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Valdivia compacta and equivalent norms

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Abstract. We prove that the dual unit ball of a Banach space X is a Corson compactum provided that the dual unit ball with respect to every equivalent norm on X is a Valdivia compactum. As a corollary we show that the dual unit ball of a Banach space X of density $\aleph_1$ is Corson if (and only if) X has a projectional resolution of the identity with respect to every equivalent norm. These results answer questions asked by M. Fabian, G. Godefroy and V. Zizler and yield a converse to Amir–Lindenstrauss' theorem.

1. Introduction. The Valdivia compact spaces form a class containing all Corson compacta which plays an important role in the theory of Banach spaces. It is closely related to projectional resolutions of the identity and Markushevich bases. These relations have been thoroughly investigated for example in [AMN], [V2], [V3], [DG], [FGZ], [K3].

The Banach spaces with dual unit ball being a special kind of Valdivia compactum (namely, such that (B_{X^*}, w^*) has a dense convex symmetric Σ-subset, see definitions below) form the largest class of spaces having a projectional resolution of the identity (PRI) (within Banach spaces of density $\aleph_1$). The fact that the well-known Amir–Lindenstrauss theorem [AL] can be extended to this class of spaces was proved in [V3]. The converse within Banach spaces of density $\aleph_1$ was observed in [FGZ].

It follows again from [V3] that for Banach spaces with dual unit ball being Corson compact the isomorphic version of Amir–Lindenstrauss' theorem holds. It means that such spaces have a PRI with respect to every equivalent norm. This follows simply from the fact that the Corson property of the dual ball is an isomorphic notion. The Valdivia property of the dual ball, as observed in [K1], is not stable with respect to equivalent norms. In the present paper we prove that the dual unit ball of a Banach space is Corson provided the dual unit ball with respect to each equivalent norm is Valdivia. Due to

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the above mentioned observation of [FGZ] it follows that the Banach spaces with Corson dual unit ball form the largest class (within spaces of density $\aleph_1$) in which the isomorphic version of Amir–Lindenstrauss' theorem holds. Some special cases of this result were proved in [FGZ].

Another motivation for our investigation comes from the study of strong non-stability properties of Valdivia compacta. For example it was proved in [K2] that a compactum K is Corson provided all continuous images of K are Valdivia. The first example of a non-Valdivia continuous image of a Valdivia compactum was given in [V4]. In this framework our result can be viewed as a result on strong non-stability of Valdivia compacta with respect to equivalent norms.

Let us start with basic definitions.

DEFINITION 1. Let Γ be a set.

(1) For $x \in \mathbb{R}^\Gamma$ we define $\operatorname{supp} x = \{\gamma \in \Gamma \mid x(\gamma) \neq 0\}$.
(2) We put $\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma \mid \operatorname{supp} x \text{ is countable}\}$.

DEFINITION 2. Let K be a compact Hausdorff space.

(1) K is called a *Corson compact* space if K is homeomorphic to a subset of $\Sigma(\Gamma)$ for some Γ.

(2) K is called a *Valdivia compact* space if K is homeomorphic to a subset K' of $\mathbb{R}^\Gamma$ for some Γ such that $K' \cap \Sigma(\Gamma)$ is dense in K'.

It turned out to be useful to introduce the following auxiliary notion.

DEFINITION 3. Let K be a compact Hausdorff space and $A \subset K$ be arbitrary. We say that A is a *Σ-subset* of K if there is a homeomorphic injection φ of K into $\mathbb{R}^\Gamma$ for some Γ such that $\varphi(A) = \varphi(K) \cap \Sigma(\Gamma)$.

In this setting a compactum K is Valdivia if it has a dense Σ-subset. In [V2], [V3] and [FGZ] the connection between Valdivia compacta and projectional resolutions of the identity and Markushevich bases was investigated. Let us now recall the definitions of these notions.

DEFINITION 4. Let X be a Banach space of density $\kappa > \aleph_0$. A *projectional resolution of the identity* (PRI) on X is an indexed family $(P_\alpha \mid \omega \leq \alpha \leq \kappa)$ of projections on X with the following properties:

(i) $P_\omega = 0$, $P_\kappa = \mathrm{Id}_X$;
(ii) $\|P_\alpha\| = 1$ for $\omega < \alpha \leq \kappa$;
(iii) $\operatorname{dens} P_\alpha X \leq \operatorname{card} \alpha$ for $\omega < \alpha \leq \kappa$;
(iv) $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ for $\omega \leq \alpha \leq \beta \leq \kappa$;
(v) $P_\alpha X = \overline{\bigcup_{\beta<\alpha} P_\beta X}$ if $\alpha \leq \kappa$ is limit.

DEFINITION 5. Let X be a Banach space.

