Contents of Volume 138, Number 3

STUDIA MATHEMATICA

Executive Editors: **Z. Ciesielski, A. Pełczyński, W. Żelazko**

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

Detailed information for authors is given on the inside back cover. Manuscripts and correspondence concerning editorial work should be addressed to

STUDIA MATHEMATICA
Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997
E-mail: studia@impan.gov.pl

Subscription information (2000): Vols. 138–143 (18 issues); $33.50 per issue.

Correspondence concerning subscription, exchange and back numbers should be addressed to

Institute of Mathematics, Polish Academy of Sciences
Publications Department
Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997
E-mail: publ@impan.gov.pl

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset using TEX at the Institute
Printed and bound by

drukarnia
herman & herman
SPÓŁKA CYWILNA
02-240 WARSZAWA UL. JAKOBINÓW 23

PRINTED IN POLAND

ISSN 0039-3223

Topological classification of strong duals to nuclear (LF)-spaces

by

TARAS BANAKH (Lviv)

Abstract. We show that the strong dual X' to an infinite-dimensional nuclear (LF)-space is homeomorphic to one of the spaces: $\mathbb{R}^\omega$, $\mathbb{R}^\infty$, $Q\times\mathbb{R}^\infty$, $\mathbb{R}^\omega\times\mathbb{R}^\infty$, or $(\mathbb{R}^\infty)^\omega$, where $\mathbb{R}^\infty=\varinjlim\mathbb{R}^n$ and $Q=[-1,1]^\omega$. In particular, the Schwartz space $\mathcal{D}'$ of distributions is homeomorphic to $(\mathbb{R}^\infty)^\omega$. As a by-product of the proof we deduce that each infinite-dimensional locally convex space which is a direct limit of metrizable compacta is homeomorphic either to $\mathbb{R}^\infty$ or to $Q\times\mathbb{R}^\infty$. In particular, the strong dual to any metrizable infinite-dimensional Montel space is homeomorphic either to $\mathbb{R}^\infty$ or to $Q\times\mathbb{R}^\infty$.

In this paper we give a complete topological classification of strong duals to nuclear (LF)-spaces. Recall that a locally convex space X is an *(LF)-space* if X is a strict inductive limit of a sequence $X_1\subset X_2\subset\ldots$ of Fréchet spaces (see [Sch, II.§6]). A topological classification of (LF)-spaces was given by P. Mankiewicz in [Ma]. This classification implies that each infinite-dimensional separable (LF)-space is homeomorphic to one of the spaces: $\mathbb{R}^\omega$, $\mathbb{R}^\infty$, or $\mathbb{R}^\omega\times\mathbb{R}^\infty$, where $\mathbb{R}^\omega$ is the countable product of lines and $\mathbb{R}^\infty=\bigoplus_{n\in\mathbb{N}}\mathbb{R}$ is a countable locally convex direct sum of lines. Note that $\mathbb{R}^\infty$ is the strong dual to $\mathbb{R}^\omega$ and vice versa, $\mathbb{R}^\omega$ is the strong dual to $\mathbb{R}^\infty$.

Recall that the *strong dual* X' to a locally convex space X is the space of all continuous linear functionals, endowed with the strong dual topology, i.e. the topology of uniform convergence on bounded subsets of X.

Below $Q=[-1,1]^\omega$ is the Hilbert cube. Our principal result is

Classification Theorem. *Suppose X is an infinite-dimensional nuclear (LF)-space. The strong dual X' to X is homeomorphic to one of the spaces*: $\mathbb{R}^\omega$, $\mathbb{R}^\infty$, $Q\times\mathbb{R}^\infty$, $\mathbb{R}^\omega\times\mathbb{R}^\infty$, *or* $(\mathbb{R}^\infty)^\omega$. *More precisely, X' is homeomorphic to*

(1) $\mathbb{R}^\omega$ *iff X is isomorphic to* $\mathbb{R}^\infty$;
(2) $\mathbb{R}^\infty$ *iff X is isomorphic to* $\mathbb{R}^\omega$;

2000 *Mathematics Subject Classification*: 57N17, 46A03, 46A11, 46A13, 46A20, 46F05.
Key words and phrases: dual space, nuclear (LF)-space, Montel space, direct limit, Hilbert cube.

