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Universal images of universal elements

by

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Abstract. We furnish several necessary and sufficient conditions for the following property: For a topological space X, a continuous selfmapping S of X and a family τ of continuous selfmappings of X, the image under S of every τ-universal element is also τ-universal. An application in operator theory, where we extend results of Bourdon, Herrero, Bes, Herzog and Lemmert, is given. In particular, it is proved that every hypercyclic operator on a real or complex Banach space has a dense invariant linear manifold with maximal algebraic dimension consisting, apart from zero, of vectors which are hypercyclic.

1. Preliminaries. Assume that X is a topological space. Denote by $C(X)$ the class of continuous selfmappings of X. Following Grosse-Erdmann [Gr], we say that a nonempty family $\tau \subset C(X)$ is *universal* when there is an element $x \in X$ such that the orbit $O(x,\tau) = \{Tx : T \in \tau\}$ is dense in X. In such a case, the element x is called τ-*universal*. $U(\tau)$ will stand for the set of τ-universal elements of X. If τ is countable, then a necessary condition for τ to be universal is, of course, the separability of X. If $T \in C(X)$ and $x \in X$, the orbit of x under T is $O(x,T) = O(x,\tau)$, where this time τ is the family of iterates $\tau = \{T^n : n \in \mathbb{N}\}$. Here $\mathbb{N}$ is the set of positive integers, $T^1 = T$, $T^2 = T \circ T$, and so on. T is called *universal* whenever this τ is universal, and an element $x \in X$ is called T-*universal* if and only if $O(x,T)$ is dense. In this case, we set $U(T) = U(\tau)$. Denote by $\mathrm{DR}(X)$ the subset of mappings $T \in C(X)$ such that the range $T(X)$ is dense in X. If T is universal, then, trivially, $T \in \mathrm{DR}(X)$. A subset $\tau \subset C(X)$ is said to be *densely universal* whenever $U(\tau)$ is dense in X. Note that $U(T)$ is always dense if T is universal, since $O(x,T) \subset U(T)$ for every T-universal

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element x, because $O(T^m x, T) = T^m(O(x,T))$ and $T^m \in \mathrm{DR}(X)$ for each $m \in \mathbb{N}$.

We should point out here that if X is a topological vector space and $\tau \subset L(X) = \{\text{operators on } X\} = \{T \in C(X) : T \text{ is linear}\}$, then the word "hypercyclicity" is preferred to "universality" (see, for instance, [GS]). In order to cause no confusion, we keep the word "universal" in this paper.

Some final notations which will be employed are the following: If $A \subset X$, $T, S \in C(X)$ and $\tau \subset C(X)$, then we set $\overline{A}$ = the closure of A, ∂A = the boundary of A, $TS = T \circ S$, $\Gamma(T) = \{S \in C(X) : TS = ST\}$ and $\Gamma(\tau) = \{S \in C(X) : TS = ST \text{ for all } T \in \tau\}$. $\mathbb{R}$ and $\mathbb{C}$ denote, as usual, the real line and the complex plane, respectively.

The aim of this note is to study the following problem: If $\tau \subset C(X)$ is universal and $S \in C(X)$, when is the image under S of every τ-universal element τ-universal? That is, when $S(U(\tau)) \subset U(\tau)$? If S commutes or "almost commutes" with each member of τ, then we provide a complete answer (Section 2). A celebrated result of Herrero for the Hilbert setting is obtained for the Banach setting in the complex case as a linear example (Section 3) by means of a proof which is easier than that in [He, Prop. 4.1]. The main tool of this proof is a result which, in turn, generalizes a statement of Herzog and Lemmert [HL, Satz 1] in the case where the dual pair (X, Y) equals (X, X^*), X being a complex Banach space. The real case is more involved and is also considered (Section 4). In both cases we prove that for every hypercyclic operator on a real or complex Banach space there exists a dense invariant linear manifold with maximal algebraic dimension consisting, apart from zero, of vectors which are hypercyclic.

2. Universal images. Firstly, we introduce the notion of "almost commutativity" on some kinds of uniformizable topological spaces. Assume that $S \in C(X)$ and $\tau = \{T_\alpha : \alpha \in I\}$ is a net in $C(X)$. If X is a metrizable topological space, then we say that S *almost commutes with* τ whenever there exists a metric d on X, compatible with its topology, such that $\lim_{\alpha \in I} d(T_\alpha Sx, ST_\alpha x) = 0$ for all $x \in X$. If X is a topological group, then we say that S *almost commutes weakly with* τ whenever the net $\{(T_\alpha Sx)(ST_\alpha x)^{-1}\}_{\alpha \in I}$ converges for every $x \in X$. It is obvious that almost commutativity implies weak almost commutativity on a metrizable topological group.

