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Partial retractions for weighted Hardy spaces

by

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Abstract. Let $1 \le p \le \infty$ and let w_0, w_1 be two weights on the unit circle such that $\log(w_0 w_1^{-1}) \in \mathrm{BMO}$. We prove that the couple $(H_p(w_0), H_p(w_1))$ of weighted Hardy spaces is a partial retract of $(L_p(w_0), L_p(w_1))$. This completes previous work of the authors. More generally, we have a similar result for finite families of weighted Hardy spaces. We include some applications to interpolation.

For $1 < p < \infty$ we can project the space $L_p(\mathbb{T})$ onto the (boundary) Hardy class H_p. This can be done by an operator independent of p, for instance, by the Riesz projection. The extreme indices $p = 1$ and $p = \infty$ cannot be included.

Though regret can hardly be allowed in connection with a true mathematical statement, the latter assertion (about the extreme indices) may evoke a sort of this feeling in some situations. The following fact proved in [13] (see [11] for a simple argument) can sometimes serve as a remedy.

For every $f \in H_1 + H_\infty$ $(= H_1)$ there is a linear operator fixing f and mapping boundedly L_1 to H_1 and L_∞ to H_∞ with norms not exceeding a universal constant.

Later, this result was extended to weighted Hardy spaces. By a *weight* we mean a nonnegative measurable function w on $\mathbb{T}$ such that $\log w \in L_1$. We put $L_p(w) = L_p(\mathbb{T}, w\,dm)$ (m is normalized Lebesgue measure on $\mathbb{T}$), and

$$L_\infty(w) = \{f : fw^{-1} \in L_\infty\}$$

equipped with the natural norm $\|f\|_{\infty,w} = \|fw^{-1}\|_\infty$. Next, let φ be an outer function satisfying $|\varphi| = w$ a.e. on $\mathbb{T}$. We introduce the weighted Hardy space $H_p(w)$, $0 < p \le \infty$, by

$$H_p(w) = \{f : f\varphi^{1/p} \in H_p\}, \qquad 0 < p < \infty,$$
$$H_\infty(w) = \{f : f\varphi^{-1} \in H_\infty\}.$$

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THEOREM A. *Let w be a weight satisfying* $\log w \in \mathrm{BMO}$. *For every $f \in H_1(w) + H_\infty$ there exists an operator $T : L_1(w) + L_\infty \to H_1(w) + H_\infty$ such that $Tf = f$, T maps $L_1(w)$ to $H_1(w)$ and L_∞ to H_∞, and the norms $\|T\|_{L_1(w)\to H_1(w)}$, $\|T\|_{L_\infty \to H_\infty}$ are controlled in terms of $\|\log w\|_{\mathrm{BMO}}$.*

This theorem was announced in [10]; the proof appeared in [14] (see [8] for simplifications). It should be noted that a detail was omitted intentionally both in [14] and in [8] (we shall return to this later). Also, Theorem 2.1 in [10] implies that, unless $\log w \in \mathrm{BMO}$, the statement of Theorem A fails. Throughout, we use the following norm on the space BMO:

$$\|g\|_{\mathrm{BMO}} = \inf\{\|\varphi\|_{L_\infty/\mathbb{C}} + \|\psi\|_{L^\infty/\mathbb{C}} : g = \varphi + \mathcal{H}\psi\},$$

where $\mathcal{H}$ is the harmonic conjugation operator.

Theorem A is directly related to the notion of a partial retract in interpolation theory. (We refer the reader to [1, 2] for standard notions and facts of this theory.) Let $(X_0, \ldots, X_N)$ and $(Y_0, \ldots, Y_N)$ be two families of compatible (quasi-)Banach spaces. Then $(Y_0, \ldots, Y_N)$ is called a *partial retract* of $(X_0, \ldots, X_N)$ if for every $y \in Y_0 + \ldots + Y_N$ there are two operators $T : X_0 + \ldots + X_N \to Y_0 + \ldots + Y_N$ and $S : Y_0 + \ldots + Y_N \to X_0 + \ldots + X_N$ such that $TSy = y$ and T (respectively, S) is bounded from X_i to Y_i (respectively, from Y_i to X_i) for $i = 0, \ldots, N$ with norms majorized by a constant independent of y. If, moreover, Y_i is a subspace of X_i for $i = 0, \ldots, N$ and the identity inclusion can always be taken as S, then we say that $(Y_0, \ldots, Y_N)$ is a *retractive subfamily* of $(X_0, \ldots, X_N)$. Thus, Theorem A asserts that $(H_1(w), H_\infty)$ is a *retractive subcouple* of $(L_1(w), L_\infty)$ if $\log w \in \mathrm{BMO}$.

It is known that if $(Y_0, \ldots, Y_N)$ is a partial retract of $(X_0, \ldots, X_N)$, then all interpolation properties of the latter $(N + 1)$-tuple are inherited by the former. This statement can be made precise if we use the notion of interpolation functors. For the case of couples, see, e.g., [8], Corollary 2.1, for the (easy) details, on which we do not dwell here. Partly, the statement is justified also by the applications of Theorem 1 that are given below.

Now, let X be a quasi-Banach lattice of measurable functions on $\mathbb{T}$. This means that X is complete in its quasinorm $\|\cdot\|$ and the conditions $x \in X$, $|y| \le x$ together with the measurability of y imply that $y \in X$ and $\|y\| \le C\|x\|$. We define the analytic subspace X_A of X as the intersection of X with the set of boundary functions for the Smirnov class. Some mild restrictions on X, which we do not mention, are needed to avoid degeneration (see, e.g., [7, 8]). Very often we have $X_A = X \cap H_r$ for small $r > 0$. In general, X_A is related to X nearly as H_p to L_p. If w is a weight, then $(L_p(w))_A = H_p(w)$.