(1) A *Markushevich basis* of X is an indexed family $(x_a, f_a)_{a\in A} \subset X \times X^*$ such that the following conditions are satisfied:
(a) $f_a(x_b) = 0$ if $a \neq b$, $f_a(x_a) = 1$ for $a, b \in A$;
(b) $\overline{\operatorname{span}}\{x_a \mid a \in A\} = X$;
(c) for every $x \in X$, $x \neq 0$ there is $a \in A$ with $f_a(x) \neq 0$.

(2) A Markushevich basis $(x_a, f_a)_{a\in A}$ is *countably* 1*-norming* if for every $x \in X$ we have $\|x\| = \sup\{f(x) \mid f \in M,\ \|f\| \leq 1\}$, where $M = \{f \in X^* \mid \{a \in A \mid f(x_a) \neq 0\} \text{ is countable}\}$.

2. Main results. Our main result is the following theorem which answers a question of [FGZ].

THEOREM 1. *Let X be a Banach space. Then the following assertions are equivalent.*

(1) *(B_{X^*}, w^*) is a Corson compact space.*
(2) *For every equivalent norm $|\cdot|$ on X the dual unit ball $(B_{(X,|\cdot|)^*}, w^*)$ has a dense convex symmetric Σ-subset.*
(3) *For every equivalent norm $|\cdot|$ on X the dual unit ball $(B_{(X,|\cdot|)^*}, w^*)$ is a Valdivia compactum.*
(4) *For every equivalent norm $|\cdot|$ on X the space $(X, |\cdot|)$ admits a countably* 1*-norming Markushevich basis.*

If the density of X is equal to $\aleph_1$, then the above conditions are also equivalent to the following one.

(5) *For every equivalent norm $|\cdot|$ on X the space $(X, |\cdot|)$ admits a projectional resolution of the identity.*

As a consequence we get the following results which are not covered by the theorems of [FGZ].

COROLLARY 1. *If Γ is an uncountable set, there is an equivalent norm on $\ell_1(\Gamma)$ such that the corresponding dual unit ball is not a Valdivia compactum.*

COROLLARY 2. *There is an equivalent norm $|\cdot|$ on $\ell_1([0, \omega_1))$ such that $(\ell_1([0, \omega_1)), |\cdot|)$ has no projectional resolution of the identity.*

Proof (of corollaries). It suffices to observe that the dual unit ball of $\ell_1(\Gamma)$ is weak* homeomorphic to $[-1, 1]^\Gamma$ which is a non-Corson Valdivia compactum. ∎

REMARK. Recently the author has been informed that Corollary 2 was independently proved, using another method, by A. Plička.

The implication (5)⇒(1) in Theorem 1 does not hold for Banach spaces of density larger than $\aleph_1$. A counterexample is the c_0 sum of $\aleph_2$ copies of the space $\ell_1([0,\omega_1))$ (cf. remarks at the end of the paper). However, we are able to prove the following partial result.

THEOREM 2. *Let Γ be an uncountable set such that* $\operatorname{card}\Gamma$ *is a regular cardinal. Then there is an equivalent norm $|\cdot|$ on $\ell_1(\Gamma)$ such that $(\ell_1(\Gamma),|\cdot|)$ has no projectional resolution of the identity.*

REMARK. In fact, a more general result holds. Suppose that X is a Banach space whose density is a regular cardinal κ. If there is a PRI, say $(P_\alpha \mid \omega \le \alpha \le \kappa)$, on X with $\bigcup_{\omega\le\alpha<\kappa} P_\alpha^* X^* \subsetneq X^*$, then there is an equivalent norm on X such that there is no PRI with respect to that norm. This can be proved by putting together the ideas of the proofs of Theorems 1 and 2. We sketch the proof at the end of the paper.

3. Auxiliary results

LEMMA 1 [K2, Proposition 2.2]. *Let K be a compact Hausdorff space and $A\subset K$ be a dense Σ-subset of K. Then:*

(1) *A is countably closed in K, i.e. $\overline{C}\subset A$ for every $C\subset A$ countable.*
(2) *A is a Fréchet–Urysohn space, i.e. whenever $x\in A$ and $C\subset A$ are such that $x\in\overline{C}$, then there are $x_n\in C$ with $x_n\to x$.*
(3) *If $G\subset K$ is a G_δ set, then $G\cap A$ is dense in G.*

LEMMA 2. *Let X be a Banach space and $K\subset S_{X^*}$ be a convex weak* compact set. Then there is a convex weak* compact set L which is weak* G_δ in B_{X^*} and $K\subset L\subset S_{X^*}$.*

Proof. Let $n\in\mathbb{N}$. The sets $(1-1/n)B_{X^*}$ and K are disjoint convex weak* compact sets, so by the Hahn–Banach theorem there are $x_n\in X$ and $c_n\in\mathbb{R}$ such that

$$\sup_{f\in(1-1/n)B_{X^*}} f(x_n) < c_n < \inf_{f\in K} f(x_n).$$

It is enough to put $L=\{f\in B_{X^*}\mid (\forall n\in\mathbb{N})(f(x_n)\ge c_n)\}$. ∎

