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Department of Mathematics
Technion, Israel Institute of Technology
Haifa 32000, Israel
E-mail: mcwikel@leeor.technion.ac.il

Department of Mathematics
Ohio University
Athens, Ohio 45701
U.S.A.
E-mail: guli@bing.math.ohiou.edu

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Weak almost periodicity of L_1 contractions and coboundaries of non-singular transformations

by

ISAAC KORNFELD (Fargo, ND) and MICHAEL LIN (Beer-Sheva)

Abstract. It is well known that a weakly almost periodic operator T in a Banach space is mean ergodic, and in the complex case, also λT is mean ergodic for every $|\lambda| = 1$. We prove that a positive contraction on L_1 is weakly almost periodic if (and only if) it is mean ergodic. An example shows that without positivity the result is false. In order to construct a contraction T on a complex L_1 such that λT is mean ergodic whenever $|\lambda| = 1$, but T is not weakly almost periodic, we prove the following: Let τ be an invertible weakly mixing non-singular transformation of a separable atomless probability space. Then there exists a complex function $\varphi \in L_\infty$ with $|\varphi(x)| = 1$ a.e. such that for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ the function $f \equiv 0$ is the only solution of the equation $f(\tau x) = \lambda\varphi(x)f(x)$. Moreover, the set of such functions φ is residual in the set of all complex unimodular measurable functions (with the L_1 topology).

1. Mean ergodicity and weak almost periodicity of L_1 contractions. Motivated by von Neumann's mean ergodic theorem, we call a linear operator T in a (real or complex) Banach space $\mathbf{B}$ *mean ergodic* if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} T^k f \text{ exists for all } f \in \mathbf{B}. \tag{1.1}$$

A mean ergodic operator T which is power-bounded (i.e., $\sup_n \|T^n\| < \infty$) induces the *ergodic decomposition* $\mathbf{B} = \{y \in \mathbf{B} : Ty = y\} \oplus \overline{(I-T)\mathbf{B}}$, and the limit in (1.1) is the projection (corresponding to this decomposition) on the subspace of fixed points.

A linear operator T is called (*weakly*) *almost periodic* if for every $f \in \mathbf{B}$ the orbit $\{T^k f\}_{k\ge 0}$ is (weakly) sequentially compact. A weakly almost periodic (WAP) operator is necessarily power-bounded, and a power-bounded operator in a reflexive space is clearly WAP. Since the closed convex hull of a weakly compact set is weakly compact [DS], a weakly almost periodic

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operator is mean ergodic (see [Y, p. 213], where the result is given in a more general context). In general, the converse is false: a mean ergodic contraction T with T^2 not mean ergodic is not weakly almost periodic (for a simple example, see [Kr, p. 84]). A mean ergodic operator need not even be power-bounded: Hille [Hi] proved that the operator defined on $L_1[0,1]$ by $Tf(x) = f(x) - \int_0^x f(y)\,dy$ is mean ergodic but not power-bounded. The existence of T *positive* mean ergodic on L_p $(1 < p < \infty)$ which is not power-bounded follows from Theorem 4.2 and section VI of [E]; furthermore, all the powers of such a T are also mean ergodic.

In this section we study the relationship between mean ergodicity and weak almost periodicity of contractions on $L_1(X,\mu)$ of a σ-finite measure space.

LEMMA 1.1. *Let T be a positive contraction of $L_1(X,\mu)$. If there is $h \in L_1$ with $Th \le h$ and $h > 0$ a.e., then T is weakly almost periodic.*

Proof. Since we can change the reference measure to any equivalent one [Kr, p. 128], we assume μ finite and $h \equiv 1$, i.e., $T1 \le 1$. Then T is also a contraction of L_2 [Kr, p. 65], so for $f \in L_2$ the sequence $\{T^k f\}$ is weakly sequentially compact in L_2, and hence (since the measure is finite) also in L_1. Since the set of functions $f \in L_1$ for which the sequence $\{T^k f\}$ is weakly sequentially compact in L_1 is closed, T is WAP.

THEOREM 1.2. *Let T be a positive contraction on $L_1(X,\mu)$. Then T is weakly almost periodic if and only if T is mean ergodic.*

Proof. Since every WAP operator on a Banach space is mean ergodic, it is enough to prove the converse.

Let T be a mean ergodic positive contraction on $L_1(X,\mu)$. If μ is σ-finite and infinite, it can be replaced by an equivalent finite measure [Kr, p. 128], so we may and do assume that the measure μ is finite. Recall ([DS, IV.8.11]) that $\{g_n\} \subset L_1(X,\mu)$ is weakly sequentially compact if and only if $\sup_n \|g_n\|_1 < \infty$ and $\sup_n \int_A |g_n|\,d\mu \to 0$ as $\mu(A) \to 0$. Thus, it is enough to show that for $0 \le f \in L_1$ we have

$$\sup_n \int_A T^n f\,d\mu \to 0 \quad \text{as } \mu(A) \to 0.$$

Let C and D denote the conservative and the dissipative parts of T, respectively [Kr]. By Helmberg's result ([Kr], p. 175), the mean ergodicity of T implies that T has a fixed point $0 \le h \in L_1$ with $C = \{h > 0\}$, and $T^{*n}1_D \to 0$ a.e. Hence $\int_D T^n f\,d\mu = \int_D f T^{*n} 1_D\,d\mu \to 0$ by Lebesgue's convergence theorem.

The subspace $L_1(C)$ is T-invariant, and T_C, the restriction of T to $L_1(C)$, has a strictly positive fixed point. By Lemma 1.1, T_C is WAP. Hence, if $0 \le g \in L_1(C)$, then $\sup_n \int_A T^n g\,d\mu \to 0$ as $\mu(A) \to 0$.

Let $0 \le f \in L_1(X,\mu)$. Fix $\varepsilon > 0$, and pick N such that

$$\|1_D T^N f\|_1 = \int_D T^N f\,d\mu < \frac{\varepsilon}{2}.$$

The function $g = 1_C T^N f$ is in $L_1(C)$. Therefore,

$$\begin{aligned}\sup_{n\ge1} \int_A T^n f\,d\mu &= \Big\{\max_{1\le n\le N} \int_A T^n f\,d\mu\Big\} \vee \Big\{\sup_{n>N} \int_A T^n f\,d\mu\Big\}\\ &= \Big\{\max_{1\le n\le N} \int_A T^n f\,d\mu\Big\}\\ &\quad \vee \Big\{\sup_{n>N} \Big[\int_A T^{n-N}(1_C T^N f)\,d\mu + \int_A T^{n-N}(1_D T^N f)\,d\mu\Big]\Big\}\\ &\le \Big\{\max_{1\le n\le N} \int_A T^n f\,d\mu\Big\} \vee \Big\{\sup_{k\ge1} \int_A T^k g\,d\mu + \frac{\varepsilon}{2}\Big\} < \varepsilon\end{aligned}$$

if $\mu(A)$ is small enough. Since $\varepsilon > 0$ is arbitrary, $\sup_n \int_A T^n f\,d\mu \to 0$ as $\mu(A) \to 0$. Hence the contraction T is WAP.