(3) $Q \times \mathbb{R}^\infty$ *iff* X *is a Fréchet space not isomorphic to* $\mathbb{R}^\omega$;

(4) $\mathbb{R}^\omega \times \mathbb{R}^\infty$ *iff* X *is isomorphic to* $Y \oplus \mathbb{R}^\infty$ *for some infinite-dimensional Fréchet space*;

(5) $(\mathbb{R}^\infty)^\omega$ *otherwise.*

The Classification Theorem implies that $\mathcal{D}'$, the space of distributions on an open set Ω in $\mathbb{R}^n$, is homeomorphic to $(\mathbb{R}^\infty)^\omega$, because $\mathcal{D}'$ is the strong dual to the space $\mathcal{D}$ of test functions on Ω which is known to be a nuclear (LF)-space (see [Sch, III.§8]).

The definition of a nuclear space can be found in [Sch, III.§7]. All we need to know about nuclear spaces is that each nuclear Fréchet space is Montel and is a projective limit of Hilbert spaces. Let us recall that a locally convex space X is *Montel* if it is reflexive and each closed bounded subset of X is compact (see [Sch, IV.§5]).

We say that a topological space X is a *direct limit of metrizable compacta* if there is a tower $X_1 \subset X_2 \subset \ldots$ of metrizable compacta in X such that $X = \bigcup_{n=1}^\infty X_n$ and X has the direct limit topology $\varinjlim X_n$ (which consists of subsets $U \subset X = \varinjlim X_n$ with $U \cap X_n$ open in X_n for every n). It is well known that the topology of a locally convex direct sum on $\mathbb{R}^\infty = \bigoplus_{n\in\mathbb{N}} \mathbb{R}$ coincides with the direct limit topology $\varinjlim \mathbb{R}^n$ with respect to the tower $\mathbb{R} \subset \mathbb{R} \oplus \mathbb{R} \subset \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \subset \ldots$ A topological characterization of the space $\mathbb{R}^\infty$ was given by K. Sakai in [Sa].

As a by-product of the proof we get

THEOREM. *For an infinite-dimensional locally convex space* X *the following conditions are equivalent*:

(1) X *is a direct limit of metrizable compacta*;

(2) X *is homeomorphic either to* $\mathbb{R}^\infty$ *or to* $Q \times \mathbb{R}^\infty$;

(3) *the strong dual* X' *to* X *is a separable Fréchet space and* X *coincides with the dual* X'' *to* X', *endowed with the topology of uniform convergence on compact subsets of* X'.

COROLLARY. *The strong dual to any infinite-dimensional metrizable Montel space is homeomorphic either to* $\mathbb{R}^\infty$ *or to* $Q \times \mathbb{R}^\infty$.

Proof. Let X be an infinite-dimensional metrizable Montel space. Then X is a separable Fréchet space [Di]. Since the closure of each bounded subset in X is compact, the strong dual topology on X' coincides with the topology of uniform convergence on compact subsets of X. Because of the reflexivity, the space X' satisfies the condition (3) of our Theorem, which implies that X' is homeomorphic either to $\mathbb{R}^\infty$ or to $Q \times \mathbb{R}^\infty$. ∎

Proof of the Theorem. All maps considered in this paper are continuous; an *embedding* is a map which is a homeomorphism onto its image.

To prove the Theorem we will need four results from infinite-dimensional topology.

A. TOPOLOGICAL CHARACTERIZATION OF $Q \times \mathbb{R}^\infty$ ([Sa]). *A topological space* X *is homeomorphic to* $Q \times \mathbb{R}^\infty$ *if and only if*

(1) X *is a direct limit of a sequence of metrizable compacta*;

(2) *for every metrizable compactum* K *every embedding* $f : B \to X$ *of a closed subset* $B \subset K$ *can be extended to an embedding* $\bar{f} : K \to X$.

B. KELLER THEOREM ([Ke] or [BP, p. 100]). *Any infinite-dimensional convex metrizable compactum in a locally convex space is homeomorphic to the Hilbert cube* Q.

A *closed* subset A of a topological space X is called a *Z-set* if any map $f : Q \to X$ of the Hilbert cube can be uniformly approximated by maps whose range misses A. An embedding $f : A \to X$ is called a *Z-embedding* if $f(A)$ is a Z-set in X.

C. ANDERSON THEOREM ([An] or [Ch, 11.2]). *For every metrizable compactum* K *every* Z*-embedding* $f : B \to Q$ *of a closed subset* $B \subset K$ *can be extended to an embedding* $f : K \to Q$.

