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On the growth of analytic semigroups along vertical lines

by

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Abstract. We construct two Banach algebras, one which contains analytic semigroups $(a^z)_{\operatorname{Re} z>0}$ such that $\|a^{1+iy}\| \to \infty$ arbitrarily slowly as $|y| \to \infty$, the other which contains ones such that $\|a^{1+iy}\| \to \infty$ arbitrarily fast.

1. Introduction. A family of elements $(a^z)_{\operatorname{Re} z>0}$ in a Banach algebra A is called an *analytic semigroup* if the map $z \mapsto a^z$ is analytic on the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and satisfies

$$a^{z+w} = a^z a^w \quad (\operatorname{Re} z > 0,\ \operatorname{Re} w > 0).$$

The structure of a Banach algebra is frequently reflected in the growth properties of its analytic semigroups (this fact has applications in recent classifications of Banach algebras, more particularly when spectral methods are not available [E3, S]). For example, in a radical Banach algebra we always have

$$\lim_{x\to\infty} \|a^x\|^{1/x} = 0. \tag{1}$$

This is a simple consequence of the spectral radius formula. On the other hand, along *vertical lines* these semigroups must actually grow. Esterle [E1] exploited this fact to give a new proof of the Wiener tauberian theorem. Building on his work, Sinclair [S, Theorem 5.6] showed that, in a radical Banach algebra, an analytic semigroup (a^z) satisfying

$$\int_{-\infty}^{\infty} \frac{\log^+ \|a^{1+iy}\|}{1+y^2}\, dy < \infty \tag{2}$$

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is identically zero. The same is also true in a Banach algebra with no zero-divisors [E4].

There are several other results of this general nature. For example, in a general Banach algebra, if an analytic semigroup (a^z) satisfies (2), then the closed subalgebra B generated by $(a^z)_{\operatorname{Re} z>0}$ is regular [EG, Theorem 1]. For semigroups such that

$$\liminf_{y\to\infty}\frac{\|a^{1+iy}\|\,\|a^{1-iy}\|}{y}=0,$$

the algebra B is weakly amenable [W, Theorem 2.3], and it fails the strong Wedderburn decomposition if it is also assumed that $\sup_{x\in(0,1)}\|a^x\|<\infty$ and $\|a^{1+iy}\|=O(|y|^k)$ as $|y|\to\infty$, for some $k\geq 0$ [W, Corollary 3.4]. If $(a^{1+iy})_{y\in\mathbb{R}}$ is relatively weakly compact, then the character space of B is at most countable, B itself is generated by its idempotents [G, Theorem 3.1], and either B is semisimple or $B^\perp:=\{b\in B: bB=0\}\neq 0$ [G, Proposition 3.2].

In the case when the ambient algebra is $L^1(\mathbb{R}^n)$ or $M(\mathbb{R}^n)$, this last result can be strengthened. Let us say that a Banach algebra A has the *Beurling–Helson property* if the only analytic semigroups (a^z) in A such that $\sup_{y\in\mathbb{R}}\|a^{1+iy}\|<\infty$ have the form $a^z=e^{cz}\delta$ for some $c\in\mathbb{R}$ (if the algebra contains an identity δ) or they reduce to zero. It was shown in [GW, Theorem 2.4] that $L^1(\mathbb{R}^n)$ and $M(\mathbb{R}^n)$ have the Beurling–Helson property. This can be viewed as an analytic-semigroup analogue of the classical theorem of Beurling and Helson [BH], that an invertible element a of $M(\mathbb{R})$ satisfying $\sup_{n\in\mathbb{Z}}\|a^n\|<\infty$ must be of the form $a=e^{ic}\delta_b$ for some $b,c\in\mathbb{R}$, where δ_b denotes the unit mass at b.

Of course, there are numerous analytic semigroups in $L^1(\mathbb{R})$ for which $(a^{1+iy})_{y\in\mathbb{R}}$ is unbounded, but all the examples we know grow at least as fast as the Poisson semigroup (P^z), for which

$$\|P^{1+iy}\|_1\asymp\log|y|\quad\text{as } |y|\to\infty.$$

This is conceivably the slowest possible growth rate for a non-zero analytic semigroup in this algebra. The analogous question for the algebra $\ell^1(\mathbb{Z})$ of absolutely convergent Fourier series is a long-standing open problem (see e.g. [K, p. 87] or [GM, p. 407]).