COROLLARY 1.3. *If T is a contraction of L_1 with mean ergodic linear modulus, then T is weakly almost periodic (and therefore mean ergodic).*

Proof. Let $|T|$ be the linear modulus of T [Kr, p. 159]. If $|T|$ is mean ergodic, then it is WAP. Then for any $f \in L_1$ we have

$$\sup_{k\ge1} \int_A |T^k f|\,d\mu \le \sup_{k\ge1} \int_A |T|^k |f|\,d\mu \to 0$$

as $\mu(A) \to 0$. Hence T is WAP.

REMARK. The mean ergodicity of T, under the assumptions of Corollary 1.3, was proved in Proposition 1.1 of [ÇL], together with a.e. convergence of the averages (1.1).

Next, we deal with non-positive contractions of L_1. We show by examples that, in general, properties of T do not yield the corresponding properties for its modulus, and Theorem 1.2 fails without the positivity assumption.

Recall that a *Dunford–Schwartz* (DS) *operator* is a contraction T of $L_1(\mu)$ (where μ is σ-finite), with $\|Tf\|_\infty \le \|f\|_\infty$, and its linear modulus $|T|$ is also DS [K, p. 161].

Lemma 1.1 and Corollary 1.3 show that a DS contraction in a *finite* measure space and its modulus are WAP. Clearly, the property of being DS is *not* invariant under changes of the reference measure. Rather, a contraction T on L_1 is DS under some change of measure if and only if *there exists a measurable function $0 < h < \infty$ a.e. with $|T|h \le h$ a.e.* (and in that case, we have a.e. convergence in (1.1) for every $f \in L_1$). The following examples will be DS contractions in an infinite measure space.

Recall also that a positive contraction T of L_1 is called *ergodic* if the only T^*-invariant functions in L_∞ are the constant functions. T is called *weakly mixing* if it is ergodic, and T^* on the complex L_∞ has no unimodular eigenvalues different from one. Thus, a non-singular transformation τ is weakly mixing if $g\circ\tau=\lambda g$, for $g\in L_\infty(\mu)$ and $|\lambda|=1$, implies $\lambda=1$ and g is constant. Weakly mixing transformations preserving an atomless infinite measure exist [KP].

EXAMPLE 1: *A mean ergodic DS contraction on a real L_1 which is not WAP.* Let μ be an infinite σ-finite measure, and let P be an ergodic and conservative positive Dunford–Schwartz operator in $L_1(\mu)$ such that P^2 is also ergodic (e.g., P is weakly mixing). Since P has no finite invariant measure, it is not mean ergodic. Let $T=-P$. Since P^2 is ergodic, $T^*g=g\in L_\infty$ implies $g=0$. Hence (by the Hahn–Banach theorem) $\overline{(I-T)L_1}=L_1$, and T is mean ergodic. Since $P^n=(-1)^nT^n$, and P is not WAP, T is not WAP. Note that the modulus of T is clearly P, so T is mean ergodic with a non-mean ergodic conservative modulus.

EXAMPLE 2: *An almost periodic DS contraction with non-WAP conservative modulus.* Let μ be an infinite σ-finite measure, and let P be an ergodic and conservative positive Dunford–Schwartz operator in $L_1(\mu)$, with P^2 ergodic, which has a cyclically moving set of period 3, i.e., a set A_0 with

$$P^*1_{A_0}=1_{A_1},\quad P^*1_{A_1}=1_{A_2},\quad P^*1_{A_2}=1_{A_0}$$

such that A_0,A_1,A_2 are disjoint, with union X (e.g., P is a Harris recurrent operator of period 3; such a P is obtained from the random walk on $\mathbb{Z}$ with steps -1 with probability $2/3$ and $+2$ with probability $1/3$. For an example of P induced by a transformation τ preserving an infinite measure, let θ be weakly mixing infinite measure preserving in (Y,ν), and in $X=\{1,2,3\}\times Y$ define $\tau(j,y)=(j\dot{+}1,\theta y)$, where $\dot{+}$ denotes addition mod 3).

Define $T=\frac12(I-P)$. Then T is clearly a DS operator in $L_1(\mu)$. Since for any contraction S on a Banach space $\|((I+S)/2)^n(I-S)\|\to0$, we have $\|T^n(I-T)\|=\frac12\|T^n(I+P)\|\to0$.

If $T^*g=g\in L_\infty$, then $P^*g=-g$, and since P^2 is ergodic, $g\equiv0$. Hence T^* has no non-zero fixed points in L_∞, so $L_1=\overline{(I-T)L_1}$. Now $\|T^n(I-T)\|\to0$ implies that $\|T^nf\|_1\to0$ for every $f\in L_1$, which shows that T is almost periodic.

Let $Q=|T|$ be the modulus of T. Then Q is DS, and obviously $Q\le\frac12(I+P)$. We prove that $Q^*1=1$, which will yield $Q=\frac12(I+P)$. For each $i=0,1,2$ we have

$$\begin{aligned}1&\ge Q^*1\ge Q^*|1_{A_i}-1_{A_{i+1}}|\\&\ge|T^*(1_{A_i}-1_{A_{i+1}})|=|\tfrac12(1_{A_i}-2\cdot1_{A_{i+1}}+1_{A_{i+2}})|\ge1_{A_{i+1}}.\end{aligned}$$

This yields $Q^*1\ge\max_{0\le i\le2}1_{A_i}=1$, which shows that $Q^*1=1$. If $0\le g\le1$ satisfies $Q^*g\le g$, then

$$0\le1-g\le1-Q^*g=Q^*(1-g)=\tfrac12(I+P^*)(1-g),$$

which yields $P^*(1-g)\ge1-g$. Since P is conservative, equality holds a.e., and therefore also $Q^*g=g$ a.e., which proves that Q is conservative.

Assume that $0\le u\in L_1$ satisfies $Qu=u$. Then $Pu=u$. But since P is ergodic and conservative with no finite invariant measure (it preserves an infinite measure), $u\equiv0$. Thus Q has no fixed points except zero, and since Q preserves integrals, it is not mean ergodic, hence not WAP.

DEFINITION. A power bounded operator T on a complex Banach space $\mathbf{B}$ is called *totally mean ergodic* (TME) if for every complex λ with $|\lambda|=1$, the operator λT is mean ergodic.

