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References

[1] A. I. Alonso and R. Obaya, *Dynamical description of bidimensional linear systems with a measurable* 2*-sheet*, J. Math. Anal. Appl. 212 (1997), 154–175.
[2] I. Assani, *Note on the equation* $y = (I - T)x$, preprint, 1997.
[3] I. Assani and J. Woś, *An equivalent measure for some nonsingular transformations and application*, Studia Math. 97 (1990), 1–12.
[4] R. C. Bradley, *On a theorem of K. Schmidt*, Statist. Probab. Lett. 24 (1995), 9–12.
[5] F. E. Browder, *On the iteration of transformations in non-compact minimal dynamical systems*, Proc. Amer. Math. Soc. 9 (1958), 773–780.
[6] H. Furstenberg, *Strict ergodicity and transformation of the torus*, Amer. J. Math. 83 (1961), 573–601.
[7] W. Gottschalk and G. Hedlund, *Topological Dynamics*, Amer. Math. Soc. Colloq. Publ. 36, Providence, RI, 1955.
[8] U. Krengel and M. Lin, *On the range of the generator of a markovian semigroup*, Math. Z. 185 (1984), 553–565.
[9] V. P. Leonov, *On the dispersion of the time averages of a stationary stochastic process*, Teor. Veroyatnost. i Primenen. 6 (1961), 93–101 (in Russian); English transl. in Theory Probab. Appl. 6 (1961).
[10] M. Lin and R. Sine, *Ergodic theory and the functional equation* $(I - T)x = y$, J. Operator Theory 10 (1983), 153–166.
[11] F. J. Martín-Reyes and A. de la Torre, *On the pointwise ergodic theorem*, Studia Math. 108 (1994), 1–4.
[12] S. Novo and R. Obaya, *An ergodic classification of bidimensional linear systems*, J. Dynam. Differential Equations 8 (1996), 373–406.
[13] —, —, *An ergodic and topological approach to almost periodic bidimensional linear systems*, in: Contemp. Math. 215, Amer. Math. Soc., 1998, 299–323.
[14] R. Sato, *Pointwise ergodic theorems in Lorentz spaces* $L(p,q)$ *for null preserving transformations*, Studia Math. 114 (1995), 227–236.

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Schauder decompositions and multiplier theorems

by

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Abstract. We study the interplay between unconditional decompositions and the R-boundedness of collections of operators. In particular, we get several multiplier results of Marcinkiewicz type for L^p-spaces of functions with values in a Banach space X. Furthermore, we show connections between the above-mentioned properties and geometric properties of the Banach space X.

1. Introduction. A number of important operators in analysis may be represented as multiplier operators with respect to a given Schauder decomposition $\{D_n\}_{n=1}^\infty$ of a Banach space X, i.e.,

$$T_\lambda(x) = \sum \lambda_k D_k x, \quad x \in X, \tag{1}$$

where $\lambda = \{\lambda_k\} \in \mathbb{C}$. The characterization of the sequences λ for which (1) defines a bounded operator T_λ on X for a given decomposition $\{D_n\}_{n=1}^\infty$ is an interesting problem. The study of this problem for the Schauder decomposition defined by the trigonometric system in $L^p(0,1)$ led J. Marcinkiewicz [Mar39] (see also [EG77]) to his famous multiplier theorem.

A similar description to that of Marcinkiewicz was obtained by G. I. Sunouchi [Sun51] for the Schauder decomposition defined by the Paley–Walsh system in $L^p(0,1)$. Vector-valued extensions of the Marcinkiewicz theorem are given in [Bou83] (see also [BG94]).

In all results mentioned above the descriptions of the sequences λ for which T_λ is bounded are given in terms of certain blockings $\Delta = \{\Delta_k\}_{k=1}^\infty$ of the Schauder decomposition $\{D_n\}_{n=1}^\infty$ (the dyadic blocking for both trigonometric and Paley–Walsh systems), which turns out to be an unconditional decomposition of X. In fact, the study of the operators given by (1) naturally

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leads to operators of the form

$$S(x)=\sum T_k\Delta_k x,\quad x\in X, \tag{2}$$

and

$$S'(x)=\sum T_k\sum_{j\in F_k}\Delta_j x,\qquad F_k\subseteq\mathbb{N},\ F_n\cap F_m=0,\ x\in X, \tag{3}$$

defined by a sequence of operators $\{T_k\}_{k=1}^{\infty}$, where the T_k are operators on X which commute with Δ_k. An obvious necessary requirement on $\{T_k\}_{k=1}^{\infty}$ for the boundedness of S is that the sequence $\{T_k\}_{k=1}^{\infty}$ is uniformly bounded; if X is a Hilbert space, this is also sufficient. In the case of general Banach spaces X an interesting property which implies boundedness of S is singled out from [BG94] under the name R(andomized)-boundedness (called the R(iesz)-property in [BG94]). The boundedness of S' (for an arbitrary choice of subsets F_n) provides the most complete analogue of a Marcinkiewicz multiplier theorem.

The present paper intends to study the relations between R-boundedness and various geometric properties of X and their further applications to (vector-valued) multipliers defined with respect to bounded Vilenkin systems, which are an immediate generalization of the Paley–Walsh system.

After collecting the necessary definitions in Section 2, we study in Section 3 the relations between R-boundedness and the various unconditional blockings of $\{D_n\}_{n=1}^{\infty}$. In Theorem 3.4 we show that R-boundedness of $\{T_k\}_{k=1}^{\infty}$ is indeed a sufficient condition for the boundedness of S. The latter result enables us to present a slightly strengthened version of the generalized Marcinkiewicz-type multiplier principle from [BG94] (see Theorem 3.5 and Corollary 3.6). It follows from these results that in order to give a satisfactory description of those λ for which T_λ is bounded, we need to study the R-boundedness of operators S defined with respect to the sequence $\{T_k\}_{k=1}^{\infty}$, where

$$T_k=\sum_{j=1}^{k}D_j,\qquad k=1,2,\ldots$$

Since in many examples an unconditional decomposition Δ is defined via a suitable sequence of mathematical expectations, we first present a proof of the vector-valued Stein inequality (Proposition 3.8) formulated in [Bou83] and apply this inequality to the partial sum projections in UMD-spaces in Theorem 3.9, showing that the partial sum projections are R-bounded. To ensure that the operator S' is also bounded, we use the so-called property (α) introduced in [Pis78] (see Definition 3.11 and Theorem 3.14). The application of the results presented in the third section to multiplier theory is discussed in Section 5.

In Section 4 we introduce the necessary notation concerning Vilenkin systems and show that our study of R-boundedness has an interesting application to the basis theory in vector-valued L^p-spaces. In fact, we show (Theorems 4.5 and 4.6) that any bounded Vilenkin system generates a Schauder decomposition of L^p_X if and only if X is a UMD-space. In the special case $X=\mathbb{C}$ we recover the result of C. Watari [Wat58] that a bounded Vilenkin system is a Schauder basis in $L^p(0,1)$ for all $1<p<\infty$ (see also [DS97]).

Finally, in the last section we present a generalized version of a multiplier theorem for bounded Vilenkin systems in vector-valued spaces L^p_X, provided X is a UMD-space with the property (α) (Theorem 5.1). Specializing our considerations to the case of the Paley–Walsh system, we give a complete characterization of Banach spaces X for which the multiplier theorem holds (Theorem 5.6). It turns out that in the Schatten classes $\mathcal{C}_p$ such a multiplier theorem fails if $p\neq 2$ (see Corollary 5.7).

