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An example of a Fréchet algebra which is a principal ideal domain

by

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Abstract. We construct an example of a Fréchet m-convex algebra which is a principal ideal domain, and has the unit disk as the maximal ideal space.

1. Introduction. In the sequel, if not stated otherwise, we consider Hausdorff locally multiplicatively convex (LMC) commutative $\mathbb{C}$-algebras with identity (denoted by 1), and we identify the set of scalar multiples of the identity with $\mathbb{C}$. A *Fréchet m-convex algebra* A is a complete metrizable LMC algebra; in this case the topology of A can be defined by an increasing sequence of algebra seminorms (see [5]).

If I is an ideal of A, we denote by I^n the ideal of A generated by all products of the form $x_1 \dots x_n$ ($x_i \in I$). We say that I is *finitely generated* if there exist elements $x_1, \dots, x_r$ in A such that $I = \sum_{i=1}^r Ax_i$, and we write $I = (x_1, \dots, x_r)$; when $r = 1$ we say that $I = (x)$ is *principal*.

As usual, A is *noetherian* (resp. *principal*) if every ideal is finitely generated (resp. *principal*).

There are many proofs of the fact that a noetherian Banach algebra is finite-dimensional, and hence semilocal (see [6], [11] for instance). For Fréchet m-convex algebras all these proofs break down; in fact the algebra of formal power series $\mathbb{C}[[X]]$ (with the topology of $\mathbb{C}^{\mathbb{N}}$) is a principal ideal domain (see also [4] and observe that all these examples are local rings).

Recall that the finiteness conditions on all ideals are somewhat rare in the LMC-context; for instance, if K is a connected compact set in $\mathbb{C}$ then the algebra $\mathcal{O}(K)$ of holomorphic germs is a principal ideal domain, but it is not metrizable. On the other hand infinite-dimensional examples of complete metrizable locally convex division algebras cannot exist, since the Gelfand–Mazur theorem is true for such algebras ([2], [13]).

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In absence of examples it is natural suspect that a Fréchet m-convex noetherian algebra is semilocal (see [12]). Here [6] is a good reference; in a final remark the authors state that this conjecture is false, they mention a future paper, but as far as we know the relevant example has never been published. Therefore we present an explicit example of a Fréchet m-convex principal ideal domain algebra with maximal ideal space homeomorphic to the closed unit disk Δ of the complex plane.

2. Preliminaries. We start with a submultiplicative sequence $a = (a_n)_{n\geq 0}$, that is,

(1) $a_n > 0$, $a_0 = 1$,
(2) $a_{n+m} \leq a_n a_m$ for all n, m in $\mathbb{N}$.

The algebra $A(a)$ is the set of all complex power series $x = \sum_{i=0}^{\infty} x_i t^i$ such that

$$\|x\| = \sum_{i=0}^{\infty} |x_i| a_i$$

is finite, with the obvious vector operations and multiplication given by the convolution product of series. Then $A(a)$ is a power series Banach algebra with identity 1 and t as a generator; if $r(a) = \lim_{n\to\infty} a_n^{1/n}$, then $r(a)$ is the spectral radius of t. Observe that $\|t^n\| = a_n$ $(n \geq 0)$; if we denote by d_n the map $x \mapsto x_n$, then we have $|d_n(x)|a_n \leq \|x\|$ for each $n \geq 0$.

Also the map $h \mapsto h(t)$ is a homeomorphism of the character space of the algebra onto the spectrum of t, and the map $z \mapsto h_z$ is a homeomorphism of the closed disk $\Delta_{r(a)} = \{z \in \mathbb{C} : |z| \leq r(a)\}$ onto the character space of $A(a)$; here we use the notation $h_z(x) = \sum_{k=0}^{\infty} x_k z^k$ (see [5], §5 for details).

We assume in what follows that

(3) $r(a) = 1$.

We denote by $\mathcal{A}(\Delta)$ the disk algebra of continuous functions on Δ which are holomorphic in Δ°; it is easy to see that, for $f \in \mathcal{A}(\Delta)$ and $|z_0| < 1$, $f(z_0) = 0$ means that f belongs to the maximal ideal $\mathcal{M}_{z_0}$ of $\mathcal{A}(\Delta)$, and this is equivalent to saying that $f = (z - z_0)g$ with $g \in \mathcal{A}(\Delta)$. Hence $\mathcal{M}_{z_0}$ is principal, generated by $z - z_0$ (of course, this argument breaks down when $|z_0| = 1$).