A weighted ergodic theorem for TME operators was obtained in [ÇLO]. Clearly, a weakly almost periodic operator is TME. In complex L_1 spaces, the next example (which depends on the main result of the next section) improves upon Example 1.

EXAMPLE 3: *A totally mean ergodic DS contraction on a complex L_1 which is not WAP.* Let τ be an invertible non-singular ergodic transformation of a separable atomless probability space $(X,\mathcal{A},\mu)$, which has an equivalent infinite σ-finite invariant measure ν (so it has no finite invariant measure), and assume that τ is weakly mixing.

Let $Pf(x)=f(\tau x)$. The invariance of ν implies that P is a positive DS operator on $L_1(\nu)$, with $P^*g(x)=g(\tau^{-1}x)$.

In the next section, we prove that under our assumptions on τ, there exists a complex measurable function φ with $|\varphi(x)|=1$ a.e. such that for every complex λ with $|\lambda|=1$, $f\equiv0$ is the only measurable solution of the equation $f(\tau x)=\lambda\varphi(x)f(x)$.

We define $Tf(x)=\varphi(x)Pf(x)$. Then $T^*g=P^*(\varphi g)=(\varphi g)\circ\tau^{-1}$. Clearly, T is DS on $L_1(\nu)$, and its modulus is P.

Fix λ with $|\lambda|=1$. To show that λT is mean ergodic, we show that its adjoint has no non-zero fixed points. Assume $\lambda T^*g=g$ for some $g\in L_\infty$. Then $\lambda\varphi(\tau^{-1}x)g(\tau^{-1}x)=g(x)$ a.e., which means $\lambda\varphi(x)g(x)=g(\tau x)$ a.e. By the choice of φ we have $g=0$ a.e. Hence $\lim_{n\to\infty}\left\|\frac1n\sum_{k=1}^n(\lambda T)^kf\right\|_1=0$ for every $f\in L_1(\nu)$. Thus, T is totally mean ergodic.

Finally, T is not WAP. If it were, then $\{|P^nf|\}=\{|T^nf|\}$ would be weakly sequentially compact, implying that P is WAP. But P is not mean ergodic, hence not WAP, since by ergodicity of τ, P has no fixed points in $L_1(\nu)$.

Note that under the change of the measure from ν to μ, the operators P and T on $L_1(\nu)$ are represented in $L_1(\mu)$ as $\widetilde{P}f(x) = h(x)f(\tau x)/h(\tau x)$ and $\widetilde{T}f(x) = \varphi(x)\widetilde{P}f(x)$ for $f \in L_1(\mu)$, where $h = d\nu/d\mu$.

A natural question is whether Theorem 1.2 can be extended to higher dimensions. A set of d commuting power-bounded operators $\{T_1, \ldots, T_d\}$ in a Banach space $\mathbf{B}$ generates an operator representation of $\mathbb{Z}_+^d$, defined by $T(k_1, \ldots, k_d) = T_1^{k_1} \ldots T_d^{k_d}$. The representation is called *mean ergodic* if

$$\lim_{n\to\infty} \frac{1}{n^d} \sum_{j=1}^{d} \sum_{k_j=1}^{n} T(k_1, \ldots, k_d) f \text{ exists for all } f \in \mathbf{B}. \tag{1.2}$$

The representation generated by mean ergodic commuting operators $\{T_j\}_{j=1}^d$ is mean ergodic ([ÇL]). However, mean ergodicity of a representation does not imply that of its generators: for a contraction T which is not mean ergodic, take $S = -\frac{1}{2}(I + T)$. Then S^* has no non-zero fixed points, hence

$$\Big\|\frac{1}{n^2}\sum_{k=1}^{n}\sum_{j=1}^{n} T^k S^j f\Big\| = \Big\|\Big(\frac{1}{n}\sum_{k=1}^{n} T^k\Big)\Big(\frac{1}{n}\sum_{j=1}^{n} S^j f\Big)\Big\| \to 0 \quad \text{for all } f \in \mathbf{B}.$$

A representation $T(k_1, \ldots, k_d)$ of $\mathbb{Z}_+^d$ is called *weakly almost periodic* if for every $f \in \mathbf{B}$ the orbit $\{T(k_1, \ldots, k_d)f : (k_1, \ldots, k_d) \in \mathbb{Z}_+^d\}$ is weakly sequentially compact. This clearly implies that each generator T_j is WAP. Hence, weak almost periodicity of a $\mathbb{Z}_+^d$ representation implies mean ergodicity of its generators, and therefore also mean ergodicity of the representation itself.

We now show that the multidimensional analogue of Theorem 1.2 is false.

EXAMPLE 4: *A representation of $\mathbb{Z}_+^2$ by positive L_1 contractions which is mean ergodic but not WAP.* Let $X_1 = \{1\} \times [0,1]$ and $X_2 = \{2\} \times [0,1]$ be two disjoint unit intervals, and $X_0 = \{0\}$. Let θ be a conservative non-singular transformation of $[0,1]$ with no absolutely continuous finite invariant measure. We define two transformations on $X = X_0 \cup X_1 \cup X_2$ (with μ the sum of Lebesgue measures on the intervals and the Dirac measure at 0) as follows:

$$\tau 0 = 0, \quad \tau(1,y) = (1,\theta y), \quad \tau(2,z) = 0;$$
$$\sigma 0 = 0, \quad \sigma(2,z) = (2,\theta z), \quad \sigma(1,y) = 0.$$

Then $\tau\sigma x = 0 = \sigma\tau x$ for every $x \in X$. Let T and S be the $L_1(X,\mu)$ preduals of the L_∞ contractions induced by τ and σ. Then T and S commute. Since $T^k S^j = TS$ for every $k > 0$ and $j > 0$, the representation generated by T and S is mean ergodic. The conservative part of T is $X_1 \cup X_0$, but every finite invariant measure for τ vanishes on X_1. Hence T is not mean ergodic (and similarly S is not mean ergodic). Therefore T is not WAP, so the representation is not WAP.

REMARKS. 1. Standard arguments show that if $\{T_1, \ldots, T_d\}$ are commuting *conservative* positive L_1 contractions, then mean ergodicity of the $\mathbb{Z}_+^d$ representation they generate implies the existence of a function $u \in L_1$ with $u > 0$ a.e. such that $T_j u = u$ for $1 \le j \le d$. In that case, as in Lemma 1.1, the representation is WAP, and each T_j is mean ergodic.

2. If we take $X = X_0 \cup X_1$ in the previous example and consider the restrictions of τ and σ to X, then the representation is mean ergodic, T is conservative and not mean ergodic, while S is mean ergodic and not conservative.