In recent time, a good deal of work has been done towards showing that for many lattices X, Y of measurable functions the interpolation properties of the couple (X_A, Y_A) are the same as those of (X, Y). See, e.g., [7], or the survey [8] and the references therein. Of course, every time we aim at proving something of this sort, it would be ideal to verify that (X_A, Y_A) is a retractive subcouple of (X, Y). However, some problems arise in connection with the latter statement even for weighted Hardy spaces.

If we want to extend Theorem A, the first question to be asked is about the couples $(L_{p_0}(w_0), L_{p_1}(w_1))$ with $p_0, p_1 \in [1, \infty)$ under the necessary (see [10]) condition $\log(w_0^{1/p_0}/w_1^{1/p_1}) \in \mathrm{BMO}$. Partial retractions can easily be constructed if $p_0 \ne p_1$. Indeed (see [14]), multiplying everything by an appropriate outer function, we reduce the problem to the case where $w_0 = w_1 := w$ (but this reduction fails if $p_0 = p_1$). Then we observe that the operator T of Theorem A maps $L_{p_0}(w)$ to $H_{p_0}(w)$ and $L_{p_1}(w)$ to $H_{p_1}(w)$ by interpolation.

Until recently, we did not know what happens if $p_0 = p_1 < \infty$, nor did we know the situation of the couple $(L^\infty(w_0), L^\infty(w_1))$ (here the necessary condition is $\log(w_0/w_1) \in \mathrm{BMO}$). The following theorem fills this gap.

THEOREM 1. *Let $1 \le p \le \infty$, and let $w_0, \ldots, w_N$ be $N + 1$ weights on $\mathbb{T}$ such that $\log(w_i w_j^{-1}) \in \mathrm{BMO}$ for all $i, j = 0, \ldots, N$. Then $(H_p(w_0), \ldots, H_p(w_N))$ is a retractive subfamily of $(L_p(w_0), \ldots, L_p(w_N))$.*

REMARKS. (i) Again, the condition $\log(w_i w_j^{-1}) \in \mathrm{BMO}$ is necessary (see [5], [10]).

(ii) Clearly, an analogue of Theorem 1 (and of Theorem A) is true for weighted Hardy spaces on a Smirnov domain G for which the conformal mapping φ of D onto G satisfies $\log|\varphi'| \in \mathrm{BMO}$. The Hardy spaces in a halfplane constitute quite a particular case of the situation described.

(iii) In the spirit of the discussion at the beginning of the paper, we remark that only the cases of $p = 1$ and $p = \infty$ are really new in Theorem 1, because in [9] it was proved that if $\log w_i \in \mathrm{BMO}$ $(i = 0, \ldots, N)$, then there is an operator R that *projects* $L_p(w_i)$ onto $H_p(w_i)$ for $i = 0, \ldots, N$ and for all $p \in (1, \infty)$ at once. However, the proof of Theorem 1 depends little on a particular value of p.

The partial retractions in Theorem A and in Theorem 1 are defined differently. The reason (probably, well-hidden behind the formalism) is that the specific formulas for the K-functionals of the pairs (L_1, L_∞) and $(L_p(w_0), L_p(w_1))$ differ in principle (in the second case we must "truncate" the weights rather than the function at which the K-functional is calculated). Combining the methods leading to Theorem A and Theorem 1, we are able to prove another extension of Theorem A.

THEOREM 2. *Let $w_1, \ldots, w_N$ be weights on $\mathbb{T}$ such that $\log w_i \in \mathrm{BMO}$, $i = 1, \ldots, N$. Then $(H_\infty, H_1(w_1), \ldots, H_1(w_N))$ is a retractive subfamily of $(L_\infty, L_1(w_1), \ldots, L_1(w_N))$.*

REMARK. In the spirit of Theorem 1, we can replace here L_∞ by $L_\infty(w_0)$ and H_∞ by $H_\infty(w_0)$; then the condition on the weights becomes $\log w_0 w_i \in BMO$, $i = 1, \ldots, N$. A change of density shows the equivalence of these two versions.

We feel that Theorems 1 and 2 are interesting in themselves, though, probably, the proof of the second is a bit too complicated against the background of the nonultimate nature of the statement (for instance, we do not know what happens in the framework of Theorem 2 if several weighted L_∞-spaces are involved). But we want to explain that these two facts are more than mere curiosities.

First, Theorem 1 (especially the case of $p = \infty$) can be used as quite a convenient technical tool, which simplifies some known constructions considerably and makes them more natural. (We note that, by itself, Theorem 1 is a relatively simple statement.) Below we give two examples of this sort. Second, we treat $(N+1)$-tuples instead of couples not merely because our methods allow us to do so, but also because, *even interpolating between a couple of spaces, technically it may be convenient to have partial retractions for triplets.* An illustration of the latter can be found in the proof of Corollary 2 below. Another one is in the relationship between Theorem A and Theorem 2.

Both in [14] and in [8], Theorem A was proved under some additional assumptions about f. Though that restricted version of Theorem A suffices for the major part of applications, it is desirable to have the result as stated above. However, doing without those assumptions seemed to require a cumbersome and rather ugly limit procedure, which was never published.