In each of the situations above, there is a minimum growth rate for $\|a^{1+iy}\|$, as if analyticity forces semigroups with small growth on $\{\operatorname{Re} z=1\}$ to be bounded there, or even zero. This appears to lend support to the possibility that, for any given Banach algebra A, all analytic semigroups which tend to infinity along $\{\operatorname{Re} z=1\}$ do so at a certain rate. Our first result shows that this is not the case. Moreover, although we do not know whether it is possible to take $A=L^1(\mathbb{R})$ or $M(\mathbb{R})$, our example does possess the Beurling–Helson property. (Our A is non-separable: see the remarks at the end of §3.)

THEOREM 1.1. *There is a Banach algebra A with the Beurling–Helson property such that, given any function $\varphi:\mathbb{R}\to\mathbb{R}$ satisfying*

$$\lim_{|y|\to\infty}\varphi(y)=\infty\quad\text{and}\quad\inf_{y\in\mathbb{R}}\varphi(y)>0,$$

there exists an analytic semigroup (a^z) in A with

$$\lim_{|y|\to\infty}\|a^{1+iy}\|=\infty\quad\text{and}\quad\|a^{1+iy}\|\leq\varphi(y)\quad(y\in\mathbb{R}).$$

The proof of Theorem 1.1 is based upon a construction of analytic semigroups of convolution operators on weighted H^1-spaces. The details of this construction are given in §2, and the proof of the theorem is completed in §3. On the way we obtain two quantitative results, Theorems 2.3 and 3.1, which may be of independent interest.

Returning, for a moment, to the case of analytic semigroups in radical Banach algebras, although the convergence in (1) may be arbitrarily slow [S, Corollary 3.13], it can *not* be arbitrarily fast [E2, Theorem 3.1]. Again, this depends heavily on analyticity [E2, Theorem 3.6]. By analogy, one might be tempted to guess that, for an analytic semigroup (a^z) in a general Banach algebra, there are constraints on the growth of $\|a^{1+iy}\|$ as $|y|\to\infty$. There is some supporting evidence for this view. For example, it was shown in [R, Corollary 1.4] that, if (2) holds, then $\|a^z\|$ is of exponential type on $\operatorname{Re} z\geq 2$. However, our second theorem demonstrates that in general $\|a^{1+iy}\|$ may indeed grow arbitrarily fast.

THEOREM 1.2. *There is a separable Banach algebra A with the Beurling–Helson property such that, given any locally bounded function $\varphi:\mathbb{R}\to\mathbb{R}$, there exists an analytic semigroup (a^z) in A with*

$$\|a^{1+iy}\|\geq\varphi(y)\quad(y\in\mathbb{R}).$$

This time the proof depends on a construction of an analytic semigroup in an infinite direct sum of tensor powers. The details are given in §4.

2. Convolution operators on weighted H^1-spaces. In this section we describe the main construction used in the proof of Theorem 1.1.

We begin by recalling the definition of the Poisson semigroup in $L^1(\mathbb{R})$. For $\operatorname{Re} z>0$, define $P^z:\mathbb{R}\to\mathbb{C}$ by

$$P^z(t)=\frac{1}{\pi}\frac{z}{z^2+t^2}\quad(t\in\mathbb{R}).$$

The following proposition summarizes some basic properties of P^z.

PROPOSITION 2.1. *Let P^z be defined as above.*

(i) *The Fourier transform satisfies $\widehat{P}^z(\xi) = e^{-z|\xi|}$ $(\xi \in \mathbb{R})$.*

(ii) *The family $(P^z)_{\operatorname{Re} z>0}$ is an analytic semigroup in $L^1(\mathbb{R})$.*

(iii) *If $f \in H^1(\mathbb{R})$ (i.e. $f \in L^1(\mathbb{R})$ and $\widehat{f}(\xi) = 0$ for all $\xi < 0$), then*

$$P^{x+iy} * f = P^x * f_y \qquad (x > 0,\ y \in \mathbb{R}),$$

where $f_y(t) = f(t-y)$.

(iv) *P^z satisfies the pointwise estimate*

$$|P^z(t)| \le \frac{8}{\pi}\left(\frac{1+|z|^2}{\operatorname{Re} z}\right)\frac{1}{1+t^2} \qquad (t \in \mathbb{R},\ \operatorname{Re} z > 0). \tag{3}$$

Proof. For (i) and (ii), see e.g. [S, Theorem 2.17].