3. The equivalence of (i) and (iii) of the next theorem is a limited multidimensional version of Theorem 1.2.

THEOREM 1.4. *Let $\{T_1, \ldots, T_d\}$ be commuting positive contractions of $L_1(X,\mu)$. Then the following conditions are equivalent:*

(i) *Each T_j is mean ergodic.*
(ii) *Each T_j is weakly almost periodic.*
(iii) *The representation of $\mathbb{Z}_+^d$ generated by $\{T_j\}_{j=1}^d$ is weakly almost periodic.*

Proof. (i)⇒(ii) follows from Theorem 1.2, and (iii)⇒(ii)⇒(i) was previously discussed. Note that we can assume that μ is finite, otherwise we change the reference measure. For the proof of (ii)⇒(iii) we need the following modification of the criterion for weak sequential compactness in L_1.

A bounded sequence $\{g_n\} \subset L_1(\mu)$, with μ finite, is weakly sequentially compact if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sup_n \int |g_n| h\, d\mu < \varepsilon$ for every measurable h with $0 \le h \le 1$ and $\int h\, d\mu < \delta$.

The "if" direction follows from the standard criterion ([DS], IV.8.11) (by applying the condition only to indicator functions).

Assume now that $\{g_n\}$ is weakly sequentially compact, and without loss of generality $\sup_n \|g_n\|_1 \le 1$. For $\varepsilon > 0$ the standard criterion yields a $\delta_1 > 0$ such that $\sup_n \int_A |g_n|\, d\mu < \varepsilon/2$ whenever $\mu(A) < \delta_1$. Let $\delta = \min\{\delta_1^2, \varepsilon^2/4\}$. If $0 \le h \le 1$ with $\int h\, d\mu < \delta$, then $\mu(\{h > \sqrt{\delta}\}) < \sqrt{\delta} \le \delta_1$. Hence for every n we have

$$\int |g_n| h\, d\mu \le \sqrt{\delta}\|g_n\|_1 + \int_{\{h>\sqrt{\delta}\}} |g_n|\, d\mu \le \frac{\varepsilon}{2}\|g_n\|_1 + \frac{\varepsilon}{2} \le \varepsilon.$$

End of the proof of Theorem 1.4. For simplicity, the proof of (ii)⇒(iii) is given for the case $d = 2$ (still assuming μ finite). Define $T_1 = T$ and $T_2 = S$. We first show that $\{T^k S^j 1 : k \ge 0,\ j \ge 0\}$ is weakly sequentially compact in L_1. Let $\varepsilon > 0$. Since S is WAP, the modified criterion yields a

$\delta_1 > 0$ such that $\sup_j \int hS^j 1\,d\mu < \varepsilon$ for $0 \le h \le 1$ with $\int h\,d\mu < \delta_1$. Since T is WAP, there is $\delta > 0$ such that $\sup_k \int_A T^k 1\,d\mu < \delta_1$ whenever $\mu(A) < \delta$. Let $\mu(A) < \delta$. Then for every k we have $\int T^{*k} 1_A\,d\mu = \int_A T^k 1\,d\mu < \delta_1$, and taking $T^{*k}1_A$ for h, we obtain

$$\sup_j \int_A T^k S^j 1\,d\mu = \sup_j \int S^j 1 T^{*k} 1_A\,d\mu < \varepsilon.$$

Hence $\{T^k S^j 1 : j \ge 0,\ k \ge 0\}$ is weakly sequentially compact in L_1 by the standard criterion. By positivity of T and S we see that $\{T^k S^j f : j \ge 0, k \ge 0\}$ is weakly sequentially compact in L_1 for every bounded f. Finally, since the set of functions $f \in L_1$ for which the sequence $\{T^k S^j f : j \ge 0, k \ge 0\}$ is weakly sequentially compact is closed in L_1, the representation is WAP.

2. Multiplicative coboundaries of non-singular transformations. The main result of this section is Theorem 2.1 about multiplicative coboundaries of a non-singular invertible weakly mixing transformation τ. When τ preserves a *finite* measure, the result was proved in [JP], but this special case is not sufficient for the construction in Example 3 of the previous section. Our result does not need even a σ-finite invariant measure. Part of the argument below is inspired by the proof of Theorem 3 in [JP], but the absence of an invariant probability requires some additional ideas.

THEOREM 2.1. *Let τ be an invertible weakly mixing non-singular transformation of a separable atomless probability space $(X, \mathcal{A}, \mu)$. There exists a complex function $\varphi \in L_\infty(X)$, $|\varphi(x)| = 1$ μ-a.e., such that for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, the equation*

$$f(\tau x) = \lambda\varphi(x) f(x) \tag{2.1}$$

has only the trivial solution $f = 0$ a.e.

Moreover, the set of such functions φ is residual in the set $U = \{f \in L_1(X) : |f(x)| \equiv 1\}$ (with the L_1 topology).

Proof. The set U, endowed with the L_1 metric and with pointwise multiplication as a group operation, becomes a complete separable metric (commutative) group.

Let $\widetilde{U}$ be the quotient group U/K, where $K \cong \mathbf{S}^1$ is the subgroup of constant unimodular complex functions, and let $\pi : U \to \widetilde{U}$ be the natural homomorphism taking every $f \in U$ to its coset $\widetilde{f} \in \widetilde{U}$.

Define $\varrho : U \to U$ by $\varrho(f) = f(\tau x)/f(x)$, $f \in U$. Then ϱ is a homomorphism of U with $\mathrm{Ker}(\varrho) = K$ (by ergodicity), and we can therefore define $\widetilde{\varrho} : \widetilde{U} \to \widetilde{U}$ by

$$\widetilde{\varrho}(\widetilde{f}) = \pi \circ \varrho(f), \quad \text{where } f \in \pi^{-1}(\widetilde{f}). \tag{2.2}$$

Clearly, $\widetilde{\varrho}$ is a homomorphism of $\widetilde{U}$, which is injective by the weak mixing property of τ. If $\{f_n\} \subset U$ converges in U (i.e., in L_1-norm) to f, then we have a.e. convergence for a subsequence, say $\{f_{n_k}\}$, and Lebesgue's theorem yields that $\|\varrho(f_{n_k}) - \varrho(f)\|_1 \to 0$. This implies continuity of ϱ. To obtain continuity of $\widetilde{\varrho}$ on $\widetilde{U}$ (with the quotient topology, in which $\widetilde{U}$ is also a complete separable metric group), let $\widetilde{f}_n \to \widetilde{f}$ in $\widetilde{U}$, which means that there exists a sequence $\{\lambda_n\}$ with $|\lambda_n| = 1$ such that $\|\lambda_n f_n - f\|_1 \to 0$. Then

$$\widetilde{\varrho}(\widetilde{f}_n) = \pi(\varrho(\lambda_n f_n)) \to \pi(\varrho(f)) = \widetilde{\varrho}(\widetilde{f}).$$

Fix $\varphi \in U$. If it does not satisfy the desired requirements, then there is a λ with $|\lambda| = 1$ for which (2.1) has a non-zero solution f (actually, there is only one such λ, by weak mixing). By ergodicity of τ, the solution f is unique up to a multiplicative constant, and its absolute value $|f|$ is constant a.e., so assume $|f| = 1$ a.e. We therefore conclude that $\lambda\varphi$ is a *multiplicative coboundary* for τ:

$$\lambda\varphi(x) = \frac{f(\tau x)}{f(x)} \quad \text{a.e.}$$

Hence, $\lambda\varphi \in \varrho U$, and $\widetilde{\varphi} = \pi \circ \varphi \in \widetilde{\varrho}\widetilde{U}$.