D. CENTRAL POINT THEOREM ([BP, V.§4]). *For every infinite-dimensional symmetric convex metrizable compactum* K *in a locally convex space*, $\frac{1}{2} \cdot K$ *is a* Z*-set in* K.

To prove the Theorem we will verify the implications (1)⇒(3)⇒(2)⇒(1).

(1)⇒(3). Suppose $X = \varinjlim X_n$ is a direct limit of a tower $0 \in X_1 \subset X_2 \subset \ldots$ of metrizable compacta with $\bigcup_{n=1}^\infty X_n = X$. First, we prove that $\{X_n : n \in \mathbb{N}\}$ is a fundamental system of bounded subsets in X, that is, every bounded subset $B \subset X$ lies in X_n for some n. Indeed, we first claim that $B \subset nX_n$ for some n. Assuming the converse we find a bounded sequence (x_n) in X such that $x_n \notin nX_n$ for every n. Then $n^{-1}x_n \notin X_n$, $n \in \mathbb{N}$, and by the definition of the direct limit topology on $X = \varinjlim X_n$, the set $\{n^{-1}x_n : n \in \mathbb{N}\}$ is closed in X. Since this set does not contain the origin, there is a neighborhood U of the origin in X such that $n^{-1}x_n \notin U$, $n \in \mathbb{N}$. On the other hand, since the sequence (x_n) is bounded, $(n^{-1}x_n)$ converges to zero [Sch, I.5.3], a contradiction. Hence $B \subset nX_n$ for some n. Since nX_n is compact, $nX_n \subset X_m$ for some m (assuming $nX_n \not\subset X_m$ for all m we would find a closed discrete subset (x_m) in nX_n such that $x_m \notin X_m$, $m \in \mathbb{N}$, which is impossible as nX_n is compact).

Thus $\{X_n : n \in \mathbb{N}\}$ is a countable fundamental family of bounded subsets in X. This implies that the linear map $E : X' \to \prod_{n=1}^\infty C(X_n)$ defined by $E : f \mapsto (f|X_n)_{n=1}^\infty$ for $f \in X'$ is a topological embedding. Here $C(X_n)$ is the Banach space of all continuous real functions on X_n. Using the fact that X has the direct limit topology $\varinjlim X_n$, we show that the image $E(X')$ is closed in $\prod_{n=1}^{\infty} C(X_n)$ (cf. Grothendieck Theorem [Sch, IV.6.2]). This implies that X', being isomorphic to a closed linear subspace in $\prod_{n=1}^{\infty} C(X_n)$, is a separable Fréchet space (let us remark that the metrizability of the compacta X_n implies the separability of the Banach spaces $C(X_n)$, see [En, 3.4.16]). Since X is infinite-dimensional, so is its dual space X'.

Next, we show that X coincides with the dual to X', endowed with the topology of compact convergence. Since the closure of any bounded subset in X is compact (being a subset of some X_n), [Sch, IV.5.5] implies that X is semireflexive, i.e. X coincides with its second dual X'' under the canonical map $X \to X''$. Finally let us show that the direct limit topology on X coincides with the topology of uniform convergence on compact subsets of X'. By [Sch, IV.6.3 and 5.2] the latter topology is the strongest topology inducing the weak topology on each bounded subset of X. Since the weak topology coincides with the original topology on each X_n and every bounded subset of X lies in some X_n, we conclude that the topology of uniform convergence on compact subsets of X' is the strongest topology inducing the original topology on each X_n, i.e. it coincides with the direct limit topology $\varinjlim X_n$.

(3)⇒(2). Suppose the strong dual space X' is a separable Fréchet space, X is semireflexive and $X = X''$ has the topology of uniform convergence on compact subsets of X'. Let $(U_n)_{n=1}^{\infty}$ be a countable base of closed convex symmetric neighborhoods of the origin in X' such that $U_{n+1} \subset \frac{1}{2}U_n$ for each $n \in \mathbb{N}$. Since X', being Fréchet, is barreled (see [Sch, II.§7]), the polars $U_n^\circ = \{x \in X : |f(x)| \le 1$ for each $f \in U_n\}$ form a fundamental system of bounded sets in X (see [Sch, IV.5.2]). The inclusion $U_{n+1} \subset \frac{1}{2}U_n$ implies $U_n^\circ \subset \frac{1}{2}U_{n+1}^\circ$ for each n. By [Sch, IV.17 and 5.2], each polar U_n° is a metrizable compactum with respect to the weak topology on X. By [Sch, IV.6.3 and 5.2], the topology of compact convergence on $X = X''$ is the strongest topology inducing the weak topology on each polar U_n°. This means that X has the direct limit topology with respect to the tower $U_1^\circ \subset U_2^\circ \subset \ldots$ of metrizable compacta (endowed with the weak topology).