2. Preliminaries. In this section we collect some of the relevant definitions and facts concerning unconditional decompositions. By E we denote a complex Banach space. The range of a linear operator T on E is denoted by $\mathrm{R}(T)$.

Definition 2.1 (cf. [LT77], Section 1.g). A. A sequence $D=\{D_k\}_{k=0}^{\infty}$ of bounded linear projections in E is called a *Schauder decomposition* of E if

(i) $D_kD_l=0$ whenever $k\neq l$,
(ii) $x=\sum_{k=0}^{\infty}D_kx$ for all $x\in E$.

The corresponding *partial sum projections* $\{P_n\}_{n=0}^{\infty}$ are defined by

$$P_n=\sum_{k=0}^{n}D_k. \tag{4}$$

If the series $\sum_{k=0}^{\infty}D_kx$ is unconditionally convergent for all $x\in E$, then D is called an *unconditional decomposition*.

Given a strictly increasing sequence $\{q_k\}_{k=0}^{\infty}$ in $\mathbb{N}$, put

$$\Delta_k=\sum_{l=q_{k-1}+1}^{q_k}D_l\qquad(k=0,1,\ldots) \tag{5}$$

(with $q_{-1}=-1$). Then the Schauder decomposition $\Delta=\{\Delta_k\}_{k=0}^{\infty}$ is called a *blocking* of D.

B. Given a Schauder decomposition $D=\{D_k\}_{k=0}^{\infty}$ of E let $E_0=\operatorname{span}\{\mathrm{R}(D_k):k\in\mathbb{N}\}$. For any sequence $\lambda=\{\lambda_k\}_{k=0}^{\infty}$ in $\mathbb{C}$ we define the linear operator $T_\lambda : E_0 \to E_0$ by

$$T_\lambda x = \sum_{k=0}^{\infty} \lambda_k D_k x. \tag{6}$$

If this operator is bounded on E_0, then it extends uniquely to a bounded linear operator (denoted by T_λ as well) on E.

By $\{\varepsilon_k\}_{k=0}^{\infty}$ we shall denote a sequence of independent symmetric $\{-1,1\}$-valued random variables on some probability space $(\Omega,\mathcal{F},P)$. If necessary, we shall use $\{\varepsilon'_k\}_{k=0}^{\infty}$ to denote a second similar sequence on some $(\Omega',\mathcal{F}',P')$. We denote by $L^p_E(\Omega)$ the Bochner space of p-integrable E-valued functions on $(\Omega,\mathcal{F},P)$.

If $D = \{D_n\}_{n=0}^{\infty}$ is an unconditional decomposition of E, then the smallest constant C_D such that

$$\Big\| \sum_{k=0}^{N} \varepsilon_k D_k x \Big\|_E \le C_D \Big\| \sum_{k=0}^{N} D_k x \Big\|_E$$

holds for all $\varepsilon_k = \pm 1$, $k = 0,1,\ldots,N$, all $N \in \mathbb{N}$ and all $x \in E$, is called the *unconditional constant* of the decomposition.

We will need the following property of unconditional decompositions, which is a well known consequence of unconditionality (see e.g. [DJT95], Lemma 1.4).

LEMMA 2.2. *Let $D = \{D_n\}_{n=0}^{\infty}$ be an unconditional Schauder decomposition of the Banach space E. Then for all $1 \le p < \infty$ we have*

$$C_D^{-1} \Big\| \sum_{k=0}^{N} D_k x \Big\|_E \le \Big\| \sum_{k=0}^{N} \varepsilon_k D_k x \Big\|_{L^p_E(\Omega)} \le C_D \Big\| \sum_{k=0}^{N} D_k x \Big\|_E \tag{7}$$

for all $x \in E$ and all $N \in \mathbb{N}$.

REMARK 2.3. If for some $1 \le p < \infty$ there exists a constant C such that (7) holds, then the decomposition D is unconditional.

3. R-boundedness. In this section we shall study in some detail the so-called R-boundedness for collections $\mathcal{T}$ of bounded linear operators on a Banach space E. This R-boundedness was already implicitly used by J. Bourgain in [Bou83] and was introduced by E. Berkson and T. A. Gillespie in their paper [BG94]. The results obtained in the present section will be used in later sections for our study of the vector-valued Vilenkin system. However, a number of these results may be of independent interest, in particular Theorem 3.9 and Theorem 3.14.

We start off by recalling the definition of R-boundedness. As usual, we denote by $\mathcal{L}(E)$ the space of bounded linear operators on a Banach space E.

DEFINITION 3.1. A collection $\mathcal{T} \subset \mathcal{L}(E)$ is said to be *randomized bounded* (*R-bounded*) if there exists a constant $M > 0$ such that

$$\Big\| \sum_{k=0}^{N} \varepsilon_k T_k x_k \Big\|_{L^2_E(\Omega)} \le M \Big\| \sum_{k=0}^{N} \varepsilon_k x_k \Big\|_{L^2_E(\Omega)} \tag{8}$$

for all $\{T_k\}_{k=0}^{N} \subset \mathcal{T}$, all $\{x_k\}_{k=0}^{N} \subset E$ and all $N \in \mathbb{N}$. The smallest constant M such that (8) holds is called the *R-bound* of $\mathcal{T}$.

We emphasize that the operators in the collections $\{T_k\}_{k=0}^{N}$ in the above definition need not be mutually distinct. Note that by Kahane's inequality (see e.g. [DJT95], Theorem 11.1) we can replace L^2_E by L^p_E, $1 \le p < \infty$, adjusting the constant M appropriately. The constant in this case will be denoted by M_p. The following lemma gives an easy method to enlarge an R-bounded collection. For the reader's convenience we include a short proof.

LEMMA 3.2. *Let $\mathcal{T} \subset \mathcal{L}(E)$ be an R-bounded collection with bound M. Then the absolute convex hull of $\mathcal{T}$ is also R-bounded. For the real absolute convex hull, the R-bound is again M. For the complex absolute convex hull the R-bound is at most $2M$.*

Proof. It is sufficient to prove the lemma for the real absolute convex hull. Since $\mathcal{T} \cup (-\mathcal{T})$ has the same R-bound as $\mathcal{T}$, we may assume that $\mathcal{T}$ is symmetric, in which case the absolute convex hull coincides with the convex hull. The result now follows from the equality

$$\operatorname{conv}(\mathcal{T}) \times \ldots \times \operatorname{conv}(\mathcal{T}) = \operatorname{conv}(\mathcal{T} \times \ldots \times \mathcal{T}),$$

where $\operatorname{conv}(\mathcal{T})$ denotes the convex hull of $\mathcal{T}$. ∎

LEMMA 3.3. *For $\mathcal{T} \subset \mathcal{L}(E)$, the following statements are equivalent:*

(i) *$\mathcal{T}$ is R-bounded.*

(ii) *For all $T_0, T_1, \ldots, T_n \in \mathcal{T}$ with $T_i \ne T_j$ $(i \ne j)$, all $x_0, x_1, \ldots, x_n \in E$ and all $n \in \mathbb{N}$ we have*

$$\Big\| \sum_{j=0}^{n} \varepsilon_j T_j x_j \Big\|_{L^2_E(\Omega)} \le M \Big\| \sum_{j=0}^{n} \varepsilon_j x_j \Big\|_{L^2_E(\Omega)},$$

for some constant M.