PROPOSITION 1. *Let X be a Banach space such that there is a weak* compact convex set $K\subset S_{X^*}$ which is not a Valdivia compact. Then there is an equivalent norm $|\cdot|$ on X such that $B_{(X,|\cdot|)^*}$ is not Valdivia.*

Proof. Let L be a convex weak* compact set, weak* G_δ in B_{X^*} such that $K\subset L\subset S_{X^*}$ (it exists due to Lemma 2). Put

$$B=\operatorname{conv}\big(K\cup(-K)\cup \tfrac12 B_{X^*}\big).$$

Then B is a convex symmetric weak* compact set such that $\frac12 B_{X^*}\subset B\subset B_{X^*}$, so there is an equivalent norm $|\cdot|$ on X such that B is its dual unit ball. To show that B is not Valdivia, we prove that $K=L\cap B$. Choose

$f\in L\cap B$. Then there are $s,t\ge 0$ with $s+t\le 1$ and $k_1,k_2\in K$, $b\in\frac12 B_X^*$ such that $f=sb+tk_1+(1-s-t)(-k_2)$. We have

$$1=\|f\|\le s\|b\|+t\|k_1\|+(1-s-t)\|k_2\|\le \frac{s}{2}+t+1-s-t=1-\frac{s}{2},$$

hence $s=0$. So $f=tk_1+(1-t)(-k_2)$. As $k_2\in K\subset L$, we get $\frac12(f+k_2)\in L$, but $\frac12(f+k_2)=\frac t2(k_1+k_2)$, so $\big\|\frac12(f+k_2)\big\|=t$, hence $t=1$.

Thus $K=L\cap B$ and therefore K is weak* G_δ in B. If B were Valdivia, K would be Valdivia as well by Lemma 1(3), contrary to assumption. ∎

REMARK. It is clear from the above proof that the equivalent norm whose dual unit ball is not Valdivia can be chosen arbitrarily close to the original one.

LEMMA 3. *Let X be a Banach space and $f_1,\dots,f_n$ be affinely independent elements of X^* with $n\ge 2$. Then there is $x\in X$ such that*

$$f_1(x)<f_2(x)=\ldots=f_{n-1}(x)<f_n(x).$$

If $n=2$ then the above condition simply means $f_1(x)<f_2(x)$.

Proof. If $n=2$ then the affine independence means $f_1\ne f_2$, so clearly there is $x\in X$ with $f_1(x)<f_2(x)$. Suppose $n\ge 3$. Then $f_2-f_1,\dots,f_n-f_1$ are linearly independent, so in particular $f_n-f_1\notin\operatorname{span}\{f_j-f_1\mid 2\le j\le n-1\}$. The latter space is finite-dimensional, and hence weak* closed, so by the Hahn–Banach theorem there is $x_1\in X$ such that $(f_j-f_1)(x_1)=0$ for $2\le j\le n-1$ and $(f_n-f_1)(x_1)=1$. Similarly there is $x_2\in X$ such that $(f_j-f_n)(x_2)=0$ for $2\le j\le n-1$ and $(f_1-f_n)(x_2)=-1$. Now it is clear that $x=x_1+x_2$ has the required property. ∎

The following lemma can be proved by straightforward calculations.

LEMMA 4. *Let X be a Banach space, $K\subset X^*$ be weak* compact and $P(K)$ be the space of Radon probabilities on K endowed with the weak* topology inherited from $C(K)^*$. Consider the mapping $\Phi: P(K)\to X^*$ defined by the formula*

$$\Phi(\mu)=\int_K f\,d\mu(f),$$

where the integral is in the weak sense, i.e.*

$$\Phi(\mu)(x)=\int_K f(x)\,d\mu(f),\qquad x\in X,\ \mu\in P(K).$$

Then Φ is a $w^*\to w^*$ continuous affine mapping of $P(K)$ onto $\overline{\operatorname{conv} K}^{w^*}$.

PROPOSITION 2. *Let X be a Banach space and $K \subset X^*$ be a weak* compact scattered set. Then*

$$\overline{\operatorname{conv} K}^{w^*} = \Big\{ \sum_{k\in K} a_k k \;\Big|\; a_k \ge 0,\ \sum_{k\in K} a_k = 1 \Big\},$$

where the sums are taken in the norm topology.

Proof. By Lemma 4, using the fact that each Radon probability on a scattered space is supported by a countable set, we get the required equality with the sums in the weak* sense. But as K is bounded, these sums converge even absolutely in norm. ■

PROPOSITION 3. *Let X be a Banach space and $K \subset X^*$ be a convex weak* compact set which is not Corson. Then there is a weak* compact convex subset of K which is not Valdivia.*

Proof. If K itself is not Valdivia, there is nothing to prove. Suppose that K is Valdivia and that $A \subset K$ is a dense Σ-subset. Let $\varphi : K \to \mathbb{R}^\Gamma$ be a homeomorphic injection with $\varphi(A) = \varphi(K) \cap \Sigma(\Gamma)$. By [K2, Proposition 2.7] there is $g \in K$ such that $\operatorname{card}\{\gamma \in \Gamma \mid \varphi(g)(\gamma) \neq 0\} = \aleph_1$. The proof will continue in several steps.