For a topological space X, a family $\tau \subset C(X)$ and a mapping $S \in C(X)$, consider the following seven properties:

(a) $S(U(\tau)) \subset U(\tau)$.
(b) $S \in \mathrm{DR}(X)$.
(c) $S(X) \cap U(\tau) \neq \emptyset$.
(d) $S(U(\tau)) \cap U(\tau) \neq \emptyset$.
(e) $\overline{S(X)} \cap U(\tau) \neq \emptyset$.
(f) $\overline{S(U(\tau))} \cap U(\tau) \neq \emptyset$.
(g) $S(X)$ is somewhere dense.

Recall that (g) means that the closure of $S(X)$ has nonempty interior. We are now ready to state our results.

THEOREM 1. *If X is a topological space, $S \in C(X)$, the family $\tau \subset C(X)$ is universal and $S \in \Gamma(\tau)$, then the six properties* (a)–(f) *are equivalent. If, in addition, τ is densely universal, then the seven properties* (a)–(g) *are equivalent.*

Proof. By hypothesis, $U(\tau)$ is not empty, so $S(U(\tau))$ is not either. It is trivial that (a) implies (d) and (d) implies (c). If (c) holds, then there is a universal element y and a point $x \in X$ such that $y = Sx$. Since $O(y,\tau)$ is dense and $ST = TS$ for every $T \in \tau$, we see that $S(O(x,\tau)) = O(Sx,\tau) = O(y,\tau)$ is dense, so $S(X)$ is dense, i.e., (b) is obtained. We now prove that (b) implies (a): Take a point $y \in S(U(\tau))$. Then there exists $x \in U(\tau)$ with $y = Sx$. Since $O(x,\tau)$ is dense and $S \in \Gamma(\tau)$, the set $O(y,\tau) = O(Sx,\tau) = S(O(x,\tau))$ is dense, so y is universal, that is, $S(U(\tau)) \subset U(\tau)$. Hence (a)–(d) are equivalent.

On the other hand, it is evident that (d) implies (f), and (f) implies (e). Let us show that (e) implies (b): Let V be a nonempty open subset of X and take a universal element y such that $y \in \overline{S(X)}$. There is $T \in \tau$ with $Ty \in V$, because $O(y,\tau)$ is dense. Since T is continuous, there exists an open subset W containing y such that $T(W) \subset V$. But there is $x \in X$ satisfying $Sx \in W$, whence $S(Tx) = TSx \in T(W) \subset V$, so $S(O(x,\tau))$ is dense and, consequently, $S(X)$ is dense. Thus, properties (a)–(f) are equivalent.

It is trivial that always (b) implies (g). Assume now that τ is densely universal. If (g) holds, then there exists a nonempty open subset $V \subset X$ such that $\overline{S(X)} \supset V$. By dense universality, there is an element $x \in U(\tau)$ with $x \in V$, so $x \in \overline{S(X)}$. Then $\overline{S(X)} \cap U(\tau) \neq \emptyset$, and this is (e). The proof is finished. ■

THEOREM 2. *Suppose that X is a topological space, $S \in C(X)$, the net $\tau = \{T_\alpha : \alpha \in I\} \subset C(X)$ is universal and that at least one of the following conditions is satisfied*:

(i) *X is metrizable and S almost commutes with τ.*
(ii) *X is a topological group and S almost commutes weakly with τ.*

Then (a)–(f) *are equivalent. If, in addition, τ is densely universal, then* (a)–(g) *are equivalent.*

Proof. It is very similar to that of Theorem 1. It suffices to take into account that, if (i) is satisfied, then, for every $x \in X$, $O(Sx,\tau) = \{T_\alpha Sx :$

$\alpha \in I\}$ is dense if and only if $S(O(x,\tau)) = \{ST_\alpha x : \alpha \in I\}$ is dense, and this is true because $\lim_{\alpha\in I} d(T_\alpha Sx, ST_\alpha x) = 0$ for some metric d compatible with the topology of X and every $x \in X$. If (ii) is satisfied, set $L(x) = \lim_{\alpha\in I}(T_\alpha Sx)(ST_\alpha x)^{-1} \in X$ for every $x \in X$. In this case, $O(Sx,\tau)$ is dense if and only if the set $\{L(x)\cdot(ST_\alpha x) : \alpha \in I\}$ is dense, and this holds if and only if $S(O(x,\tau))$ is dense. ∎

Note that the second statement of Theorems 1, 2 holds in the case where T is universal and, respectively, $S \in \Gamma(T)$ (Theorem 1) and S either almost commutes (case (i)) or almost commutes weakly (case (ii)) with $\{T^n : n \in \mathbb{N}\}$ (Theorem 2).

