STUDIA MATHEMATICA 138 (2) (2000)

The continuity of Lie homomorphisms

by

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Abstract. We prove that the separating space of a Lie homomorphism from a Banach algebra onto a Banach algebra is contained in the centre modulo the radical.

The ideas in this paper were inspired by a result which is contained in [14]. Suppose that A and B are (complex) Banach algebras and let $\theta : A \to B$ be a Lie homomorphism between A and B, that is, a linear map satisfying $\theta([x,y]) = [\theta(x), \theta(y)]$, x, $y \in A$, where, as usual, $[x,y]$ denotes the commutator $xy - yx$. If θ is bijective and B is a C*-algebra, then the separating space of θ (see below) is contained in the centre of B [14; Proposition 1.9 together with Proposition 2.7]. This has recently been extended to the case of semisimple Banach algebras A and B, but θ still being bijective [4]. Both results rely essentially on the notion of the "weak radical" of the Lie algebras which are canonically associated with A and B. The interrelation of this nonassociative device with the surrounding associative structure seems to be not well understood; in particular, it is not known whether the weak radical coincides with the (associative) centre of a Banach algebra, even in the semisimple case. Moreover, to obtain their result in [4], Berenguer and Villena have to appeal to the rather deep structure theory of Lie isomorphisms in the spirit of Herstein and to use it in a somewhat technical way. (For a large class of C*-algebras including all von Neumann algebras their result follows directly from the structure theory contained in [12]. For a good account on "Herstein's Programme", see [3].)

We felt that, since one is starting from an associative context and only considering the derived Lie algebra structure, it should be possible, and much simpler, to avoid the concept of the weak radical altogether, and also

2000 *Mathematics Subject Classification*: Primary 46H40; Secondary 16W10, 17B60, 46H70, 47B47, 47B48.

Key words and phrases: Lie homomorphisms, Banach algebras, spectrally bounded maps.

This work was started while the second-named author was a *Marie Curie Fellow* of the European Commission.

to replace the structural approach by some more direct argument. In fact, it turns out that it suffices to merely use a fundamental result, underlying also the previous theorems, which has proven its usefulness in related automatic continuity problems (see Lemma A below). At the same time, we remove all assumptions on the Banach algebras A and B and, what appears to be even more important, do not have to assume the injectivity of θ. Dropping the injectivity immediately rules out the application of the structure theory for Lie isomorphisms. On the other hand, it is not possible to omit the surjectivity of θ (as illustrated by an example below) so that we obtain the optimal result on the continuity of Lie homomorphisms.

THEOREM. *Let A and B be Banach algebras and let $\theta : A \to B$ be a surjective Lie homomorphism. Then the separating space $\mathcal{S}(\theta)$ is contained in $\mathcal{Z}(B)$, the centre modulo the radical of B.*

To establish the above theorem, we first recollect a few notions and known results. Extending the concept introduced in [11] we call a linear mapping $T : E \to B$ into a (unital) Banach algebra B defined on a *subspace* E of a (unital) Banach algebra A *spectrally bounded* if there is a constant $M \geq 0$ such that

$$r(Tx) \leq Mr(x) \quad (x \in E).$$

Here, and henceforth, $r(x)$ denotes the spectral radius of a Banach algebra element x.

Spectrally bounded derivations are intimately tied to the noncommutative Singer–Wermer conjecture [7], [9]. There has been some recent progress in understanding the structure of spectrally bounded mappings on Banach algebras (see [6], [8], [15], and the references therein). Their use in automatic continuity theory stems from the following result due to the first-named author [1], [2], which we state in a slightly more general form that we shall need subsequently.

As is standard, we denote the separating space of a linear mapping T between normed spaces by $\mathcal{S}(T)$, i.e.,

$$\mathcal{S}(T) = \{y \mid y = \lim_{n\to\infty} Tx_n \text{ for some sequence } x_n \to 0\}.$$

The separating space is a closed subspace of the range space.