Moreover, if $\mathcal{T} = \{T_n : n \in \mathbb{N}\}$, then (i) *and* (ii) *are equivalent to*

(iii) *For all $x_0, x_1, \ldots, x_n \in E$ and all $n \in \mathbb{N}$ we have*

$$\Big\| \sum_{j=0}^{n} \varepsilon_j T_j x_j \Big\|_{L^2_E(\Omega)} \le M \Big\| \sum_{j=0}^{n} \varepsilon_j x_j \Big\|_{L^2_E(\Omega)},$$

for some constant M.

Proof. The implication (i)⇒(ii) is obvious.

(ii)⇒(i). Take $T_j \in \mathcal{T}$ $(j = 0, 1, \ldots, n)$ arbitrary and $x_0, x_1, \ldots, x_n \in E$. Let $S_0, S_1, \ldots, S_m$ be the *distinct* operators in $\{T_0, T_1, \ldots, T_n\}$. For $0 \le k \le m$ let $F_k = \{j : T_j = S_k\}$. Then

$$\Big\|\sum_{j=0}^{n} \varepsilon_j T_j x_j\Big\|_{L^2_E(\Omega)} = \Big\|\sum_{k=0}^{m} S_k \sum_{j\in F_k} \varepsilon_j x_j\Big\|_{L^2_E(\Omega)}. \tag{9}$$

Since for fixed $\omega' \in \Omega'$ the random variables $\{(\varepsilon_k'(\omega')\varepsilon_j : j \in F_k) : k = 0, 1, \ldots, m\}$ and $\{\varepsilon_j : j = 0, 1, \ldots, n\}$ have the same distribution, it follows that

$$\begin{aligned}\Big\|\sum_{k=0}^{m} \varepsilon_k'(\omega') S_k \sum_{j\in F_k} \varepsilon_j x_j\Big\|_{L^2_E(\Omega)} &= \Big\|\sum_{k=0}^{m}\sum_{j\in F_k} \varepsilon_k'(\omega')\varepsilon_j S_k x_j\Big\|_{L^2_E(\Omega)}\\ &= \Big\|\sum_{k=0}^{m}\sum_{j\in F_k} \varepsilon_j S_k x_j\Big\|_{L^2_E(\Omega)}\\ &= \Big\|\sum_{k=0}^{m} S_k \sum_{j\in F_k} \varepsilon_j x_j\Big\|_{L^2_E(\Omega)},\end{aligned}$$

and similarly

$$\Big\|\sum_{k=0}^{m} \varepsilon_k'(\omega') \sum_{j\in F_k} \varepsilon_j x_j\Big\|_{L^2_E(\Omega)} = \Big\|\sum_{k=0}^{m}\sum_{j\in F_k} \varepsilon_j x_j\Big\|_{L^2_E(\Omega)}.$$

Using these observations in combination with the hypothesis (ii) on the S_k's, we find that

$$\begin{aligned}\Big\|\sum_{k=0}^{m} S_k \sum_{j\in F_k} \varepsilon_j x_j\Big\|_{L^2_E(\Omega)} &= \Big(\int_{\Omega'} \Big\|\sum_{k=0}^{m} \varepsilon_k'(\omega') S_k \sum_{j\in F_k} \varepsilon_j x_j\Big\|^2_{L^2_E(\Omega)} dP'(\omega')\Big)^{1/2}\\ &= \Big(\int_{\Omega} \Big\|\sum_{k=0}^{m} \varepsilon_k' S_k \sum_{j\in F_k} \varepsilon_j(\omega) x_j\Big\|^2_{L^2_E(\Omega')} dP(\omega)\Big)^{1/2}\\ &\overset{(*)}{\le} M\Big(\int_{\Omega} \Big\|\sum_{k=0}^{m} \varepsilon_k' \sum_{j\in F_k} \varepsilon_j(\omega) x_j\Big\|^2_{L^2_E(\Omega')} dP(\omega)\Big)^{1/2}\\ &= M\Big(\int_{\Omega'} \Big\|\sum_{k=0}^{m} \varepsilon_k'(\omega') \sum_{j\in F_k} \varepsilon_j x_j\Big\|^2_{L^2_E(\Omega)} dP'(\omega')\Big)^{1/2}\\ &= M\Big\|\sum_{j=0}^{n} \varepsilon_j x_j\Big\|_{L^2_E(\Omega)}.\end{aligned}$$

For $(*)$ we have used the inequality in the hypothesis (ii). This, together with (9), shows that $\mathcal{T}$ is R-bounded.

Now assume $\mathcal{T} = \{T_n : n \in \mathbb{N}\}$. It is clear that only the implication (iii)⇒(ii) needs proof. So take $T_{k_0}, T_{k_1}, \ldots, T_{k_n} \in \mathcal{T}$ $(T_{k_i} \ne T_{k_j}$ whenever $i \ne j)$ and $x_{k_0}, x_{k_1}, \ldots, x_{k_n} \in E$. We have to show that

$$\Big\|\sum_{j=0}^{n} \varepsilon_j T_{k_j} x_{k_j}\Big\|_{L^2_E(\Omega)} \le M\Big\|\sum_{j=0}^{n} \varepsilon_j x_{k_j}\Big\|_{L^2_E(\Omega)}. \tag{10}$$

Since the quantities in (10) are invariant under permutations of the k_j's, we may assume that $k_0 < k_1 < \ldots < k_n$. But then (10) follows immediately from (iii) by choosing $x_k = 0$ if $k \ne k_j$. ∎

The following two theorems show the relevance of R-boundedness in connection with unconditional decompositions.

THEOREM 3.4. *Let $\{\Delta_k\}_{k=0}^{\infty}$ be an unconditional Schauder decomposition of the Banach space X. Suppose that $\mathcal{T} \subset \mathcal{L}(X)$ is R-bounded (with R-bound M). If $\{T_k\}_{k=0}^{\infty} \subset \mathcal{T}$ such that $\Delta_k T_k = T_k \Delta_k$ for all $k \in \mathbb{N}$, then the series*

$$Sx := \sum_{k=0}^{\infty} T_k \Delta_k x$$

is convergent in X for all $x \in X$, and defines a bounded linear operator $S : X \to X$ with $\|S\| \le K$ (where K only depends on M and the unconditional constant of $\{\Delta_k\}_{k=0}^{\infty}$).