Now we show that X is homeomorphic either to $\mathbb{R}^\infty$ or to $Q \times \mathbb{R}^\infty$. There are two cases:

1. All polars U_n° are finite-dimensional. In this case X is a direct limit of finite-dimensional metrizable compacta and by [Ba], X is homeomorphic (even isomorphic) to $\mathbb{R}^\infty$.

2. One of the polars U_n° is infinite-dimensional. Without loss of generality, $\dim U_1^\circ = \infty$. Then all U_n° are infinite-dimensional. Since X is a direct limit of metrizable compacta, to prove that X is homeomorphic to $Q \times \mathbb{R}^\infty$ it suffices to verify the second condition of the Characterization Theorem A. Fix a metrizable compactum K and an embedding $f : B \to X = \varinjlim U_n^\circ$ of a closed subset $B \subset K$. By compactness, $f(B) \subset U_n^\circ$ for some n. Since $U_n^\circ \subset \frac{1}{2}U_{n+1}^\circ$, by the Central Point Theorem D, U_n° is a Z-set in U_{n+1}°. This implies $f : B \to U_{n+1}^\circ$ is a Z-embedding. By the Keller Theorem B, the infinite-dimensional convex compactum U_{n+1}° is homeomorphic to the Hilbert cube Q. Thus the Anderson Theorem C is applicable and we can extend the Z-embedding $f : B \to U_{n+1}^\circ$ to an embedding $\bar{f} : K \to U_{n+1}^\circ \subset X$. Thus the second condition of the Characterization Theorem A is satisfied, and consequently X is homeomorphic to $Q \times \mathbb{R}^\infty$.

The trivial implication (2)⇒(1) completes the proof of the Theorem. ■

A lemma concerning Fréchet nuclear spaces. We say that a map $f : Y \to X$ between topological spaces is *homeomorphic to a trivial bundle* if there are a topological space F and a homeomorphism $h : Y \to X \times F$ such that $\mathrm{pr} \circ h = f$, where $\mathrm{pr} : X \times F \to X$ stands for the natural projection. Evidently, all fibers $f^{-1}(y)$ are then homeomorphic to F.

LEMMA. *If X is a closed linear subspace of a Fréchet nuclear space Y, then the dual operator $E' : Y' \to X'$ to the embedding operator $E : X \to Y$ is homeomorphic to a trivial bundle.*

Proof. First note that Y, being a Fréchet nuclear space, is a Montel space (see Corollary 2 from [Sch, III.7.2] and [Sch, IV.5.6]). The Lemma will be proven in three steps.

STEP 1. *We show that for every compact subset $K \subset X'$ there is a continuous map $g : K \to Y'$ such that $E' \circ g = \mathrm{id}$.*

By [Sch, IV.§1], the polar $K^\circ = \{x \in X : |f(x)| < 1$ for all $f \in K\}$ is a neighborhood of the origin in X. Because Y is a Fréchet nuclear space, by [Sch, III.7.3], there is a linear continuous operator $P : Y \to H$ into a Hilbert space H such that $X \cap P^{-1}(B) \subset K^\circ$, where B is the closed unit ball in H. Denote by H_X the closure of $P(X)$ in H and by $P_X : X \to H_X$ the restriction of P onto X. Let $\mathrm{Pr} : H \to H_X$ be the operator of orthogonal projection onto H_X. Hence we get a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{E} & Y \\ \Big\downarrow{\scriptstyle P_X} & & \Big\downarrow{\scriptstyle P} \\ H_X & \xleftarrow{\mathrm{Pr}} & H \end{array}$$

which induces the dual diagram

$$\begin{array}{ccc} X' & \xleftarrow{E'} & Y' \\ \Big\uparrow{\scriptstyle P_X'} & & \Big\uparrow{\scriptstyle P'} \\ H_X' & \xrightarrow{\mathrm{Pr}'} & H' \end{array}$$

Observe that the dual operator P'_X is injective (because $P_X(X)$ is dense in H_X). Thus we may consider the map $g = P' \circ \mathrm{Pr}' \circ (P'_X)^{-1}|K : K \to Y'$. Evidently, $E' \circ g = \mathrm{id}$.