STEP 1. We construct $f_n \in K$ and $x_n \in X$ for $n \in \mathbb{N}$ satisfying the following conditions:

(i) $\|f_n - g\| < 1/n$;
(ii) f_n does not belong to the affine envelope of $\{g\} \cup \{f_i \mid 1 \le i < n\}$;
(iii) $g(x_k) < f_n(x_k) < f_1(x_k)$ for $1 \le k < n$;
(iv) $g(x_n) < f_1(x_n) = \ldots = f_{n-1}(x_n) < f_n(x_n)$.

As K is not Corson, it cannot be a singleton, so there is $h \in K \setminus \{g\}$. By convexity the segment $[g,h]$ lies in K. Choose $f_1 \in (g,h]$ with $\|f_1 - g\| < 1$, and $x_1 \in X$ such that $g(x_1) < f_1(x_1)$. These f_1 and x_1 clearly satisfy (i), (ii) and (iv), and condition (iii) gives no restriction if $n = 1$.

Suppose we have already constructed $f_1, \ldots, f_n$ and $x_1, \ldots, x_n$ satisfying (i)–(iv). To construct f_{n+1} and x_{n+1}, denote by M_n the affine envelope of $\{g, f_1, \ldots, f_n\}$. As M_n is finite-dimensional and K is not Corson, there is $h_0 \in K \setminus M_n$. We take auxiliary functionals $h_1, \ldots, h_n$ such that for $j = 1, \ldots, n$ we have

$$(*) \qquad h_j \in [h_{j-1}, f_j) \quad \text{and} \quad h_j(x_j) > g(x_j).$$

This is possible, as $f_j(x_j) > g(x_j)$ by (iv).

Moreover, we have

$$(**) \qquad h_j(x_i) > g(x_i) \quad \text{for } 1 \le i \le j \le n.$$

We prove this by induction on j. If $j = 1$, then $h_1(x_1) > g(x_1)$ by $(*)$. Suppose we have proved $(**)$ up to some $j < n$. Let $1 \le i \le j+1$. If $i = j+1$, then $h_{j+1}(x_i) > g(x_i)$ by $(*)$. Further, if $i < j+1$, then $i \le j$, and so, by the induction hypothesis, we have $h_j(x_i) > g(x_i)$. Moreover, $f_{j+1}(x_i) > g(x_i)$ by (iii), and therefore $(*)$ yields $h_{j+1}(x_i) > g(x_i)$ as well, which completes the proof of $(**)$.

In particular we have $h_n(x_i) > g(x_i)$ for $1 \le i \le n$. Further, $g(x_i) < f_1(x_i)$ for $1 \le i \le n$ by (iv). So there is $f_{n+1} \in [h_n, g)$ such that $f_{n+1}(x_i) < f_1(x_i)$ (and, of course, $f_{n+1}(x_i) > g(x_i)$) for $1 \le i \le n$ and $\|f_{n+1} - g\| < 1/(n+1)$. By convexity of K we have $f_{n+1} \in K$ and clearly $f_{n+1} \notin M_n$. Hence the conditions (i)–(iii) are satisfied also for f_{n+1}. Finally, we can choose $x_{n+1} \in X$ satisfying (iv) due to Lemma 3. This completes the construction.

STEP 2. We construct a copy of $[0, \omega_1]$ in K. To this end we refine the construction of [K2, Proposition 2.7]. Put

$$G = \{h \in K \mid (\forall n \in \mathbb{N})(h(x_n) = g(x_n))\}.$$

This is clearly a closed G_δ subset of K. Further put

$$I = \operatorname{supp} \varphi(g) = \{\gamma \in \Gamma \mid \varphi(g)(\gamma) \neq 0\}.$$

By the choice of g we have $\operatorname{card} I = \aleph_1$. Fix an enumeration $I = \{i_\alpha \mid \alpha < \omega_1\}$. For $\alpha < \omega_1$ we construct $g_\alpha \in G \cap A$, $y_\alpha \in X$ and $J_\alpha \subset \Gamma$ such that the following conditions are satisfied:

(a) $i_\alpha \in J_{\alpha+1}$, $\bigcup_{\beta<\alpha} \operatorname{supp} \varphi(g_\beta) \subset J_\alpha$, J_α is countable;
(b) $J_\alpha \subset J_{\alpha+1}$, $\operatorname{supp} \varphi(g_\alpha) \cap I \subsetneq J_{\alpha+1} \cap I$;
(c) $J_\alpha = \bigcup_{\beta<\alpha} J_\beta$ if α is limit;
(d) $\varphi(g_\alpha)(i) = \varphi(g)(i)$ for $i \in J_\alpha$;
(e) $g_\alpha(y_\beta) = g(y_\beta)$ if $\beta < \alpha$;
(f) $g_\alpha = \lim_{\beta<\alpha} g_\beta$ if α is limit;
(g) $g(y_\alpha) > \sup_{\beta\le\alpha} g_\beta(y_\alpha)$.

