either to $(\mathbb{R}^\infty)^\omega$ or to $Q \times (\mathbb{R}^\infty)^\omega$. In the first case the proof is complete; in the second, observe that $Q \times (\mathbb{R}^\infty)^\omega \approx ([0,1] \times \mathbb{R}^\infty)^\omega \approx (\mathbb{R}^\infty)^\omega$ (because $[0,1] \times \mathbb{R}^\infty \approx \mathbb{R}^\infty$, see [Sa]). ■

QUESTION. Is the Classification Theorem still valid for Montel (LF)-spaces? For separable (LF)-spaces?

Note that the answer to the first question is affirmative provided the Lemma is valid for Montel Fréchet spaces.

PROBLEM. Classify topologically strong duals to separable (reflexive) Fréchet spaces.

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Interpolation on families of characteristic functions

by

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Abstract. We study a problem of interpolating a linear operator which is bounded on some family of characteristic functions. A new example is given of a Banach couple of function spaces for which such interpolation is possible. This couple is of the form $\overline{\Phi} = (B, L^\infty)$ where B is an arbitrary Banach lattice of measurable functions on a σ-finite nonatomic measure space (Ω, Σ, μ). We also give an equivalent expression for the norm of a function f in the real interpolation space $(B, L^\infty)_{\theta,p}$ in terms of the characteristic functions of the level sets of f.

1. Introduction. Our goal in this paper is to study interpolation problems for linear operators acting on spaces of functions, in the case where these operators satisfy boundedness conditions only on a given family of characteristic functions rather than on all functions in the spaces. It has been shown by the second author [G1, G2, G5] that, under some additional restrictions, the boundedness of a linear operator T on a given family $\{\chi_E : E \in \mathcal{E}\}$ of characteristic functions from a couple of Lorentz spaces into a couple of Banach spaces implies the boundedness of T in the real interpolation spaces on the family of functions having all their level sets in $\mathcal{E}$. Moreover, a generalization of this interpolation theorem was obtained in [G3, G4, G5] in the form of a norm estimate in the real interpolation space for the Pettis integral of a mapping in terms of certain given estimates of the integrand. The method of interpolating from characteristic functions was used in the papers mentioned above and in [G6] to study the behavior of different linear operators, mainly the Fourier transform and the embedding operator.

In this paper we describe another situation where such interpolation from characteristic functions is possible. Here, instead of a couple of Lorentz

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spaces, we consider couples of the form (B, L^∞) where B is an arbitrary Banach lattice of measurable functions on some nonatomic σ-finite measure space. The structure of the paper is as follows. In Section 2 we gather the necessary definitions, pose a general problem of interpolating from a family of characteristic functions, and formulate the main result (Theorem 1). In Section 3 we discuss the problem of reconstructing a function from the characteristic functions of its level sets. We begin the proof of Theorem 1 in Section 4. Our proof is based on some ideas from [G5]. We adapt them to the case of a couple of Banach lattices and get some estimates for the K-functional (see Theorem 3) and for the norm in the interpolation space (see Theorem 4). In Section 5 we obtain an equivalent expression for the norm of a nonnegative function f in the interpolation space $(B, L^\infty)_{\theta,p}$ in terms of the characteristic functions of the level sets of f. This result is needed to complete our proof of Theorem 1 at the end of Section 5, but it also seems to be of independent interest.

2. Necessary definitions and main results. Throughout this paper $\overline{A} = (A_0, A_1)$ will denote a Banach couple, i.e. a pair of Banach spaces continuously embedded into a Hausdorff topological vector space Z. We will use the standard notation $\Sigma(\overline{A}) = A_0 + A_1$, $\Delta(\overline{A}) = A_0 \cap A_1$, and $\overline{A}_{\theta,p} = (A_0, A_1)_{\theta,p}$. The symbol $\Sigma^\circ(\overline{A})$ will stand for the closure of $\Delta(\overline{A})$ in $\Sigma(\overline{A})$. Similarly, A_i° will denote the closure of $\Delta(\overline{A})$ in A_i where $i = 0, 1$. We refer to [BL] for more information concerning the real interpolation method.

The symbol m will stand for Lebesgue measure on the half-line $\mathbb{R}^+$. Let (Ω, Σ, μ) be a nonatomic complete σ-finite measure space. A Banach space Φ of (equivalence classes of μ-almost everywhere equal) real measurable functions on Ω such that if $f \in \Phi$ and $|g(\omega)| \le |f(\omega)|$ almost everywhere, then $g \in \Phi$ and $\|g\|_\Phi \le \|f\|_\Phi$, is called a *Banach lattice on* (Ω, Σ, μ), or a *Banach function lattice*, or simply a *Banach lattice*.

Let $\overline{\Phi} = (\Phi_0, \Phi_1)$ be a Banach couple of Banach lattices on (Ω, Σ, μ). Suppose that X is a locally convex topological vector space in which the Banach spaces A_0 and A_1 of the couple $\overline{A}$ introduced above are continuously embedded. The space X need not necessarily be contained in the Hausdorff topological vector space Z used in the original definition of $\overline{A}$. Suppose that $T : \overline{\Phi} \to X$ is a continuous linear operator. Let $\mathcal{E}$ be a family of measurable sets $E \in \Sigma$ such that $\chi_E \in \Delta(\overline{\Phi})$ and

$$\|T\chi_E\|_{A_i} \le c_i \|\chi_E\|_{\Phi_i} \tag{1}$$

for $i = 0, 1$ and for some positive constants c_i. Consider the class $\mathcal{M}_{\mathcal{E}}$ of all nonnegative measurable functions f on Ω such that the level set

$$E(y, f) = \{x : f(x) > y\}$$

belongs to $\mathcal{E}$ for each $y > 0$.