By the open mapping theorem for homomorphisms of complete separable metric groups (which is obtained from [Ke, p. 213], or can be proved as in the classical vector space case, using a right-invariant metric), if $\mathrm{Im}(\widetilde{\varrho})$ is not of first category, then it is a closed (and open) subgroup of $\widetilde{U}$, and $\widetilde{\varrho}$ is an open mapping onto its image. In that case, $\widetilde{\varrho}$, being injective, is a bijection, and the inverse mapping $\widetilde{\varrho}^{-1}$ is continuous on $\mathrm{Im}(\widetilde{\varrho})$.

The proof of the theorem will be obtained from the central Lemma 2.2 below. We first introduce the necessary notation.

Let $(\Omega, \mathcal{B}, m)$ be a probability space. For a measurable function $h : \Omega \to [0, 1)$, its *distribution function* $\mathcal{D}_h$ is defined by $\mathcal{D}_h(t) := m(\{x \in \Omega : h(x) \le t\})$, for $t \in [0, 1]$. If $f(x) = \exp(2\pi i h(x))$, we also call $\mathcal{D}_h(t)$ the distribution function of f; in other words, $\mathcal{D}_f(t) = m(\{x \in \Omega : \arg f(x) \le 2\pi t\})$. The distribution function of a probability measure ν on $[0, 1]$ is $\mathcal{D}_\nu(t) := \nu([0, t])$.

Denote by $\mathcal{M}_n$ the set of all (discrete) probability measures on $[0, 1) \cong \mathbf{S}^1$ concentrated on the set of points $\{j/n : 0 \le j < n\}$. Every probability $\nu \in \mathcal{M}_n$ can (and will) be naturally identified with the n-dimensional vector $\{\nu_j\}_{j=0}^{n-1}$, with $\nu_j = \nu(j/n)$. By this identification, the distance $d_n(\nu, \nu') = \|\nu - \nu'\|$ between two probabilities $\nu, \nu' \in \mathcal{M}_n$ equals the ℓ_1^n-distance between the corresponding vectors: $d_n(\nu, \nu') = \sum_{j=0}^{n-1} |\nu_j - \nu'_j|$. We denote by $l^{(n)}$ the uniform probability in $\mathcal{M}_n$, i.e., $l^{(n)}(j/n) = 1/n$, $0 \le j < n$. The Dirac measure at 0, i.e., the point-mass probability concentrated at zero, will be denoted by δ, and $\delta \in \mathcal{M}_n$ for every n.

LEMMA 2.2. *There exists a sequence $\{f_n\}$ of functions in U such that*

(2.3) $$\pi\left(\frac{f_n(\tau x)}{f_n(x)}\right) \to \mathbf{1} \quad \text{in } \widetilde{U};$$

(2.4) *for all $\{\lambda_n\}$ with $|\lambda_n| = 1$, $\mathcal{D}_{\lambda_n f_n}(t) \nrightarrow \mathcal{D}_\delta(t)$ in measure.*

We first deduce Theorem 2.1 from this lemma.

Assume that $\operatorname{Im}(\widetilde{\varrho})$ is not of first category. We saw that in that case $\widetilde{\varrho}$ is a bijection onto its image with continuous inverse. The sequence $\{f_n\}$ from the lemma satisfies $\widetilde{\varrho}(\widetilde{f}_n) \to 1$ by (2.3). Continuity of $\widetilde{\varrho}^{-1}$ yields $\widetilde{f}_n \to 1$, which means that there exists a sequence $\{\lambda_n\}$ with $|\lambda_n| = 1$ such that $\|\lambda_n f_n - 1\|_1 \to 0$, which contradicts (2.4). Hence $\widetilde{\varrho}$ is not surjective, so $\operatorname{Im}(\widetilde{\varrho})$ is a set of first category in $\widetilde{U}$, i.e., $\operatorname{Im}(\widetilde{\varrho}) = \bigcup_{i=1}^{\infty} \widetilde{N}_i$, with each $\widetilde{N}_i$ nowhere dense in $\widetilde{U}$. It follows that $\pi^{-1}(\operatorname{Im}(\widetilde{\varrho})) = \bigcup_{i=1}^{\infty} \pi^{-1}(\widetilde{N}_i)$ is a set of first category in U. Then every $\varphi \in V = U - \pi^{-1}(\operatorname{Im}(\widetilde{\varrho}))$ satisfies the desired requirements, since otherwise, as we have seen, $\widetilde{\varphi} = \pi \circ \varphi$ would be in $\widetilde{\varrho}\widetilde{U}$.

Proof of Lemma 2.2. Let ϕ be the Radon–Nikodym derivative $d\mu \circ \tau / d\mu$. Without loss of generality we can (and will) assume that

(2.5) $$1/2 \le \phi(x) \le 2 \quad \mu\text{-a.e.},$$

since otherwise the measure μ can be replaced by another measure, $\widetilde{\mu}$, equivalent to it,

$$\widetilde{\mu} = \sum_{n=-\infty}^{\infty} 2^{-|n|-1} \mu \circ \tau^n,$$

whose Radon–Nikodym derivative $\widetilde{\phi}(x) = d\widetilde{\mu} \circ \tau / d\widetilde{\mu}$ obviously takes values between 1/2 and 2.

We fix n, and construct the function f_n. To simplify notation, we will often suppress the subscript n from the parameters of the construction.

First, we apply the "non-singular version" of the Rokhlin lemma (see, for example, [W]) to the transformation τ, and get a tower $\{\tau^k A\}_{0 \le k < n}$ with base A, i.e., A is measurable and satisfies

(2.6) $$\tau^k A \cap \tau^j A = \emptyset, \quad 0 \le k \ne j < n; \quad \mu\Big(\bigcup_{k=0}^{n-1} \tau^k A\Big) > 1 - \frac{1}{n}.$$

It can be shown [W, p. 94] that (even in the general non-singular case) we can have a tower which also satisfies

(2.7) $$\mu(\tau^{n-1} A) < 1/n.$$

In the sequel, $\mathcal{T}$ will denote the collection of sets $\{\tau^k A\}_{0 \le k < n}$, as well as their union.