We claim that g is continuous. First, observe that $X \cap P^{-1}(B) \subset K^\circ$ implies $(P_X)^{-1}(K) \subset B^\circ_X$, where B°_X is the dual unit ball of H'_X. Since $K \subset X'$ is (weakly) compact, $\widetilde{K} = (P'_X)^{-1}(K) \subset B^\circ_X$ is weakly closed and bounded. As H_X is a Hilbert space, $\widetilde{K}$ endowed with the weak topology is compact. Since P'_X is weakly continuous and the weak topology on K coincides with the original one, $P'_X : \widetilde{K} \to K$ is a homeomorphism. This implies that $g : K \to Y'$ is continuous with respect to the weak topology on Y'. Consequently, $g(K)$ is weakly compact in Y'. Since Y is a Montel space, so is its strong dual Y' (see [Sch, IV.5.9]). Hence, $g(K)$ is compact in the strong dual topology of Y', which implies that the weak and the strong topologies coincide on $g(K)$. Consequently, $g : K \to Y'$ is continuous with respect to the strong dual topology on Y'.

STEP 2. *We construct a continuous map* $g : X' \to Y'$ *such that* $E' \circ g = \mathrm{id}$.

Since Y is a metrizable Montel space, so is its closed subspace X. By the Corollary, the strong dual X' to X has the direct limit topology with respect to a tower $\emptyset = K_0 \subset K_1 \subset K_2 \subset \ldots$ of metrizable compacta in X'. To construct the required map $g : X' \to Y'$ it suffices to construct a sequence $\{g_n : K_n \to Y\}_{n=0}^\infty$ of maps such that $g_{n+1}|K_n = g_n$ and $E' \circ g_n = \mathrm{id}$ for each $n \in \mathbb{N}$. Then the map $g : X' \to Y'$ defined by $g|K_n = g_n$, $n \in \mathbb{N}$, will be continuous and will satisfy $E' \circ g = \mathrm{id}$.

The sequence (g_n) will be constructed by induction. Let $g_0 : K_0 \to Y'$ be a unique map (recall that $K_0 = \emptyset$). Suppose for some $n \ge 0$ the maps $g_0, \ldots, g_n$ have been constructed. As we proved in Step 1, there is a map $f : K_{n+1} \to Y'$ with $E' \circ f = \mathrm{id}$. Consider the map $h : K_n \to \mathrm{Ker}(E') \subset Y'$ defined by $h(x) = g_n(x) - f(x)$ for $x \in K_n$.

Since the kernel $\mathrm{Ker}(E') \subset Y'$ of E' is a locally convex space and K_{n+1} is metrizable, by the Dugundji Theorem [BP, II.§3], the map h can be extended to a map $\overline{h} : K_{n+1} \to \mathrm{Ker}(E')$. Then the map $g_{n+1} : K_{n+1} \to Y'$ defined by $g_{n+1}(x) = \overline{h}(x) + f(x)$ for $x \in K_{n+1}$ extends g_n and satisfies $E' \circ g_{n+1} = E' \circ f = \mathrm{id}$. Thus g_{n+1} is constructed, which completes the inductive step.

STEP 3. *We define a homeomorphism* $h : Y' \to X' \times \mathrm{Ker}(E')$ *such that* $\mathrm{pr} \circ h = E'$, *where* $\mathrm{pr} : X' \times \mathrm{Ker}(E') \to X'$ *is the projection.*

Let $g : X' \to Y'$ be a map with $E' \circ g = \mathrm{id}$ constructed in Step 2. Define a homeomorphism $h : Y' \to X' \times \mathrm{Ker}(E')$ letting $h(y) = (E'(y), y - g \circ E'(y))$ for $y \in Y'$. Evidently, $\mathrm{pr} \circ h = E'$. Since h has a continuous inverse h^{-1} defined by $h^{-1}(x,z) = g(x) + z$ for $(x,z) \in X' \times \mathrm{Ker}(E')$, h is a homeomorphism between the operator E' and the trivial bundle pr. ∎

Proof of the Classification Theorem. Suppose X is an infinite-dimensional nuclear (LF)-space. We consider five cases.