For (iii), we compute Fourier transforms: given $\xi \in \mathbb{R}$, we have

$$(P^{x+iy} * f)\widehat{\ }(\xi) = \widehat{P}^{x+iy}(\xi)\widehat{f}(\xi) = e^{-(x+iy)|\xi|}\widehat{f}(\xi)$$

and

$$(P^x * f_y)\widehat{\ }(\xi) = \widehat{P}^x(\xi)\widehat{f}_y(\xi) = e^{-x|\xi|}e^{-iy\xi}\widehat{f}(\xi).$$

As $f \in H^1(\mathbb{R})$, we have $\widehat{f}(\xi) = 0$ for all $\xi < 0$ and so the two right-hand sides are equal.

Finally, for (iv), we divide the proof into three cases. If $|t| \le |z|/2$, then

$$\frac{|z^2+t^2|}{1+t^2} \ge \frac{|z|^2-t^2}{1+t^2} \ge \frac{|z|^2-|z|^2/4}{1+|z|^2/4} \ge \frac{3}{4}\frac{|z|^2}{1+|z|^2} \ge \frac{3}{4}\frac{|z|(\operatorname{Re} z)}{1+|z|^2}.$$

If $|z|/2 \le |t| \le 2|z|+1$, then

$$\frac{|z^2+t^2|}{1+t^2} \ge \frac{2|t|(\operatorname{Re} z)}{1+t^2} \ge \frac{|z|(\operatorname{Re} z)}{1+(2|z|+1)^2} \ge \frac{1}{8}\frac{|z|(\operatorname{Re} z)}{1+|z|^2}.$$

Lastly, if $|t| \ge 2|z|+1$, then

$$\frac{|z^2+t^2|}{1+t^2} \ge \frac{t^2-|z|^2}{2t^2} \ge \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^2 \ge \frac{3}{8}\frac{|z|(\operatorname{Re} z)}{1+|z|^2}.$$

Hence, in all cases,

$$|P^z(t)| = \frac{1}{\pi}\frac{|z|}{|z^2+t^2|} \le \frac{|z|}{\pi}\frac{8(1+|z|^2)}{|z|(\operatorname{Re} z)}\frac{1}{1+t^2} \le \frac{8}{\pi}\left(\frac{1+|z|^2}{\operatorname{Re} z}\right)\frac{1}{1+t^2},$$

as claimed. ∎

A *weight* is a Borel function $\omega : \mathbb{R} \to [1,\infty)$ satisfying

$$\omega(s+t) \le \omega(s)\omega(t) \qquad (s,t \in \mathbb{R}).$$

Given a weight ω, we define

$$L^1_\omega = \Big\{f \in L^1(\mathbb{R}) : \|f\|_\omega := \int_{-\infty}^{\infty} |f(t)|\omega(t)\,dt < \infty\Big\}.$$

The condition on ω ensures that $(L^1_\omega, \|\cdot\|_\omega)$ is a Banach algebra under convolution.

PROPOSITION 2.2. *The Poisson semigroup $(P^z)_{\operatorname{Re} z>0}$ is an analytic semigroup in the Banach algebra L^1_ω if and only if*

$$I(\omega) := \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\omega(t)}{1+t^2}\,dt < \infty.$$

Proof. The "only if" is clear, since $\|P^1\|_\omega = I(\omega)$.

For the "if", assume that $I(\omega) < \infty$. Then, for $\operatorname{Re} z > 0$, we have

$$\int_{-\infty}^{\infty} |P^z(t)|\omega(t)\,dt \le \frac{8}{\pi}\left(\frac{1+|z|^2}{\operatorname{Re} z}\right)\int_{-\infty}^{\infty} \frac{1}{1+t^2}\omega(t)\,dt$$
$$= 8\left(\frac{1+|z|^2}{\operatorname{Re} z}\right)I(\omega) < \infty,$$

so $P^z \in L^1_\omega$.

Also, given z, w with $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$,

$$|P^z(t) - P^w(t)| \le \frac{|z-w|}{\pi}\frac{|zw|+t^2}{|z^2+t^2||w^2+t^2|}$$
$$\le \pi|z-w|\left(1+\frac{1}{|zw|}\right)(1+t^2)|P^z(t)||P^w(t)|,$$

whence, using (3) once more,

$$\|P^z - P^w\|_\omega \le 64|z-w|\left(1+\frac{1}{|zw|}\right)\left(\frac{1+|z|^2}{\operatorname{Re} z}\right)\left(\frac{1+|w|^2}{\operatorname{Re} w}\right)I(\omega).$$

In particular, $z \mapsto P^z$ is continuous as a map into L^1_ω.