In the present paper we deal with the following problem.

PROBLEM 1. *What are sufficient conditions guaranteeing that inequalities* (1) *imply that for each* $\theta \in (0, 1)$ *and* $p \in [1, \infty]$ *we have*

$$\|Tf\|_{\overline{A}_{\theta,p}} \le c_{\theta,p} \|f\|_{\overline{\Phi}_{\theta,p}} \tag{2}$$

for all $f \in \overline{\Phi}_{\theta,p} \cap \mathcal{M}_{\mathcal{E}}$ *and for a constant* $c_{\theta,p}$ *depending only on* c_0, c_1, θ, *and* p?

In general, condition (1) alone is not sufficient to imply estimate (2) (see [G5], Example 2.1). In order to formulate an additional condition which does enable us to obtain (2), we first need to recall the following notion: If B is a Banach space, then the topological space (B^*, weak^*) is called an *angelic space* if for any bounded set H in B^* the weak* closure of H coincides with the set of weak* limits of sequences in H. This definition is a special case of the notion of angelic space introduced by Fremlin (see [F, P]). The class of Banach spaces B satisfying the above condition will be denoted by AN. Talagrand has shown that all reflexive, separable, and more generally all weakly compactly generated Banach spaces belong to the class AN. In fact, this is true for even more general classes of Banach spaces. For details of this and other related results we refer to [E], pp. 564–565 and [T], Théorème 3.6 (p. 415) and Théorème 6.4 (p. 427).

The following condition was considered in [G5]:

$$\Sigma^\circ(\overline{A}) \in \text{AN}. \tag{3}$$

This condition plays an important role in problems concerning interpolation on characteristic functions. Condition (3) is not very restrictive. For example, any of the conditions (i)–(v) below implies (3):

(i) one of the spaces A_0 or A_1 is separable;
(ii) one of the spaces A_0 or A_1 is reflexive;
(iii) one of the spaces $\overline{A}_{\theta,p}$ with $\theta \in (0, 1)$ and $p \in (1, \infty)$ is separable;
(iv) one of the spaces $\overline{A}_{\theta,p}$ with $\theta \in (0, 1)$ and $p \in (1, \infty)$ is reflexive;
(v) one of the spaces $\overline{A}_{\theta,p}$ with $\theta \in (0, 1)$ and $p \in (1, \infty)$ is in AN.

Indeed, it is known that (i) implies (iii) and (ii) implies (iv) for all θ and p as above (see [B], p. 31 and p. 40). Since separable and reflexive spaces are in AN, conditions (iii) and (iv) imply (v). Suppose now that condition (v) holds. Since $\overline{A}_{\theta,p}$ is dense in $\Sigma^\circ(\overline{A})$, the space $(\Sigma^\circ(\overline{A})^*, \text{weak}^*)$ is continuously embedded into the space $(\overline{A}^*_{\theta,p}, \text{weak}^*)$. Applying the "angelic lemma" (see [F], p. 28 and p. 31), we get condition (3). This shows that each of the conditions (i)–(v) implies condition (3).

It was shown in [G5] that if the couple $\overline{A}$ satisfies condition (3) and $\overline{\Phi}$ is a couple of different Lorentz spaces, then (2) holds. In the present paper we provide another class of examples of couples $\overline{A}$ and $\overline{\Phi}$ for which the interpolation theorem from the characteristic functions holds. This is done in the following theorem, which is the main result of this paper.

THEOREM 1. *Let $\overline{A}$ be a Banach couple continuously embedded into a locally convex Hausdorff topological vector space X and satisfying condition* (3). *Let $\overline{\Phi}$ be a Banach couple of lattices on a nonatomic complete σ-finite measure space (Ω, Σ, μ) such that $\Phi_0 = B$ where B is an arbitrary Banach function lattice and $\Phi_1 = L^\infty$. Let $\mathcal{E}$ be some subset of Σ such that $\chi_E \in B$ for all $E \in \mathcal{E}$. Suppose a linear operator T is given such that T maps both B and L^∞ into X continuously and*

$$\|T\chi_E\|_{A_0} \le c_0 \|\chi_E\|_B \tag{4}$$

and

$$\|T\chi_E\|_{A_1} \le c_1 \|\chi_E\|_{L^\infty} \tag{5}$$

for all $E \in \mathcal{E}$. Then

$$\|Tf\|_{\overline{A}_{\theta,p}} \le c_{\theta,p} \|f\|_{(B,L^\infty)_{\theta,p}} \tag{6}$$

for all $0 < \theta < 1$, $1 \le p \le \infty$, and all nonnegative functions f satisfying $E(y,f) \in \mathcal{E}$ for all $y > 0$.

The theory of Pettis integration will be an important tool in the proof of this theorem. We conclude this section by recalling some definitions and results from this theory: Let (S, Λ, ν) be a σ-finite measure space and λ be a mapping of S into a locally convex linear topological space C. For every functional $\gamma \in C^*$ let $\gamma \circ \lambda$ denote the composition of the mappings γ and λ.

DEFINITION 1. The mapping λ is called *weakly measurable* if the function $\gamma \circ \lambda$ is Λ-measurable for every $\gamma \in C^*$. The mapping λ is called *Pettis integrable in* C if λ is weakly measurable, $\int_S |\gamma \circ \lambda| \, d\nu < \infty$ for every $\gamma \in C^*$, and if, for each $G \in \Lambda$, there exists an element $x_G \in C$ such that

$$\gamma(x_G) = \int_G \gamma \circ \lambda \, d\nu \quad \text{for all } \gamma \in C^*.$$

The element x_G is called the *Pettis integral of* λ *over* G and is denoted by $C\text{-}\int_G \lambda \, d\nu$.