1. X is isomorphic to $\mathbb{R}^\infty$. Then X' is homeomorphic (even isomorphic) to $\mathbb{R}^\omega$.

2. X is isomorphic to $\mathbb{R}^\omega$. Then X' is homeomorphic (even isomorphic) to $\mathbb{R}^\infty$.

3. X is a Fréchet space, not isomorphic to $\mathbb{R}^\omega$. Since X is nuclear, it is Montel. Then the Corollary yields that X' is homeomorphic either to $\mathbb{R}^\infty$ or to $Q \times \mathbb{R}^\infty$. Assuming that X' is homeomorphic to $\mathbb{R}^\infty$ and applying a result of [Ba], we see that X' is isomorphic to $\mathbb{R}^\infty$. Then X is isomorphic to $\mathbb{R}^\omega$, a contradiction, which shows that X' is homeomorphic to $Q \times \mathbb{R}^\infty$.

4. X is isomorphic to $Y \oplus \mathbb{R}^\infty$ for some infinite-dimensional Fréchet space Y. Then Y, being a closed linear subspace of a metrizable Montel space $X \cong Y \oplus \mathbb{R}^\infty$, is metrizable and Montel. (Here "$\cong$" means "is isomorphic to", and "$\approx$" means "is homeomorphic to"). By the Corollary, Y' is homeomorphic either to $\mathbb{R}^\infty$ or to $Q \times \mathbb{R}^\infty$. Then X', being isomorphic to $(Y \oplus \mathbb{R}^\infty)' \cong Y' \oplus \mathbb{R}^\omega$, is homeomorphic either to $\mathbb{R}^\infty \times \mathbb{R}^\omega$ or to $Q \times \mathbb{R}^\infty \times \mathbb{R}^\omega$. In the first case the proof is complete. In the second, we use the homeomorphy of $\mathbb{R}^\omega$ and $Q \times \mathbb{R}^\omega$ (see [BP, p. 319]).

5. X is not isomorphic to $Y \oplus \mathbb{R}^\infty$, where Y is a Fréchet space. Write X as a strict inductive limit $\underrightarrow{\mathrm{ind}}\, X_n$ of a sequence $\{0\} = X_0 \subset X_1 \subset X_2 \subset \ldots$ of Fréchet spaces. Without loss of generality, we may assume that each X_n has infinite codimension in X_{n+1} (otherwise X is isomorphic either to X_n or to $X_n \times \mathbb{R}^\infty$ for some n). By [Sch, III.7.4] the spaces X_n are nuclear as closed linear subspaces of the nuclear space X. Next, the X_n's, being metrizable and nuclear, are Montel.

By the duality between inductive and projective topologies, the strong dual space X' can be identified with the projective limit $\underleftarrow{\lim}\, X'_n$ of the dual sequence

$$\{0\} = X_0 \xleftarrow{E'_0} X'_1 \xleftarrow{E'_1} X'_2 \xleftarrow{E'_2} X'_3 \longleftarrow \ldots,$$

where $E'_n : X'_{n+1} \to X'_n$ are the dual operators to the embedding operators $E_n : X_n \to X_{n+1}$. By the Lemma, each E'_n is homeomorphic to the trivial bundle $\mathrm{pr} : X'_n \times \mathrm{Ker}(E'_n) \to X'_n$. This implies that the projective limit $\underleftarrow{\lim}\, X'_n$ is homeomorphic to $\prod_{n=0}^\infty \mathrm{Ker}(E'_n)$.

We have to prove that this product is homeomorphic to $(\mathbb{R}^\infty)^\omega$. Since each X_n has infinite codimension in X_{n+1}, we conclude that each X_n, $n \ge 1$, is infinite-dimensional and $\dim \mathrm{Ker}(E'_n) = \infty$ for $n \ge 0$. By the Corollary, the strong dual X'_n to each X_n, $n \ge 1$, is a direct limit of metrizable compacta. Then $\mathrm{Ker}(E'_n)$, being closed in X'_{n+1}, is also a direct limit of metrizable compacta. By the Theorem, each $\mathrm{Ker}(E'_n)$ is homeomorphic to $\mathbb{R}^\infty$ or to $Q \times \mathbb{R}^\infty$. This implies that the product $\prod_{n=0}^\infty \mathrm{Ker}(E'_n)$ is homeomorphic either to $(\mathbb{R}^\infty)^\omega$ or to $Q \times (\mathbb{R}^\infty)^\omega$. In the first case the proof is complete; in the second, observe that $Q \times (\mathbb{R}^\infty)^\omega \approx ([0,1] \times \mathbb{R}^\infty)^\omega \approx (\mathbb{R}^\infty)^\omega$ (because $[0,1] \times \mathbb{R}^\infty \approx \mathbb{R}^\infty$, see [Sa]). ■

QUESTION. Is the Classification Theorem still valid for Montel (LF)-spaces? For separable (LF)-spaces?