Proof. Take $x \in X$ and $0 \le m \le n$ in $\mathbb{N}$. Then

$$\begin{aligned}\Big\|\sum_{k=m}^{n} T_k \Delta_k x\Big\|_X &= \Big\|\sum_{k=m}^{n} \Delta_k T_k \Delta_k x\Big\|_X\\ &= \Big\|\sum_{k=0}^{n} \Delta_k\Big(\sum_{j=m}^{n} \Delta_j T_j \Delta_j x\Big)\Big\|_X\\ &\le C_\Delta \Big\|\sum_{k=0}^{n} \varepsilon_k \Delta_k\Big(\sum_{j=m}^{n} \Delta_j T_j \Delta_j x\Big)\Big\|_{L^2_X(\Omega)}\\ &= C_\Delta \Big\|\sum_{k=m}^{n} \varepsilon_k T_k \Delta_k x\Big\|_{L^2_X(\Omega)} \le C_\Delta M \Big\|\sum_{k=m}^{n} \varepsilon_k \Delta_k x\Big\|_{L^2_X(\Omega)}\\ &\le C_\Delta^2 M \Big\|\sum_{k=m}^{n} \Delta_k x\Big\|_X.\end{aligned}$$

Since $x = \sum_{k=0}^{\infty} \Delta_k x$, the result now follows immediately. ∎

THEOREM 3.5. *Let $D = \{D_k\}_{k=0}^{\infty}$ be a Schauder decomposition of E. Let $\{q_k\}_{k=0}^{\infty}$ be a strictly increasing sequence in $\mathbb{N}$ and let $\Delta = \{\Delta_k\}_{k=0}^{\infty}$ be the corresponding blocking of D. Let $K > 0$ and let Λ_K be the set of all complex sequences $\lambda = \{\lambda_k\}_{k=0}^{\infty}$ such that*

- $|\lambda_k| \leq K$ *for all* $k \in \mathbb{N}$,
- $\sum_{l=q_{k-1}+1}^{q_k-1} |\lambda_{l+1} - \lambda_l| \leq K$ *for all* $k \in \mathbb{N}$.

Then the following are equivalent:

(i) *$\{T_\lambda : \lambda \in \Lambda_K\} \subset \mathcal{L}(E)$ with $\|T_\lambda\| \leq CK$ for all $\lambda \in \Lambda_K$ and some constant $C > 0$.*

(ii) *The blocking Δ is unconditional and there exists a constant $M > 0$ such that*

$$\Big\| \sum_{k=0}^{N} \varepsilon_k P_{m_k} x_k \Big\|_{L^2_X(\Omega)} \leq M \Big\| \sum_{k=0}^{N} \varepsilon_k x_k \Big\|_{L^2_X(\Omega)}, \tag{11}$$

for all $N \in \mathbb{N}$, all $\{x_k\}_{k=0}^{\infty} \subset E$ with $x_k \in \mathrm{R}(\Delta_k)$ and all $\{m_k\}_{k=0}^{\infty}$ such that $q_{k-1} + 1 \leq m_k \leq q_k$ for $k \in \mathbb{N}$.

Here the P_m and T_λ are defined in (4) *and* (6).

Proof. (i)$\Rightarrow$(ii). By choosing the sequence $\lambda_k \in \{-1, 0, 1\}$ to be constant on each of the blocks of Δ, the unconditionality follows immediately. Let $\{m_k\}_{k=0}^{\infty}$ be given as specified and take $N \in \mathbb{N}$. For fixed $\omega \in \Omega$, a suitable choice of $\lambda_k^\omega \in \{-1, 0, 1\}$ gives the operator

$$T_{\lambda^\omega} = \sum_{k=0}^{N} \varepsilon_k(\omega) P_{m_k} \Delta_k.$$

From (i), with $K = 1$, we get

$$\Big\| \sum_{k=0}^{N} \varepsilon_k(\omega) P_{m_k} \Delta_k x \Big\|_E \leq C \Big\| \sum_{k=0}^{N} \Delta_k x \Big\|_E.$$

Integration over Ω and an application of Lemma 2.2 gives the result.

(ii)$\Rightarrow$(i). Take $x \in \operatorname{span}\{\mathrm{R}(\Delta_k) : k \in \mathbb{N}\}$. Using summation by parts, we can write

$$T_\lambda x = \sum_{k=0}^{\infty} \lambda_k D_k x = \sum_{k=0}^{\infty} \lambda_{q_k} \Delta_k x + \sum_{k=0}^{\infty} \Big(\sum_{l=q_{k-1}+1}^{q_k-1} (\lambda_l - \lambda_{l+1}) P_l \Big) \Delta_k x$$

(finite sum). Since the blocking is unconditional, the norm of the first term is bounded by $KC_\Delta \|x\|$.

For the second term we have

$$\begin{aligned}
&\Big\| \sum_{k=0}^{\infty} \Big(\sum_{l=q_{k-1}+1}^{q_k-1} (\lambda_l - \lambda_{l+1}) P_l \Big) \Delta_k x \Big\|_E \\
&\quad \overset{(*)}{\leq} C_\Delta \Big\| \sum_{k=0}^{\infty} \varepsilon_k \Big(\sum_{l=q_{k-1}+1}^{q_k-1} (\lambda_l - \lambda_{l+1}) P_l \Big) \Delta_k x \Big\|_{L^2_E(\Omega)} \\
&\quad \overset{(**)}{\leq} 2KMC_\Delta \Big\| \sum_{k=0}^{\infty} \varepsilon_k \Delta_k x \Big\|_{L^2_E(\Omega)} \\
&\quad \overset{(*)}{\leq} 2KMC_\Delta^2 \Big\| \sum_{k=0}^{\infty} \Delta_k x \Big\|_E = 2KMC_\Delta^2 \|x\|_E.
\end{aligned}$$

Here $(*)$ follows from the unconditionality of the blocking and $(**)$ follows from an appropriate modification of Lemma 3.2 in combination with (11). This gives the desired bound on $T_\lambda x$. ∎

If the partial sum operators of the Schauder decomposition $\{D_k\}$ are R-bounded, then (11) is certainly satisfied. This gives the generalized multiplier principle as formulated by E. Berkson and T. A. Gillespie.

COROLLARY 3.6 ([BG94], Theorem (4.4)′). *Let $D = \{D_k\}_{k=0}^{\infty}$ be a Schauder decomposition of the Banach space E. Let $\{q_k\}_{k=0}^{\infty}$ be a strictly increasing sequence in $\mathbb{N}$ and let $\Delta = \{\Delta_k\}_{k=0}^{\infty}$ be the corresponding blocking of D. Suppose that*

(i) *the partial sum projections of D are R-bounded,*
(ii) *Δ is an unconditional decomposition.*

For any sequence $\{\lambda_k\}_{k=0}^{\infty}$ in $\mathbb{C}$ for which there exists a constant $K > 0$ such that

(iii) $|\lambda_k| \leq K$ *for all* $k \geq 0$,
(iv) $\sum_{l=q_{k-1}+1}^{q_k-1} |\lambda_{l+1} - \lambda_l| \leq K$ *for all* $k \geq 0$,

the operator T_λ defined by (6) *is bounded and $\|T_\lambda\| \leq CK$, where C depends only on the decompositions Δ and D.*

REMARK 3.7. We note that in Theorem 3.5 condition (11) is in general strictly weaker than the R-boundedness of the collection $\{P_m\}_{m=0}^{\infty}$. Indeed, take $X = \mathcal{C}_1$ as in Example 3.10 below and $D_k = \Delta_k = R_k$.

In the situation of the above corollary, it follows in particular that if $\sup_{k \geq 0} (q_k - q_{k-1}) < \infty$, then the Schauder decomposition D itself is unconditional. It should be noted that in general the partial sum operators of a Schauder basis are not R-bounded. It was pointed out to us by P. Wojtaszczyk that in [KP79] N. J. Kalton and N. T. Peck have constructed a reflexive Banach space which has an unconditional Schauder decomposition into two-dimensional subspaces, but which does not have any unconditional basis. By choosing an appropriate basis in each of these two-dimensional subspaces we obtain a Schauder basis of E whose partial sum projections are not R-bounded. Indeed, as observed above, this would imply that this basis is unconditional.