Now under the hypothesis (3) it is clear that $A(a)$ can be algebraically identified with the subalgebra of $\mathcal{A}(\Delta)$ of the functions whose Taylor series at 0 belong to $A(a)$, with the above norm defined by the sequence a; as usual we denote by $\widehat{x}$ the function defined by $\widehat{x}(z) = h_z(x)$, and by M_z the maximal ideal $\operatorname{Ker}(h_z)$ of $A(a)$, for $|z| \leq 1$. Furthermore, let $G : A(a) \to C(\mathbb{R})$ be the map defined by $G(x)(s) = h_{e^{is}}(x)$. The following fact is very well known:

LEMMA 2.1. *$G(A(a)) \subset C^k(\mathbb{R})$ if the sequence n^k/a_n $(n \geq 0)$ is bounded.*

We also remark the following fact:

LEMMA 2.2. *If $G(A(a)) \subset C^\infty(\mathbb{R})$ and if $x \in M_z^{n+1}$ for $z = e^{is}$, then $G(x)^{(k)}(s) = 0$ when $0 \leq k \leq n$.*

Proof. We denote by $\mathcal{I}_s$ the maximal ideal of $C^\infty(\mathbb{R})$ of the functions vanishing at $s \in \mathbb{R}$; clearly $G(M_z) \subset \mathcal{I}_s$, hence $G(x) \in \mathcal{I}_s^{n+1}$ and observe that a function $f \in \mathcal{I}_s^{n+1}$ satisfies $f^{(k)}(s) = 0$, $k = 0, 1, \ldots, n$.

We now choose a particular sequence a in such a way that the functions in $G(A(a))$ are C^∞ and belong to a quasi-analytic class.

3. A quasi-analytic algebra. Let $\mathbb{R}^+ = \{s \in \mathbb{R} : s \geq 0\}$; we start with the function $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\Omega(s) = \begin{cases} (e-1)s + 1 & \text{for } 0 \leq s \leq 1, \\ e^{s/(1+\ln(s))} & \text{for } 1 \leq s. \end{cases}$$

This map has the following properties:

(a) Ω is increasing, $\Omega(0) = 1$.
(b) If $u, v \geq 1$ then $\Omega(u + v) \leq \Omega(u)\Omega(v)$.
(c) $\lim_{s\to\infty} \Omega(s)^{1/s} = 1$.

DEFINITION 3.1. For $\xi, s \geq 0$ let

$$m_\xi(s) = s^\xi/\Omega(s), \qquad M(\xi) = \sup_{s\geq 0} m_\xi(s).$$

REMARK 3.2. We have the following properties:

(1) For each $\xi \geq 0$, $M(\xi)$ is finite. Also $M(0) = 1$ and $M(\xi) \geq e^{-1}$ for each $\xi \geq 0$.
(2) If $\xi \geq 1$, then $M(\xi) = \sup_{s\geq 1} m_\xi(s)$.
(3) M is increasing for $\xi \geq 1$.
(4) For each $n \geq 1$ we have $M(n)^2 \leq M(n-1)M(n+1)$; more generally, $M(\xi)^2 \leq M(\xi + \varepsilon)M(\xi - \varepsilon)$ for $0 \leq \varepsilon \leq \xi$, $1 \leq \xi$.

In particular the sequence $M(n)$ is log-convex.

DEFINITION 3.3. For each $x \neq 0$ in $\mathbb{R}^+$, set

$$\tau(x) = \sup_{n\geq 0} \frac{x^n}{M(n)}, \qquad q(x) = \sup_{s\geq 0} \frac{x^s}{M(s)}.$$

Here τ is the associated map of the sequence $\Omega(n)$ ([7], V.2.3), and q is an auxiliary function (see below).

REMARKS 3.4. (a) τ and q are increasing maps.
(b) For every $x > 0$ we have $1 \leq \tau(x) \leq q(x)$.