DEFINITION 2. Let $(G, \mathcal{F}, \mu)$ be a finite measure space and C be a Banach space. It is said that C has the *μ-Pettis integral property* if every bounded weakly measurable function $\xi : G \to C$ is Pettis integrable in C.

It is known that if C is a Banach space then

$$C \in \mathrm{AN} \Rightarrow C \text{ has the } \mu\text{-Pettis integral property for every probability measure } \mu \tag{7}$$

(see [E], pp. 564–565). The next lemma is standard and we leave it as an exercise for the reader.

LEMMA 1. *Let λ be a weakly measurable mapping of $(0,\infty)$ into a Banach space C such that the mapping $\lambda\chi_{[a,b]}$ is Pettis integrable in C for every compact interval $[a,b] \subset (0,\infty)$. Suppose that $\|\lambda(y)\|_C \le \phi(y)$ for all $y > 0$ where $\phi \in L^1(0,\infty)$. Then the mapping λ is Pettis integrable in C and*

$$\Big\| C\text{-}\int_E \lambda(y)\, dy \Big\|_C \le \int_E \phi(y)\, dy$$

for every Lebesgue measurable subset E of $(0,\infty)$.

3. How to reconstruct a function from its level sets. Suppose f is a nonnegative measurable function on a measure space (Ω, Σ, μ) as in Section 2. Let τ_f be a mapping of the half-line $\mathbb{R}^+ = (0,\infty)$ into the space of measurable functions on Ω defined by $\tau_f(y) = \chi_{E(y,f)}$, $y > 0$, where $E(y,f) = \{f > y\}$ are the level sets of the function f. If Φ is a Banach lattice on (Ω, Σ, μ) and $f \in \Phi$, then $\chi_{E(y,f)} \in \Phi$ for all $y > 0$ since

$$0 \le \chi_{E(y,f)} \le y^{-1} f. \tag{8}$$

Hence, τ_f maps $\mathbb{R}^+$ into Φ.

The function f can always be reconstructed from the mapping τ_f by pointwise integration:

$$f = \int_0^\infty \tau_f(y)\, dy. \tag{9}$$

This follows easily from Tonelli's Theorem. Now let Φ be a Banach lattice as above. Our next goal is to obtain the Pettis integral version of the representation (9). First we study the measurability problem.

LEMMA 2. *For every Banach lattice Φ and every nonnegative measurable function $f \in \Phi$, the mapping $\tau_f : \mathbb{R}^+ \to \Phi$ is weakly measurable.*

Proof. Let γ be an arbitrary functional in Φ^*. We have to show that the function $\gamma \circ \tau_f$ is a measurable function. It suffices to show that $\gamma \circ \tau_f(y)$ is a measurable function of y on $(1/n, \infty)$ for each positive integer n. For each such n let $L^\infty|_{E(1/n,f)}$ denote the subspace of $L^\infty(\Omega, \Sigma, \mu)$ of those functions which vanish μ-a.e. on $\Omega \backslash E(1/n, f)$. It follows from (8) that this space is continuously embedded into Φ. So the restriction γ_n of γ to this space is given by $\gamma_n(\psi) = \int_\Omega \psi \, d\nu_n$ for some finitely additive measure ν_n which is defined on the measurable subsets of $E(1/n, f)$ and vanishes on all such subsets having μ-measure zero. We can represent ν_n as a difference of two nonnegative finitely additive measures ν_n^1 and ν_n^2, i.e. $\nu_n = |\nu_n| - (|\nu_n| - \nu_n) = \nu_n^1 - \nu_n^2$. Now for each $y \in (1/n, \infty)$ we have

$\chi_{E(y,f)} \in L^\infty|_{E(1/n,f)}$ and so

$$\nu_n(E(y,f)) = \nu_n^1(E(y,f)) - \nu_n^2(E(y,f)).$$

Both functions on the right are monotonic on $(1/n, \infty)$. Thus their difference is measurable. This completes the proof of Lemma 2.

Next we study the Pettis integrability of the mapping τ_f where $f \geq 0$ belongs to an arbitrary Banach lattice Φ. Let $E \subset \mathbb{R}^+$ be a Lebesgue measurable set. Define $f_E(\omega) = \int_E \tau_f(y)(\omega)\,dy$ for each $\omega \in \Omega$, where the integral should be understood in the pointwise sense. It is clear that $f_E(\omega) = m(E \cap [0, f(\omega)])$, from which we can see that f_E is a measurable function. Since $f_E \leq f$, we have $f_E \in \Phi$.

LEMMA 3. *Let Φ be a Banach lattice on (Ω, Σ, μ) and $f \geq 0$ be a function in Φ. Let $0 < a < b < \infty$. Let E be an arbitrary Lebesgue measurable subset of $[a,b]$. Then*

$$\gamma(f_E) = \int_E \gamma \circ \tau_f\,dy \quad \text{for all } \gamma \in \Phi^*. \tag{10}$$