Next, we partition the space X into finitely many subsets B_r, $1 \le r \le m_n$, on each of which the Radon–Nikodym derivative $\phi = d\mu \circ \tau / d\mu$ is "approximately constant". More precisely, for each r, $1 \le r \le m_n$, we want to have

(2.8) $$\frac{\sup\{\phi(x) : x \in B_r\}}{\inf\{\phi(x) : x \in B_r\}} < 1 + n^{-2}.$$

In order to ensure (2.8), it is enough to choose for m_n any m satisfying $4^{1/m} < 1 + n^{-2}$, and take $B_r = \phi^{-1}(\Delta_r)$ for $1 \le r \le m$, where $\Delta_r = \big[\frac{1}{2} \cdot 4^{(r-1)/m}, \frac{1}{2} \cdot 4^{r/m}\big)$ for $1 \le r < m$, and $\Delta_m = \big[\frac{1}{2} \cdot 4^{(m-1)/m}, 2\big]$. Note that the sets $\{B_r\}_{1 \le r \le m}$ are disjoint, and, by (2.5), their union is the entire space X mod 0.

For each k, $0 \le k < n$, denote by ξ_k the partition of the set $A_k = \tau^k A \subset \mathcal{T}$ into the sets $A_{k,r} = \tau^k A \cap B_r$, $1 \le r \le m$. Every partition ξ_k can be "pulled down" to the base A of the tower $\mathcal{T}$, to yield a partition $\tau^{-k}\xi_k$ of A. Let $\eta = \bigvee_{k=0}^{n-1} \tau^{-k}\xi_k$ be the common refinement of all these partitions of A, i.e., the elements of η are all possible (non-empty mod 0) intersections of the form $\bigcap_{k=0}^{n-1} \tau^{-k} C_k$, where each C_k is an element of ξ_k, $0 \le k < n$. Denote by $s = s_n$ the number of elements of η, and by $E_1, \ldots, E_s$ the elements themselves.

We can now consider the tower $\mathcal{T}$ as the disjoint union of the "columns" $\mathcal{C}_p$, $1 \le p \le s$, where $\mathcal{C}_p = \bigcup_{k=0}^{n-1} \tau^k E_p$. The construction of the desired function f_n will be carried out on each column $\mathcal{C}_p$ separately. In order to further simplify notation, we will now consider p fixed (n is still kept fixed), and suppress not only the subscript n, but also the subscript p from the parameters of the construction.

Partition the base set $E = E_p$ of the column $\mathcal{C} = \mathcal{C}_p$ into n subsets $D_1, \ldots, D_n$ of equal measure: for each t, $1 \le t \le n$, $\mu(D_t) = \frac{1}{n}\mu(E)$. This allows us to consider the column $\mathcal{C}$ as the disjoint union of the subcolumns $\mathcal{C}^{(t)}$, $1 \le t \le n$, where $\mathcal{C}^{(t)} = \bigcup_{k=0}^{n-1} \tau^k D_t$.

We are now ready to define the function f_n on the column $\mathcal{C}$. We set

$$f_n(x) = \exp(2\pi i h_n(x)),$$

where the function h_n is defined on $\mathcal{C} = \bigcup_{t=1}^{n} \bigcup_{k=0}^{n-1} \tau^k D_t$ by

$$h_n(x) = \frac{k+t-1}{n} \pmod 1 \quad \text{for } x \in \tau^k D_t.$$

In other words, on every subcolumn $\mathcal{C}^{(t)}$ the function h_n takes all the values j/n, $0 \le j \le n-1$, in cyclic order, starting with zero on the base of the subcolumn $\mathcal{C}^{(1)}$, starting with $1/n$ on the base of the subcolumn $\mathcal{C}^{(2)}$, and so on. This means that on each "horizontal" set $E^{(k)} := \tau^k E$ of the column $\mathcal{C}$ all the values j/n, $0 \le j \le n-1$, are assumed. Let

$$E^{(k,j)} = \{x \in E^{(k)} : h_n(x) = j/n\}, \quad 0 \le k < n,\ 0 \le j < n.$$

For $0 \le k < n$, denote by $\nu^{(k)}$ the probability measure on $[0,1)$ concentrated on the points j/n, $0 \le j < n$ (so, $\nu^{(k)}$ is in $\mathcal{M}_n$), given by the probability vector $\{\nu_j^{(k)}\}_{j=0}^{n-1}$, where $\nu_j^{(k)} = \mu(E^{(k,j)})/\mu(E^{(k)})$. Note that the measure $\nu^{(0)}$ (the measure corresponding to the "bottom" level $E^{(0)} = E$ of the column $\mathcal{C}$) is exactly the uniform measure $l^{(n)}$.

It follows from (2.8) that τ maps every level $E^{(k)}$ of the column $\mathcal{C}$ to the next level $E^{(k+1)}$ "almost linearly". This guarantees that for every k, $0 \le k < n$, the measure $\nu^{(k)}$ is close, in the sense of the metric d_n, to the uniform measure $l^{(n)}$. To be precise, (2.8) yields that for every k, $0 \le k < n$,

$$\frac{\max_j \nu_j^{(k)}}{\min_j \nu_j^{(k)}} \le (1+n^{-2})^k \le (1+n^{-2})^n \le 1 + 2n^{-1}.$$

This, in turn, implies that $|\nu_j^{(k)} - 1/n| \le 2/n^2$ for each j, so

$$d_n(\nu^{(k)}, l^{(n)}) \le 2/n, \qquad 0 \le k < n. \tag{2.9}$$

The above construction can be carried out on every column $\mathcal{C} = \mathcal{C}_p$, $1 \le p \le s$. So, we have the function f_n defined on the entire tower $\mathcal{T}$. Outside the tower $\mathcal{T}$ the function f_n is defined to be zero.

From now on, the subscript p will be "reinstalled" (still keeping n fixed). In particular, the set $E^{(k)}$ (the kth level of the column $\mathcal{C} = \mathcal{C}_p$) will now be denoted by $E_p^{(k)}$, and the measure $\nu^{(k)}$ corresponding to $E_p^{(k)}$ will be denoted by $\nu^{(k,p)}$.