(c) For every $x>0$ we have $q(x) \le e(x+1)\tau(x)$; in fact,

$$q(x) = \max\left\{ \sup_{0\le s\le 1} \frac{x^s}{M(s)}, \sup_{s\ge 1} \frac{x^s}{M(s)} \right\}$$

and then observe that $q(x) \le e$ for $0<x<1$. On the other hand, if $x\ge 1$ then $\sup_{0\le s\le 1} x^s/M(s) \le ex$ and since M is increasing in $[1,\infty)$, we have the inequalities

$$\sup_{s\ge 1} \frac{x^s}{M(s)} \le \sup_{s\ge 1} \frac{x^s}{M([s])} \le x \sup_{s\ge 1} \frac{x^{[s]}}{M([s])} \le x\tau(x)$$

(where [ ] means "integral part"). Hence for $x\ge 1$ we have

$$q(x) \le \max\{ex, x\tau(x)\} \le ex\tau(x).$$

This yields the required inequality for all $x>0$.

(d) $\int_0^\infty \frac{\ln \tau(x)}{1+x^2}\,dx < \infty$ is equivalent to $\int_0^\infty \frac{\ln q(x)}{1+x^2}\,dx < \infty$. This is a consequence of (c) above and the fact that $\ln(e(x+1))/(1+x^2)$ is integrable in $[0,\infty)$.

LEMMA 3.5. *If* $x \ge e^3$, *then* $\ln q(x) = x/(1+\ln(x))$.

Proof. The function $C(x) = e^x/(1+x)$ is continuous and convex for $x\ge 0$; if $C^*(x)$ denotes the conjugate function, we have the following identity for $\xi \ge 1$:

$$\ln M(\xi) = \sup_{x\ge 1}\{\xi\ln(x) - \ln \Omega(x)\} = \sup_{x\ge 1}\left\{\xi\ln(x) - \frac{x}{1+\ln(x)}\right\}$$
$$= \sup_{x\ge 1}\{\xi\ln(x) - C(\ln(x))\} = \sup_{s\ge 0}\{\xi s - C(s)\} = C^*(\xi).$$

We need the following

AUXILIARY LEMMA. *We have*

$$\sup_{\xi\ge 0}\{y\xi - C^*(\xi)\} = \sup_{\xi\ge 1}\{y\xi - C^*(\xi)\} \quad \textit{for } y \ge 2.$$

Proof. Set $\phi(x) = C'(x) = xe^x/(1+x)^2$. Thus

$$C(x) = \int_0^x \phi(u)\,du + 1, \qquad C^*(x) = \int_0^x \phi^{-1}(u)\,du - 1$$

(see [14], 1.10.11). Since ϕ^{-1} is increasing,

$$\frac{1}{1-\xi}\int_\xi^1 \phi^{-1}(u)\,du \le \phi^{-1}(1) \le y$$

because $1 \le 2e^2/9 = \phi(2)$. So

$$-\int_0^\xi \phi^{-1}(u)\,du \le (1-\xi)y - \int_0^1 \phi^{-1}(u)\,du$$

and hence $\sup_{0\le\xi\le 1}\{y\xi - C^*(\xi)\} \le y - C^*(1)$.

Thus for $y\ge 2$ we have $\sup_{\xi\ge 1}\{y\xi - C^*(\xi)\} = \sup_{\xi\ge 0}\{y\xi - C^*(\xi)\} = C^{**}(y) = C(y)$ (see [9], I.15, for instance).

Finally, for $x \ge e^3$ we have

$$\ln q(x) = \sup_{\xi\ge 0}\{\xi\ln(x) - \ln M(\xi)\}$$
$$= \max\{\sup_{0\le\xi\le 1}\{\xi\ln(x) - \ln M(\xi)\}, \sup_{\xi\ge 1}\{\xi\ln(x) - C^*(\xi)\}\}$$
$$= \max\left\{\sup_{0\le\xi\le 1}\{\xi\ln(x) - \ln M(\xi)\}, \frac{x}{1+\ln(x)}\right\} = \frac{x}{1+\ln(x)}.$$

COROLLARY 3.6. *The integral* $\int_0^\infty \frac{\ln \tau(x)}{1+x^2}\,dx$ *is divergent.*

Proof. By 3.4(d), it suffices to prove the assertion for $\ln q(x)$, and this follows from the previous lemma, since

$$\int_0^\infty \frac{\ln q(x)}{1+x^2}\,dx \ge \int_{e^3}^\infty \frac{\ln q(x)}{1+x^2}\,dx = \int_{e^3}^\infty \frac{x}{(1+\ln(x))(1+x^2)}\,dx$$
$$\ge \int_{e^3}^\infty \frac{x}{4x^2\ln(x)}\,dx = \int_{e^3}^\infty \frac{1}{4x\ln(x)}\,dx = \infty.$$

3.7. *A quasi-analytic Banach algebra of power series.* If we set $M_n = M(n)$, where M is the function of 3.1, then the class $C\{M_n\}$ of all $f \in C^\infty(\mathbb{R})$ such that $\|f^{(n)}\|_\infty \le k_f K_f^n M_n$ (for some constans k_f, K_f depending only of f) is a quasi-analytic class, by 3.2(4), 3.6 and the Denjoy–Carleman theorem (see [7], V.2.6 and V.2.4; or [10], 19.11).