Next we discuss an extension to general unconditional decompositions of an inequality due to E. Stein in the case of martingale decompositions in scalar valued L^p-spaces (See [Ste70], IV.3, Theorem 8). In [Ste70] this inequality is given as a square function estimate, but this is in the scalar-valued situation equivalent to the formulation given below, via the Khinchin inequality. The vector-valued version for martingale decompositions is formulated without proof in [Bou85]. For the sake of completeness we include a proof of this result (Proposition 3.8).

We first introduce some notation. Given a probability space $(S,\mathcal{A},\mu)$ and an increasing sequence $\mathcal{A}_0\subset\mathcal{A}_1\subset\ldots$ of sub-σ-algebras of $\mathcal{A}$, we denote by $\mathbb{E}(\cdot|\mathcal{A}_j)$ and $\mathbb{E}^X(\cdot|\mathcal{A}_j)$ the conditional expectation operators with respect to $\mathcal{A}_j$ in $L^p(S)$ and $L^p_X(S)$ respectively ($1<p<\infty$), where X is a Banach space (for information concerning conditional expectations in spaces of vector-valued functions we refer the reader to [DU77]). Furthermore, we recall that X is called a *UMD-space* if there exists a constant $C_2(X)$ (the *UMD-constant* of X) such that

$$\Big\|\alpha_0\mathbb{E}^X(f|\mathcal{A}_0)+\sum_{j=1}^n\alpha_j\{\mathbb{E}^X(f|\mathcal{A}_j)-\mathbb{E}^X(f|\mathcal{A}_{j-1})\}\Big\|_{L^2_X(S)}\le C_2(X)\|f\|_{L^2_X(S)}$$

for all choices of $\alpha_j=\pm1$, for all $f\in L^2_X(S)$, for all $n=1,2,\ldots$ and for all $(S,\mathcal{A},\mu)$ and $\{\mathcal{A}_j\}_{j=0}^\infty$ as above. We note that in this definition of the UMD-property $L^2_X(S)$ may be replaced by any $L^p_X(S)$ with $1<p<\infty$ (replacing $C_2(X)$ by $C_p(X)$; we refer to e.g. [Bur83], [Bou83] for more on UMD-spaces).

PROPOSITION 3.8 ([Bou85], vector-valued Stein inequality). *Let X be a UMD space and $E=L^p_X(S)$, $1<p<\infty$, for some probability space $(S,\mathcal{A},\mu)$. Then for any increasing sequence $\{\mathcal{A}_n\}_{n=0}^\infty$ of sub-σ-algebras of $\mathcal{A}$, all $f_0,f_1,\ldots,f_n\in E$ and all $n\in\mathbb{N}$ we have*

$$\Big\|\sum_{k=0}^n\varepsilon_k\mathbb{E}^X(f_k|\mathcal{A}_k)\Big\|_{L^2_E(\Omega)}\le C\Big\|\sum_{k=0}^n\varepsilon_kf_k\Big\|_{L^2_E(\Omega)}$$

(*where C depends only on p and the UMD-constant of X*).

Proof. By Kahane's inequality it is sufficient to show that there exists a constant $C>0$ such that

$$\Big\|\sum_{k=0}^n\varepsilon_k\mathbb{E}^X(f_k|\mathcal{A}_k)\Big\|_{L^p_E(\Omega)}\le C\Big\|\sum_{k=0}^n\varepsilon_kf_k\Big\|_{L^p_E(\Omega)}.$$

Without loss of generality we may assume that $\mathcal{A}_n\nearrow\mathcal{A}$ (if not, then we replace $\mathcal{A}$ by the σ-algebra generated by the $\mathcal{A}_n$'s).

For $n=0,1,\ldots$ we define the sub-σ-algebras $\mathcal{F}_n\subset\mathcal{F}$ by

$$\mathcal{F}_n=\sigma\{\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_n\},$$

i.e., $\mathcal{F}_n$ is the sub-σ-algebra of $\mathcal{F}$ generated by the functions $\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_n$. As above, we may assume that $\mathcal{F}_n\nearrow\mathcal{F}$. We denote by $\mathcal{F}\otimes\mathcal{A}$ the product σ-algebra of $\mathcal{F}$ and $\mathcal{A}$ in $\Omega\times S$. Now define the sub-σ-algebras $\mathcal{G}_n$ of $\mathcal{F}\otimes\mathcal{A}$ in $\Omega\times S$ by

$$\begin{cases}\mathcal{G}_{2n}=\mathcal{F}_n\otimes\mathcal{A}_n & (n\ge0),\\ \mathcal{G}_{2n-1}=\mathcal{F}_{n-1}\otimes\mathcal{A}_n & (n\ge1).\end{cases}$$

Then $\mathcal{G}_0\subset\mathcal{G}_1\subset\ldots\nearrow\mathcal{F}\otimes\mathcal{A}$. If $F\in L^p_X(\Omega\times S,\mathcal{F}\otimes\mathcal{A},P\otimes\mu)$, then $\mathbb{E}^X(F|\mathcal{G}_n)\to F$ as $n\to\infty$ in norm (see e.g. [DU77], Theorem V.2.1) and hence the series

$$\mathbb{E}^X(F|\mathcal{G}_0)+\sum_{n=1}^\infty\{\mathbb{E}^X(F|\mathcal{G}_n)-\mathbb{E}^X(F|\mathcal{G}_{n-1})\}=F$$

is norm convergent in $L^p_X(\Omega\times S)$. Since X is a UMD-space, this series converges unconditionally and hence

$$Q(F)=\mathbb{E}^X(F|\mathcal{G}_0)+\sum_{n=1}^\infty\{\mathbb{E}^X(F|\mathcal{G}_{2n})-\mathbb{E}^X(F|\mathcal{G}_{2n-1})\}$$

defines a bounded linear operator $Q:L^p_X(\Omega\times S)\to L^p_X(\Omega\times S)$ with $\|Q\|\le C_p(X)$.

If we take $g\in L^p(\Omega)$ and $f\in L^p_X(S)$ then

$$Q(gf)=\mathbb{E}(g|\mathcal{F}_0)\mathbb{E}^X(f|\mathcal{A}_0)+\sum_{n=1}^\infty\{\mathbb{E}(g|\mathcal{F}_n)-\mathbb{E}(g|\mathcal{F}_{n-1})\}\mathbb{E}^X(f|\mathcal{A}_n).$$

Now take $f_0,f_1,\ldots,f_n\in L^p_X(S)$ and let $F=\sum_{k=0}^n\varepsilon_kf_k$. It follows from

$$Q(\varepsilon_kf_k)=\varepsilon_k\mathbb{E}^X(f_k|\mathcal{A}_k),\quad 0\le k\le n,$$

that

$$Q(F)=\sum_{k=0}^n\varepsilon_k\mathbb{E}^X(f_k|\mathcal{A}_k).$$

Consequently,

$$\Big\|\sum_{k=0}^n\varepsilon_k\mathbb{E}^X(f_k|\mathcal{A}_k)\Big\|_{L^p_X(\Omega\times S)}\le\|Q\|\Big\|\sum_{k=0}^n\varepsilon_kf_k\Big\|_{L^p_X(\Omega\times S)}.$$

