x \in (0,\infty)$. Then M_γ is bounded from $\Lambda^p(v)$ into $\Lambda^q(w)$ if and only if there exists a positive constant C such that*

$$r^{\gamma/n}\Big(\int_0^r w(t)\,dt\Big)^{1/q} \le C\Big(\int_0^r v(t)\,dt\Big)^{1/p} \tag{1.4}$$

and

$$\Big(\int_r^\infty t^{q(\gamma/n-1)} w(t)\,dt\Big)^{1/q}\Big(\int_0^r \Big(t^{-1}\int_0^t v(y)\,dy\Big)^{-p'} v(t)\,dt\Big)^{1/p'} \le C \tag{1.5}$$

hold for all $r \in (0,\infty)$.

REMARKS 1.3. (i) If $\gamma = 0$, then (1.4) and (1.5) coincide with the conditions in [S, Theorem 2].

(ii) If v satisfies $t^{-1}\int_0^t v(\tau)\,d\tau \le Cv(t)$ for all $t > 0$ (in particular, if v is nondecreasing), then (1.5) can be replaced by

$$\Big(\int_r^\infty t^{q(\gamma/n-1)} w(t)\,dt\Big)^{1/q}\Big(\int_0^r (v(t))^{1-p'}\,dt\Big)^{1/p'} \le C. \tag{1.6}$$

Indeed, while (1.5) obviously implies (1.6), the condition (1.6) is (by [OK]) equivalent to saying that the inequality

$$\Big(\int_0^\infty \Big(t^{\gamma/n-1}\int_0^t \varphi(y)\,dy\Big)^q w(t)\,dt\Big)^{1/q} \le C\Big(\int_0^\infty \varphi^p(t)v(t)\,dt\Big)^{1/p} \tag{1.7}$$

holds for all $\varphi \in \mathfrak{M}^+(0,\infty)$, which of course implies that (1.7) holds for all $\varphi \in \mathfrak{M}^+(0,\infty;\downarrow)$, and the latter yields (1.5) by [S, Theorem 2].

2. Proof of Theorem 1.1. Our point of departure will be the following two estimates involving $(M_\gamma f)^*$:

$$\sup_{t>0} t^{1-\gamma/n}(M_\gamma f)^*(t) \le C\int_{\mathbb{R}^n} |f(y)|\,dy \tag{2.1}$$

and

$$\sup_{t>0}(M_\gamma f)^*(t) \le C\sup_{t>0} t^{\gamma/n} f^*(t), \tag{2.2}$$

where C depends only on n and γ.

Observe that inequalities (2.1) and (2.2) amount to saying that the operator M_γ is bounded from $L^1(\mathbb{R}^n)$ into $L^{n/(n-\gamma),\infty}(\mathbb{R}^n)$, and from $L^{n/\gamma,\infty}(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$, respectively. In contrast to this, the Riesz potential I_γ, while also bounded from $L^1(\mathbb{R}^n)$ into $L^{n/(n-\gamma),\infty}(\mathbb{R}^n)$, is only bounded from $L^{n/\gamma,1}(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$. This is equivalent (cf. [BS, Chapter 4, Theorem 4.11]) to the inequality

$$(I_\gamma f)^*(t) \le C\Big(t^{\gamma/n-1}\int_0^t f^*(y)\,dy + \int_t^\infty y^{\gamma/n-1} f^*(y)\,dy\Big), \quad t\in(0,\infty).$$

Together with the elementary estimate $(M_\gamma f)(x) \le C(I_\gamma|f|)(x)$, $x\in\mathbb{R}^n$, $\gamma\in(0,n)$, we arrive at

$$(M_\gamma f)^*(t) \le C\Big(t^{\gamma/n-1}\int_0^t f^*(y)\,dy + \int_t^\infty y^{\gamma/n-1} f^*(y)\,dy\Big), \quad t\in(0,\infty). \tag{2.3}$$

This latter estimate is not as sharp as (1.2). Indeed, it is easily seen that

$$\tau^{\gamma/n} f^{**}(\tau) \le \Big(t^{\gamma/n-1}\int_0^t f^*(y)\,dy + \int_t^\infty y^{\gamma/n-1} f^*(y)\,dy\Big) \quad \text{for all } \tau\in[t,\infty).$$

On the other hand, for any function f on $\mathbb{R}^n$ satisfying $f^*(t) = t^{-\gamma/n}$, $t\in(0,\infty)$, the right hand side of (1.2) is finite while the right side of (2.3) is not. In fact, the right hand side of (1.2) is the least nonincreasing majorant of the first summand on the right hand side of (2.3).