Note that the answer to the first question is affirmative provided the Lemma is valid for Montel Fréchet spaces.

PROBLEM. Classify topologically strong duals to separable (reflexive) Fréchet spaces.

References

[An] R. D. Anderson, *On topological infinite deficiency*, Michigan Math. J. 14 (1967), 365–389.

[Ba] T. Banakh, *On linear topological spaces (linearly) homeomorphic to* $\mathbb{R}^\infty$, Mat. Stud. 9 (1998), 99–101.

[BP] C. Bessaga and A. Pełczyński, *Selected Topics in Infinite-Dimensional Topology*, PWN, Warszawa, 1975.

[Ch] T. A. Chapman, *Lectures on Hilbert Cube Manifolds*, CBMS Regional Conf. Ser. in Math. 28, Amer. Math. Soc., 1976.

[Di] J. Dieudonné, *Sur les espaces de Montel métrisables*, C. R. Acad. Sci. Paris 238 (1954), 194–195.

[En] R. Engelking, *General Topology*, PWN, Warszawa, 1977.

[Ke] O. M. Keller, *Die homoimorphie der kompakten konvexen Mengen in Hilbertschen Raum*, Math. Ann. 105 (1931), 748–758.

[Ma] P. Mankiewicz, *On topological, Lipschitz, and uniform classification of LF-spaces*, Studia Math. 52 (1974), 109–142.

[Sa] K. Sakai, *On* $\mathbb{R}^\infty$*-manifolds and* Q^∞*-manifolds*, Topology Appl. 18 (1984), 69–79.

[Sch] H. H. Schaefer, *Topological Vector Spaces*, Macmillan, New York, 1966.

Department of Mathematics
Lviv University
Universytetska 1
Lviv, 79000, Ukraine
E-mail: tbanakh@franko.lviv.ua

Received June 15, 1998 (4122)

STUDIA MATHEMATICA 138 (3) (2000)

Interpolation on families of characteristic functions

by

MICHAEL CWIKEL (Haifa) and
ARCHIL GULISASHVILI (Athens, Ohio)

Abstract. We study a problem of interpolating a linear operator which is bounded on some family of characteristic functions. A new example is given of a Banach couple of function spaces for which such interpolation is possible. This couple is of the form $\overline{\Phi} = (B, L^\infty)$ where B is an arbitrary Banach lattice of measurable functions on a σ-finite nonatomic measure space (Ω, Σ, μ). We also give an equivalent expression for the norm of a function f in the real interpolation space $(B, L^\infty)_{\theta,p}$ in terms of the characteristic functions of the level sets of f.

1. Introduction. Our goal in this paper is to study interpolation problems for linear operators acting on spaces of functions, in the case where these operators satisfy boundedness conditions only on a given family of characteristic functions rather than on all functions in the spaces. It has been shown by the second author [G1, G2, G5] that, under some additional restrictions, the boundedness of a linear operator T on a given family $\{\chi_E : E \in \mathcal{E}\}$ of characteristic functions from a couple of Lorentz spaces into a couple of Banach spaces implies the boundedness of T in the real interpolation spaces on the family of functions having all their level sets in $\mathcal{E}$. Moreover, a generalization of this interpolation theorem was obtained in [G3, G4, G5] in the form of a norm estimate in the real interpolation space for the Pettis integral of a mapping in terms of certain given estimates of the integrand. The method of interpolating from characteristic functions was used in the papers mentioned above and in [G6] to study the behavior of different linear operators, mainly the Fourier transform and the embedding operator.

In this paper we describe another situation where such interpolation from characteristic functions is possible. Here, instead of a couple of Lorentz

2000 *Mathematics Subject Classification*: Primary 46B70, 46G10, 46B42; Secondary 46E30.

The research of the first named author was supported by the Fund for Promotion of Research at the Technion.