LEMMA A. *Let T be a spectrally bounded linear map defined on a subspace E of a Banach algebra. Then $\mathcal{S}(T) \cap T(E)$ consists of quasinilpotent elements.*

The proof given for the case where T is defined on a Banach algebra in [1; Theorem 1] or [2; Theorem 5.5.1] takes over verbatim. It uses the subharmonicity of the spectral radius in an essential way. An alternative argument can be found in [10].

As a consequence of a slight refinement of the statement in Lemma A and Zemánek's characterisation of the radical, every spectrally bounded linear map from a closed subspace of a Banach algebra onto a Banach algebra B has its separating space in the radical of B. In particular, every surjective Jordan homomorphism from a Banach algebra onto a semisimple Banach algebra is automatically continuous, which extends the classical result due to Johnson.

Since *every* linear mapping between commutative Banach algebras is a Lie homomorphism, the optimal result that one can hope for states that the separating space is contained in the centre, if the image algebra is assumed to be semisimple. For an arbitrary Banach algebra B, the *centre modulo the radical*, $\mathcal{Z}(B)$, is defined as the inverse image of the centre of the Banach algebra $B/\mathrm{rad}(B)$, where $\mathrm{rad}(B)$ denotes the Jacobson radical of B. Thus, $b \in \mathcal{Z}(B)$ if and only if $[x, b] \in \mathrm{rad}(B)$ for all $x \in B$. This is in fact equivalent to the spectral boundedness of the (left) multiplication by b on B (see [13], [9]).

Besides Lemma A, the second main ingredient in the proof of the above theorem will therefore be a characterisation of the elements in $\mathcal{Z}(B)$ derived from the Jacobson Density Theorem [2; Theorem 4.2.5] and Kaplansky's description of locally algebraic operators [2; Theorem 4.2.7]. By δ_b, b in an algebra B, we denote the inner derivation $\delta_b : x \to [x, b]$, and by $\mathrm{inn}(B)$ the Lie algebra of all inner derivations on B.

PROPOSITION. *Let B be a (unital) Banach algebra. Then $b \in B$ belongs to $\mathcal{Z}(B)$ if and only if $r(\delta_{[x,b]}) = 0$ for all $x \in B$.*

Proof. Suppose that $b \in \mathcal{Z}(B)$. Then $[x, b] \in \mathrm{rad}(B)$ for all $x \in B$; in particular, $r([x, b]) = 0$. Since left and right multiplication commute it follows that

$$r(\delta_{[x,b]}) \leq 2r([x, b]) = 0.$$

Now assume that $r(\delta_{[x,b]}) = 0$ for all $x \in B$. Then $\lim_{k\to\infty} \delta^k_{[x,b]} = 0$ for all $x \in B$, and this is all we need in the sequel.

Let π be an irreducible representation of B as bounded linear operators on a Banach space E. As an inner derivation leaves the kernel of π invariant, we obtain an induced inner derivation on the irreducibly acting algebra $\pi(B)$ with the property that $\delta^k_{[\pi(x),\pi(b)]} \circ \pi = \pi \circ \delta^k_{[x,b]}$ for all $k \in \mathbb{N}$. Since π is continuous it follows that $\delta^k_{[\pi(x),\pi(b)]}\pi(y)\xi \to 0$ as $k \to \infty$ for each $y \in B$ and each $\xi \in E$.

Suppose that, for some $\xi \in E$, the vectors ξ, $\eta = \pi(b)\xi$, and $\pi(b)\eta$ are linearly independent. By Jacobson's Density Theorem, there is $x \in B$ such that $\pi(x)\xi = 0$, $\pi(x)\eta = \xi$, and $\pi(x)\pi(b)\eta = \eta$. Putting $c = [x, b]$ we have