Since $\|Q\| \le C_p(X)$, it now follows via Fubini's theorem that
$$\Big\|\sum_{k=0}^{n}\varepsilon_k\mathbb{E}^X(f_k|\mathcal{A}_k)\Big\|_{L^p_E(\Omega)} \le C_p(X)\Big\|\sum_{k=0}^{n}\varepsilon_k f_k\Big\|_{L^p_E(\Omega)}. \ \blacksquare$$

THEOREM 3.9. *Let X be a UMD space and let $\{\Delta_k\}_{k=0}^{\infty}$ be an unconditional Schauder decomposition with unconditional constant C_Δ. Let $P_n = \sum_{k=0}^{n}\Delta_k$. Then*
$$\Big\|\sum_{k=0}^{n}\varepsilon_k P_k x_k\Big\|_{L^2_X(\Omega)} \le C_2(X)C_\Delta^2\Big\|\sum_{k=0}^{n}\varepsilon_k x_k\Big\|_{L^2_X(\Omega)}$$
for all $x_0, x_1, \ldots, x_n \in X$ and all $n \in \mathbb{N}$. Consequently, the collection $\{P_n\}_{n\in\mathbb{N}}$ is R-bounded.

Proof. Let $C_\Delta \ge 1$ be the unconditional constant of the decomposition $\{\Delta_k\}_{k=0}^{\infty}$. Then
$$C_\Delta^{-1}\Big\|\sum_{k=m}^{n}\Delta_k x\Big\|_X \le \Big\|\sum_{k=m}^{n}\varepsilon_k'\Delta_k x\Big\|_{L^2_X(\Omega')} \le C_\Delta\Big\|\sum_{k=m}^{n}\Delta_k x\Big\|_X$$
for all $x \in X$ and all $0 \le m \le n$ in $\mathbb{N}$. This implies, in particular, that for each $x \in X$ the series $\Phi(x) = \sum_{k=0}^{\infty}\varepsilon_k'\Delta_k x$ is norm convergent in $E = L^2_X(\Omega')$, and
$$C_\Delta^{-1}\|x\|_X \le \|\Phi(x)\|_{L^2_X(\Omega')} \le C_\Delta\|x\|_X.$$
Hence $\Phi : X \to L^2_X(\Omega')$ is an isomorphism.

For $n \in \mathbb{N}$ define
$$\mathcal{F}_n' = \sigma\{\varepsilon_0', \varepsilon_1', \ldots, \varepsilon_n'\}.$$
It is straightforward to verify that
$$\mathbb{E}^X(\Phi(x)|\mathcal{F}_n') = \Phi(P_n x)$$
for all $x \in X$ and $n \in \mathbb{N}$, where $P_n = \sum_{k=0}^{n}\Delta_k$. For $x_0, x_1, \ldots, x_n \in X$ it now follows via Proposition 3.8 that
$$\begin{aligned}
\Big\|\sum_{k=0}^{n}\varepsilon_k P_k x_k\Big\|_{L^2_X(\Omega)} &= \Big(\int_\Omega\Big\|\sum_{k=0}^{n}\varepsilon_k(\omega)P_k x_k\Big\|_X^2\,dP(\omega)\Big)^{1/2}\\
&\le C_\Delta\Big(\int_\Omega\Big\|\sum_{k=0}^{n}\varepsilon_k(\omega)\Phi(P_k x_k)\Big\|_{L^2_X(\Omega')}^2\,dP(\omega)\Big)^{1/2}\\
&= C_\Delta\Big\|\sum_{k=0}^{n}\varepsilon_k\Phi(P_k x_k)\Big\|_{L^2_E(\Omega)}^2\\
&= C_\Delta\Big\|\sum_{k=0}^{n}\varepsilon_k\mathbb{E}^X(\Phi(x_k)|\mathcal{F}_k')\Big\|_{L^2_E(\Omega)}^2\\
&\le C_\Delta C_2(X)\Big\|\sum_{k=0}^{n}\varepsilon_k\Phi(x_k)\Big\|_{L^2_E(\Omega)}^2\\
&= C_\Delta C_2(X)\Big(\int_\Omega\Big\|\sum_{k=0}^{n}\varepsilon_k(\omega)\Phi(x_k)\Big\|_{L^2_X(\Omega')}^2\,dP(\omega)\Big)^{1/2}\\
&\le C_\Delta^2 C_2(X)\Big(\int_\Omega\Big\|\sum_{k=0}^{n}\varepsilon_k(\omega)x_k\Big\|_X^2\,dP(\omega)\Big)^{1/2}\\
&= C_\Delta^2 C_2(X)\Big\|\sum_{k=0}^{n}\varepsilon_k x_k\Big\|_{L^2_X(\Omega)}.
\end{aligned}$$

This proves the desired inequality. The final statement of the corollary is now an immediate consequence of Lemma 3.3. ■

Let $\{\Delta_k\}_{k=0}^{\infty}$ be an unconditional decomposition of the UMD space X. In connection with Theorem 3.9 it is a natural question whether the collection
$$\mathcal{S} = \Big\{\sum_{k\in F}\Delta_k : F \subset \mathbb{N},\ F \text{ finite}\Big\} \tag{12}$$
is R-bounded. Even if X is a UMD-space, this need not be the case as shown by the following example.

EXAMPLE 3.10. Let H be a separable Hilbert space and let $X = \mathcal{C}_p$, $1 \le p < \infty$, be the Schatten p-class of compact operators on H. Take a fixed orthonormal basis $\{e_n\}_{n=0}^{\infty}$ in H. For $m, n \in \mathbb{N}$ we define the matrix units $E_{mn} \in \mathcal{C}_p$ by $E_{mn}(x) = \langle x, e_n\rangle e_m$ for all $x \in H$. For $m \in \mathbb{N}$ we define the row projections $R_m : \mathcal{C}_p \to \mathcal{C}_p$ and column projections $C_m : \mathcal{C}_p \to \mathcal{C}_p$ by $R_m(A) = E_{mm}A$ and $C_m(A) = AE_{mm}$, $A \in \mathcal{C}_p$, respectively. It is easy to see that $\{R_m\}_{m=0}^{\infty}$ and $\{C_m\}_{m=0}^{\infty}$ are both unconditional decompositions of $\mathcal{C}_p$ and that $C_n R_m = R_m C_n$ for all $m, n \in \mathbb{N}$.

If $1 < p < \infty$, then $\mathcal{C}_p$ is a UMD-space (see e.g. [Bou85]). We claim that the collection
$$\mathcal{R} = \Big\{\sum_{k\in F}R_k : F \subset \mathbb{N},\ F \text{ finite}\Big\}$$
is not R-bounded if $p \ne 2$. Indeed, assuming that $\mathcal{R}$ is R-bounded, it follows from Theorem 3.4 (with $\mathcal{T} = \mathcal{R}$, $\Delta_k = C_k$) that for any choice of finite subsets $F_n \subset \mathbb{N}$ the operator
$$A \mapsto \sum_{n=0}^{\infty}\Big(\sum_{m\in F_n}R_m\Big)C_n(A)$$
is bounded on $\mathcal{C}_p$. But this would imply that the matrix units $\{E_{mn} : m, n \in \mathbb{N}\}$ form an unconditional basis in $\mathcal{C}_p$, which is false if $p \ne 2$ (see [KP70]).