In the framework of Theorem 2, a similar problem also arises. Technically, it is convenient to assume that $|f|^{1/2}\chi_{\{|f|>\lambda\}} \in L_1(w_1 \wedge \ldots \wedge w_N)$ (where f is a function which is going to be a fixed point of the operator T in the definition of a retractive subfamily). However, here this problem is settled in a "more regular" way: if we mean *all* values of N, the above condition is not a restriction. Indeed, we can always enlarge the collection of weights so as to ensure the integrability of $w_1 \wedge \ldots \wedge w_N \wedge w_{N+1}$ (say, by taking $w_{N+1} \equiv 1$). Then the above condition is fulfilled for every $f \in L_1(w_1) + \ldots + L_1(w_{N+1}) + L_\infty$.

Thus, the proof of Theorem 2 for $N = 2$ presented below is, apparently, the only *complete* proof of Theorem A available in writing.

We pass to the applications of Theorem 1 promised above. They pertain to the collections $(X_{0,A}, \ldots, X_{N,A})$ for general Banach lattices $X_0, \ldots, X_N$ of measurable functions.

A quasi-Banach lattice X of measurable functions on $\mathbb{T}$ is said to be BMO-*regular* if for every $x \in X$ there exists $w \in X$ with $|w| \ge x$, $\|w\| \le C\|x\|$, and $\|\log w\|_{\mathrm{BMO}} \le C$, where $C > 0$ is a constant depending only on X. This w is called a BMO-*majorant* of x. See [7–8] for a discussion of BMO-regular lattices. In particular, such are $L_p(w)$ if $0 < p \le \infty$ and $\log w \in \mathrm{BMO}$, and also all lattices X such that the Hilbert transformation is a bounded operator on X^α for some $\alpha > 0$.

It is easily seen that if X is BMO-regular, then X embeds in L_r for some $r > 0$. We shall assume that this embedding is continuous. It should be mentioned that $X_A = X \cap H_r$ with this r.

Next, let $(X_0, \ldots, X_N)$ be a collection of compatible Banach spaces, and let Y_i be a closed subspace of X_i, $i = 0, \ldots, N$. We say that the $(N+1)$-tuple $(Y_0, \ldots, Y_N)$ is K-*closed* in $(X_0, \ldots, X_N)$ if every decomposition $y = x_0 + \ldots + x_N$ of a vector $y \in Y_0 + \ldots + Y_N$ can be modified to $y = y_0 + \ldots + y_N$ where $y_i \in Y_i$ and $\|y_i\| \le C\|x_i\|$ (C is independent of the vectors involved).

This notion is well known for couples (see, e.g., [12], [6] and the survey [8]), in which case it is intimately related to K-functionals (whence the term has come). If K-closedness occurs, then (apparently) all $(N+1)$-tuple real interpolation theories that may ever arise (see, e.g., [3]) can easily be carried from $(X_0, \ldots, X_N)$ over to $(Y_0, \ldots, Y_N)$.

COROLLARY 1. *If $X_0, \ldots, X_N$ are* BMO-*regular quasi-Banach lattices, then the $(N+1)$-tuple $(X_{0,A}, \ldots, X_{N,A})$ is K-closed in $(X_0, \ldots, X_N)$.*

Proof. Let $f \in X_{0,A} + \ldots + X_{N,A}$ be represented as $f = g_0 + \ldots + g_N$ with $g_i \in X_i$, $i = 0, \ldots N$. We fix a BMO-majorant w_i for g_i in X_i ($i = 0, \ldots, N$) and treat this representation as a decomposition in the sum $L_\infty(w_0) + \ldots + L_\infty(w_1)$ of the corresponding weighted L_∞-spaces. It remains to apply Theorem 1 for $p = \infty$, obtaining an operator $T : L_\infty(w_i) \to H_\infty(w_i)$ ($i = 0, \ldots, N$) that fixes f, etc. ∎

REMARK. In fact, the proof gives $f_i \in X_{i,A}$ such that $f = f_0 + \ldots + f_N$ and $|f_i| \le c w_i$ ($i = 0, \ldots, N$). For couples, this was proved in [8] in a different way. It is possible to do without Theorem 1 also for $(N+1)$-tuples, but the argument will be far less compact.

The next corollary is stated for couples for simplicity, but the same argument is applicable to finite collections of spaces, in the framework of the complex interpolation theory presented in [4].

COROLLARY 2. *Let X_0, X_1 be* BMO-*regular Banach lattices, and let $0 < \theta < 1$. If the norm of the lattice $X_0^{1-\theta} X_1^\theta$ is order absolutely continuous, then we have the following formula for complex interpolation spaces*:

$$(X_{0,A}, X_{1,A})_\theta = ((X_0, X_1)_\theta)_A.$$

This statement first appeared in [8]. The short argument that follows (note the use of triplets in it) should be compared with the bulky proof in that paper. (The first author wishes he had had Theorem 1 at the time of writing the paper [8]...)