Proof. Let $\gamma \in \Phi^*$. By estimate (8), we have

$$\|\tau_f\|_\Phi \leq y^{-1}\|f\|_\Phi. \tag{11}$$

It follows from (11) and Lemma 2 that $\gamma \circ \tau_f \in L^1(E)$. Using (8) again, we see that the subspace $L^\infty|_{E(a,f)}$ of $L^\infty(\Omega, \Sigma, \mu)$ of those functions which vanish almost everywhere on the complement of $E(a,f)$ is continuously embedded into Φ. So the restriction γ_a of a functional $\gamma \in \Phi^*$ is generated by a finitely additive signed measure ν_a on (Ω, Σ) which vanishes on all measurable subsets of the complement of the set $E(a,f)$ and also on all measurable subsets of zero μ-measure of the set $E(a,f)$. Moreover, $\nu_a = \nu_a^1 - \nu_a^2$ where ν_a^i is a nonnegative finitely additive measure for $i = 1,2$. Hence $\gamma_a = \gamma_a^1 - \gamma_a^2$ where γ_a^i is a nonnegative bounded linear functional on $L^\infty|_{E(a,f)}$ for $k = 1,2$. Now we get

$$\int_E \gamma \circ \tau_f\,dy = \int_E \gamma_a^1 \circ \tau_f\,dy - \int_E \gamma_a^2 \circ \tau_f\,dy. \tag{12}$$

Consider a subdivision of $[a,b]$ by $N+1$ points c_m,

$$a = c_0 < c_1 < \ldots < c_{N-1} < c_N = b.$$

This can be chosen so that $m(E \cap [c_{n-1}, c_n]) = m(E)/N$ for $n = 1, \ldots, N$. Since $f_E = \sum_{n=1}^N f_{E\cap[c_{n-1},c_n]}$ and

$$\frac{m(E)}{N}\chi_{E(c_n,f)} \leq f_{E\cap[c_{n-1},c_n]} \leq \frac{m(E)}{N}\chi_{E(c_{n-1},f)}$$

we deduce that

$$\frac{m(E)}{N}\sum_{n=1}^N \gamma_a^i(\tau_f(c_n)) \leq \gamma_a^i(f_E) \leq \frac{m(E)}{N}\sum_{n=1}^N \gamma_a^i(\tau_f(c_{n-1})).$$

But by choosing N sufficiently large, both of the sums appearing in this estimate can be made arbitrarily close to $\int_E \gamma_a^i \circ \tau_f\,dy$. (The reasoning here is of course the same as in the proof of the Riemann integrability of a monotonic function.) Therefore,

$$\gamma_a^i(f_E) = \int_E \gamma_a^i \circ \tau_f\,dy \tag{13}$$

for $i = 1,2$. Now (10) follows from (12) and (13). This completes the proof of Lemma 3.

LEMMA 4. *Let Φ be a Banach lattice and $f \geq 0$ be a function in Φ. If the mapping τ_f is Pettis integrable in Φ, then for every Lebesgue measurable set $E \subset \mathbb{R}^+$ we have*

$$f_E = \Phi\text{-}\int_E \tau_f\,dy. \tag{14}$$

Proof. Consider the following sequence of functions: $g_n = f_{E\cap[1/n,n]}$, $n \geq 1$. By Lemma 3 we have

$$g_n = \Phi\text{-}\int_{E\cap[1/n,n]} \tau_f\,dy.$$

Since the Pettis integral is countably additive (see [DU], p. 53),

$$\lim_{n\to\infty} g_n = \Phi\text{-}\int_E \tau_f\,dy \tag{15}$$

where the convergence on the left side is in the space Φ. Denote the limit in (15) by g.

It remains to prove that $g = f_E$ a.e. This follows by a straightforward argument (cf. e.g. [KPS], p. 41) which we reproduce here for the reader's convenience: Clearly we have the pointwise inequalities $0 \leq g_m \leq g_{m+1} \leq \lim_{n\to\infty} g_n = f_E$. First we claim that $g_n \leq g$ a.e. for all n. If not, then for some n there exists a set A with $\mu(A) > 0$ such that $g_n - g > 0$ on A and thus for each $m > n$ we have

$$0 < \|(g_n - g)\chi_A\|_\Phi \leq \|(g_m - g)\chi_A\|_\Phi \leq \|g_m - g\|_\Phi,$$

which is clearly a contradiction. Taking pointwise limits, we deduce that $f_E \leq g$ a.e. Now suppose that $f_E < g$ on some set A with $\mu(A) > 0$. Then $0 < \|(g - f_E)\chi_A\|_\Phi \leq \|(g - g_n)\chi_A\|_\Phi$ for all n, which again is of course a contradiction. Therefore, $f_E = g$ a.e. and hence (14) follows from (15). This completes the proof of Lemma 4.

LEMMA 5. *Suppose that a function $f \geq 0$ is given such that $\tau_f(y) \in \Phi$ for all $y > 0$. Assume*

$$\int_0^\infty \|\tau_f(y)\|_\Phi \, dy < \infty. \tag{16}$$

Then $f \in \Phi$, the mapping τ_f is Pettis integrable, and moreover,

$$f_E = \Phi\text{-}\int_E \tau_f(y)\, dy \tag{17}$$

for every Lebesgue measurable subset E of $(0,\infty)$.

Proof. Let us show that $f \in \Phi$. Indeed, since Φ is a Banach lattice and

$$f \leq \sum_{k=-\infty}^{\infty} 2^{k+1}\chi_{E(2^k,f)},$$

we have

$$\|f\|_\Phi \leq 4 \sum_{k=-\infty}^{\infty} 2^{k-1}\|\chi_{E(2^k,f)}\|_\Phi \leq 4\int_0^\infty \|\chi_{E(y,f)}\|_\Phi \, dy.$$

Using (16), we get $f \in \Phi$. Now Lemma 5 follows from Lemmas 1–4.

4. Interpolation from characteristic functions in Banach lattices. The first theorem in this section gives an estimate for the K-functional, provided some estimates for the characteristic functions of the level sets of a function are known.