Observe that, in particular, Theorem 1 proves that if S is somewhere dense but does not have dense range then it cannot commute with a family τ with $U(\tau)$ dense in X. Furthermore, examples showing that Theorems 1, 2 are false if we omit the commuting property of S can be given. For this, we follow an example in [GS]. Let (a_n) be a dense sequence in $\mathbb{R}^2$, and $\{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$. For each positive integer n, let $b_n \in \mathbb{R}^2$ be of norm one and orthogonal to a_n, and define $T_n : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T_n(\lambda e_1 + \mu e_2) := \lambda a_n + n\mu b_n.$$

Then $\tau := \{T_n : n \geq 1\}$ satisfies:

(i) $U(\tau) = \operatorname{span}\{e_1\} \setminus \{0\}$ (see [GS, Section 1]).

(ii) If $S : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, then $SU(T) \subset U(T)$ if and only if e_1 is an eigenvector of S of nonzero eigenvalue.

Now, consider $S(\lambda e_1 + \mu e_2) := \lambda e_1$. Then we have:

(a) $SU(\tau) = U(\tau)$.

(b) S does not "almost commute weakly" (in particular, it does not "almost commute", and hence it does not commute) with τ. Indeed, given $x = e_1$ and $n \in \mathbb{N}$, we have $T_nSx - ST_nx = T_n(e_1) - S(a_n) = a_n - S(a_n)$. Since (a_n) is dense, there are subsequences $(a_{n(j)})$ and $(a_{m(j)})$ converging respectively to 0 and e_2. But then $S(a_{n(j)}) \to 0$ and $S(a_{m(j)}) \to S(e_2) = 0$ $(j \to \infty)$, whence $T_{n(j)}Sx - ST_{n(j)}x \to 0$ and $T_{m(j)}Sx - ST_{m(j)}x \to e_2$ $(j \to \infty)$, so $(T_nSx - ST_nx)$ does not converge.

(c) $S(\mathbb{R}^2)$ $(= \operatorname{span}\{e_1\})$ is nowhere dense.

On the other hand, note also that a selfmapping S may have dense range and yet not preserve universal elements: take for instance τ as above, and the mapping $S(\lambda e_1 + \mu e_2) := \mu e_1 + \lambda e_2$.

3. A linear example: c-dimensional universal manifolds. In this section, X will stand for a *complex* Banach space. We need some background on general spectral theory (see, for instance, [Do, Chapter 1] or [Ru, Chapter 10]). If $T \in L(X)$ and T^* is its adjoint, then $\sigma(T) = \sigma(T^*)$. The *point spectrum* $\sigma_p(T)$ of T is the set of eigenvalues of T, that is, the set of $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not one-to-one, I being the identity operator. Denote by $\mathcal{F}(T)$ the family of all functions which are analytic on some neighbourhood of $\sigma(T)$. Hence $\mathcal{F}(T) = \mathcal{F}(T^*)$. Let $f \in \mathcal{F}(T)$ and γ be a positively oriented Jordan cycle surrounding $\sigma(T)$ such that both γ and its geometric interior are contained in the domain of analyticity of f. Then the operator $f(T)$ is defined by the following equation, where the integral exists as a limit of Riemann sums in the norm of $L(X)$:

$$f(T) = \frac{1}{2\pi i}\cdot\oint_\gamma f(\lambda)(\lambda I - T)^{-1}\,d\lambda.$$

$f(T)$ depends only on f. It happens that if $f(z)$ has power series expansion $f(z) = \sum_{n=0}^\infty a_n z^n$ valid in a neighbourhood of $\sigma(T)$, then the series $\sum_{n=0}^\infty a_n T^n$ converges to $f(T)$ in the norm of $L(X)$. Then, in this sense, the notion of $f(T)$ extends the definition $P(T) = \sum_{n=0}^m a_n T^n$ (with $T^0 = I$) when $P(z)$ is the polynomial $P(z) = \sum_{n=0}^m a_n z^n$.