P r o o f. Under the conditions of Corollary 2, we have

$$(X_0, X_1)_\theta = X_0^{1-\theta} X_1^\theta \quad \text{with equality of norms.}$$

We will denote the latter space by Z. Only the inclusion $Z_A \subset (X_{0,A}, X_{1,A})_\theta$ requires a proof. We take a norm-one function x in Z_A and find positive $x_i \in X_i$ such that $\|x_i\| = 1$ and $|x| \le 2x_0^{1-\theta} x_1^\theta$. Then, for each i, we fix a BMO-majorant w_i for x_i in the space X_i, and consider the following triplet of spaces:

$$L_\infty(w_0), \quad L_\infty(w_1), \quad L_\infty(w_0^{1-\theta} w_1^\theta).$$

By Theorem 1, there exists an operator T that fixes x and maps $L_\infty(w_i)$ to $H_\infty(w_i)$ $(i = 0, 1)$ and $L_\infty(w_0^{1-\theta} w_1^\theta)$ to $H_\infty(w_0^{1-\theta} w_1^\theta)$ with all relevant norms bounded by a constant that depends eventually on the initial lattices X_i $(i = 0, 1)$ only. Let

$$e_\varepsilon = \{\varepsilon \le w_i \le \varepsilon^{-1},\ i = 0, 1\}, \quad \varepsilon > 0,$$

and let $y_\varepsilon = x\chi_{e_\varepsilon}$. From the specific formula for T given in the proof of Theorem 1 below (namely, from the weak* continuity of the $\xi_{n,k}$'s), it follows that $Ty_\varepsilon \to Tx = x$ a.e. as $\varepsilon \to 0$. Since T is bounded on $L_\infty(w_0^{1-\theta} w_1^\theta)$, we see that

$$|Ty_\varepsilon| \le C w_0^{1-\theta} w_1^\theta$$

uniformly in ε. Thus, since the norm of Z is order absolutely continuous, $Ty_\varepsilon \to x$ in the norm of Z. On the other hand, $y_\varepsilon \in L_\infty(w_0) \cap L_\infty(w_1)$. Consequently,

$$\|y_\varepsilon\|_{(L_\infty(w_0), L_\infty(w_1))_\theta} \le 2.$$

Thus by interpolation,

$$\|Ty_\varepsilon\|_{(H_\infty(w_0), H_\infty(w_1))_\theta} \le C,$$

and, finally,

$$\|Ty_\varepsilon\|_{(X_{0,A}, X_{1,A})_\theta} \le C.$$

We have proved that every norm-one element of Z_A can be approximated in this space by an element of the C-ball of $(X_{0,A}, X_{1,A})_\theta$, within any given accuracy. In a standard way, this implies the inclusion announced at the beginning of the proof. ■

It is routine to write out more applications of Theorems 1 and 2 (see [13] and [14] for some relevant patterns).

Now, we pass to the proofs of our main results.

In the sequel C will denote a positive constant which is independent of particular functions but may depend on the index p and on the BMO norms of the logarithms of the weights under consideration.

The following lemma is taken from [9] (see the arguments preceding formula (6) in that paper).

LEMMA 1. *Let w be a weight such that $\log w \in \mathrm{BMO}$. Then there exists a sequence $\{\varphi_n\}_{n\in\mathbb{Z}}$ of functions in H_∞ satisfying*

$$|\varphi_n| \le C \min\{(w/2^n)^8,\ (2^n/w)^4\}, \quad \forall n \in \mathbb{Z}, \tag{1}$$

$$\sum_{n\in\mathbb{Z}} \varphi_n = 1. \tag{2}$$

Note that by (1) we have

$$|\varphi_n| \le C, \quad \forall n \in \mathbb{Z}. \tag{3}$$

Also note that (1) implies

$$\sum_{n\in\mathbb{Z}} |\varphi_n|^{1/4} \le C. \tag{4}$$

Indeed, let $e_k = \{2^k \le w < 2^{k+1}\}$, $k \in \mathbb{Z}$. Then from (3) and (1) we deduce that

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} |\varphi_n|^{1/4} &= \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\varphi_n|^{1/4} \chi_{e_k} \\
&\le C \sum_{n\in\mathbb{Z}} \Big(\sum_{k\le n} 2^{-n} 2^k \chi_{e_k} + \sum_{k>n} 2^n 2^{-k} \chi_{e_k} \Big) \\
&= C \sum_{k\in\mathbb{Z}} \Big(2^k \sum_{n\ge k} 2^{-n} + 2^{-k} \sum_{n<k} 2^n \Big) \chi_{e_k} \\
&\le C \sum_{k\in\mathbb{Z}} \chi_{e_k} = C.
\end{aligned}$$

REMARK. In support of our claim that Theorem 1 is a convenient tool in the study of interpolation of Hardy-type subspaces, we note that the entire way to Lemma 1 from total ignorance is reasonably short. As shown in [9], this lemma is not difficult modulo the K-closedness property for the couple $(H_1(w_0), H_1(w_1))$ provided that $\log w_0, \log w_1 \in \mathrm{BMO}$. The proof of the latter involves a trick, but otherwise is easy (see [10] and also [8] for a simplification). If Lemma 1 is known, the proof of Theorem 1 is not difficult either, as will be shown immediately.

Proof of Theorem 1. Multiplying by an appropriate outer function, we reduce the theorem to the case where $w_0 = 1$. Then the condition on the weights turns into $\log w_i \in \mathrm{BMO}$, $i = 1, \ldots, N$. For notational simplicity, we will only consider the case of $N = 2$, the general one being treated similarly.