Next, for each triangle Δ in $\{\operatorname{Re} z > 0\}$, we have $\int_\Delta P^z\,dz = 0$ by Cauchy's theorem. Hence, by Morera's theorem, $z \mapsto P^z$ is analytic as a map into L^1_ω.

Finally, the semigroup property is automatic since (P^z) is already a semigroup in $L^1(\mathbb{R})$. ∎

Given a weight ω, we define

$$H^1_\omega = L^1_\omega \cap H^1(\mathbb{R}) = \{f \in L^1_\omega : \widehat{f}(\xi) = 0 \text{ for all } \xi < 0\}.$$

Observe that H^1_ω is a closed ideal in L^1_ω.

THEOREM 2.3. *Let ω be a weight satisfying*

$$I(\omega) := \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\omega(t)}{1+t^2}\,dt < \infty.$$

For $\operatorname{Re} z > 0$, *define* $T^z : H^1_\omega \to H^1_\omega$ *by*

$$T^z(f) = P^z * f \qquad (f \in H^1_\omega).$$

Then $(T^z)_{\operatorname{Re} z>0}$ *is an analytic semigroup in* $\mathcal{L}(H^1_\omega)$, *the Banach algebra of bounded linear operators on* H^1_ω, *and*

$$\frac{1}{4I(\omega)^2} \le \frac{\|T^{1+iy}\|}{\omega(y)} \le I(\omega) \qquad (y \in \mathbb{R}). \tag{4}$$

Proof. From Proposition 2.2, it follows immediately that (T^z) is an analytic semigroup in $\mathcal{L}(H^1_\omega)$. It remains to prove the estimates (4).

We begin with the upper bound. Let $y \in \mathbb{R}$. Then, given $f \in H^1_\omega$,

$$\|T^{1+iy}f\|_\omega = \|P^{1+iy} * f\|_\omega = \|P^1 * f_y\|_\omega \le \|P^1\|_\omega \|f_y\|_\omega = I(\omega)\|f_y\|_\omega,$$

where the second equality is from Proposition 2.1(iii). Now

$$\begin{aligned}\|f_y\|_\omega &= \int_{-\infty}^{\infty} |f(t-y)|\omega(t)\,dt = \int_{-\infty}^{\infty} |f(s)|\omega(s+y)\,ds \\ &\le \int_{-\infty}^{\infty} |f(s)|\omega(s)\omega(y)\,ds = \omega(y)\|f\|_\omega.\end{aligned}$$

As this holds for each $f \in H^1_\omega$, we conclude that

$$\|T^{1+iy}\| \le I(\omega)\omega(y) \qquad (y \in \mathbb{R}).$$

Now for the lower bound. Let

$$g(t) = \frac{1}{\pi}\frac{1}{(t+i)^2} \qquad (t \in \mathbb{R}).$$

Then $|g(t)| = P^1(t)$, so $g \in L^1_\omega$ with $\|g\|_\omega = \|P^1\|_\omega = I(\omega)$. Further, a simple computation with Fourier transforms shows that $g \in H^1_\omega$. Therefore, using Proposition 2.1(iii) once more, we have

$$T^{1+iy}g = P^{1+iy} * g = P^1 * g_y = h_y \qquad (y \in \mathbb{R}),$$

where $h = P^1 * g$. Thus, for $y \in \mathbb{R}$,

$$\begin{aligned}\|T^{1+iy}g\|_\omega &= \int_{-\infty}^{\infty} |h(t-y)|\,\omega(t)\,dt = \int_{-\infty}^{\infty} |h(s)|\omega(s+y)\,ds \\ &\ge \int_{-\infty}^{\infty} |h(s)|\frac{\omega(y)}{\omega(-s)}\,ds = \omega(y)\int_{-\infty}^{\infty} |h(s)|\frac{1}{\omega(s)}\,ds \\ &\ge \omega(y)\frac{\|h\|_1^2}{\|h\|_\omega},\end{aligned}$$

the last inequality resulting from an application of Cauchy–Schwarz. Now another calculation with Fourier transforms shows that

$$h(t) = (P^1 * g)(t) = \frac{1}{\pi}\frac{1}{(t+2i)^2} \qquad (t \in \mathbb{R}).$$