Estimate (2.1) is well known (cf. e.g. [T, Chapter VI, (2.19)]). The proof of (2.2) is easy. Indeed, for every cube $Q\subset\mathbb{R}^n$, the Hardy–Littlewood inequality ([BS, Chapter 2, Theorem 2.2]) yields

$$|Q|^{\gamma/n-1}\int_Q |f(y)|\,dy \le |Q|^{\gamma/n-1}\int_0^{|Q|} t^{\gamma/n} f^*(t) t^{-\gamma/n}\,dt \le \frac{n}{n-\gamma}\sup_{t>0} t^{\gamma/n} f^*(t),$$

and (2.2) follows.

Proof of Theorem 1.1. Fix $t\in(0,\infty)$ and let $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We may assume that

$$\sup_{t<\tau<\infty} \tau^{\gamma/n} f^{**}(\tau) < \infty, \tag{2.4}$$

otherwise (1.2) holds trivially. Then, by the Hardy–Littlewood inequality,

$$\int_E |f(x)|\,dx \le \int_0^t f^*(y)\,dy < \infty$$

for every set $E\subset\mathbb{R}^n$ of measure at most t. In particular, if we put $E = \{x\in\mathbb{R}^n : |f(x)| > f^*(t)\}$, then $|E|\le t$ (cf. [BS, Chapter 2, (1.18)]), and so f is integrable over E. In other words, the function

$$g_t(x) = \max\{|f(x)| - f^*(t), 0\}\operatorname{sgn} f(x), \quad x\in\mathbb{R}^n,$$

belongs to $L^1(\mathbb{R}^n)$. Next, the function

$$h_t(x) = \min\{|f(x)|, f^*(t)\}\operatorname{sgn} f(x), \quad x\in\mathbb{R}^n,$$

satisfies

$$h_t^*(\tau) = \min\{f^*(\tau), f^*(t)\}, \quad \tau\in(0,\infty).$$

Hence,

$$\begin{aligned}\sup_{\tau>0}\tau^{\gamma/n}h_t^*(\tau) &= \max\{\sup_{0<\tau<t}\tau^{\gamma/n}f^*(t), \sup_{t\le\tau<\infty}\tau^{\gamma/n}f^*(\tau)\}\\ &= \sup_{t\le\tau<\infty}\tau^{\gamma/n}f^*(\tau) \le \sup_{t<\tau<\infty}\tau^{\gamma/n}f^{**}(\tau),\end{aligned} \tag{2.5}$$

which, together with (2.4), implies that $h_t\in L^{n/\gamma,\infty}(\mathbb{R}^n)$. Furthermore, $f = g_t + h_t$, and

$$g_t^*(\tau) = \chi_{(0,t)}(\tau)(f^*(\tau) - f^*(t)), \quad \tau\in(0,\infty). \tag{2.6}$$

Therefore, using [BS, Chapter 2, Proposition 1.7], (2.1), (2.2), (2.6) and (2.5), we get

$$\begin{aligned}(M_\gamma f)^*(t) &\le (M_\gamma g_t)^*(t/2) + (M_\gamma h_t)^*(t/2)\\ &\le C\Big(\Big(\frac{t}{2}\Big)^{\gamma/n-1}\int_0^\infty g_t(y)\,dy + \sup_{\tau>0}\tau^{\gamma/n}h_t^*(\tau)\Big)\\ &\le C\Big(t^{\gamma/n-1}\int_0^t (f^*(\tau) - f^*(t))\,d\tau + \sup_{t<\tau<\infty}\tau^{\gamma/n}f^{**}(\tau)\Big)\\ &\le C\sup_{t<\tau<\infty}\tau^{\gamma/n}f^{**}(\tau),\end{aligned}$$

and (1.2) follows.

We now prove (1.3). Let $\varphi\in\mathfrak{M}^+(0,\infty;\downarrow)$. Putting $f(x) = \varphi(\omega_n|x|^n)$, $x\in\mathbb{R}^n\setminus\{0\}$, we have $f^* = \varphi$ a.e. on $(0,\infty)$. Moreover, given $y\in\mathbb{R}^n$, denote by $B(|y|)$ the ball in $\mathbb{R}^n$, centered at the origin and having radius $|y|$. Then, for every $x,y\in\mathbb{R}^n$ such that $|y|>|x|$, we have

$$\begin{aligned}(M_\gamma f)(x) &\ge c|B(|y|)|^{\gamma/n-1}\int_{B(|y|)} f(z)\,dz\\ &= c(\omega_n|y|^n)^{\gamma/n-1}\int_0^{\omega_n|y|^n} f^*(\tau)\,d\tau = cH(\omega_n|y|^n),\end{aligned}$$

where $c = \omega_n^{1-\gamma/n}2^{\gamma-n}$ and $H(t) = t^{\gamma/n-1}\int_0^t\varphi(\tau)\,d\tau$ for $t\in(0,\infty)$.