$$\pi(c)\xi = (\pi(x)\pi(b) - \pi(b)\pi(x))\xi = \xi$$

and
$$\pi(c)\eta = (\pi(x)\pi(b) - \pi(b)\pi(x))\eta = 0.$$
Consequently,
$$\delta_{\pi(c)}\pi(b)\xi = (\pi(b)\pi(c) - \pi(c)\pi(b))\xi = \eta$$
whence
$$\delta^2_{\pi(c)}\pi(b)\xi = (\delta_{\pi(c)}\pi(b)\pi(c) - \pi(c)\delta_{\pi(c)}\pi(b))\xi = \eta.$$
It follows by induction that $\delta^k_{\pi(c)}\pi(b)\xi = \eta$ for all $k \in \mathbb{N}$. However, this contradicts the hypothesis $\delta^k_c \to 0$ as observed above. Hence, for every $\xi \in E$, the vectors ξ, $\pi(b)\xi$, and $\pi(b)^2\xi$ are linearly dependent, wherefore, by Kaplansky's Theorem, there exist $\alpha, \beta \in \mathbb{C}$ such that $\pi(b)^2 = \alpha\pi(b) + \beta$ or $\pi(b)$ is a multiple of the identity. In fact, in the first case, we may assume that $\alpha = 0$, upon replacing b by $b' = b - \alpha/2$, which satisfies $\pi(b')^2 = \beta + \alpha^2/4$ and yields the same commutator as b.

If $\pi(b)$ is not a multiple of the identity but $\pi(b)^2 = \beta$, then there exist $\xi \in E$ nonzero and $x \in B$ such that $\pi(x)\xi = \xi$ and $\pi(x)\pi(b)\xi = \xi + \pi(b)\xi$. Putting $\eta = \pi(b)\xi$ and $c = [x, b]$ we get
$$\pi(c)\xi = (\pi(x)\pi(b) - \pi(b)\pi(x))\xi = \xi + \eta - \eta = \xi$$
and
$$\begin{aligned}\pi(c)\eta &= (\pi(x)\pi(b) - \pi(b)\pi(x))\eta \\ &= \pi(x)\pi(b)^2\xi - \pi(b)(\xi + \pi(b)\xi) = \beta\pi(x)\xi - \eta - \beta\xi = -\eta.\end{aligned}$$
Therefore,
$$\begin{aligned}\delta_{\pi(c)}\pi(x)\eta &= (\pi(x)\pi(c) - \pi(c)\pi(x))\eta \\ &= -\pi(x)\eta - \pi(c)(\xi + \eta) = -(\xi + \eta) - \xi + \eta = -2\xi\end{aligned}$$
whence
$$\begin{aligned}\delta^2_{\pi(c)}\pi(x)\eta &= (\delta_{\pi(c)}\pi(x)\pi(c) - \pi(c)\delta_{\pi(c)}\pi(x))\eta \\ &= 2\xi + 2\pi(c)\xi = 4\xi.\end{aligned}$$
By induction, we obtain $\delta^k_{\pi(c)}\pi(x)\eta = (-2)^k\xi$ for all $k \in \mathbb{N}$, again contradicting the hypothesis $\delta^k_c \to 0$.

This shows that $\pi(b)$ has to be a multiple of the identity for all irreducible representations π. As a result, $b \in \mathcal{Z}(B)$. ∎

Remark. The above proof was inspired by that of [2; Theorem 5.2.1] which has its roots in a result of Le Page. In fact, the Proposition extends the quoted result since the spectrum of $\delta_{[x,b]}$ is always contained in the set of differences of two spectral values of $[x, b]$. Thus, if the spectrum of the latter consists of a singleton, the spectrum of $\delta_{[x,b]}$ reduces to zero wherefore the Proposition yields $b \in \mathcal{Z}(B)$.

The following standard result from spectral theory (see, e.g., [14; Lemma 3.1]) will be needed.