Put $J_0 = \{i_0\}$ and choose $g_0 \in G \cap A$ such that $\varphi(g_0)(i_0) = \varphi(g)(i_0)$. This is possible due to Lemma 1(3), as the set $\{h \in G \mid \varphi(g)(i_0) = \varphi(h)(i_0)\}$ is G_δ in K. Further, we can find $y_0 \in X$ with $g_0(y_0) < g(y_0)$ (as $g_0 \neq g$).

Suppose we have constructed g_β, J_β, y_β for $\beta \le \alpha$. As $\operatorname{supp} \varphi(g_\alpha)$ is at most countable and $\operatorname{card} I = \aleph_1$, there is some $j \in I \setminus \operatorname{supp} \varphi(g_\alpha)$. Put $J_{\alpha+1} = J_\alpha \cup \operatorname{supp} \varphi(g_\alpha) \cup \{i_\alpha, j\}$. Then $J_{\alpha+1}$ is clearly countable, and (a)–(c) are again satisfied. Since

$$M = \{h \in G \mid \varphi(h)(i) = \varphi(g)(i) \text{ for } i \in J_{\alpha+1} \text{ and } h(y_\beta) = g(y_\beta) \text{ for } \beta \le \alpha\}$$

is a G_δ subset of K, we can choose $g_{\alpha+1} \in M \cap A$ by Lemma 1(3), and in this way (d)–(f) are again satisfied. To find $y_{\alpha+1}$ satisfying (g), it is enough to show, by the Hahn–Banach theorem, that

$$g \notin \overline{\operatorname{conv}\{g_\beta \mid \beta \le \alpha+1\}}^{w^*}.$$

We have

$$\overline{\operatorname{conv}\{g_\beta \mid \beta \le \alpha+1\}}^{w^*} = \operatorname{conv}(\overline{\operatorname{conv}\{g_\beta \mid \beta \le \alpha\}}^{w^*} \cup \{g_{\alpha+1}\}).$$

If $g = tg_{\alpha+1} + (1-t)h$ for some $h \in \overline{\operatorname{conv}\{g_\beta \mid \beta \le \alpha\}}^{w^*}$ and $t \in [0,1]$, then

$$g(y_\alpha) = tg_{\alpha+1}(y_\alpha) + (1-t)h(y_\alpha) = tg(y_\alpha) + (1-t)h(y_\alpha).$$

As $h(y_\alpha) \le \sup_{\beta\le\alpha} g_\beta(y_\alpha) < g(y_\alpha)$ by the induction hypothesis, we get $t = 1$, hence $g_{\alpha+1} = g$, which is impossible since $g \notin A$ and $g_{\alpha+1} \in A$.

Now suppose that α is limit and we have constructed $g_\beta, J_\beta, y_\beta$ for every $\beta < \alpha$. Put $J_\alpha = \bigcup_{\beta<\alpha} J_\beta$. It is easy to check that the net $(\varphi(g_\beta) \mid \beta < \alpha)$ converges in $\mathbb{R}^\Gamma$ to the point p such that $p(i) = \varphi(g)(i)$ if $i \in J_\alpha$ and $p(i) = 0$ otherwise. As φ is a homeomorphism and G is compact, the net $(g_\beta \mid \beta < \alpha)$ weak* converges to a point of G. Denote the limit by g_α. Then clearly (a)–(d) and (f) are again satisfied. The validity of (e) follows easily from the definition of weak* convergence. To find y_α satisfying (g), it is enough to show, by the Hahn–Banach theorem, that

$$g \notin \overline{\operatorname{conv}\{g_\beta \mid \beta \le \alpha\}}^{w^*}.$$

Suppose that this is not the case. As $\{g_\beta \mid \beta \le \alpha\}$ is clearly countable and compact, by Proposition 2 there are $c_\beta \ge 0$ such that $\sum_{\beta\le\alpha} c_\beta = 1$ and $g = \sum_{\beta\le\alpha} c_\beta g_\beta$. It follows that for every $\gamma < \alpha$ we have

$$\begin{aligned} g(y_\gamma) &= \sum_{\beta\le\alpha} c_\beta g_\beta(y_\gamma) = \sum_{\beta\le\gamma} c_\beta g_\beta(y_\gamma) + \sum_{\gamma<\beta\le\alpha} c_\beta g_\beta(y_\gamma) \\ &\le \Big(\sum_{\beta\le\gamma} c_\beta\Big) \sup_{\beta\le\gamma} g_\beta(y_\gamma) + \Big(\sum_{\gamma<\beta\le\alpha} c_\beta\Big) g(y_\gamma). \end{aligned}$$

As $\sup_{\beta\le\gamma} g_\beta(y_\gamma) < g(y_\gamma)$ by the induction hypothesis, we see that $c_\beta = 0$ for $\beta \le \gamma$. As $\gamma < \alpha$ was arbitrary, we obtain $c_\beta = 0$ for all $\beta < \alpha$, hence $g_\alpha = g$, which is impossible since $g_\alpha \in A$ and $g \notin A$. This completes the construction.