In particular, $|h| \le |g|$, so $\|h\|_\omega \le \|g\|_\omega = I(\omega)$. Also, $\|h\|_1 = 1/2$. Hence, we obtain the lower bound

$$\|T^{1+iy}\| \ge \omega(y)\frac{1/4}{I(\omega)^2} \qquad (y \in \mathbb{R}). \ \blacksquare$$

REMARK. By taking $g(t) = (n/\pi)(t+ni)^{-2}$ $(n \ge 1)$ in the proof above, the lower bound in (4) may be improved to

$$\sup_{n\ge1} \frac{n^2}{(n+1)^2}\frac{1}{I(\omega)^{2n}}.$$

We shall need one further result. Recall from §1 that a non-unital Banach algebra has the Beurling–Helson property if it contains no non-zero analytic semigroups (a^z) such that $\sup_{y\in\mathbb{R}} \|a^{1+iy}\| < \infty$.

THEOREM 2.4. *Let* ω *be a weight satisfying* $I(\omega) < \infty$. *Let* B_ω *be the closed subalgebra of* $\mathcal{L}(H^1_\omega)$ *generated by the semigroup* $(T^z)_{\operatorname{Re} z>0}$ *of Theorem 2.3. Then* B_ω *is non-unital, and it has the Beurling–Helson property if and only if* $\sup_{y\in\mathbb{R}} \omega(y) = \infty$.

Proof. For $\xi > 0$, choose $f \in H^1_\omega$ with $\widehat{f}(\xi) \ne 0$, and define $\chi_\xi : B_\omega \to \mathbb{C}$ by

$$\chi_\xi(S) = (\widehat{Sf})(\xi)/\widehat{f}(\xi) \qquad (S \in B_\omega).$$

This definition of χ_ξ does not depend on the choice of f. Indeed, given $f_1, f_2 \in H^1_\omega$, we have $T^z(f_1) * f_2 = P^z * f_1 * f_2 = f_1 * T^z(f_2)$ for all z, whence $S(f_1) * f_2 = f_1 * S(f_2)$ for all $S \in B_\omega$, which in turn implies $\widehat{Sf_1}(\xi)\widehat{f}_2(\xi) = \widehat{f}_1(\xi)\widehat{Sf_2}(\xi)$, as claimed. From this, it readily follows that χ_ξ is a character on B_ω. Further, for each z with $\operatorname{Re} z > 0$,

$$\lim_{\xi\to\infty} \chi_\xi(T^z) = \lim_{\xi\to\infty} e^{-z|\xi|} = 0,$$

whence

$$\lim_{\xi\to\infty} \chi_\xi(S) = 0 \qquad (S \in B_\omega). \tag{5}$$

In particular, this implies that B_ω is non-unital.

If ω is bounded, then by Theorem 2.3 the semigroup (T^z) is bounded on $\operatorname{Re} z = 1$, but not identically zero, and so B_ω fails the Beurling–Helson property.

Conversely, suppose that ω is unbounded. Let $(S^z)_{\operatorname{Re} z>0}$ be an analytic semigroup in B_ω which is bounded on $\operatorname{Re} z = 1$. For each $\xi > 0$, the function $h_\xi(z) = \chi_\xi(S^z)$ is holomorphic on $\operatorname{Re} z > 0$ and satisfies $h_\xi(z+w) = h_\xi(z)h_\xi(w)$ there. Therefore, either $h_\xi \equiv 0$, or it is of the form $h_\xi(z) = e^{-\alpha_\xi z}$, for some $\alpha_\xi \in \mathbb{R}$. The arguments of [GW, §1 and §2] show that there are two possible cases:

CASE 1: $h_\xi \equiv 0$ for all $\xi > 0$. In this case, as L^1_ω is semisimple, it follows that $S^z \equiv 0$.

CASE 2: $h_\xi(z) = e^{-(b\xi+c)z}$ for all $\xi > 0$, where $b \ge 0$ and $c \in \mathbb{R}$. Notice that (5) implies that actually $b > 0$. Hence, given $f \in H^1_\omega$ and $\operatorname{Re} z > 0$,

$$\widehat{S^z f}(\xi) = e^{-(b\xi+c)z}\widehat{f}(\xi) = e^{-cz}e^{-b|\xi|z}\widehat{f}(\xi) = e^{-cz}\widehat{T^{bz}f}(\xi) \quad (\xi > 0).$$

It follows that the analytic semigroup (S^z) is none other than $(e^{-cz}T^{bz})$. In particular, $\|S^{1+iy}\| = e^{-c}\|T^{b(1+iy)}\|$. But the latter is bounded below by a multiple of $\omega(by)$ (the proof is similar to that of the lower bound (4)), and by hypothesis $\sup_{y\in\mathbb{R}} \omega(y) = \infty$. This contradicts the fact that (S^z) is bounded on $\operatorname{Re} z = 1$, and so this case cannot arise.