In [Bo] it is proved that if $x \in U(T)$ and P is a nonzero polynomial, then $P(T)x \in U(T)$. The key idea is to show that $P(T)$ has dense range whenever T is universal. Herzog and Lemmert [HL, Satz 1] show that the kernel of $P(T^*)$ is trivial if T is universal and P is a nonzero polynomial. This, when applied to $P(T^*) - \lambda I$ for every $\lambda \in \mathbb{C}$, tells us in fact that $\sigma_p(P(T^*)) = \emptyset$ for each nonconstant polynomial P. By the Hahn–Banach theorem and by the fact $P(T^*) = P(T)^*$, $P(T)$ has dense range (see also [Ki] for the case $P(T) = T - \lambda I$), and this is all that is needed. As a consequence, the set $M := \operatorname{span}(O(x,T)) = \{P(T)x : P \text{ is a polynomial}\}$ is a dense T-invariant linear manifold such that $M \setminus \{0\} \subset U(T)$. Herrero [He, Prop. 4.1] showed that if T is a universal operator on a complex Hilbert space X, Ω is a connected analytic Cauchy domain including $\sigma(T)$ and $f \in H^2(\partial\Omega)$ (= the closure in $L^2(\partial\Omega)$, with respect to linear Lebesgue measure, of the rational functions with poles outside $\overline{\Omega}$) is nonzero, then $f(T)(U(T)) \subset U(T)$. We state a more general result (Theorem 3(2)) which is valid for any complex Banach space. Furthermore, the proof provided here—based upon part (1) of Theorem 3, which in turn extends Herzog–Lemmert's result for the dual pair (X, X^*), X being a complex Banach space—is easier than that of Proposition 4.1 of [He]. Recall that a (real or complex) separable Banach space X supports a universal operator if and only if $\dim X = \infty$ (see [Ro], [An] and [Be]) if and only if $\dim X = c$ (= the cardinality of the continuum).

THEOREM 3. *Suppose that X is a complex Banach space and that $T \in L(X)$ is universal. Assume that $f \in \mathcal{F}(T)$, $D(f)$ being its domain of analyticity. We have:*

(1) *If f is nonconstant on every connected component of $D(f)$, then $\sigma_{\mathrm{p}}(f(T^*)) = \emptyset$.*

(2) *If f is not identically zero and $D(f)$ is connected, then $f(T)(U(T)) \subset U(T)$.*

(3) *There exists a dense T-invariant linear manifold M with maximal algebraic dimension (i.e., $\dim M = c$) such that $M \setminus \{0\} \subset U(T)$.*

Proof. Since T is universal, X must be separable and infinite-dimensional, so $\dim X = c$.

(1) If f is nonconstant on every connected component of $D(f)$ then, by a special version of the spectral mapping theorem [Ru, Theorem 10.33], $\sigma_{\mathrm{p}}(f(T^*)) = f(\sigma_{\mathrm{p}}(T^*))$. But, since T is universal, $\sigma_{\mathrm{p}}(T^*) = \emptyset$, so $\sigma_{\mathrm{p}}(f(T^*)) = \emptyset$.

(2) Assume that $D(f)$ is connected and $f \neq 0$. If f is nonconstant then, by (1), $\sigma_{\mathrm{p}}(f(T^*)) = \emptyset$. But $f(T^*) = (f(T))^*$, so $\sigma_{\mathrm{p}}(f(T)^*) = \emptyset$. In particular, $0 \notin \sigma_{\mathrm{p}}(f(T)^*)$, so $f(T)^*$ is one-to-one. From the Hahn–Banach theorem, we get $f(T) \in \mathrm{DR}(X)$. Since $zf(z) = f(z)z$ and $(f \cdot g)(T) = f(T)g(T)$ $(g \in \mathcal{F}(T))$, we have $f(T) \in \Gamma(T)$. By Theorem 1, we obtain $f(T)(U(T)) \subset U(T)$. If $f = \lambda \neq 0$ is constant, then $f(T) = \lambda I$, in which case $f(T)(U(T)) = \{\lambda x : x \in U(T)\} = U(T)$.