Now it is easy to observe that the net $(g_\alpha \mid \alpha < \omega_1)$ converges to $g = g_{\omega_1}$ and that $\{g_\alpha \mid \alpha \le \omega_1\}$ is a homeomorphic copy of $[0, \omega_1]$.

STEP 3. Put $L = \{f_n \mid n \in \mathbb{N}\} \cup \{g_\alpha \mid \alpha \le \omega_1\}$. Note that L is homeomorphic to the non-Valdivia compactum constructed in [V4]. Further let $H = \overline{\operatorname{conv} L}^{w^*}$. We claim that H is not a Valdivia compactum.

STEP 4. We show that each f_n is a G_δ point of H. Clearly it suffices to prove that

$$\{f_n\} = \{h \in H \mid h(x_n) = f_n(x_n)\}.$$

The inclusion "$\subset$" is obvious, so let us prove the reverse one. Let h be in the set on the right-hand side. Since L is a scattered compactum, by Proposition 2 there are $a_\alpha \ge 0$ and $b_k \ge 0$ such that $\sum_{\alpha\le\omega_1} a_\alpha + \sum_{k\in\mathbb{N}} b_k = 1$ and $h = \sum_{\alpha\le\omega_1} a_\alpha g_\alpha + \sum_{k\in\mathbb{N}} b_k f_k$. We have

$$\begin{aligned} f_n(x_n) = h(x_n) &= \sum_{\alpha\le\omega_1} a_\alpha g_\alpha(x_n) + \sum_{k\in\mathbb{N}} b_k f_k(x_n) \\ &= \sum_{\alpha\le\omega_1} a_\alpha g(x_n) + \sum_{k=1}^{n-1} b_k f_k(x_n) + b_n f_n(x_n) + \sum_{k=n+1}^{\infty} b_k f_k(x_n). \end{aligned}$$

Now, $g(x_n) < f_n(x_n)$ and $f_k(x_n) < f_n(x_n)$ for $k < n$ by condition (iv) from Step 1. For $k > n$ by (iii) and (iv) from Step 1 we have $f_k(x_n) < f_1(x_n) \le f_n(x_n)$. So necessarily $h = f_n$. This finishes the proof of Step 4.

STEP 5. Now we show that g_α is a G_δ point of H for every $\alpha < \omega_1$. In fact, we prove that

$$\{g_\alpha\} = \{h \in H \mid h(x_1) = g(x_1) \ \&\ (\forall\beta < \alpha)(h(y_\beta) = g(y_\beta)) \ \&\ h(y_\alpha) = g_\alpha(y_\alpha)\}.$$

Clearly the set on the right-hand side is G_δ in H and contains g_α. Conversely, let h be in the set. By Proposition 2 there are $a_\beta \ge 0$ and $b_n \ge 0$ with $\sum_{\beta\le\omega_1} a_\beta + \sum_{n\in\mathbb{N}} b_n = 1$ such that

$$h = \sum_{\beta\le\omega_1} a_\beta g_\beta + \sum_{n\in\mathbb{N}} b_n f_n.$$

By the choice of h we have

$$\begin{aligned} g(x_1) = h(x_1) &= \sum_{\beta\le\omega_1} a_\beta g_\beta(x_1) + \sum_{n\in\mathbb{N}} b_n f_n(x_1) \\ &= g(x_1) \sum_{\beta\le\omega_1} a_\beta + \sum_{n\in\mathbb{N}} b_n f_n(x_1). \end{aligned}$$

By (iii) from Step 1 we have $f_n(x_1) > g(x_1)$ for $n \ge 2$, and $f_1(x_1) > g(x_1)$ by (iv). It follows that $b_n = 0$ for all $n \in \mathbb{N}$. Using again the choice of h we get, for each $\beta < \alpha$,

$$\begin{aligned} g(y_\beta) = h(y_\beta) &= \sum_{\gamma\le\omega_1} a_\gamma g_\gamma(y_\beta) = \sum_{\gamma\le\beta} a_\gamma g_\gamma(y_\beta) + \sum_{\beta<\gamma\le\omega_1} a_\gamma g_\gamma(y_\beta) \\ &\le \sup_{\gamma\le\beta} g_\gamma(y_\beta) \sum_{\gamma\le\beta} a_\gamma + g(y_\beta) \sum_{\beta<\gamma\le\omega_1} a_\gamma. \end{aligned}$$

By (g) from Step 2 we have $\sup_{\gamma\le\beta} g_\gamma(y_\beta) < g(y_\beta)$, so $a_\gamma = 0$ for $\gamma \le \beta$. As $\beta < \alpha$ was arbitrary, it follows that $a_\gamma = 0$ for all $\gamma < \alpha$.