Let now $a = (a_n)_{n\ge 0}$ be the sequence $a_n = \Omega(n)$, where Ω is the function of the previous section. The properties (a), (b) and (c) of Ω show that $B = A(a)$ is a Banach algebra of power series. We claim that $G(B) \subset C\{M_n\}$. In fact, we first observe that for every $x \in B$, the map $G(x)$ is C^∞: it is clear that each sequence $(n^k/a_n)_{n\ge 0}$ is bounded for $k\ge 0$. We now assume that $x = \sum_{n\ge 0} x_n t^n$ belongs to B; then for every $k\ge 0$ we have

$$\|G(x)^{(k)}\|_\infty \le \sum_{n\ge 0} |x_n| n^k \le \sum_{n\ge 0} |x_n| a_n \frac{n^k}{a_n}$$
$$\le \sum_{n\ge 0} |x_n| a_n \sup_{n\ge 1}\frac{n^k}{a_n} \le \|x\| M_k.$$

In other words we have shown

PROPOSITION 3.7.1. *The algebra B defined above is a quasi-analytic Banach algebra of power series.*

The following property will be used in the final construction (6.2 below).

PROPOSITION 3.7.2. *If $M \subset B$ is a maximal ideal, then $\bigcap_{n=1}^{\infty} M^n = 0$.*

Proof. If $M = M_z$, $|z| < 1$, then $x \in M^n$ for all n means that for all complex derivatives of $\widehat{x}$ we must have $\widehat{x}^{(n)}(z) = 0$. Since $\widehat{x}$ is holomorphic in $|z| < 1$, we obtain $x = 0$ (this argument holds for any power series Banach algebra $A(a)$ as a subalgebra of the disk algebra $\mathcal{A}(\Delta)$). Now, if $|z| = 1$, say $z = e^{is}$, then $x \in M^n$ implies that all the derivatives of order $k \leq n-1$ of the function $G(x)$ vanish at $s \in \mathbb{R}$. Hence in this case $x \in M^n$ for all n again implies $x = 0$ because $G(x) \in C\{M_n\}$.

4. Fréchet algebras of power series

4.1. *Submultiplicative sequences.* We assume that $W = \{a(p),\ p \geq 0\}$ is a sequence of submultiplicative sequences such that

(1) $a(p) \leq a(p+1)$ for each $p \geq 0$, i.e., $a_n(p) \leq a_n(p+1)$ for all $n, p \geq 0$,

(2) $\lim_{n\to\infty} a_n(p)^{1/n} = 1$ for each $p \geq 0$.

Then for each $p \geq 0$ we have a Banach algebra $A(a(p))$ of power series and continuous inclusion maps $A(a(p+1)) \subset A(a(p))$ for $p \geq 0$; this system produces a matrix algebra $F(W) = A(a_{p,n})$ of complex power series where $a_{p,n} = a_n(p)$ in the sense of [3]. Explicitly,

$$F(W) = \Big\{ \sum_{n=0}^{\infty} x_n t^n : \|x\|_p = \sum_{n=0}^{\infty} |x_n| a_n(p) < \infty \text{ for every } p \geq 0 \Big\}$$

with the topology defined by the family $\|\ \|_p$, $p \geq 0$, of norms. Note that t is a topological generator of $F(W)$, hence all characters of $F(W)$ are continuous ([1]); the character space of $F(W)$ can be identified with the closed disk $\Delta = \{z \in \mathbb{C} : |z| \leq 1\}$ as in the Banach algebra situation of §2. Therefore the set of units of $F(W)$ is open (see [6]) and hence every maximal ideal is closed. Since every Fréchet m-convex algebra which is a field is necessarily isomorphic to $\mathbb{C}$ (see [13]), every (closed) maximal ideal is the kernel of a continuous character. We write each character h_z of $F(W)$ in the form $\widehat{x}(z) = h_z(x) = \sum_{n=0}^{\infty} x_n z^n$, and denote by $M_z = \operatorname{Ker}(h_z)$ the associated maximal ideal. It follows that there is a continuous homomorphism $i : F(W) \to \mathcal{A}(\Delta)$ defined by $i(x)(z) = \widehat{x}(z)$ and $i(M_z) \subset \mathcal{M}_z$ for all $z \in \Delta$.