To summarize, we have shown that the only analytic semigroup in B_ω which is bounded on $\operatorname{Re} z = 1$ is identically zero. Therefore B_ω has the Beurling–Helson property. ■

3. Analytic semigroups of slow growth. The goal of this section is to prove Theorem 1.1. The idea is to relax the constraint of being a weight function in Theorem 2.3, without losing too much control over the growth of the semigroup. The main quantitative result is as follows.

THEOREM 3.1. *There is a Banach algebra A with the Beurling–Helson property such that, given an unbounded even function $\psi : \mathbb{R} \to \mathbb{R}$, increasing on $[0,\infty)$, and satisfying*

$$\psi(0) \ge e \quad \text{and} \quad I(\psi) := \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\psi(t)}{1+t^2}\,dt < \infty,$$

there exists an analytic semigroup (a^z) in A with

$$\frac{1}{4I(\psi)^3}\psi\left(\frac{y}{\log^+|y| + \log 2I(\psi)}\right) \le \|a^{1+iy}\| \le \psi(y) \quad (y \in \mathbb{R}).$$

Proof. As our algebra A, we take the ℓ^1-direct sum (with multiplication defined coordinatewise) of the Banach algebras B_ω of Theorem 2.4, where ω ranges over all weights satisfying $I(\omega) < \infty$ and $\sup_{y\in\mathbb{R}}\omega(y) = \infty$. As each B_ω has the Beurling–Helson property, so too does A.

Let Ω be the set of even weight functions increasing on $[0,\infty)$. Define $\widetilde{\psi} : \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{\psi}(t) = \sup\{\omega(t) : \omega \in \Omega,\ \omega \le \psi\} \quad (t \in \mathbb{R}).$$

It is easily checked that $\widetilde{\psi}$ is itself in Ω, and that $I(\widetilde{\psi}) \le I(\psi) < \infty$. Further, we claim that

$$\widetilde{\psi}(y) \ge \psi\left(\frac{y}{\log^+|y| + \log(2I(\psi))}\right) \quad (y \in \mathbb{R}), \tag{6}$$

and in particular, that $\widetilde{\psi}$ is unbounded. Assuming this for the moment, let $(T^z)_{\operatorname{Re} z>0}$ be the analytic semigroup furnished by Theorem 2.3, and set

$$a^z = I(\widetilde{\psi})^{-z}T^z \quad (\operatorname{Re} z > 0).$$

Then $(a^z)_{\operatorname{Re} z>0}$ is an analytic semigroup in $B_{\widetilde{\psi}}$, and hence in A, satisfying

$$\frac{1}{4I(\psi)^3}\widetilde{\psi}(y) \le \|a^{1+iy}\| \le \psi(y) \quad (y \in \mathbb{R}).$$

Combining this with the estimate (6) yields the inequality stated in the theorem.

It remains to justify the claim (6). Let $s \ge 0$ and define

$$\omega(t) = \min(e^{|t|/s}, \psi(s)) \quad (t \in \mathbb{R}).$$

It is readily checked that $\omega \in \Omega$. Also

$$|t| < s \Rightarrow \omega(t) \le e \le \psi(0) \le \psi(t), \qquad |t| \ge s \Rightarrow \omega(t) \le \psi(s) \le \psi(t),$$

so $\omega \le \psi$. Therefore also $\omega \le \widetilde{\psi}$. In particular, we have

$$\widetilde{\psi}(s\log\psi(s)) \ge \omega(s\log\psi(s)) = \psi(s). \tag{7}$$

Now

$$I(\psi) \ge \frac{1}{\pi}\int_{|t|\ge s} \frac{\psi(t)}{1+t^2}\,dt \ge \frac{2}{\pi}\int_s^\infty \frac{\psi(s)}{1+t^2}\,dt = \frac{2}{\pi}\psi(s)\arctan(1/s).$$