THEOREM 2. *Let $\overline{A}$ be a Banach couple, continuously embedded into a locally convex topological vector space X, and $\overline{\Phi}$ be a couple of Banach lattices on (Ω, Σ, μ). Suppose $\mathcal{E}$ is a family of measurable sets E such that $\chi_E \in \Delta(\overline{\Phi})$. Let $T : \overline{\Phi} \to X$ be a continuous linear operator such that condition (1) is satisfied for all $E \in \mathcal{E}$. Suppose also that $\Sigma^\circ(\overline{A}) \in \mathrm{AN}$. Then for every function $f \in \Sigma(\overline{\Phi}) \cap \mathcal{M}_{\mathcal{E}}$ we have*

$$Tf \in \Sigma^\circ(\overline{A}) \tag{18}$$

and

$$K(t, Tf; A_0, A_1) \leq c\int_0^\infty \min\{\|\tau_f(y)\|_{\Phi_0}, t\|\tau_f(y)\|_{\Phi_1}\}\, dy \tag{19}$$

for all $t > 0$, provided the integral in (19) is finite for at least one value (and therefore for all values) of $t > 0$. The constant c above depends only on the constants c_i in (1).

Proof. We begin by studying the vector-valued function λ_f defined by

$$\lambda_f(y) = T\chi_{E(y,f)}. \tag{20}$$

This takes values in $A_0 \cap A_1$ since $\chi_{E(y,f)} \in \Phi_0 \cap \Phi_1$ for all $y > 0$. Our first goal will be to prove that λ_f is weakly measurable in the space $A_0^\circ + A_1^\circ$, which is easily seen to be exactly the same as the space $\Sigma^\circ(\overline{A})$.

By Lemma 2, the vector-valued function $y \mapsto \chi_{E(y,f)}$ is weakly measurable in $\Sigma(\overline{\Phi})$. Therefore, since $T : \Sigma(\overline{\Phi}) \to X$ is continuous, the function λ_f is weakly measurable in X. Since X^* separates points on $\Sigma^\circ(\overline{A})$, the set $X^* \subset \Sigma^\circ(\overline{A})^*$ is weak* dense in $\Sigma^\circ(\overline{A})^*$. Now consider the set

$$P = \{\gamma \in \Sigma^\circ(\overline{A})^* : \gamma(T\chi_{E(y,f)}) \text{ is a measurable function of } y\}.$$

P is weak* dense in $\Sigma^\circ(\overline{A})^*$ since $X^* \subset P$. Thus in order to prove that $P = \Sigma^\circ(\overline{A})^*$ it is enough to check that P is a weak* closed set. By a theorem of Krein–Shmulian (see [DS], p. 429), it is sufficient to prove that the set $P \cap B(0,r)$ is weak* closed for every ball $B(0,r)$ in $\Sigma^\circ(\overline{A})$ of radius r centered at 0. This set is bounded in $\Sigma^\circ(\overline{A})$. Using the condition $\Sigma^\circ(\overline{A}) \in \mathrm{AN}$, we see that every element of the weak* closure of the set $P \cap B(0,r)$ can be represented as the weak* limit of a sequence in $P \cap B(0,r)$. It follows that $P \cap B(0,r)$ is weak* closed. As already explained, this gives $P = \Sigma^\circ(\overline{A})^*$. Thus, the mapping λ_f defined by (20) is weakly measurable in $\Sigma^\circ(\overline{A})$.

Given an arbitrary number $t > 0$ we equip the space $\Sigma^\circ(\overline{A})$ with the equivalent norm

$$\|a\|^{(t)}_{\Sigma^\circ(\overline{A})} = K(t, a; A_0^\circ, A_1^\circ).$$

Since the integral in (19) is finite, Lemma 5 implies that the mapping $y \mapsto \chi_{E(y,f)}$ is Pettis integrable in $\Sigma(\overline{\Phi})$ and

$$f = \Sigma(\overline{\Phi})\text{-}\int_0^\infty \chi_{E(y,f)}\, dy. \tag{21}$$

It follows from (21) and the continuity of the operator T from $\Sigma(\overline{\Phi})$ into X that the mapping λ_f is Pettis integrable in X and

$$Tf = X\text{-}\int_0^\infty \lambda_f(y)\, dy. \tag{22}$$

Since T satisfies (1) we obtain

$$\|\lambda_f(y)\|^{(t)}_{\Sigma^\circ(\overline{A})} \leq c\min\{\|\chi_{E(y,f)}\|_{\Phi_0}, t\|\chi_{E(y,f)}\|_{\Phi_1}\} \tag{23}$$

for all $y > 0$ where the constant $c > 0$ does not depend on y and t. We have already shown that the mapping

$$\lambda_f : (0,\infty) \to (\Sigma^\circ(\overline{A}), \|\cdot\|^{(t)}_{\Sigma^\circ(\overline{A})}) \tag{24}$$

is weakly measurable. Now (3), (7), (23), and the condition $f \in \Sigma(\overline{\Phi}) \cap \mathcal{M}_{\mathcal{E}}$ imply that the mapping (24) is Pettis integrable in the space $\Sigma^\circ(\overline{A})$ over every compact interval $[a,b] \subset (0,\infty)$. By (23), the norm of (24) has an integrable majorant. It follows from Lemma 1 that λ_f is Pettis integrable in $\Sigma^\circ(\overline{A})$ over all of $(0,\infty)$. Next, using (22), we get

$$Tf = \Sigma^\circ(\overline{A})\text{-}\int_0^\infty \lambda_f(y)\,dy. \tag{25}$$

This proves (18). Moreover, (23), (25) and Lemma 1 imply that

$$\begin{aligned} K(t,Tf;A_0,A_1) &\le K(t,Tf;A_0^\circ,A_1^\circ) = \|Tf\|^{(t)}_{\Sigma^\circ(\overline{A})} \\ &\le c\int_0^\infty \min\{\|\chi_{E(y,f)}\|_{\Phi_0}, t\|\chi_{E(y,f)}\|_{\Phi_1}\}\,dy. \end{aligned}$$

This gives estimate (19) and so completes the proof of Theorem 2.