(3) Fix a vector $x \in U(T)$ and define $M = \{f(T)x : f \text{ is entire}\}$. By (2), $M \setminus \{0\} \subset U(T)$. It is obvious that M is a linear manifold. M is T-invariant since $zf(z)$ is entire for each entire function $f(z)$. It is dense because $M \supset O(x,T)$. Finally, the linear spaces M and {entire functions} are evidently algebraically isomorphic, so $\dim M = c$. ■

4. The real case. The proof of the statement (3) in Theorem 3 is based on the statement (2), which involves the fact that the underlying Banach space is complex, because (2) is based in turn on properties arising from the Cauchy-type definition of the operators $f(T)$. Nevertheless, we will be able to show a result which is analogous to Theorem 3(2, 3) in the real case. This is the aim of this section.

J. P. Bes [Bs] has recently proved that if X is a *real* Banach space, $T \in L(X)$ is universal and P is a nonzero polynomial with real coefficients, then $P(T)$ has dense range. Then, as in the proof of [Bo], we have $P(T)(U(T)) \subset U(T)$ and $M := \mathrm{span}(O(x,T)) = \{P(T)x : P \text{ is a polynomial with real coefficients}\}$ is a dense T-invariant linear manifold such that $M \setminus \{0\} \subset U(T)$.

If $f(t) = \sum_{j=0}^{\infty} a_j t^j$ is a real entire function (i.e., $f(t) \in \mathbb{R}$ for every $t \in \mathbb{R}$ or, equivalently, $a_j \in \mathbb{R}$ for every $j \geq 0$) and $T \in L(X)$, where X is a real Banach space, then it is a standard exercise to prove that the series $\sum_{j=0}^{\infty} a_j T^j$ converges in the norm of $L(X)$, and $f(T)$ is therefore defined in a natural way as $f(T) = \sum_{j=0}^{\infty} a_j T^j$. Part (b) of the following theorem is derived from (a) by using Theorem 1. Part (c) is obtained from (b) in the same way as (3) is obtained from Theorem 3(2) by considering this time $M = \{f(T)x : f \text{ is real entire}\}$, x being a vector chosen in $U(T)$.

THEOREM 4. *Suppose that X is a real Banach space and that $T \in L(X)$ is universal. We have:*

(a) *If f is a real nonzero entire function, then $f(T) \in \mathrm{DR}(X)$.*

(b) *If f is a real nonzero entire function, then $f(T)(U(T)) \subset U(T)$.*

(c) *There exists a dense T-invariant linear manifold M such that $\dim M = c$ and $M \setminus \{0\} \subset U(T)$.*

The remainder of this section is devoted to proving (a). Note that Bes's result is the special case when f = a nonzero polynomial. Denote by $H(\mathbb{C})$ the space of entire functions, endowed with the topology of uniform convergence on compact subsets of $\mathbb{C}$. The following three results are well known when X is a complex Banach space, even if f_n $(n \in \mathbb{N})$, f, g are analytic on a domain distinct from $\mathbb{C}$. We give an independent proof for the real case in Lemma 1, which of course works in the complex case as well. The proof of Lemma 2 is straightforward from the definitions and it is left to the reader. Lemma 3 is a consequence of Lemma 2. From now on, $\|\cdot\|$ will stand for the norm either in X or in $L(X)$, without distinction.

LEMMA 1. *Assume that X is a real Banach space and that f_n $(n \in \mathbb{N})$, f are real entire functions satisfying $f_n \to f$ $(n \to \infty)$ in $H(\mathbb{C})$. If $T \in L(X)$, then $f_n(T) \to f(T)$ $(n \to \infty)$ in the norm of $L(X)$.*

Proof. Suppose that $f_n(z) = \sum_{j=0}^{\infty} a_j^{(n)} z^j$, $f(z) = \sum_{j=0}^{\infty} a_j z^j$ $(a_j, a_j^{(n)} \in \mathbb{R};\ j \in \{0,1,2,\ldots\},\ n \in \mathbb{N})$. It is well known that (f_n) converges to f in $H(\mathbb{C})$ if and only if

$$\sup\{|a_0^{(n)} - a_0|, |a_j^{(n)} - a_j|^{1/j} : j \in \mathbb{N}\} \to 0 \quad (n \to \infty).$$