Let $\{\varphi_n\}_{n\in\mathbb{Z}}$ (resp. $\{\psi_n\}_{n\in\mathbb{Z}}$) be the sequence obtained in Lemma 1 with $w = w_1$ (resp. $w = w_2$). We fix a function $f \in H_p + H_p(w_1) + H_p(w_2)$. Given any $n, k \in \mathbb{Z}$, we choose a positive linear functional $\xi_{n,k} \in (L_p)^*$ of norm at most 1 such that

$$\xi_{n,k}(|f|\cdot|\varphi_n|^{1/2}|\psi_k|^{1/2}) = 2^{-1}\||f|\cdot|\varphi_n|^{1/2}|\psi_k|^{1/2}\|_{L_p}. \tag{5}$$

If $p=\infty$, we also require that $\xi_{n,k}$ be w^*-continuous. This is needed, for instance, in the proof of Corollary 2. Now for any $g\in L_p+L_p(w_1)+L_p(w_2)$, we define

$$T_{n,k}(g) = \frac{\xi_{n,k}(\operatorname{sgn}(\bar f)g|\varphi_n|^{1/2}|\psi_k|^{1/2})}{2^{-1}\||f|\cdot|\varphi_n|^{1/2}|\psi_k|^{1/2}\|_{L_p}}, \quad \forall n,k\in\mathbb{Z},$$

and

$$T(g) = \sum_{n,k\in\mathbb{Z}} T_{n,k}(g)\varphi_n\psi_k f.$$

We are going to check that this operator T satisfies all requirements of Theorem 1. First, observe that by (2) (applied to both $\{\varphi_n\}$ and $\{\psi_n\}$) and (5), we see that $Tf=f$. Then we must check the boundedness of T from $L_p(w_i)$ into $H_p(w_i)$. Note that since the terms in the sum defining Tg are all analytic functions, if this sum converges in $L_p(w_i)$ (with respect to the w^*-topology in the case of $p=\infty$), Tg must belong to $H_p(w_i)$.

We first consider the case where $p<\infty$. Fix $g\in L_p+L_p(w_1)+L_p(w_2)$. By Hölder's inequality, (3), and (4), we have

$$|Tg|^p \le C\sum_{n,k\in\mathbb{Z}} |T_{n,k}(g)|^p|\varphi_n|^{3p/4}|\psi_k|^{3p/4}|f|^p. \tag{6}$$

Thus if $g\in L_p$, then

$$\begin{aligned}\|Tg\|_{L_p}^p &\le C\sum_{n,k\in\mathbb{Z}} |T_{n,k}(g)|^p\||\varphi_n|^{1/2}|\psi_k|^{1/2}|f|\|_{L_p}^p\\ &\le C\sum_{n,k\in\mathbb{Z}} |\xi_{n,k}(\operatorname{sgn}(\bar f)g|\varphi_n|^{1/2}|\psi_k|^{1/2})|^p\\ &\le C\sum_{n,k\in\mathbb{Z}}\int_{\mathbb{T}}(|g|\cdot|\varphi_n|^{1/2}|\psi_k|^{1/2})^p\\ &\le C\|g\|_{L_p}^p \quad \text{(by (4))}.\end{aligned}$$

Therefore, T maps L_p into H_p and $\|T: L_p\to H_p\|\le C$. Now let $g\in L_p(w_1)$. By (3) and (1), we get

$$|\varphi_n|^{p/4}w_1 \le C|\varphi_n|^{1/4}w_1 \le C2^n;$$

combining this with (6) yields

$$\begin{aligned}\|Tg\|_{L_p(w_1)}^p &\le C\sum_{n,k\in\mathbb{Z}} |T_{n,k}(g)|^p 2^n\int_{\mathbb{T}}|\varphi_n|^{p/2}|\psi_k|^{p/2}|f|^p\\ &\le C\sum_{n,k\in\mathbb{Z}}\int_{\mathbb{T}}|g|^p(2^n|\varphi_n|^{p/4})|\varphi_n|^{p/4}|\psi_k|^{p/2};\end{aligned}$$

however, again by (3) and (1),

$$2^n|\varphi_n|^{p/4} \le C2^n|\varphi_n|^{1/8} \le Cw_1.$$

Therefore,

$$\begin{aligned}\|Tg\|_{L_p(w_1)}^p &\le C\sum_{n,k\in\mathbb{Z}}\int_{\mathbb{T}}|g|^p w_1|\varphi_n|^{p/4}|\psi_k|^{p/2}\\ &= \int_{\mathbb{T}}|g|^p w_1\sum_{n,k\in\mathbb{Z}}|\varphi_n|^{p/4}|\psi_k|^{p/2}\\ &\le C\|g\|_{L_p(w_1)}^p \quad \text{(by (4))};\end{aligned}$$

so $T: L_p(w_1)\to H_p(w_1)$ is bounded and of norm $\le C$. Similarly, changing the roles of w_1 and w_2, we prove the boundedness of T from $L_p(w_2)$ to $H_p(w_2)$.

The case of $p=\infty$ is dealt with in a similar (and slightly easier) way. Indeed, let $g\in L_\infty$. Then by (3) and (4),

$$\begin{aligned}|Tg| &\le C\sum_{n,k\in\mathbb{Z}}|T_{n,k}(g)|\cdot|\varphi_n|\cdot|\psi_k|\cdot|f|\\ &= C\sum_{n,k\in\mathbb{Z}}|\xi_{n,k}(\operatorname{sgn}(\bar f)g|\varphi_n|^{1/2}|\psi_k|^{1/2})|\cdot|\varphi_n|^{1/2}|\psi_k|^{1/2}\\ &\le C\sum_{n,k\in\mathbb{Z}}\|g\|_{L_\infty}|\varphi_n|^{1/2}|\psi_k|^{1/2}\\ &\le C\|g\|_{L_\infty} \quad \text{(by (4))}.\end{aligned}$$

Thus T maps L_∞ into H_∞ and is of norm $\le C$. Much as in the above case of $p<\infty$, we check that T is simultaneously bounded from $L_\infty(w_i)$ to $H_\infty(w_i)$ for $i=1,2$. Therefore, the proof of Theorem 1 is complete. ■

Proof of Theorem 2. In a sense, the argument is a combination of the proof of Theorem 1 and a proof presented in [11] (see also [8]). We shall construct the required partial retraction for functions $f\in H_\infty+H_1(w_1)+\ldots+H_1(w_N)$ satisfying the additional condition $|f|^{1/2}\chi_{\{|f|>\lambda\}}\in L_1(w_1\wedge\ldots\wedge w_N)$ for all $\lambda>0$. It has already been explained that, really, this is not a restriction. Again, we only consider the typical case of $N=2$ (see, however, additional hints at the end of the proof).