In what follows we assume the following additional properties:

(3) For each $p \geq 0$ there exists a constant C_p such that

$$(n+1)a_n(p) \leq C_p a_{n+1}(p+1) \qquad \text{for every } n \geq 0.$$

(4) Each sequence $a(p)$, $p \geq 0$, is increasing.

EXAMPLES 4.2. We give two concrete examples of sequences W having the properties 4.1(1)–(4):

(a) Set $a_0(p) = 1$ and $a_n(p) = (2n)^p$ for $n > 0$, $p \geq 0$. All the verifications are almost trivial (here we can take $C_p = 1$ for all p).

(b) Fix $\lambda \in (0,1)$ and define $a_n(p) = e^{pn^\lambda}$ for all $n, p \geq 0$. As before, all the verifications are easy, with the possible exception of 4.1(3); this can be obtained (with $C_p = (\lambda e)^{-\lambda^{-1}}$ for all p) from the inequality $n+1 \leq (\lambda e)^{-\lambda^{-1}} e^{(n+1)^\lambda}$. This inequality follows from elementary calculus applied to evaluate the maximum of the function $f(x) = xe^{-x^\lambda}$ for $x \geq 1$, attained at $x = \lambda^{-\lambda^{-1}}$.

REMARK 4.3. The above properties imply that for each $p \geq 0$ we have

$$\sum_{i=0}^{n} a_i(p) \leq C_p a_{n+1}(p+1).$$

In fact, $\sum_{i=0}^{n} a_i(p) \leq (n+1)a_n(p) \leq C_p a_{n+1}(p+1)$.

4.4. *Auxiliary operators.* Assuming 4.1(3), (4) we can define, for each $z \in \Delta$, a continuous linear operator $T_z : F(W) \to F(W)$ in three steps:

(i) Set $T_z(t^n) = \sum_{k+h=n-1} z^h t^k$ if $n \geq 1$, and $T_z(1) = 0$. Observe that if $n \geq 1$ then

$$\begin{aligned} \|T_z(t^n)\|_p &\leq \sum_{k+h=n-1} |z|^h a_k(p) \leq 1 + a_1(p) + \ldots + a_{n-1}(p) \\ &\leq C_p a_n(p+1) = C_p \|t^n\|_{p+1}, \end{aligned}$$

by 4.3; obviously this also holds for $n = 0$.

(ii) We now define T_z on the polynomials in t by linearity; in this case, the definition gives: if $x = \sum_{n=0}^{N} x_n t^n$ then $T_z(x) = \sum_{n=0}^{N} x_n T_z(t^n)$. Hence by (i) we have

$$\begin{aligned} \|T_z(x)\|_p &\leq \sum_{n=1}^{N} |x_n| C_p \|t^n\|_{p+1} = C_p \sum_{n=1}^{N} |x_n| a_n(p+1) \\ &\leq C_p \|x\|_{p+1}. \end{aligned}$$

(iii) The above inequalities show that T_z can be extended by continuity to a continuous linear map $A(a(p+1)) \to A(a(p))$ for all $p \geq 0$. This completes the definition of T_z.

An explicit formula for T_z is not necessary here, but we recall that after a rearrangement of the absolutely convergent series $T_z(x) = \sum_{n\geq 0} x_n T_z(t^n)$

we obtain without difficulty

$$T_z(x) = \sum_{n\geq 0}\Big(\sum_{j\geq n+1} x_j z^{j-(n+1)}\Big)t^n.$$

We state the following properties of these operators:

LEMMA 4.4.1. *For every $x \in F(W)$ and every $z \in \Delta$,*

(i) $$x = (t-z)T_z(x) + h_z(x),$$

(ii) $$x = \sum_{k=0}^{n-1} h_z(T_z^k(x))(t-z)^k + T_z^n(x)(t-z)^n \qquad (n \geq 1).$$

Proof. (i) This is clear when $x = t^n, n \geq 0$. Hence by linearity the equality holds when x is a polynomial in t. Finally by continuity the result follows for all $x \in F(W)$.