Consequently,

$$(M_\gamma f)(x) \ge c \sup_{\tau>\omega_n|x|^n} H(\tau),$$

whence (1.3) follows on taking rearrangements.

Finally, to see that the expression $\sup_{t<\tau<\infty}\tau^{\gamma/n}f^{**}(\tau)$ can be replaced by $(t^{\gamma/n}f^{**}(t)+\sup_{t<\tau<\infty}\tau^{\gamma/n}f^*(\tau))$ in (1.2), take $f\in L^1_{\rm loc}(\mathbb{R}^n)$ and $t\in(0,\infty)$. Then

$$\begin{aligned}\sup_{t<\tau<\infty}\tau^{\gamma/n-1}\int_t^\tau f^*(y)\,dy &\le \sup_{t<\tau<\infty}\tau^{\gamma/n-1}\int_t^\tau[\sup_{t<y<\infty}y^{\gamma/n}f^*(y)]y^{-\gamma/n}\,dy\\ &\le \frac{n}{n-\gamma}\sup_{t<y<\infty}y^{\gamma/n}f^*(y),\end{aligned}$$

and the result follows since

$$\sup_{t<\tau<\infty}\tau^{\gamma/n}f^{**}(\tau)\le t^{\gamma/n}f^{**}(t)+\sup_{t<\tau<\infty}\tau^{\gamma/n-1}\int_t^\tau f^*(y)\,dy.$$

The same assertion for (1.3) is a consequence of the elementary fact that $f^{**}\ge f^*$. ■

3. Proof of Theorem 1.2. We begin by proving a weighted norm inequality for the operator R_γ defined at $\varphi\in\mathfrak{M}^+(0,\infty;\downarrow)$ by

$$(R_\gamma\varphi)(t)=\sup_{t<\tau<\infty}\tau^{\gamma/n}\varphi(\tau),\quad t\in(0,\infty).$$

LEMMA 3.1. *Let $n\in\mathbb{N}$, $\gamma\in[0,n)$, $1<p\le q<\infty$, and let w,v be nonnegative measurable functions on $(0,\infty)$ with v satisfying $\int_0^x v(t)\,dt<\infty$ for every $x\in(0,\infty)$. Then there is a positive constant C such that the inequality*

$$\Big(\int_0^\infty[(R_\gamma\varphi)(t)]^q w(t)\,dt\Big)^{1/q}\le C\Big(\int_0^\infty\varphi^p(t)v(t)\,dt\Big)^{1/p} \tag{3.1}$$

holds for all $\varphi\in\mathfrak{M}^+(0,\infty;\downarrow)$ if and only if (1.4) *holds for all $r\in(0,\infty)$.*

Proof. *Necessity*: Since, for any $r\in(0,\infty)$, $(R_\gamma\chi_{(0,r)})(\tau)=r^{\gamma/n}\chi_{(0,r)}(t)$, $t\in(0,\infty)$, the necessity of (1.4) follows by testing (3.1) on $\varphi=\chi_{(0,r)}$.

Sufficiency: If $\gamma=0$ and $\varphi\in\mathfrak{M}^+(0,\infty;\downarrow)$, then $(R_\gamma\varphi)(t)=\varphi(t)$, and the assertion follows from [S, Remark (i), p. 148]. Let $\gamma\in(0,n)$. With no loss of generality, we assume that $w\ne0$ on a set of positive measure. Then (1.4) entails $\int_0^\infty v(t)\,dt=\infty$. Consequently, there is an increasing sequence $\{r_k\}_{k\in\mathbb{Z}}$ in $(0,\infty)$ such that

$$\int_0^{r_k} v(t)\,dt=2^k,\quad k\in\mathbb{Z}. \tag{3.2}$$

It clearly suffices to verify (3.1) for continuous φ having compact support in $[0,\infty)$ and $\varphi\not\equiv0$. For such φ, the set $A\subset\mathbb{Z}$ given by $A=\{k\in\mathbb{Z}:(R_\gamma\varphi)(r_{k-1})>(R_\gamma\varphi)(r_k)\}$ is not empty. Take $k\in A$ and define

$$z_k=\begin{cases}0 & \text{if } (R_\gamma\varphi)(t)=(R_\gamma\varphi)(r_{k-1}),\ t\in(0,r_{k-1}),\\ \min\{r_j:(R_\gamma\varphi)(r_j)=(R_\gamma\varphi)(r_{k-1})\} & \text{otherwise.}\end{cases}$$

Then we obtain

$$(R_\gamma\varphi)(t)=(R_\gamma\varphi)(r_{k-1}),\quad k\in A,\ t\in[z_k,r_{k-1}].$$