Finally, we note that if $X = \mathcal{C}_1$, then the collection $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$, with $P_m = \sum_{k=0}^{m} R_k$, is not R-bounded. Indeed, if $\mathcal{P}$ is R-bounded, then Theorem 3.4 (with $\mathcal{T} = \mathcal{P}$, $\Delta_k = C_k$) implies that the operator

$$A \mapsto \sum_{n=0}^{\infty} P_n C_n(A),$$

which is the triangular truncation operator, is bounded on $\mathcal{C}_1$, which is false (see e.g. [GK70]). This shows that the result of Theorem 3.9 (and hence of Proposition 3.8) does not hold in general if X is not a UMD-space.

There is an interesting class of Banach spaces in which the collection $\mathcal{S}$, as defined by (12), is R-bounded. We recall the following definition.

Definition 3.11 ([Pis78]). A Banach space X has *property* (α) if there exists a constant $\alpha > 0$ such that

$$\Big\| \sum_{i,j=1}^{n} \alpha_{ij}\varepsilon_i\varepsilon_j' x_{ij} \Big\|_{L^2_X(\Omega\times\Omega')} \le \alpha \Big\| \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j' x_{ij} \Big\|_{L^2_X(\Omega\times\Omega')}$$

for all $x_{ij} \in X$ $(i,j = 1,\ldots,n)$, all $\alpha_{ij} = \pm 1$ $(i,j = 1,\ldots,n)$ and all $n \in \mathbb{N}\setminus\{0\}$.

Remark 3.12. (i) In the definition of property (α) we may replace $\alpha_{ij} = \pm 1$ by $\alpha_{ij} = 0, 1$.

(ii) Any Banach space with local unconditional structure and finite cotype has property (α). This follows from a combination of Proposition 2.1 in [Pis78] and e.g. Theorem 14.1 in [DJT95] (we also refer the reader to that book for relevant definitions). In particular, any Banach lattice with finite cotype has (α).

(iii) Property (α) and the UMD-property are independent: any infinite-dimensional L^1-space has (α) but is not UMD; the Schatten classes $\mathcal{C}_p$ $(1 < p < \infty$ and $p \neq 2)$ are UMD spaces, but do not have (α).

(iv) If the Banach space X has (α), then $L^p_X(S)$ has (α) as well for any σ-finite measure space (S,Σ,μ) and $1 \le p < \infty$.

Lemma 3.13. *Let X be a Banach space, let $\mathcal{T} \subset \mathcal{L}(X)$ be R-bounded and suppose that X has property (α). Then there exists a constant $K \ge 0$ such that*

$$\Big\| \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j' T_{ij} x_{ij} \Big\|_{L^2_X(\Omega\times\Omega')} \le K \Big\| \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j' x_{ij} \Big\|_{L^2_X(\Omega\times\Omega')}$$

for all $T_{ij} \in \mathcal{T}$, all $x_{ij} \in X$ $(i,j = 1,\ldots,n)$ and all $n \in \mathbb{N}\setminus\{0\}$.

Proof. Let $\{\alpha_{ij}\}_{i,j=1}^{\infty}$ be a sequence of independent symmetric $\{-1,1\}$-valued random variables on some probability space $(\Omega'',\mathcal{F}'',P'')$. Since X has (α), it follows that

$$\Big\| \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j' T_{ij} x_{ij} \Big\|_{L^2_X(\Omega\times\Omega')} \le \alpha \Big\| \sum_{i,j=1}^{n} \alpha_{ij}(\omega'')\varepsilon_i\varepsilon_j' T_{ij} x_{ij} \Big\|_{L^2_X(\Omega\times\Omega')}$$

for all $\omega'' \in \Omega''$. Integration over Ω'' yields

$$\begin{aligned}
&\Big\| \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j' T_{ij} x_{ij} \Big\|_{L^2_X(\Omega\times\Omega')} \\
&\le \alpha \Big(\int_{\Omega''} \Big\| \sum_{i,j=1}^{n} \alpha_{ij}(\omega'')\varepsilon_i\varepsilon_j' T_{ij} x_{ij} \Big\|^2_{L^2_X(\Omega\times\Omega')} dP''(\omega'') \Big)^{1/2} \\
&= \alpha \Big(\int_{\Omega\times\Omega'} \Big\| \sum_{i,j=1}^{n} \alpha_{ij}\varepsilon_i(\omega)\varepsilon_j'(\omega') T_{ij} x_{ij} \Big\|^2_{L^2_X(\Omega'')} dP(\omega)\,dP'(\omega') \Big)^{1/2}.
\end{aligned}$$

For fixed ω and ω', $\{\alpha_{ij}\varepsilon_i(\omega)\varepsilon_j'(\omega')\}$ is a sequence of independent symmetric $\{-1,1\}$-valued random variables, so we can use the R-boundedness of $\mathcal{T}$ to get

$$\begin{aligned}
&\Big\| \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j' T_{ij} x_{ij} \Big\|_{L^2_X(\Omega\times\Omega')} \\
&\le \alpha M \Big(\int_{\Omega\times\Omega'} \Big\| \sum_{i,j=1}^{n} \alpha_{ij}\varepsilon_i(\omega)\varepsilon_j'(\omega') x_{ij} \Big\|^2_{L^2_X(\Omega'')} dP(\omega)\,dP'(\omega') \Big)^{1/2} \\
&\le \alpha^2 M \Big(\int_{\Omega''} \Big\| \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j' x_{ij} \Big\|^2_{L^2_X(\Omega\times\Omega')} dP''(\omega'') \Big)^{1/2} \\
&= \alpha^2 M \Big\| \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j' x_{ij} \Big\|^2_{L^2_X(\Omega\times\Omega')},
\end{aligned}$$

where M denotes the R-bound. This proves the lemma with $K = \alpha^2 M$. ∎

Theorem 3.14. *Let X be a Banach space that has property (α), let $\Delta = \{\Delta_k\}_{k=0}^{\infty}$ be an unconditional Schauder decomposition and let $\mathcal{T} \subset \mathcal{L}(X)$ be an R-bounded collection of operators. Let*

$$\mathcal{S} = \Big\{ \sum_{k=0}^{\infty} T_k\Delta_k : T_k \in \mathcal{T} \text{ such that } \Delta_k T_k = T_k\Delta_k \text{ for all } k \in \mathbb{N} \Big\}.$$

Then $\mathcal{S}$ is R-bounded.