Define a measure $\nu_{\mathcal{T}} \in \mathcal{M}_n$ by

$$\nu_{\mathcal{T}}(j/n) = \mu_{\mathcal{T}}(\{x \in \mathcal{T} : h_n(x) = j/n\}), \qquad 0 \le j < n,$$

where $\mu_{\mathcal{T}}$ is the conditional measure $\mu/\mu(\mathcal{T})$. In other words, $\nu_{\mathcal{T}} = \mu_{\mathcal{T}} \circ h_n^{-1}$.

It is clear that the measure $\nu_{\mathcal{T}}$ is a convex combination of the measures $\nu^{(k,p)}$, $0 \le k < n$, $1 \le p \le s$:

$$\nu_{\mathcal{T}} = \sum_{k=0}^{n-1} \sum_{p=1}^{s} \alpha_p^{(k)} \nu^{(k,p)},$$

where $\alpha_p^{(k)} = \nu_{\mathcal{T}}(E_p^{(k)})$.

Due to (2.9), the triangle inequality yields

$$d_n(\nu_{\mathcal{T}}, l^{(n)}) = \|\nu_{\mathcal{T}} - l^{(n)}\|_1 \le \sum_{k=0}^{n-1} \sum_{p=1}^{s} \alpha_p^{(k)} \|\nu^{(k,p)} - l^{(n)}\|_1 \le \frac{2}{n}. \tag{2.10}$$

Thus, for each n we have defined f_n, using a Rokhlin tower $\mathcal{T} = \mathcal{T}_n$, and have constructed a probability $\nu_{\mathcal{T}_n}$ which satisfies (2.10).

Hence, for every $t \in [0,1]$ we have $|\mathcal{D}_{\nu_{\mathcal{T}_n}}(t) - \mathcal{D}_{l^{(n)}}(t)| \le \|\nu_{\mathcal{T}_n} - l^{(n)}\|_1 \to 0$ as $n \to \infty$. Since $\{l^{(n)}\}$ converges in distribution to l, the uniform (Lebesgue) measure on $[0,1)$, also the measures $\nu_{\mathcal{T}_n}$ converge in distribution to l.

Finally, define a measure $\nu^{(n)} \in \mathcal{M}_n$ by

$$\nu^{(n)}(j/n) = \mu(\{x \in \mathcal{T}_n : h_n(x) = j/n\}), \qquad 0 \le j < n.$$

In other words, $\nu^{(n)} = \mu \circ h_n^{-1}$.

It follows from (2.6) that $d_n(\nu^{(n)}, \nu_{\mathcal{T}_n}) \to 0$ as $n \to \infty$. This shows that also the probability measures $\nu^{(n)}$ converge in distribution to the Lebesgue measure on $[0,1)$. Since the distribution functions $\mathcal{D}_{f_n}$ and $\mathcal{D}_{\nu^{(n)}}$ are the same, (2.4) follows. Note also that (2.7) and the construction of the functions h_n guarantee that

$$\mu(\{x \in X : |h_n(\tau x) - h_n(x)| \ge 1/n\}) \to 0$$

as $n \to \infty$. This yields (2.3) and therefore completes the proofs of Lemma 2.2 and Theorem 2.1.

REMARK. Equation (2.1) for $\lambda = 1$ is sometimes called the *multiplicative cohomological equation* for τ, and any $\varphi \in U$ which is of the form $\varphi(x) = f(\tau x)/f(x)$ for some $f \in U$ is called a *multiplicative coboundary* for τ. A measurable function $u(x)$ which is of the form $u(x) = h(\tau x) - h(x)$ for some real (complex) measurable function is called an *additive real* (*complex*) *coboundary* for τ, and h is called a *transfer function* for u. Obviously, if u is an additive real coboundary, then $\varphi(x) = \exp(2\pi i u(x))$ is a multiplicative coboundary, but the converse is false in general. This shows that constructing functions which are not multiplicative coboundaries (as done in Theorem 2.1) can be harder than constructing functions which are not additive real coboundaries.

The relationship between additive real coboundaries and multiplicative coboundaries has been studied in [HOkOs], [Sc], [MSc], [He-1], [He-2]. Some recent results on cohomology of non-singular transformations can be found in [IY].

In [JP], a unimodular function φ is called a *weak* (*multiplicative*) *coboundary* for τ if for some complex λ with $|\lambda| = 1$, the function $\lambda\varphi$ is a multiplicative coboundary. In this terminology, Theorem 2.1 says that *if τ is weakly mixing, then the set of weak multiplicative coboundaries is of first category in the set of all complex measurable unimodular functions, with the L_1 topology.*

THEOREM 2.3. *Let τ be an invertible ergodic non-singular transformation of a separable atomless probability space $(X, \mathcal{A}, \mu)$. Then the set of its multiplicative coboundaries is of first category in the set of complex unimodular functions, with the L_1 topology.*

Proof. We use the notation in the proof of Theorem 2.1. If $\widetilde{f} = \widetilde{g}$, then obviously $\varrho(f) = \varrho(g)$, so we can define $\widehat{\varrho} : \widetilde{U} \to U$ by $\widehat{\varrho}(\widetilde{f}) = \varrho(f)$. It can be checked that $\widehat{\varrho}$ is continuous, and injective (by ergodicity).

Ergodicity of τ is sufficient for the proof of Lemma 2.2, which in fact gives a sequence $\{f_n\} \subset U$ such that $f_n(\tau x)/f_n(x) \to 1$ in L_1-norm, while $\mathcal{D}_{f_n} \not\to \mathcal{D}_\delta$ in measure.

The set of multiplicative coboundaries is exactly the image of $\widehat{\varrho}$. If this image is not of first category in U, then it is a closed subgroup of U, and $\widehat{\varrho}$ is an open mapping onto it (again by the open mapping theorem). As in the proof of Theorem 2.1, we use the sequence $\{f_n\}$ to show that the image of $\widehat{\varrho}$ must be of first category.

REMARK. For τ probability preserving, Theorem 2.3 was proved in [JP].

The following theorem shows that under the assumptions of Theorem 2.3 the set of multiplicative coboundaries, though of first category, is dense in U.

THEOREM 2.4. *Let τ be an ergodic non-singular transformation of an atomless probability space $(X, \mathcal{A}, \mu)$. Then the set of its multiplicative coboundaries is dense in the set of complex unimodular functions, with the L_1 topology.*

Proof. For a complex number $z \neq 0$, denote by $\operatorname{Arg}(z)$ its unique angle in $[0, 2\pi)$.