Lemma B. *Let E, F be Banach spaces and let $S \in \mathcal{L}(E)$, $R \in \mathcal{L}(F)$. If there is a linear surjective mapping $T : E \to F$ satisfying $TS = RT$, then $r(R) \le r(S)$.*

Proof. Take λ in the spectrum of R such that $|\lambda| = r(R)$. Since $\lambda - R$ is a topological divisor of zero, the mapping $\lambda - R$ cannot be surjective. This and the identity $T(\lambda - S) = (\lambda - R)T$ imply that $\lambda - S$ cannot be surjective either, as T is onto. Hence, λ belongs to the spectrum of S so that $r(R) = |\lambda| \le r(S)$. ∎

We are now ready for the proof of our main result.

Proof of the Theorem. Step 1. We first show that every $b \in \mathcal{S}(\theta)$ satisfies $r(\delta_b) = 0$. To this end, define $T : \operatorname{inn}(A) \to \operatorname{inn}(B)$ by $\delta_a \mapsto \delta_{\theta(a)}$, $a \in A$. Since θ is surjective, this is a well-defined surjective Lie homomorphism because
$$\begin{aligned}[T(\delta_{a_1}), T(\delta_{a_2})] &= [\delta_{\theta(a_1)}, \delta_{\theta(a_2)}] = \delta_{[\theta(a_2),\theta(a_1)]} \\ &= \delta_{\theta([a_2,a_1])} = T(\delta_{[a_2,a_1]}) = T([\delta_{a_1}, \delta_{a_2}])\end{aligned}$$
for all $a_1, a_2 \in A$. For all $x \in A$, we have
$$\theta\delta_a(x) = \theta([x, a]) = [\theta(x), \theta(a)] = \delta_{\theta(a)}\theta(x),$$
that is, $\theta\delta_a = \delta_{\theta(a)}\theta$. By Lemma B, it follows that $r(\delta_{\theta(a)}) \le r(\delta_a)$ for all $a \in A$. Consequently, T is spectrally bounded (with constant $M = 1$) as a linear mapping from $\operatorname{inn}(A) \subseteq \mathcal{L}(A)$ into $\mathcal{L}(B)$. Thus, $r(\delta_b) = 0$ for all $\delta_b \in \mathcal{S}(T)$ by Lemma A.

Let $b \in \mathcal{S}(\theta)$ and take a sequence $(x_n)_{n\in\mathbb{N}}$ in A such that $x_n \to 0$ and $\theta(x_n) \to b$. Then $\delta_{x_n} \to 0$ and $\delta_{\theta(x_n)} \to \delta_b$ wherefore $\delta_b \in \mathcal{S}(T)$. Consequently, $r(\delta_b) = 0$ as claimed.

Step 2. The surjectivity of θ yields that $\mathcal{S}(\theta)$ is a Lie ideal in B. Hence, by Step 1, for all $b \in \mathcal{S}(\theta)$ and all $x \in B$, we have $r(\delta_{[x,b]}) = 0$. By the Proposition, each $b \in \mathcal{S}(\theta)$ belongs to $\mathcal{Z}(B)$, which completes the proof. ∎

Remark. In the case of C*-algebras A and B, we can avoid the Proposition, and hence all the representation theory, in Step 2. Let $b \in \mathcal{S}(\theta)$. Then $c = [b, b^*]$ is a self-adjoint element in $\mathcal{S}(\theta)$. By Step 1, $r(\delta_c) = 0$ and since $\exp(it\delta_c)$ is an isometry for every real t, this entails that $\delta_c = 0$ [5; 10.17]. Therefore, c is central and $[b, [b, b^*]] = 0$ yields that $r(c) = r([b, b^*]) = 0$ by the Kleinecke–Shirokov Theorem. But a central quasinilpotent element in B has to be zero, wherefore b itself is normal. This shows that every element in $\mathcal{S}(\theta)$ is necessarily normal.