We now consider the case $\overline{\Phi} = (B, L^\infty)$ where B is a Banach lattice of measurable functions on (Ω,Σ,μ). In this case the estimate in Theorem 2 has the (slightly simpler) form

$$K(t,Tf;A_0,A_1) \le c\int_0^\infty \min\{\|\chi_{E(y,f)}\|_B, t\}\,dy. \tag{26}$$

This estimate will be used in the next theorem.

THEOREM 3. *Suppose $\overline{\Phi} = (B, L^\infty)$ as above and all conditions in Theorem 2 are satisfied. Then for every function $f \in \Sigma(\overline{\Phi}) \cap \mathcal{M}_\mathcal{E}$ and all numbers θ and p such that $0<\theta<1$ and $1 \le p < \infty$ we have $Tf \in \overline{A}_{\theta,p}$ and*

$$\|Tf\|_{\overline{A}_{\theta,p}} \le c_{\theta,p}\Big\{\int_0^\infty y^{p-1}\|\chi_{E(y,f)}\|_B^{p(1-\theta)}\,dy\Big\}^{1/p} \tag{27}$$

provided the integral in (27) is finite. The constant $c_{\theta,p}$ in (27) depends only on θ, p, and the constants c_i in (1). Analogously, for $p=\infty$ we have $Tf \in \overline{A}_{\theta,\infty}$ and

$$\|Tf\|_{\overline{A}_{\theta,\infty}} \le c_\theta \sup_{y>0}\{y\|\chi_{E(y,f)}\|_B^{1-\theta}\} \tag{28}$$

provided the supremum in (28) is finite. The constant c_θ in (28) depends only on θ and the constants c_i in (1).

Proof. Define $\phi(y) = \|\chi_{E(y,f)}\|_B$. We shall use the notation $D(\cdot,\phi)$ for the distribution function of the function ϕ, i.e. for all $t>0$ we set $D(t,\phi) = m(\{y : \phi(y) > t\})$. Using well known properties of the distribution function, estimate (26), Hardy's inequality (see [SW], Lemma 3.14 on p. 196), and

the fact that ϕ is nonincreasing, in the case $1 \le p < \infty$ we get

$$\begin{aligned} \|Tf\|^p_{\overline{A}_{\theta,p}} &\le c\int_0^\infty t^{-\theta p-1}\,dt\Big(\int_0^\infty \min\{\phi(y),t\}\,dy\Big)^p \\ &\le c_1\int_0^\infty t^{-\theta p-1}\,dt\Big(\int_{\{y:\phi(y)\le t\}} \phi(y)\,dy\Big)^p \\ &\quad + c_1\int_0^\infty t^{(1-\theta)p-1}D(t,\phi)^p\,dt \\ &\le c_1\int_0^\infty t^{-\theta p-1}\Big(\int_0^t D(z,\phi)\,dz\Big)^p dt \\ &\quad + c_1\int_0^\infty t^{(1-\theta)p-1}D(t,\phi)^p\,dt \\ &\le \alpha_{\theta,p}\int_0^\infty t^{(1-\theta)p-1}D(t,\phi)^p\,dt \\ &= \frac{\alpha_{\theta,p}}{1-\theta}\int_0^\infty y^{p-1}\phi(y)^{(1-\theta)p}\,dy, \end{aligned} \tag{29}$$

which proves (27).

Next we will prove estimate (28). From (26) we get

$$\begin{aligned} \|Tf\|_{\overline{A}_{\theta,\infty}} &\le c\sup_{t>0} t^{-\theta}\int_0^\infty \min\{\phi(y),t\}\,dy \\ &\le c\sup_{t>0} t^{-\theta}\int_{\{y:\phi(y)\le t\}} \phi(y)\,dy + c\sup_{t>0} t^{1-\theta}D(t,\phi) \\ &= c\sup_{t>0} t^{-\theta}\int_0^t D(z,\phi)\,dz + c\sup_{t>0} t^{1-\theta}D(t,\phi) \\ &\le c\sup_{t>0} t^{-\theta}\int_0^t z^{\theta-1}\sup_{y>0} y^{\theta-1}D(y,\phi)\,dz + c\sup_{t>0} t^{1-\theta}D(t,\phi) \\ &= c\Big(\frac{1}{\theta}+1\Big)\sup_{t>0} t^{1-\theta}D(t,\phi). \end{aligned} \tag{30}$$

By well known properties of the distribution function, since ϕ is nonincreasing, this last expression is equal to $c(1/\theta+1)\sup_{y>0} y\phi(y)^{1-\theta}$. (Cf. also Lemma 3.8 on p. 191 in [SW].) This establishes (28) and completes the proof of Theorem 3.

5. Spaces $(B, L^\infty)_{\theta,p}$ and the level sets of functions. In this section we give an equivalent expression for the norm of a nonnegative function f in the interpolation space $(B, L^\infty)_{\theta,p}$ in terms of the level sets $E(y,f)$. This result enables us to complete the proof of Theorem 1.