Given $\varepsilon \in (0,1)$, there exists $N \in \mathbb{N}$ such that the latter expression is less than $\varepsilon/(3(1+\|T\|))$ for all $n \geq N$. Then

$$\begin{aligned}\|f_n(T) - f(T)\| &= \Big\|\sum_{j=0}^{\infty}(a_j^{(n)} - a_j)T^j\Big\| \\ &\leq |a_0^{(n)} - a_0| + \sum_{j=1}^{\infty}|a_j^{(n)} - a_j| \cdot \|T\|^j \\ &< \frac{\varepsilon}{3(1+\|T\|)} + \sum_{j=1}^{\infty}\left(\frac{\varepsilon\|T\|}{3(1+\|T\|)}\right)^j,\end{aligned}$$

hence

$$\|f_n(T) - f(T)\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3-\varepsilon} < \varepsilon \quad \text{for all } n \geq N,$$

as required. ∎

LEMMA 2. *If X is a real Banach space, f, g are real entire functions and $T \in L(X)$, then $(f \cdot g)(T) = f(T)g(T)$.*

LEMMA 3. *If X is a real Banach space and $T \in L(X)$, then the operator $e^T = \sum_{j=0}^{\infty}(1/j!)T^j$ is invertible and $(e^T)^{-1} = e^{-T}$.*

The next auxiliary statement comes from the Weierstrass factorization theorem and from the fact that if a is a nonreal zero of a real entire function f, then $\overline{a}$ is also a zero of f with the same multiplicity.

LEMMA 4. *If f is a nonconstant real entire function with infinitely many zeros, then there exist $m \in \{0, 1, 2, \ldots\}$, a real entire function g, a sequence $(a_n) \subset \mathbb{C} \setminus \{0\}$ and a sequence (P_n) of polynomials with real coefficients satisfying*

$$f(z) = z^m e^{g(z)} \prod_{n=0}^{\infty} q_n(z) e^{P_n(z)}$$

uniformly in compact subsets of $\mathbb{C}$, where $q_n(z) = 1 - a_n^{-1}z$ if $a_n \in \mathbb{R}$ and $q_n(z) = 1 - 2\operatorname{Re}(a_n^{-1})z + |a_n|^{-2}z^2$ if $a_n \in \mathbb{C} \setminus \mathbb{R}$.

Let us finish the proof of Theorem 4(a). Fix a real nonzero entire function f and an operator $T \in L(X)$. We may assume that f has infinitely many zeros (the other cases are easier and left to the reader). With the notation of Lemma 4, we have $f(z) = z^m e^{g(z)} G_n(z) R_n(z)$ $(z \in \mathbb{C},\ n \in \mathbb{N})$, where $G_n(z) = q_1(z) \ldots q_n(z) e^{P_1(z)+\ldots+P_n(z)}$ and $R_n(z) = \prod_{k=n+1}^{\infty} q_k(z) e^{P_k(z)}$, in such a way that for each fixed n the infinite product R_n converges in $H(\mathbb{C})$, and $R_n(z) \to 1$ $(n \to \infty)$ in $H(\mathbb{C})$. From Lemmas 1, 2 one gets $f(T) = T^m e^{g(T)} G_n(T) R_n(T)$ for all $n \in \mathbb{N}$ and $R_n(T) \to I$ $(n \to \infty)$ in the norm of $L(X)$. Therefore there exists $N \in \mathbb{N}$ with $\|R_N(T) - I\| < 1$, so $R_N(T)$ is invertible. Thus, $R_N(T) \in \mathrm{DR}(X)$. We have $f(T) = T^m e^{g(T)} G_N(T) R_N(T)$. Observe that T (so T^m) has dense range because it is universal. The operator $e^{g(T)}$ has dense range since it is invertible by Lemma 3. Note that $G_N(T) = q_1(T) \ldots q_N(T) e^{P_1(T)+\ldots+P_N(T)}$ again by Lemma 2. The operator $e^{P_1(T)+\ldots+P_N(T)}$ is invertible, so of dense range. Finally, each $q_k(T)$ has either the form $I - a_k^{-1}T$ (with $a_k \in \mathbb{R} \setminus \{0\}$) or the form $I - 2\operatorname{Re}(a_k^{-1})T + |a_k|^{-2}T^2$ (with $a_k \in \mathbb{C} \setminus \mathbb{R}$), i.e., $g_k(T)$ is a nonzero polynomial in T with real coefficients. But from Bes's theorem each $q_k(T)$ has dense range. Consequently, $G_N(T) \in \mathrm{DR}(X)$ and one concludes that $f(T)$ has also dense range. ∎