By Lemma 1 (applied to $w=w_1/w_2$), we choose a sequence $\{\psi_k\}_{k\in\mathbb{Z}}\subset H_\infty$ such that

$$|\psi_k| \le C\min\left\{\left(\frac{2^k w_2}{w_1}\right)^8, \left(\frac{w_1}{2^k w_2}\right)^4\right\}, \quad k\in\mathbb{Z}, \tag{7}$$

$$\sum_{k\in\mathbb{Z}}\psi_k = 1. \tag{8}$$

We put

$$a_k = \min\left\{\left(\frac{2^k w_2}{w_1}\right)^8, \left(\frac{w_1}{2^k w_2}\right)^4\right\}, \quad k \in \mathbb{Z}. \tag{9}$$

Then

$$\sup_{k\in\mathbb{Z}} \|a_k\|_{L_\infty} \le 1 \quad \text{and} \quad \sup_{k\in\mathbb{Z}} \|\log a_k\|_{\mathrm{BMO}} < \infty.$$

Now, we take a function $f \in H_\infty + H_1(w_1) + H_1(w_2)$ satisfying $|f|^{1/2}\chi_{\{|f|>\lambda\}} \in L_1(w_1 \wedge w_2)$ for all $\lambda > 0$.

LEMMA 2. *With the above notation, for any $k \in \mathbb{Z}$ and $\lambda > 0$ there exist two analytic functions g_k, h_k such that $f\psi_k = g_k + h_k$ and*

$$|g_k| \le C\lambda|\psi_k| \min\left\{\frac{|f|}{\lambda}, \frac{\lambda}{|f|}\right\}, \quad \int_{\mathbb{T}} |h_k|^{1/2} w_1 \le C \int_{\{|f|>\lambda\}} |f|^{1/2} a_k^{1/2} w_1.$$

Proof. Since the BMO-norms of the functions $\log(w_1 a_k^{1/2})$ are uniformly bounded, by [8, Lemma 3.1] we may choose $C > 0$, $0 < \varrho < 1$, and some functions u_k such that

$$C^{-1} w_1 \le u_k \le C w_1, \quad |\mathcal{H}((u_k a_k^{1/2})^\varrho)| \le C(u_k a_k^{1/2})^\varrho, \tag{10}$$

where $\mathcal{H}$ denotes the Hilbert transformation on the unit circle $\mathbb{T}$. From now on and till the end of the proof of the lemma, we assume that k is fixed. Choosing an integer l such that $l/2 > 1/\varrho$, we put

$$\alpha = \max\left\{1, \left(\frac{|f|}{\lambda}\right)^{1/l}\right\}, \quad F = \frac{(u_k a_k^{1/2})^\varrho + i\mathcal{H}((u_k a_k^{1/2})^\varrho)}{\alpha(u_k a_k^{1/2})^\varrho + i\mathcal{H}(\alpha(u_k a_k^{1/2})^\varrho)}.$$

Then it is easy to check that F is an analytic function in H_∞, and, by (10),

$$|F| \le C/\alpha \le C. \tag{11}$$

We define

$$G = 1 - (1 - F^{2l})^l, \quad g_k = G f \psi_k, \quad h_k = (1 - G) f \psi_k.$$

Clearly, $f\psi_k = g_k + h_k$, and the g_k's and h_k's are analytic. By (11) and the definition of α,

$$\begin{aligned}|g_k| &\le C|f| \cdot |\psi_k|/\alpha^{2l} = C|f| \cdot |\psi_k| \min\{1, (\lambda/|f|)^2\} \\ &= C\lambda|\psi_k| \min\{|f|/\lambda, \lambda/|f|\}.\end{aligned}$$

As for h_k, by (11) and the choice of l ($l/2 > 1/\varrho$), we have

$$\begin{aligned}\int_{\mathbb{T}} |h_k|^{1/2} w_1 &= \int_{\mathbb{T}} |1 - G|^{1/2} |f|^{1/2} |\psi_k|^{1/2} w_1 \\ &\le C \int_{\{|f|>\lambda\}} |f|^{1/2} |\psi_k|^{1/2} w_1 + C \int_{\{|f|\le\lambda\}} |1 - F|^{1/\varrho} |f|^{1/2} |\psi_k|^{1/2} w_1.\end{aligned}$$