Moreover, by the definition of A, the supremum appearing in the definition of $(R_\gamma\varphi)(r_{k-1})$ is attained in $[r_{k-1},r_k)$. Therefore, for every $k\in A$ and $t\in[z_k,r_k)$, we have

$$(R_\gamma\varphi)(t)\le(R_\gamma\varphi)(r_{k-1})=\sup_{r_{k-1}<\tau<r_k}\tau^{\gamma/n}\varphi(\tau)\le r_k^{\gamma/n}\varphi(r_{k-1}). \tag{3.3}$$

Thus, by (3.3), (1.4), (3.2), monotonicity of φ and the inequality $q\ge p$, we get

$$\begin{aligned}\Big(\int_0^\infty[(R_\gamma\varphi)(t)]^q w(t)\,dt\Big)^{1/q} &= \Big(\sum_{k\in A}\int_{z_k}^{r_k}[(R_\gamma\varphi)(t)]^q w(t)\,dt\Big)^{1/q}\\ &\le\Big(\sum_{k\in A}r_k^{q\gamma/n}\varphi^q(r_{k-1})\int_0^{r_k}w(t)\,dt\Big)^{1/q}\\ &\le C\Big(\sum_{k\in A}\varphi^q(r_{k-1})\Big(\int_0^{r_k}v(t)\,dt\Big)^{q/p}\Big)^{1/q}\\ &\le 4^{1/p}C\Big(\sum_{k\in A}\varphi^q(r_{k-1})\Big(\int_{r_{k-2}}^{r_{k-1}}v(t)\,dt\Big)^{q/p}\Big)^{1/q}\\ &\le 4^{1/p}C\Big(\sum_{k\in A}\Big(\int_{r_{k-2}}^{r_{k-1}}\varphi^p(t)v(t)\,dt\Big)^{q/p}\Big)^{1/q}\\ &\le 4^{1/p}C\Big(\int_0^\infty\varphi^p(t)v(t)\,dt\Big)^{1/p},\end{aligned}$$

and (3.1) follows. ■

Theorem 1.2 is a consequence of Theorem 1.1, Lemma 3.1 and the characterization in [S, Theorem 2] of the inequality

$$\Big(\int_0^\infty\Big(\frac1t\int_0^t\varphi(s)\,ds\Big)^q w(t)\,dt\Big)^{1/q}\le C\Big(\int_0^\infty\varphi^p(t)v(t)\,dt\Big)^{1/p}$$

for all $\varphi\in\mathfrak{M}^+(0,\infty;\downarrow)$.

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Istituto di Matematica
Facoltà di Architettura
Università di Firenze
Via dell'Agnolo 14
50122 Firenze, Italy
E-mail: cianchi@cesit1.unifi.it

Department of Mathematics
Brock University
St. Catharines, Ontario, Canada
E-mail: rkerman@spartan.ac.brocku.ca

Mathematical Institute
Czech Academy of Sciences
Žitná 25
115 67 Praha 1, Czech Republic
E-mail: opic@math.cas.cz

Department of Mathematical Analysis
Faculty of Mathematics and Physics
Charles University
Sokolovská 83
186 75 Praha 8, Czech Republic
E-mail: pick@karlin.mff.cuni.cz

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On pointwise estimates for maximal and singular integral operators

by

A. K. LERNER (Odessa)

Abstract. We prove two pointwise estimates relating some classical maximal and singular integral operators. In particular, these estimates imply well-known rearrangement inequalities, L^p_ω and BLO-norm inequalities.

Introduction. For a locally integrable function f on $\mathbb{R}^n$, define the Hardy–Littlewood and Fefferman–Stein maximal functions by

$$Mf(x) = \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f(y)|\,dy,$$

$$f^{\#}(x) = \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\,dy,$$

where $f_Q = |Q|^{-1}\int_Q f$, the supremum is taken over all cubes Q containing x, and $|Q|$ denotes the Lebesgue measure of Q.

We also define the Calderón–Zygmund maximal singular integral operator by

$$T^*f(x) = \sup_{\varepsilon>0} \Big| \int_{|x-y|>\varepsilon} f(y)k(x-y)\,dy \Big|,$$

where the kernel $k(x)$ satisfies the standard conditions:

$$(1)\qquad \begin{gathered} |k(x)| \le \frac{c}{|x|^n}, \qquad \int_{R_1<|x|<R_2} k(x)\,dx = 0 \quad (0<R_1<R_2<\infty), \\ |k(x) - k(x-y)| \le \frac{c|y|^\alpha}{|x|^{n+\alpha}} \quad (|y| \le |x|/2,\ \alpha>0). \end{gathered}$$

Let ω be a non-negative, locally integrable function. Given a measurable set E, let $\omega(E) = \int_E \omega(x)\,dx$. We say that ω satisfies *Muckenhoupt's*

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