We add here that property (a) in Theorem 4 can also be achieved by using complexifications. Given a real Banach space X and a continuous linear operator A on X, denote by $\widetilde{A}$ and $\widetilde{X}$ their complexifications (i.e., $\widetilde{X} := X + iX$, $\widetilde{A}(x + iy) = Ax + iAy$, $x + iy \in \widetilde{X}$). Now, let f be real, entire and nonconstant (that $f(T)$ has dense range follows immediately if $f \neq 0$ is constant). Since T is universal, by a recent result of J. Bonet and A. Peris [BP],

$$\sigma_{\mathrm{p}}((\widetilde{T})^*) = \emptyset.$$

Thus, by the spectral mapping theorem [Ru, Theorem 10.33],

$$\sigma_{\mathrm{p}}(f((\widetilde{T})^*)) = f(\sigma_{\mathrm{p}}((\widetilde{T})^*)) = \emptyset.$$

In particular, $f(\widetilde{T})^* = f((\widetilde{T})^*)$ is one-to-one, and so $\widetilde{f(T)} = f(\widetilde{T})$ has dense range. Hence $f(T)$ must have dense range. This proof also yields that $f(T) \in \mathrm{DR}(X)$ (so $f(U(T)) \subset U(T)$) for any real analytic function that extends holomorphically to some open neighbourhood Ω of the spectrum $\sigma(\widetilde{T})$ (and is nonconstant on every component of Ω).

5. Final remarks. 1. Theorem 1 is related to 1.3.1 and 1.3.2 of [Gr], which tell us in particular that (1) $U(\{TS : T \in \tau\}) = S^{-1}(U(\tau))$ and (2) if $S \in \mathrm{DR}(X)$ then $U(\tau) \subset U(\{ST : T \in \tau\})$, without the assumption of commutativity. For example observe that under the suppositions of Theorem 1, property (2) includes the part "(b) implies (a)" of this theorem.

2. In connection with Theorems 3(3) and 4(c), note that Bourdon–Bes's manifold $M = \operatorname{span}(O(x, T))$ had just countable dimension. On the other hand, Read [Re] gave an extreme example for that statement; namely, he constructed an operator T on the sequence space l^1 such that $U(T) = l^1 \setminus \{0\}$.

3. For other results related to universality and analytic transforms of operators, see, for instance, [HS] and [Sc].

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Partial retractions for weighted Hardy spaces

by

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Abstract. Let $1 \le p \le \infty$ and let w_0, w_1 be two weights on the unit circle such that $\log(w_0 w_1^{-1}) \in \mathrm{BMO}$. We prove that the couple $(H_p(w_0), H_p(w_1))$ of weighted Hardy spaces is a partial retract of $(L_p(w_0), L_p(w_1))$. This completes previous work of the authors. More generally, we have a similar result for finite families of weighted Hardy spaces. We include some applications to interpolation.

For $1 < p < \infty$ we can project the space $L_p(\mathbb{T})$ onto the (boundary) Hardy class H_p. This can be done by an operator independent of p, for instance, by the Riesz projection. The extreme indices $p = 1$ and $p = \infty$ cannot be included.

Though regret can hardly be allowed in connection with a true mathematical statement, the latter assertion (about the extreme indices) may evoke a sort of this feeling in some situations. The following fact proved in [13] (see [11] for a simple argument) can sometimes serve as a remedy.

For every $f \in H_1 + H_\infty$ $(= H_1)$ there is a linear operator fixing f and mapping boundedly L_1 to H_1 and L_∞ to H_∞ with norms not exceeding a universal constant.

Later, this result was extended to weighted Hardy spaces. By a *weight* we mean a nonnegative measurable function w on $\mathbb{T}$ such that $\log w \in L_1$. We put $L_p(w) = L_p(\mathbb{T}, w\,dm)$ (m is normalized Lebesgue measure on $\mathbb{T}$), and

$$L_\infty(w) = \{f : fw^{-1} \in L_\infty\}$$

equipped with the natural norm $\|f\|_{\infty,w} = \|fw^{-1}\|_\infty$. Next, let φ be an outer function satisfying $|\varphi| = w$ a.e. on $\mathbb{T}$. We introduce the weighted Hardy space $H_p(w)$, $0 < p \le \infty$, by

$$H_p(w) = \{f : f\varphi^{1/p} \in H_p\}, \quad 0 < p < \infty,$$
$$H_\infty(w) = \{f : f\varphi^{-1} \in H_\infty\}.$$

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