We use the choice of h once more to get

$$g_\alpha(y_\alpha) = h(y_\alpha) = \sum_{\alpha\le\beta\le\omega_1} a_\beta g_\beta(y_\alpha) = a_\alpha g_\alpha(y_\alpha) + \sum_{\alpha<\beta\le\omega_1} a_\beta g(y_\alpha).$$

As $g(y_\alpha) > g_\alpha(y_\alpha)$ (by (g) from Step 2), we get $a_\beta = 0$ for $\beta > \alpha$. Therefore $h = g_\alpha$. This completes the proof of Step 5.

STEP 6: CONCLUSION. Suppose that there is a dense Σ-subset B of H. By Step 4 each f_n is a G_δ point of H, hence $f_n \in B$ (Lemma 1(3)). As $f_n \to g$ (even in norm), we get $g \in B$ by Lemma 1(1). On the other hand, g_α is a G_δ point of H for $\alpha < \omega_1$ by Step 5, hence $g_\alpha \in B$ by Lemma 1(3). Now, g is in the closure of $\{g_\alpha \mid \alpha < \omega_1\}$ but it is the limit of no sequence from this set, hence $g \notin B$ by Lemma 1(2). This contradiction finishes the proof. ∎

4. Proofs of the main results

Proof of Theorem 1. The implications (1)⇒(2)⇒(3) are trivial, (4)⇒(3) is proved in [K1, Lemma 3], and (1)⇒(4) follows for example from [V3].

To prove (3)⇒(1), let X be a Banach space with (B_{X^*}, w^*) being Valdivia but not Corson. Let $A \subset B_{X^*}$ be a dense Σ-subset. If $S_{X^*} \subset A$, then by Lemma 1(1) and a corollary to the Josefson–Nissenzweig theorem [D, Chapter XII, Exercise 2(i)] we would get $B_{X^*} \subset A$, and so B_{X^*} would be Corson. So there is $f \in S_{X^*} \setminus A$. Apply Lemma 2 to get a convex weak* compact set $L \subset S_{X^*}$ which is weak* G_δ in B_{X^*} and contains f. By Lemma 1(3), $L \cap A$ is dense in L, so L is Valdivia, and as $f \in L \setminus A$, L is not Corson. By Proposition 3 there is $K \subset L$ convex weak* compact, non-Valdivia. Finally by Proposition 1 there is an equivalent norm on X such that the corresponding dual unit ball is not Valdivia. This completes the proof.

The implication (1)⇒(5) follows from [V1]. If X has density $\aleph_1$, then (5)⇒(3) by [FGZ]. ∎

Proof of Theorem 2. First let us define some auxiliary notions. If κ is an uncountable cardinal, put $\Sigma_\kappa(I) = \{x \in \mathbb{R}^I \mid \operatorname{card}\operatorname{supp} x < \kappa\}$, and call a compact space K *κ-Corson* if it is homeomorphic to a subset of $\Sigma_\kappa(I)$ for a set I. Further, call K *κ-Valdivia* if it is homeomorphic to some $K' \subset \mathbb{R}^I$ with $K' \cap \Sigma_\kappa(I)$ dense in K'.

Next suppose that κ is regular. It is easy to check that $\Sigma_\kappa(I)$ is "$<\kappa$-closed" in $\mathbb{R}^I$, i.e. $\overline{A} \subset \Sigma_\kappa(I)$ whenever $A \subset \Sigma_\kappa(I)$ and $\operatorname{card} A < \kappa$. In particular, it is countably closed. Further, it is not hard to check that the space $[0,\kappa]$ is not κ-Corson.

Now, we show that the dual unit ball of $X = \mathcal{C}_0[0,\kappa)$ is not κ-Valdivia. Suppose the converse; let $h : B_{X^*} \to \mathbb{R}^I$ be a homeomorphism with $A = h^{-1}(\Sigma_\kappa(I))$ dense in B_{X^*}. By the previous paragraph A is countably compact, so it is easy to see ([K2, Lemma 2.3]) that it contains each G_δ point of B_{X^*}. Now observe that the point $\xi_n = \frac{1}{2}(\delta_n - \delta_{n+1})$ (where δ_x is the Dirac measure supported at x) is G_δ for every $n \in \mathbb{N}$. Indeed, let $f = \chi_{\{n\}} - \chi_{\{n+1\}} \in X$. Then the set $\{\mu \in B_{X^*} \mid (\mu, f) = 1\}$ is weak* G_δ in B_{X^*}, and it is easy to check that it is the segment $\{t\delta_n - (1-t)\delta_{n+1} \mid t \in [0,1]\}$, and clearly ξ_n is a G_δ point of this segment. Therefore $\xi_n \in A$. Further we have $\xi_n \xrightarrow{w^*} 0$, so $0 \in A$ as A is countably closed. Moreover, it is clear that δ_x is a G_δ point of B_{X^*} whenever $x < \kappa$ is an isolated ordinal. So $\delta_x \in A$ for x isolated. If $y < \kappa$ is limit, then $\delta_y = \lim_{x<y} \delta_x$, hence $\delta_y \in A$ by the previous paragraph. Finally, the mapping $\varphi : [0,\kappa] \to B_{X^*}$ defined by

$$\varphi(x) = \begin{cases} \delta_x, & x < \kappa, \\ 0, & x = \kappa, \end{cases}$$

is a homeomorphic embedding of $[0,\kappa]$ into A, which contradicts the previous paragraph.