The next to the last term is already good; so it remains to estimate the last one. To this end, by (7), (9)–(11), the boundedness of $\mathcal{H}$ on the (unweighted) space $L_{1/\varrho}$, and the observation that $\alpha = 1$ on the set $\{|f| \le \lambda\}$, we get

$$\begin{aligned}&\int_{\{|f|\le\lambda\}} |1 - F|^{1/\varrho} |f|^{1/2} |\psi_k|^{1/2} w_1 \\ &\le C\lambda^{1/2} \int_{\{|f|\le\lambda\}} |1 - F|^{1/\varrho} a_k^{1/2} u_k \\ &\le C\lambda^{1/2} \int_{\{|f|\le\lambda\}} \frac{|\mathcal{H}((\alpha - 1)(u_k a_k^{1/2})^\varrho)|^{1/\varrho}}{u_k a_k^{1/2}} a_k^{1/2} u_k \\ &\le C\lambda^{1/2} \int_{\mathbb{T}} ((\alpha - 1)(u_k a_k^{1/2})^\varrho)^{1/\varrho} \\ &\le C\lambda^{1/2} \int_{\{|f|>\lambda\}} (|f|/\lambda)^{1/(l\varrho)} u_k a_k^{1/2} \\ &\le C\lambda^{1/2} \int_{\{|f|>\lambda\}} (|f|/\lambda)^{1/2} u_k a_k^{1/2} \\ &\le C \int_{\{|f|>\lambda\}} |f|^{1/2} a_k^{1/2} w_1.\end{aligned}$$

Thus, we have obtained the desired estimate for h_k, and so have finished the proof of Lemma 2.

Now we continue the proof of Theorem 2. Putting $\lambda = 2^n$ ($n \in \mathbb{Z}$), we obtain two corresponding functions $g_{n,k}$ and $h_{n,k}$ as in Lemma 2. Next, let

$$\varphi_{n,k} = g_{n+1,k} - g_{n,k}, \quad n, k \in \mathbb{Z}.$$

Then $\varphi_{n,k}$ is in H_∞; also note that $\varphi_{n,k} = h_{n,k} - h_{n+1,k}$. It is easy to see that

$$\sum_{n\in\mathbb{Z}} \varphi_{n,k} = f\psi_k, \quad \forall k \in \mathbb{Z};$$

so by (8),

$$\sum_{n,k\in\mathbb{Z}} \varphi_{n,k} = f. \tag{12}$$

On the other hand, by Lemma 2,

$$|\varphi_{n,k}| \le C2^n |\psi_k| \min\{|f|/2^n, 2^n/|f|\}, \tag{13}$$

and

$$\int_{\mathbb{T}} |\varphi_{n,k}|^{1/2} w_1 \le C \int_{\{|f|>2^n\}} |f|^{1/2} a_k^{1/2} w_1. \tag{14}$$

Now we can define the desired operator T. Setting $\Omega_n = \{|f| > 2^n\}$, for any $g \in L_\infty + L_1(w_1) + L_1(w_2)$ we define

$$T_{n,k}(g) = \frac{\int_{\Omega_n} \operatorname{sgn}(\bar f) g |f|^{-1/2} a_k^{1/2} w_1}{\int_{\Omega_n} |f|^{1/2} a_k^{1/2} w_1}, \qquad n, k \in \mathbb{Z},$$

which is understood as 0 if $|\Omega_n| = 0$, and

$$T(g) = \sum_{n,k\in\mathbb{Z}} T_{n,k}(g)\varphi_{n,k}.$$

By (12), $Tf = f$ (observe that if $|\Omega_n| = 0$, then $\varphi_{n,k} = 0$). We are going to estimate various norms of T. Let $g \in L_\infty$ with $\|g\|_{L_\infty} \le 1$. Then

$$|T_{n,k}(g)| \le \|g\|_{L_\infty} \frac{\int_{\Omega_n} |f|^{-1}|f|^{1/2} a_k^{1/2} w_1}{\int_{\Omega_n} |f|^{1/2} a_k^{1/2} w_1} \le 2^{-n};$$

so by (13), (4) (applied to $\{\psi_k\}$) and the argument leading to (4) (applied to the sum over n), we get

$$|Tg| \le \sum_{n,k\in\mathbb{Z}} 2^{-n}|\varphi_{n,k}| \le C \sum_{k\in\mathbb{Z}} |\psi_k| \sum_{n\in\mathbb{Z}} \min\{|f|/2^n, 2^n/|f|\} \le C.$$

Since all $\varphi_{n,k}$'s are analytic, as an operator from L_∞ to H_∞, T is bounded and its norm is majorized by a constant C.

Next, let $g \in L_1(w_1)$. Then by (13), (14),

$$\begin{aligned}
\int_{\mathbb{T}} |T(g)|w_1 &\le \sum_{n,k\in\mathbb{Z}} |T_{n,k}(g)| \int_{\mathbb{T}} |\varphi_{n,k}|w_1 \\
&\le C \sum_{n,k\in\mathbb{Z}} |T_{n,k}(g)| 2^{n/2} \int_{\Omega_n} |f|^{1/2} a_k^{1/2} w_1 \\
&\le C \sum_{n,k\in\mathbb{Z}} 2^{n/2} \int_{\Omega_n} |g|\cdot|f|^{-1/2} a_k^{1/2} w_1;
\end{aligned}$$

by the argument proving (4), we get

$$\sum_{k\in\mathbb{Z}} a_k^{1/2} \le C;$$

on the other hand, setting $e_j = \{2^j \le |f| < 2^{j+1}\}$, we have

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} 2^{n/2} \int_{\Omega_n} |g|\cdot|f|^{-1/2} w_1 &\le C \sum_{n\in\mathbb{Z}} 2^{n/2} \sum_{j\ge n} \int_{e_j} |g| 2^{-j/2} w_1 \\
&\le C \sum_{j\in\mathbb{Z}} \int_{e_j} |g| w_1 2^{-j/2} \sum_{n\le j} 2^{n/2} \\
&\le C \int_{\mathbb{T}} |g| w_1.
\end{aligned}$$

Combining the preceding inequalities, we get the boundedness of T as an operator from $L_1(w_1)$ into $H_1(w_1)$.