If $s \ge 1$ then $\arctan(1/s) \ge \pi/4s$ and so we have $\psi(s) \le 2I(\psi)s$. On the other hand, if $0 \le s \le 1$ then $\psi(s) \le \psi(1) \le 2I(\psi)$. Therefore, in all cases,

$$\log\psi(s) \le \log^+ s + \log(2I(\psi)).$$

Substituting into (7), we obtain

$$\widetilde{\psi}(s(\log^+ s + \log(2I(\psi)))) \ge \psi(s) \quad (s \ge 0). \tag{8}$$

Now let $y \in \mathbb{R}$, and choose $s \ge 0$ so that

$$s(\log^+ s + \log(2I(\psi))) = |y|.$$

Then certainly $s \le |y|$, and so also

$$s \ge \frac{|y|}{\log^+|y| + \log(2I(\psi))}.$$

Substituting into (8), and using the fact that both ψ and $\widetilde{\psi}$ are even functions, we get

$$\widetilde{\psi}(y) \geq \psi(s) \geq \psi\left(\frac{y}{\log^+|y| + \log(2I(\psi))}\right),$$

which proves (6). ∎

Finally, we can prove the qualitative theorem stated in the introduction.

Proof of Theorem 1.1. Let A be the Banach algebra of Theorem 3.1. It has the Beurling–Helson property. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$\lim_{|y|\to\infty} \varphi(y) = \infty \quad \text{and} \quad \inf_{y\in\mathbb{R}} \varphi(y) = \delta > 0.$$

Define $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(t) = \frac{e}{\delta}\min(\inf_{|y|\geq|t|} \varphi(y),\ \delta + |t|^{1/2}) \quad (t \in \mathbb{R}).$$

Then ψ satisfies the hypotheses of Theorem 3.1, so there exists an analytic semigroup $(b^z)_{\operatorname{Re} z>0}$ in A such that

$$\frac{1}{4I(\psi)^3}\psi\left(\frac{y}{\log^+|y| + \log 2I(\psi)}\right) \leq \|b^{1+iy}\| \leq \psi(y) \quad (y \in \mathbb{R}).$$

In particular, since $\lim_{|y|\to\infty} \psi(y) = \infty$, it follows that

$$\lim_{|y|\to\infty} \|b^{1+iy}\| = \infty.$$

Also, since $\psi(y) \leq (e/\delta)\varphi(y)$ for all y, we have

$$\|b^{1+iy}\| \leq \frac{e}{\delta}\varphi(y) \quad (y \in \mathbb{R}).$$

Thus, taking $a^z = (\delta/e)^z b^z$, we obtain an analytic semigroup in A satisfying

$$\lim_{|y|\to\infty} \|a^{1+iy}\| = \infty \quad \text{and} \quad \|a^{1+iy}\| \leq \varphi(y) \quad (y \in \mathbb{R}). \ ∎$$

REMARKS. (1) The condition $\inf_{y\in\mathbb{R}} \varphi(y) > 0$ in Theorem 1.1 is necessary. Indeed, suppose, if possible, that (a^z) is an analytic semigroup (in any Banach algebra) such that $\lim_{|y|\to\infty} \|a^{1+iy}\| = \infty$ but $\inf_{y\in\mathbb{R}} \|a^{1+iy}\| = 0$. Then there exists $y_0 \in \mathbb{R}$ with $a^{1+iy_0} = 0$. It follows that $a^{x+iy_0} = a^{x-1}a^{1+iy_0} = 0$ for all $x \geq 1$. By the identity principle, $a^z \equiv 0$, which is a contradiction.

(2) The algebra A constructed above is not separable. We do not know whether there exists a separable Banach algebra satisfying the conclusions of Theorem 1.1. Of course, given any countable family Φ of functions φ as in Theorem 1.1, the closed subalgebra A_0 of A generated by the corresponding analytic semigroups (a^z) is separable. The problem is that, whatever the choice of Φ, there is always another function φ_0 tending to infinity at infinity such that $\varphi_0 \leq \varphi$ for *no* $\varphi \in \Phi$, and it is then unclear whether A_0 contains an analytic semigroup corresponding to φ_0.

4. Analytic semigroups of rapid growth. We now turn to the proof of Theorem 1.2. It is based upon the construction given in the following proposition. Here and in what follows, given a Banach algebra B, we write $\widehat{\bigotimes}^n B$ for the nth projective tensor power of B.