As X has weight κ, it is well known that X is a quotient of $\ell_1([0,\kappa))$ (cf. [HHZ, p. 71, Theorem 91 and the following remarks]). Let Q denote the quotient mapping. Then Q^* is a linear, isometric, weak*-to-weak* homeomorphic embedding. Put $K = Q^*(B_{X^*})$. This is a weak* compact convex symmetric set. Further, let

$$B = \operatorname{conv}([-1/2, 1/2]^{[0,\kappa]} \cup (K \times \{1\}) \cup (K \times \{-1\})).$$

This is the dual unit ball of an equivalent norm on $\ell_1([0,\kappa])$ and it is easy to see (using the same idea as in the proof of Proposition 1) that B is not κ-Valdivia.

Finally, note that, if a Banach space Y of density κ has a PRI, then the dual unit ball (B_{Y^*}, w^*) is κ-Valdivia. Indeed, if $(P_\alpha \mid \omega \le \alpha \le \kappa)$ is a PRI, let $I_\alpha \subset (P_{\alpha+1} - P_\alpha)Y$ be a dense set with $\operatorname{card} I_\alpha \le \operatorname{card}\alpha$ and put $I = \bigcup_{\omega\le\alpha<\kappa} I_\alpha$; it is not hard to check that $h(\xi)(i) = \xi(i)$ for $\xi \in B_{Y^*}$ and $i \in I$ is an embedding witnessing that B_{Y^*} is κ-Valdivia (cf. [FGZ, Lemma 2]).

Now it follows that $\ell_1([0,\kappa])$ with the equivalent norm constructed above has no PRI. ∎

REMARK. The proof of Theorem 2 indicates a simpler proof of the result of [K1]. Also it shows an easy proof of Corollaries 1 and 2, which does not use the theorems nor Proposition 3 but only Proposition 1.

Sketch of the proof of the remark after Theorem 2. We use the terminology introduced in the proof of Theorem 2. The first step is to observe that, under our assumptions, the dual unit ball B_{X^*} is κ-Valdivia but not κ-Corson. In the same way as in the proof of Theorem 1, find $L \subset S_{X^*}$ convex, weak* compact, weak* G_δ in B_{X^*} which is κ-Valdivia and not κ-Corson. Using an analogue of Proposition 3 we get $K \subset L$ convex, weak* compact, which is not κ-Valdivia. Finally, using the idea of Proposition 1 we get a dual unit ball with respect to an equivalent norm which is not κ-Valdivia. It remains to observe that there is no PRI with respect to this norm. ■

We finish by discussing some natural open questions in this area.

As said in the introduction, Valdivia compacta are closely related to PRI on Banach spaces. However, the following question seems to be still open.

QUESTION 1. Suppose that the dual unit ball of the Banach space X is a Valdivia compactum. Does X have a PRI?

It is natural to ask what happens if the space in question has density greater than $\aleph_1$. Unfortunately, the usual definition of PRI does not allow one to characterize Banach spaces with Corson dual unit ball in terms of PRI. It can be proved, for example, that a Banach space of density κ, where κ is a successor cardinal, has a PRI with respect to every equivalent norm if and only if the dual unit ball of X is κ-Corson. We do not know what happens for limit cardinals. But it seems not to be impossible to characterize spaces with Corson dual ball with a notion stronger than PRI. Let us call a family of projections *strong PRI* if it satisfies the same conditions as PRI except for condition (iii) which is replaced by

(iii′) $\operatorname{dens} P_\alpha X = \operatorname{card} \alpha$ for $\omega < \alpha \le \kappa$.

It is easy to check that a space of density $\aleph_1$ has a strong PRI whenever it has a PRI, and that spaces with Corson dual ball have even strong PRI. So we can formulate the following question.

QUESTION 2. Suppose that a Banach space X (of density greater than $\aleph_1$) has a strong PRI with respect to every equivalent norm. Is then the dual unit ball of X Corson?

Our last question is concerned with (non-)stability with respect to taking subspaces.

QUESTION 3. Suppose that B_{Y^*} is a Valdivia compactum for each subspace Y of a Banach space X. Is then B_{X^*} necessarily Corson?

Let us remark that a partial affirmative answer to Question 3 (for $X = C(K)$ with K being a continuous image of a Valdivia compactum) is given in [K4].

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