(ii) By induction,

$$\begin{aligned} x &= \sum_{k=0}^{n-1} h_z(T_z^k(x))(t-z)^k + T_z^n(x)(t-z)^n \\ &= \sum_{k=0}^{n-1} h_z(T_z^k(x))(t-z)^k + (h_z(T_z^n(x)) + T_z^{n+1}(x)(t-z))(t-z)^n \\ &= \sum_{k=0}^{n} h_z(T_z^k(x))(t-z)^k + T_z^{n+1}(x)(t-z)^{n+1}. \end{aligned}$$

PROPOSITION 4.5. *For every maximal ideal M_z in $F(W)$, M_z^n is principal, generated by $(t-z)^n$.*

Proof. It is enough to show that M_z is generated by $t-z$, and this follows quickly from 4.4.1(i).

PROPOSITION 4.6. *Let P be a prime ideal in $F(W)$; then either*

(a) *P is a maximal ideal, or else*
(b) *for some $z \in \Delta$, we have $P \subset \bigcap_{n=1}^{\infty} M_z^n$.*

Proof. Clearly $P \subset M_z$ for some $z \in \Delta$. Supose that $x \in P$ and $P \neq M_z$; we assert that if $x = (t-z)^n y_n$, then $x = (t-z)^{n+1}y_{n+1}$. In fact, if $x = (t-z)^n y_n$, then either $t-z \in P$ (hence $M_z = P$) or else $y_n \in P$. In the latter case we obtain $y_n = (t-z)y_{n+1}$ and hence $x = (t-z)^{n+1}y_{n+1}$. Finally, if $x \in P \subsetneq M_z$, $x = (t-z)y$ then $x \in \bigcap_{n=1}^{\infty} M_z^n$ by 4.5. Note that by 4.4.1(ii), $y_n = T^n(x)$ and $y_{n+1} = T(y_n) = T^{n+1}(x)$.

REMARK 4.7. Recall from 4.1 that $i(M_z) \subset \mathcal{M}_z$. Hence if $|z| < 1$ then necessarily $P = 0$ in (b) above.

5. Algebraic lemmas. In the construction of examples we use some characterizations of a principal ideal domain by properties of prime ideals. For the reader's convenience we include the proof of the following results (which are known).

LEMMA 5.1. *Let A be a commutative $\mathbb{C}$-algebra with identity, and $\mathcal{N}$ be the set of non-principal ideals of A, ordered by inclusion. Then:*

(i) *$\mathcal{N}$ is inductive.*
(ii) *If A is an integral domain, then every maximal element $I \in \mathcal{N}$ is a prime ideal.*

Proof. (i) Routine. (ii) Assume that $ab \in I$, and $a, b \notin I$. The ideal $I + (a)$ is proper: otherwise we can write $1 = y + xa$ with $y \in I$ and $x \in A$, hence $b = by + x(ab) \in I$, a contradiction. Then by the maximality of I, the ideal $I + (a)$ is principal, with a generator c, and we have $c = y + xa$, $y \in I$, $x \in A$, and also $y = uc, a = vc$, for u and v in A; note that $1 = u + xv$.

Clearly $(c) = (y) + (a)$. On the other hand we can see that $I = (y) + Ja$ where $J = \{z \in A : za \in I\}$; in fact, if $y' \in I \subset Ac$ then $y' = a_1 y + a_2 a$, which implies $a_2 a \in I$, hence $a_2 \in J$. Then we have $I \subset (y) + Ja$, and the opposite inclusion is trivial.

Note that $I \subset J$ and $I \neq J$ (since $b \in J, b \notin I$); also $A \neq J$ (since $a \notin I$). Hence J is principal, with a generator g. Since $I = Ay + Ja$ it follows that $I = (y) + Aag = (y) + Ja$.

Finally, let H be the ideal generated by $\{u, gv\}$; we claim that $I \subset H$: first observe that $y = cu \in H$, and then $ag = cvg \in H$. But $I \neq H$, since $g = gu + xgv \in H$, $g \notin I$. Hence the ideal H is principal, generated by an element $s \in A$. Now we have $(sc) = Hc = \{uc, gvc\} = \{y, ag\} = (y) + Ja = I$. This gives a contradiction.