Finally, to prove that T maps $L_1(w_2)$ into $H_1(w_2)$ boundedly, we observe that by (13), (7), and (9),

$$|\varphi_{n,k}|^{1/2} w_2 \le C 2^{n/2} |\psi_k|^{1/2} w_2 \le C 2^{n/2} 2^{-k} w_1;$$

so, if $g \in L_1(w_2)$, then

$$\begin{aligned}
\int_{\mathbb{T}} |T(g)|w_2 &\le \sum_{n,k\in\mathbb{Z}} |T_{n,k}(g)| \int_{\mathbb{T}} |\varphi_{n,k}|w_2 \\
&\le C \sum_{n,k\in\mathbb{Z}} |T_{n,k}(g)| 2^{n/2} 2^{-k} \int_{\mathbb{T}} |\varphi_{n,k}|^{1/2} w_1 \\
&\le C \sum_{n,k\in\mathbb{Z}} 2^{n/2} 2^{-k} |T_{n,k}(g)| \int_{\Omega_n} |f|^{1/2} a_k^{1/2} w_1 \qquad \text{(by (14))} \\
&\le C \sum_{n,k\in\mathbb{Z}} 2^{n/2} 2^{-k} \int_{\Omega_n} |g|\cdot|f|^{-1/2} a_k^{1/2} w_1.
\end{aligned}$$

Now by (9) and an argument similar to the proof of (4), we see that

$$\sum_{k\in\mathbb{Z}} 2^{-k} a_k^{1/2} w_1 \le C w_2,$$

so, as before for $\int_{\mathbb{T}} |Tg|w_1$, we deduce that

$$\int_{\mathbb{T}} |T(g)|w_2 \le C \sum_{n\in\mathbb{Z}} 2^{n/2} \int_{\Omega_n} |g|\cdot|f|^{-1/2} w_2 \le C \int_{\mathbb{T}} |g| w_2.$$

Thus $T: L_1(w_2) \to H_1(w_2)$ is bounded and of norm $\le C$. Therefore, we have completed the proof of Theorem 2 for $N = 2$ under the additional assumption $|f|^{1/2}\chi_{\{|f|>\lambda\}} \in L_1(w_1 \wedge w_2)$.

We give some hints to the proof for $N = 3$. Let $f \in H_\infty + H_1(w_1) + H_1(w_2) + H_1(w_3)$ satisfy the condition $|f|^{1/2}\chi_{\{|f|>\lambda\}} \in L_1(w_1 \wedge w_2 \wedge w_3)$. Along with $\{\psi_k\}$, we find a sequence $\{\xi_k\} \subset H_\infty$ satisfying (7) and (8) with w_2 replaced by w_3. Next, along with the a_k (see (9)) we define b_k in a similar way, again with w_3 in place of w_2. Then the decomposition in Lemma 2 will change in the following way: $f\psi_k\xi_j = g_{k,j} + h_{k,j}$, where

$$|g_{k,j}| \le C\lambda |\psi_k \xi_j| \min\{|f|/\lambda, \lambda/|f|\},$$

$$\int_{\mathbb{T}} |h_{k,j}|^{1/2} w_1 \le C \int_{\{|f|>\lambda\}} |f|^{1/2} a_k^{1/2} b_j^{1/2} w_1.$$

We omit the further details. ∎

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An example of a Fréchet algebra which is a principal ideal domain

by

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Abstract. We construct an example of a Fréchet m-convex algebra which is a principal ideal domain, and has the unit disk as the maximal ideal space.

1. Introduction. In the sequel, if not stated otherwise, we consider Hausdorff locally multiplicatively convex (LMC) commutative $\mathbb{C}$-algebras with identity (denoted by 1), and we identify the set of scalar multiples of the identity with $\mathbb{C}$. A *Fréchet m-convex algebra* A is a complete metrizable LMC algebra; in this case the topology of A can be defined by an increasing sequence of algebra seminorms (see [5]).

If I is an ideal of A, we denote by I^n the ideal of A generated by all products of the form $x_1 \ldots x_n$ ($x_i \in I$). We say that I is *finitely generated* if there exist elements $x_1, \ldots, x_r$ in A such that $I = \sum_{i=1}^r Ax_i$, and we write $I = (x_1, \ldots, x_r)$; when $r = 1$ we say that $I = (x)$ is *principal*.

As usual, A is *noetherian* (resp. *principal*) if every ideal is finitely generated (resp. *principal*).

There are many proofs of the fact that a noetherian Banach algebra is finite-dimensional, and hence semilocal (see [6], [11] for instance). For Fréchet m-convex algebras all these proofs break down; in fact the algebra of formal power series $\mathbb{C}[[X]]$ (with the topology of $\mathbb{C}^{\mathbb{N}}$) is a principal ideal domain (see also [4] and observe that all these examples are local rings).

Recall that the finiteness conditions on all ideals are somewhat rare in the LMC-context; for instance, if K is a connected compact set in $\mathbb{C}$ then the algebra $\mathcal{O}(K)$ of holomorphic germs is a principal ideal domain, but it is not metrizable. On the other hand infinite-dimensional examples of complete metrizable locally convex division algebras cannot exist, since the Gelfand–Mazur theorem is true for such algebras ([2], [13]).

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