Consequently, $[\delta_b, \delta_{b^*}] = \delta_{[b^*,b]} = 0$ and

$$r(\delta_{b+b^*}) = r(\delta_b + \delta_{b^*}) \le r(\delta_b) + r(\delta_{b^*}) \quad (b \in \mathcal{S}(\theta)).$$

If we define $\theta^* : A \to B$ by $\theta^*(a) = \theta(a^*)^*$, $a \in A$, we obtain a surjective Lie homomorphism such that $\mathcal{S}(\theta) = \mathcal{S}(\theta^*)^*$. It follows that, if $b \in \mathcal{S}(\theta)$, then $r(\delta_{b^*}) = 0$ by applying Step 1 to θ^* in place of θ. By the above, $r(\delta_{b+b^*}) = 0$ for every $b \in \mathcal{S}(\theta)$, which implies $b + b^*$ central as before. Similarly, we have $b - b^*$ central, from which we conclude that every element b in $\mathcal{S}(\theta)$ belongs to the centre of B as claimed.

Simple examples show that the surjectivity in the Theorem cannot be dropped.

EXAMPLE. Let A be an infinite-dimensional unital commutative Banach algebra and take two linearly independent linear functionals f, g on A such that $f|_{\ker g}$ and $g|_{\ker f}$ are unbounded. For instance, let $\{e_i \mid i \in \mathbb{N} \dot\cup \mathbb{I}\}$ be a vector space basis, indexed by the disjoint union of $\mathbb{N}$ and some suitable set $\mathbb{I}$ and contained in the unit ball of A, and put $f(e_n) = n$ for all $n \in \mathbb{N}$ and $f(e_i) = 0$ for all $i \in \mathbb{I}$, while g is defined by $g(e_n) = n$ for all $n \in \mathbb{N}$ even and $g(e_i) = 0$ for all other $i \in \mathbb{N} \dot\cup \mathbb{I}$. Let $B = (K(H) \oplus K(H))[1]$ be the unitisation of the direct sum of the compact operators $K(H)$ on the separable infinite-dimensional Hilbert space H with itself. Fix $c \in K(H)$ nonzero. Then $\theta(a) = (f(a)c, g(a)c)$, $a \in A$, defines a Lie homomorphism from A into B such that $\mathcal{S}(\theta) \cong \mathbb{C}^2$ is not contained in $Z(B) = \mathbb{C}1$.

For each Banach algebra A, the centre modulo the radical, $\mathcal{Z}(A)$, clearly is a closed Lie ideal of A. We denote by $\widehat{A} = A/\mathcal{Z}(A)$ the canonically associated Banach Lie algebra obtained by quotienting out $\mathcal{Z}(A)$. Every surjective Lie homomorphism $\theta : A \to B$ induces a surjective Lie homomorphism $\widehat{\theta} : \widehat{A} \to \widehat{B}$ since $\theta(\mathcal{Z}(A)) \subseteq \mathcal{Z}(B)$. This is derived from the Proposition and the proof of the Theorem as follows. If $a \in \mathcal{Z}(A)$ then $r(\delta_{[x,a]}) = 0$ for all $x \in A$. As observed above,

$$r(\delta_{[\theta(x),\theta(a)]}) = r(\delta_{\theta([x,a])}) \le r(\delta_{[x,a]}) = 0,$$

wherefore $r(\delta_{[y,\theta(a)]}) = 0$ for all $y \in B$ by the surjectivity of θ. Hence, by the Proposition, $\theta(a) \in \mathcal{Z}(B)$. Since $\widehat{\theta}$ is continuous if and only if $\mathcal{S}(\theta) \subseteq \mathcal{Z}(B)$, we obtain the following immediate consequence of the Theorem.

COROLLARY. *Let θ be a surjective Lie homomorphism from a Banach algebra A onto a Banach algebra B. Then the induced Lie homomorphism $\widehat{\theta}$ from the Banach Lie algebra $\widehat{A}$ onto $\widehat{B}$ is continuous.*

This result was obtained in [4] under the hypothesis that both A and B are semisimple and that θ is bijective.

Acknowledgements. We are indebted to Angel Rodríguez Palacios for a helpful discussion in the course of the preparation of this paper.

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Received June 21, 1999
Revised version October 19, 1999 (4346)