PROPOSITION 4.1. *Let B be a Banach algebra. Let A be the ℓ^1-direct sum of the Banach algebras $\widehat{\bigotimes}^n B$ $(n \geq 1)$, with multiplication defined coordinatewise. Then, given an analytic semigroup $(b^z)_{\operatorname{Re} z>0}$ in B, and an entire function $f : \mathbb{C} \to \mathbb{C}$ with $f(0) = 0$, there exists an analytic semigroup $(a^z)_{\operatorname{Re} z>0}$ in A satisfying*

$$\|a^{1+iy}\| \geq |f(\|b^{1+iy}\|)| \quad (y \in \mathbb{R}).$$

Proof. For $\operatorname{Re} z > 0$, define

$$a^z = (|c_n|^z(\overbrace{b^z \otimes \cdots \otimes b^z}^{n}))_{n\geq1},$$

where $(c_n)_{n\geq1}$ are the Taylor coefficients of the entire function f. Note that

$$\sum_{n\geq1} \| |c_n|^z(b^z \otimes \cdots \otimes b^z)\| = \sum_{n\geq1} |c_n|^{\operatorname{Re} z}\|b^z\|^n,$$

and the right-hand side converges locally uniformly for $\operatorname{Re} z > 0$ because $|c_n|^{1/n} \to 0$. Thus $a^z \in A$ $(\operatorname{Re} z > 0)$, and the map $z \mapsto a^z$ is continuous. Indeed, it is even analytic, by the same argument as in Proposition 2.2. Thus $(a^z)_{\operatorname{Re} z>0}$ is an analytic semigroup in A. Finally, for $z = 1 + iy$ we have

$$\|a^{1+iy}\| = \sum_{n\geq1} |c_n|\,\|b^{1+iy}\|^n \geq \Big|\sum_{n\geq1} c_n\|b^{1+iy}\|^n\Big| = |f(\|b^{1+iy}\|)| \quad (y \in \mathbb{R}). \ ∎$$

Proof of Theorem 1.2. We apply the preceding proposition to a particular case.

For B, we take $L^1(\mathbb{R})$. Note that $\widehat{\bigotimes}^n L^1(\mathbb{R}) = L^1(\mathbb{R}^n)$ for each n, so the algebra A of the proposition is just the ℓ^1-direct sum of $\{L^1(\mathbb{R}^n) : n \geq 1\}$. In particular, since each $L^1(\mathbb{R}^n)$ has the Beurling–Helson property [GW, Theorem 2.4], so too does A. Also, A is clearly separable.

For (b^z) we take the Gauss semigroup:

$$b^z(t) = \frac{1}{\sqrt{4\pi z}}e^{-t^2/4z} \quad (t \in \mathbb{R},\ \operatorname{Re} z > 0).$$

By [S, Theorem 2.15], $(b^z)_{\operatorname{Re} z>0}$ is an analytic semigroup in $L^1(\mathbb{R})$, and

$$\|b^{1+iy}\|_1 = (1 + y^2)^{1/4} \quad (y \in \mathbb{R}).$$

Suppose now that $\varphi : \mathbb{R} \to \mathbb{R}$ is a locally bounded function. We claim that there exists an entire function $f : \mathbb{C} \to \mathbb{C}$ with non-negative Taylor coefficients such that

$$f(0) = 0 \quad \text{and} \quad f((1+y^2)^{1/4}) \geq \varphi(y) \quad (y \in \mathbb{R}).$$

If so, then by Proposition 4.1, there exists an analytic semigroup (a^z) in A such that

$$\|a^{1+iy}\| \geq |f(\|b^{1+iy}\|)| \geq f((1+y^2)^{1/4}) \geq \varphi(y) \quad (y \in \mathbb{R}),$$

as desired.

It remains to construct f. First, since φ is locally bounded, we can use a partition-of-unity argument to build a continuous function $\psi : \mathbb{R} \to \mathbb{R}$ such that

$$\psi(t) \geq \max(\varphi(t), \varphi(-t)) + 1 \quad (t \in \mathbb{R}).$$

Then, by Carleman's theorem [C], there exists an entire function g such that

$$g(0) = 0 \quad \text{and} \quad |g(s) - \psi(\sqrt{s^4 - 1})| \leq 1 \quad (s \in \mathbb{R},\ |s| \geq 1).$$

Finally, we let f be the entire function obtained from g by replacing the Taylor coefficients with their absolute values. Then f has the properties required since $f(0) = 0$ and

$$f((1+y^2)^{1/4}) \geq |g((1+y^2)^{1/4})| \geq \psi(|y|) - 1 \geq \varphi(y) \quad (y \in \mathbb{R}). \ \blacksquare$$

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