THEOREM 4. *Let B be a Banach function lattice on a nonatomic σ-finite measure space (Ω, Σ, μ). Then for every $\theta \in (0,1)$ and $p \in [1,\infty)$ there exist positive constants $\alpha_{\theta,p}$ and $\beta_{\theta,p}$ such that, for every measurable function $f : \Omega \to [0,\infty)$,*

$$(31)\quad \alpha_{\theta,p} \int_0^\infty \|\chi_{E(y,f)}\|_B^{(1-\theta)p} y^{p-1}\,dy \le \|f\|^p_{(B,L^\infty)_{\theta,p}} \le \beta_{\theta,p} \int_0^\infty \|\chi_{E(y,f)}\|_B^{(1-\theta)p} y^{p-1}\,dy.$$

In the case $p = \infty$ there exist positive constants α_θ and β_θ such that

$$(32)\quad \alpha_\theta \sup_{y>0} y \|\chi_{E(y,f)}\|_B^{1-\theta} \le \|f\|_{(B,L^\infty)_{\theta,\infty}} \le \beta_\theta \sup_{y>0} y\|\chi_{E(y,f)}\|_B^{1-\theta}.$$

REMARK 1. The inequalities in (31) and (32) should be understood in the following sense. The finiteness of the norm of a function in the interpolation space in (31) or (32) implies the finiteness of the corresponding integral or supremum and the validity of the estimates from below. Similarly, if the integral in (31) or the supremum in (32) is finite, then the function is in the corresponding interpolation space and the estimates from above hold.

Proof (of Theorem 4). Let $1 \le p < \infty$. Our first goal is to prove that

$$(33)\quad \|f\|^p_{(B,L^\infty)_{\theta,p}} \le \beta_{\theta,p} \int_0^\infty \|\chi_{E(y,f)}\|_B^{(1-\theta)p} y^{p-1}\,dy$$

provided the integral is finite. This can be considered as a special case of the estimate (27) obtained in Theorem 3, if we choose $A_0 = \Phi_0 = B$ and $A_1 = \Phi_0 = L^\infty$, let $\mathcal{E}$ consist of all measurable sets for which $\chi_E \in B$, take $X = B + L^\infty$, and let T be the identity operator. However, we cannot prove this estimate by immediately applying Theorems 2 and 3 in this special case, since we do not know whether the condition $\Sigma^\circ(\overline{A}) \in \mathrm{AN}$ holds. Instead, we shall use a (simpler) alternative argument. For each fixed $t > 0$ consider the space $A = (\Sigma(\Phi), \|\cdot\|^{(t)}_{\Sigma(\overline{\Phi})})$ as in Theorem 2. Since B and L^∞ are Banach lattices, so is A. Moreover, $\tau_f(y) \in A$ and

$$(34)\quad \|\tau_f(y)\|^{(t)}_{\Sigma(\overline{\Phi})} \le \min\{\|\chi_{E(t,f)}\|_B, t\}$$

for all $y > 0$ and $t > 0$. The function on the right side of (34) appears in the first line of the estimates (29) and so it is integrable on $(0,\infty)$ for each choice of fixed $t > 0$, in view of the subsequent estimates in (29) and the finiteness of the integral in (33). Applying Lemma 5 with A in the role of the Banach lattice Φ and then using Lemma 1, we obtain

$$(35)\quad f \in B + L^\infty$$

and

$$(36)\quad K(t,f;B,L^\infty) \le \int_0^\infty \min\{\|\chi_{E(y,f)}\|_B, t\}\,dy.$$

(Note that (35) and (36) are analogous to (18) and (19) (or (26)) respectively, in the special case discussed above.) Using (35), (36) and, once more, the estimates (29) of the proof of Theorem 3, we obtain (33) and so prove the upper estimate for $\|f\|^p_{(B,L^\infty)_{\theta,p}}$ in (31). The upper estimate in (32) is proved in a similar way.

In order to finish the proof of Theorem 4 we need to obtain the lower estimate in (31) and (32). Suppose $f \in (B, L^\infty)_{\theta,p}$ where $1 \le p \le \infty$. Define $v(t) = t^{-1}K(t,f)$. The function v is nonincreasing on $(0,\infty)$ because $v(t) = \inf_{f_0+f_1=f}\{t^{-1}\|f_0\|_B + \|f_1\|_\infty\}$. Moreover, $f \in (B,L^\infty)_{\theta,p}$ implies $t^{-\theta}K(t,f) \le M$ where M is some positive constant. Hence $v(t) \le Mt^{-1+\theta}$ and $\lim_{t\to 0} v(t) = 0$. Denote by $\widetilde{v}$ the following variant of the distribution function of v: $\widetilde{v}(y) = m\{t \in \mathbb{R}^+ : v(t) \ge y\}$. The function $\widetilde{v}$ is nonincreasing and left-continuous.

We need some simple and known facts about the K-functional for the couple (B, L^∞), which we will formulate as a separate lemma.

LEMMA 6. *If f is a nonnegative function in $(B,L^\infty)_{\theta,p}$ for some $\theta \in (0,1)$ and $p \in [1,\infty]$, then for each $y > 0$ we have $\chi_{E(y,f)} \in B$ and*

$$(37)\quad \|f\chi_{E(y,f)}\|_B \le K(\|\chi_{E(y,f)}\|_B, f; B, L^\infty).$$

Proof. Fix any $y > 0$. Since $\sup_{t>0} t^{-\theta} K(t,f;B,L^\infty) < \infty$, there exists some decomposition $f = g + h$ of f with $g \in B$ and $\|h\|_{L^\infty} < y$. It follows that

$$|g|\chi_{E(y,f)} = |f - h|\chi_{E(y,f)} \ge (y - \|h\|_{L^\infty})\chi_{E(y,f)}.$$

Since B is a Banach lattice, the function on the left must be an element of B and therefore so is the function on the right and consequently also $\chi_{E(y,f)}$.