LEMMA 5.2. *Let A be a $\mathbb{C}$-algebra which is an integral domain such that every prime ideal in A is principal. Then A is a principal ideal domain.*

Proof. If there exist non-principal ideals in A, let I be a maximal element of the set $\mathcal{N}$ of non-principal ideals of A. By 5.1(ii), I is prime, hence principal, which is impossible.

COROLLARY 5.3. *Let A be a $\mathbb{C}$-algebra which is an integral domain such that*

(i) *every non-null prime ideal is maximal, and*
(ii) *every maximal ideal is principal.*

Then A is a principal ideal domain.

6. Example of a Fréchet m-convex principal ideal domain. Let $W = (a(p))_{p\geq 0}$ be a sequence of submultiplicative sequences with the properties 4.1(3), (4) (see for instance 4.2) and let $a = (a_n)_{n\geq 0}$ be the submultiplicative sequence $a_n = \Omega(n)$ of 3.7. We define $\widetilde{a}_n(p) = a_n a_n(p)$ for each $n, p \geq 0$; more explicitly $\widetilde{a}_0(p) = 1$, and $\widetilde{a}_n(p) = e^{n/(1+\ln(n))} a_n(p)$ for all $n, p \geq 0$.

LEMMA 6.1. *The sequence* $\widetilde{W} = (\widetilde{a}(p))_{p\geq 0}$ *has the following properties*:

(a) *Each sequence* $\widetilde{a}(p)$ *is submultiplicative.*

(b) $\widetilde{W}$ *has the properties* 4.1(1)–(4).

(c) *For each* $p \geq 0$ *we have* $a_n \leq \widetilde{a}_n(p)$ *for all* $n \geq 0$.

Proof. Routine verification.

We denote by B (as in 3.7) the Banach algebra $A(a)$ of power series; observe that 6.1(c) gives an inclusion $j : F(\widetilde{W}) \to B$.

Now we consider the algebra $F(\widetilde{W})$, which is a Fréchet m-convex algebra with character space Δ by 4.1(2). On the other hand, we have the following

THEOREM 6.2. $F(\widetilde{W})$ *is a principal ideal domain.*

Proof. We apply 5.3; in fact, every maximal ideal in $F(\widetilde{W})$ is principal by 6.1(b) and 4.5. Also every prime ideal $P \subset F(\widetilde{W})$ is either maximal or null: this is clear when $P \subset \widetilde{M}_z$, $|z| < 1$, and when $|z| = 1$ we have $j(\widetilde{M}_z) \subset M_z \subset B$, hence 4.6(b) gives

$$j(P) \subset j\Big(\bigcap_{n=1}^{\infty} \widetilde{M}_z^n\Big) \subset \bigcap_{n=1}^{\infty} M_z^n = 0$$

by 3.7.2.

We have thus constructed the promised example; in fact, a collection of examples can be obtained according to the selection of sequences W of §4.

REMARK 6.3. *The algebra* $F(\widetilde{W})$ *is also a Montel space.*

Proof. For each $p \geq 0$ we have the linear isometry $j_p : \ell^1 \to A(a(p))$ given by $j_p(\sum_{n\geq 0} x_n t^n) = \sum_{n\geq 0} x_n a_n(p)^{-1} t^n$, and the inclusion map $i : A(a(p+1)) \to A(a(p))$ gives a map $\Phi : \ell^1 \to \ell^1$ by the rule $i \circ j_{p+1} = j_p \circ \Phi$. Now the property 4.1(3) and the submultiplicativity of the sequence $a(p)$ imply that Φ is compact, because it is the transpose of the map $\Psi : c_0 \to c_0$ given by $\Psi(s)_n = s_n a_n(p) a_n(p+1)^{-1}$, which is compact. Indeed, if we define a sequence of finite rank maps $\Psi_n : c_0 \to c_0$ by $\Psi_n(s)_k = s_k a_k(p) a_k(p+1)^{-1}$ when $k \leq n$, and 0 otherwise, then

$$\|\Psi_n - \Psi\| \leq \frac{C_p a_1(p)}{n+1},$$

hence Ψ_n converges to Ψ in norm, and so Ψ is compact. Finally $\Phi = \Psi^*$, and also the inclusion map $A(a(p+1)) \to A(a(p))$ is compact.

Now, it follows from [8] that $F(\widetilde{W})$ is also a Montel space.

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