Assume first that τ has no invariant probability absolutely continuous with respect to μ. By [Kr, p. 141], there exists $h_0 \in L_\infty$, $h_0 > 0$ a.e., with

$$\left\| \frac{1}{N} \sum_{n=1}^{N} h_0(\tau^n x) \right\|_\infty \to 0.$$

Define $A_j = \{x : h_0(x) > 1/j\}$. Then $\{A_j\}$ is an increasing sequence with $\bigcup_j A_j = X$, and

$$\lim_N \left\| \frac{1}{N} \sum_{n=1}^{N} 1_{A_j}(\tau^n x) \right\|_\infty \to 0 \quad \text{for all } j \geq 1. \tag{2.11}$$

For $\varphi \in U$, let $\psi(x) = \operatorname{Arg}(\varphi(x))$. Then $\|\psi 1_{A_j} - \psi\|_1 \to 0$. For each j we have $\lim_N \left\| \frac{1}{N} \sum_{n=1}^{N} (\psi 1_{A_j})(\tau^n x) \right\|_\infty \to 0$, so by the Yosida mean ergodic theorem [Kr, Theorem 2.1.3], $\psi 1_{A_j}$ is in the L_∞-closure of $\{h \circ \tau - h : h \in L_\infty\}$. Hence there exists a sequence of bounded *real* measurable functions $\{h_k\}$ such that $h_k \circ \tau - h_k \to \psi$ in L_1-norm. Then (using a.e. convergence along subsequences) $\exp(ih_k) \circ \tau / \exp(ih_k) \to \exp(i\psi) = \varphi$ in L_1-norm.

We now prove the theorem when τ does have a finite invariant measure $\nu \ll \mu$. This implies that the conservative part C is not null, and the ergodicity of τ implies that C is the support of ν, while almost every point

of the dissipative part D is eventually moved by τ into C (we do not assume invertibility, so D need not be null). For $j \geq 1$, let $D_j = \{x \in D : \tau^{j-1}x \notin C,\ \tau^j x \in C\}$. Then $A_j := \bigcup_{i=1}^{j} D_i$ is increasing to D, and $\{A_j\}$ satisfies (2.11) (since $\sum_{n=1}^{\infty} 1_{D_i}(\tau^n x) \leq 1$). This yields, as before, that every bounded f supported in D is the L_1-limit of a sequence $\{h_k \circ \tau - h_k\}$, with each $h_k \in L_\infty(X, \mu)$.

Let $f \in L_\infty(X, \mu)$ with $f = 0$ a.e. on D. If there is a bounded function h' defined on C, with $h' \circ \tau - h' = f$ a.e. on C, we define h on all of X by $h = h'$ on C, and by $h(x) = h'(\tau^j x)$ for $x \in D_j$. Then $h \circ \tau - h = f$ a.e. on X. We first assume that ν is the restriction of μ to C. The restriction of τ to C preserves ν, and is also ergodic, so, by the mean ergodic theorem (and denseness of $L_\infty(C, \nu)$ in $L_1(C, \nu)$), the set $\{g \circ \tau - g : g \in L_\infty(C)\}$ is L_1-dense in the set of ν-integrable functions (on C) with zero integral. Now let $f \in L_1(\mu)$ with $\int f\,d\nu = 0$ and $f = 0$ a.e. on D. Using the previously shown extension of transfer functions from C to X, we obtain a sequence of bounded functions $\{h_k\}$ with $h_k \circ \tau - h_k \to f$ in L_1-norm. Combining this with what was obtained for bounded f supported on D, we see that for any f bounded with $\int_C f\,d\mu = \int f\,d\nu = 0$ there is a sequence $\{h_k\} \in L_\infty$ such that $h_k \circ \tau - h_k \to f$ in L_1-norm. Fix $\varphi \in U$, and $\alpha := \int \operatorname{Arg}(\varphi(x))\,d\nu = \int_C \operatorname{Arg}(\varphi(x))\,d\mu$. Since μ is atomless, there is a set $A \subset C$ with $\mu(A) = \alpha/(2\pi)$. Define $\psi(x) = \operatorname{Arg}(\varphi(x)) - 2\pi 1_A(x)$. Then $\exp(i\psi(x)) = \varphi(x)$, and $\int_C \psi\,d\mu = 0$. If $h_k \circ \tau - h_k \to \psi$ in L_1-norm, then, as before, $\exp(ih_k) \circ \tau / \exp(ih_k) \to \exp(i\psi) = \varphi$ in L_1-norm.

Finally, we drop the assumption that $\mu|_C$ is invariant, and let μ' be an equivalent probability such that its restriction to C is invariant. We saw that $\varphi \in U$ is the limit in $L_1(\mu')$ of a sequence of multiplicative coboundaries, and therefore it is also the limit in $L_1(\mu)$ of the same sequence.

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North Dakota State University
Fargo, ND 58105, U.S.A.
E-mail: kornfeld@plains.nodak.edu

Ben-Gurion University of the Negev
Beer-Sheva, Israel
E-mail: lin@cs.bgu.ac.il

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Universal images of universal elements

by

LUIS BERNAL-GONZÁLEZ (Sevilla)

Abstract. We furnish several necessary and sufficient conditions for the following property: For a topological space X, a continuous selfmapping S of X and a family τ of continuous selfmappings of X, the image under S of every τ-universal element is also τ-universal. An application in operator theory, where we extend results of Bourdon, Herrero, Bes, Herzog and Lemmert, is given. In particular, it is proved that every hypercyclic operator on a real or complex Banach space has a dense invariant linear manifold with maximal algebraic dimension consisting, apart from zero, of vectors which are hypercyclic.

1. Preliminaries. Assume that X is a topological space. Denote by $C(X)$ the class of continuous selfmappings of X. Following Grosse-Erdmann [Gr], we say that a nonempty family $\tau \subset C(X)$ is *universal* when there is an element $x \in X$ such that the orbit $O(x,\tau) = \{Tx : T \in \tau\}$ is dense in X. In such a case, the element x is called τ-*universal*. $U(\tau)$ will stand for the set of τ-universal elements of X. If τ is countable, then a necessary condition for τ to be universal is, of course, the separability of X. If $T \in C(X)$ and $x \in X$, the orbit of x under T is $O(x,T) = O(x,\tau)$, where this time τ is the family of iterates $\tau = \{T^n : n \in \mathbb{N}\}$. Here $\mathbb{N}$ is the set of positive integers, $T^1 = T$, $T^2 = T \circ T$, and so on. T is called *universal* whenever this τ is universal, and an element $x \in X$ is called T-*universal* if and only if $O(x,T)$ is dense. In this case, we set $U(T) = U(\tau)$. Denote by $\mathrm{DR}(X)$ the subset of mappings $T \in C(X)$ such that the range $T(X)$ is dense in X. If T is universal, then, trivially, $T \in \mathrm{DR}(X)$. A subset $\tau \subset C(X)$ is said to be *densely universal* whenever $U(\tau)$ is dense in X. Note that $U(T)$ is always dense if T is universal, since $O(x,T) \subset U(T)$ for every T-universal

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