Now let $f = f_0 + f_1$ be any decomposition of f with $f_0 \in B$ and $f_1 \in L^\infty$. Then

$$\|f\chi_{E(y,f)}\|_B \le \|f_0\chi_{E(y,f)}\|_B + \|f_1\chi_{E(y,f)}\|_B \le \|f_0\|_B + \|\chi_{E(y,f)}\|_B \|f_1\|_{L^\infty}.$$

We obtain (37) by taking the infimum over all such decompositions of f, and so complete the proof of Lemma 6.

REMARK 2. It is rather easy to extend the previous proof to obtain equivalent formulæ for the K-functional of the couple (B, L^∞), in particular when additional hypotheses are made about B. Various authors have considered this problem. In many cases one has $K(t,f) \simeq \|\chi_{F_t} f\|_B$ for a suitable set F_t depending on t and f. A discussion of the case where B is an arbitrary Banach lattice and (Ω, Σ, μ) is an arbitrary measure space can be found in [CJM] (see also [CP]).

Now we continue the proof of Theorem 4. For every $y > 0$, Lemma 6 gives

$$v(\|\chi_{E(y,f)}\|_B) \ge \frac{\|\chi_{E(y,f)} f\|_B}{\|\chi_{E(y,f)}\|_B} \ge \frac{y\|\chi_{E(y,f)}\|_B}{\|\chi_{E(y,f)}\|_B} = y. \tag{38}$$

It follows from the definition of $\widetilde{v}$ that

$$c > 0,\ \alpha > 0,\ v(c) \ge \alpha \Rightarrow \widetilde{v}(\alpha) \ge c. \tag{39}$$

Putting $c = \|\chi_{E(y,f)}\|_B$ and $\alpha = y$ in (39) and using (38), we get

$$\widetilde{v}(y) \ge \|\chi_{E(y,f)}\|_B. \tag{40}$$

Since $\widetilde{v}(y) = m(\{t \in \mathbb{R}^+ : v(t) > y\})$ for almost all y, we get, using (40) and standard properties of distribution functions (cf. the last step in (29)),

$$\begin{aligned}\|f\|^p_{(B,L^\infty)_{\theta,p}} &= \int_0^\infty K(t,f;B,L^\infty)^p t^{-\theta p-1}\,dt \\ &= \int_0^\infty v(t)^p t^{(1-\theta)p-1} dt = (1-\theta)\int_0^\infty \widetilde{v}(y)^{(1-\theta)p} y^{p-1}\,dy \\ &\ge (1-\theta)\int_0^\infty \|\chi_{E(y,f)}\|_B^{(1-\theta)p} y^{p-1}\,dy.\end{aligned}$$

This gives the estimate of the norm of f from below in (31).

In order to deal with the case $p = \infty$ let us first observe that (cf. the step immediately after (30))

$$\sup_{t>0}\{t^{1-\theta}v(t)\} = \sup_{y>0}\{y\widetilde{v}(y)^{1-\theta}\}.$$

Thus we have

$$\begin{aligned}\|f\|_{(B,L^\infty)_{\theta,\infty}} &= \sup_{t>0}\{t^{-\theta}K(t,f;B,L^\infty)\} = \sup_{t>0}\{t^{1-\theta}v(t)\} \\ &= \sup_{y>0}\{y\widetilde{v}(y)^{1-\theta}\} \ge \sup_{y>0}\{y\|\chi_{E(y,f)}\|_B^{1-\theta}\}.\end{aligned}$$

This gives the estimate of the norm of f from below in (32) and completes the proof of Theorem 4.

Proof of Theorem 1. Theorem 1 immediately follows from Theorems 3 and 4.

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Weak almost periodicity of L_1 contractions and coboundaries of non-singular transformations

by

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Abstract. It is well known that a weakly almost periodic operator T in a Banach space is mean ergodic, and in the complex case, also λT is mean ergodic for every $|\lambda| = 1$. We prove that a positive contraction on L_1 is weakly almost periodic if (and only if) it is mean ergodic. An example shows that without positivity the result is false. In order to construct a contraction T on a complex L_1 such that λT is mean ergodic whenever $|\lambda| = 1$, but T is not weakly almost periodic, we prove the following: Let τ be an invertible weakly mixing non-singular transformation of a separable atomless probability space. Then there exists a complex function $\varphi \in L_\infty$ with $|\varphi(x)| = 1$ a.e. such that for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ the function $f \equiv 0$ is the only solution of the equation $f(\tau x) = \lambda\varphi(x)f(x)$. Moreover, the set of such functions φ is residual in the set of all complex unimodular measurable functions (with the L_1 topology).

1. Mean ergodicity and weak almost periodicity of L_1 contractions. Motivated by von Neumann's mean ergodic theorem, we call a linear operator T in a (real or complex) Banach space $\mathbf{B}$ *mean ergodic* if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} T^k f \text{ exists for all } f \in \mathbf{B}. \tag{1.1}$$

A mean ergodic operator T which is power-bounded (i.e., $\sup_n \|T^n\| < \infty$) induces the *ergodic decomposition* $\mathbf{B} = \{y \in \mathbf{B} : Ty = y\} \oplus \overline{(I-T)\mathbf{B}}$, and the limit in (1.1) is the projection (corresponding to this decomposition) on the subspace of fixed points.

A linear operator T is called (*weakly*) *almost periodic* if for every $f \in \mathbf{B}$ the orbit $\{T^k f\}_{k\ge 0}$ is (weakly) sequentially compact. A weakly almost periodic (WAP) operator is necessarily power-bounded, and a power-bounded operator in a reflexive space is clearly WAP. Since the closed convex hull of a weakly compact set is weakly compact [DS], a weakly almost periodic

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