Proof. Without loss of generality we may assume that $0 \in \mathcal{T}$. Take $S_1,\ldots,S_n \in \mathcal{S}$ which are of the form

$$S_k = \sum_{j=0}^{N} T_{jk} \Delta_j$$

with $T_{jk} \in \mathcal{T}$ such that $T_{jk}\Delta_j = \Delta_j T_{jk}$ ($j = 0, 1, \ldots, N$ and $k = 1, \ldots, n$) for some $N \in \mathbb{N}$. Take $x_1, \ldots, x_n \in X$. Then

$$\begin{aligned}
\Big\| \sum_{k=1}^{n} \varepsilon_k S_k x_k \Big\|_{L^2_X(\Omega)} &= \Big\| \sum_{j=0}^{N} \Big(\sum_{k=1}^{n} \varepsilon_k T_{kj} \Delta_j x_k \Big) \Big\|_{L^2_X(\Omega)} \\
&\le C_\Delta \Big\| \sum_{j=0}^{N} \varepsilon'_j \Big(\sum_{k=1}^{n} \varepsilon_k T_{kj} \Delta_j x_k \Big) \Big\|_{L^2_X(\Omega\times\Omega')} \\
&\le K C_\Delta \Big\| \sum_{j=0}^{N} \sum_{k=1}^{n} \varepsilon_k \varepsilon'_j \Delta_j x_k \Big\|_{L^2_X(\Omega\times\Omega')} \\
&= K C_\Delta \Big\| \sum_{j=0}^{N} \varepsilon_k \varepsilon'_j \Delta_j \Big(\sum_{k=1}^{n} x_k \Big) \Big\|_{L^2_X(\Omega\times\Omega')} \\
&\le K C_\Delta^2 \Big\| \sum_{k=1}^{n} \varepsilon_k x_k \Big\|_{L^2_X(\Omega)},
\end{aligned}$$

where K is the constant from Lemma 3.13. The result follows by letting $N \to \infty$. ∎

The above result can now be applied to the collection (12). Indeed, applying Theorem 3.14 with $\mathcal{T} = \{0, I\}$ we get the following corollary.

COROLLARY 3.15. *Let $\{\Delta_k\}_{k=0}^{\infty}$ be an unconditional Schauder decomposition of a Banach space X with (α). Then*

$$\mathcal{S} = \Big\{ \sum_{k\in G} \Delta_k : G \subset \mathbb{N} \Big\}$$

is R-bounded.

It follows from Corollary 3.15 and considerations presented in Example 3.10 that $\mathcal{C}_p$, $p \ne 2$, does not have property (α). The latter fact was established by G. Pisier in [Pis78] by different methods.

For later reference we collect below some additional results concerning R-boundedness. The proof of the next simple lemma is left to the reader.

LEMMA 3.16. *Let X be a Banach space and let $\mathcal{T}, \mathcal{S} \subset \mathcal{L}(X)$ be two collections which are R-bounded. Then the collection*

$$\mathcal{S}\mathcal{T} = \{ST : S \in \mathcal{S},\ T \in \mathcal{T}\}$$

is R-bounded.

We will use the following notation. Let X be a Banach space and $E = L^p_X(S)$ for some σ-finite measure space (S, Σ, μ) and $1 \le p < \infty$. For $\phi \in L^\infty(S)$ we denote by M_ϕ the multiplication operator on $L^p_X(S)$ given by $(M_\phi f)(s) = \phi(s) f(s)$ μ-a.e. on S for all $f \in L^p_X(S)$. Note that $M_\phi \in \mathcal{L}(E)$ and $\|M_\phi\| = \|\phi\|_\infty$. If $T \in \mathcal{L}(E)$ and $\phi, \psi \in L^\infty(S)$, then we denote the operator $M_\phi T M_\psi$ also simply by $\phi T \psi$.

LEMMA 3.17. *Let $E = L^p_X$ as above. Then the collection*

$$\{M_\phi : \phi \in L^\infty(S),\ \|\phi\|_\infty \le 1\} \tag{13}$$

is R-bounded in E.

Proof. Take $\phi_k \in L^\infty(S)$ with $\|\phi_k\|_\infty \le 1$ and $f_k \in L^p_X(S)$, $k = 0, 1, \ldots, n$. It follows from Kahane's contraction principle that

$$\begin{aligned}
\Big\| \sum_{k=0}^{n} \varepsilon_k M_{\phi_k} f_k \Big\|_{L^p_E(\Omega)} &= \Big(\int_S \Big\| \sum_{k=0}^{n} \varepsilon_k \phi_k(s) f_k(s) \Big\|^p_{L^p_X(\Omega)} d\mu(s) \Big)^{1/p} \\
&\le 2 \Big(\int_S \Big\| \sum_{k=0}^{n} \varepsilon_k f_k(s) \Big\|^p_{L^p_X(\Omega)} d\mu(s) \Big)^{1/p} \\
&= 2 \Big\| \sum_{k=0}^{n} \varepsilon_k f_k \Big\|_{L^p_E(\Omega)},
\end{aligned}$$

which proves the lemma. ∎

The following corollary is now an immediate consequence of the two lemmas above.

COROLLARY 3.18. *Let $E = L^p_X(S)$ as above. If $\mathcal{T} \subset \mathcal{L}(E)$ is R-bounded, then the collection*

$$\{\phi T \psi : \phi, \psi \in L^\infty(S) \text{ with } \|\phi\|_\infty, \|\psi\|_\infty \le 1,\ T \in \mathcal{T}\}$$

is R-bounded as well.

4. The Vilenkin system in UMD-spaces. We start this section by recalling some facts and introducing some notation concerning the Vilenkin systems (for more information we refer the reader to e.g. [SWS90]). For any $p \in \mathbb{N}$, $p \ge 2$, we denote by $\mathbb{Z}_p$ the cyclic group $\mathbb{Z}/(p) = \{0, 1, \ldots, p-1\}$. Let $\boldsymbol{p} = (p_1, p_2, \ldots)$ be a sequence of natural numbers $p_k \ge 2$ and let $G_{\boldsymbol{p}} = \prod_{k=1}^{\infty} \mathbb{Z}_{p_k}$, equipped with the product topology and the normalized Haar measure. As is well known, the dual group $\widehat{G}_{\boldsymbol{p}}$ of $G_{\boldsymbol{p}}$ can be identified with the collection of all sequences $\boldsymbol{n} = (n_1, n_2, \ldots)$ with $n_k \in \{0, 1, \ldots, p_k - 1\}$ for all k and $n_k \ne 0$ for only finitely many values of k (see e.g. [SWS90], Appendix 0.7). The pairing between $G_{\boldsymbol{p}}$ and $\widehat{G}_{\boldsymbol{p}}$ is given by

$$\langle \boldsymbol{\theta}, \boldsymbol{n} \rangle = \psi_{\boldsymbol{n}}(\boldsymbol{\theta}), \tag{14}$$

where

$$\psi_{\boldsymbol{n}}(\boldsymbol{\theta}) = \prod_{k=1}^{\infty} \phi_k^{n_k}(\boldsymbol{\theta}) \quad \text{for all } \boldsymbol{n} = (n_1, n_2, \ldots) \in \widehat{G}_{\boldsymbol{p}} \tag{15}$$

and

$$\phi_k(\boldsymbol{\theta}) = e^{2\pi i \theta_k / p_k} \quad \text{for all } \boldsymbol{\theta} = (\theta_1, \theta_2, \ldots) \in G_{\boldsymbol{p}}. \tag{16}$$

The characters $\{\psi_{\boldsymbol{n}} : \boldsymbol{n} \in \widehat{G}_{\boldsymbol{p}}\}$ form a complete orthonormal system in $L^2(G_{\boldsymbol{p}})$, which is called the *Vilenkin system* corresponding to $\boldsymbol{p} = (p_1, p_2, \ldots)$. If $p_k = 2$ for all $k \in \mathbb{N}$, the system is called the *Paley–Walsh system*. In this case we use the notation $\mathbb{D} = G_{\boldsymbol{p}}$, the dyadic group. For $\boldsymbol{m}, \boldsymbol{n} \in \widehat{G}_{\boldsymbol{p}}$ we define $\boldsymbol{m} < \boldsymbol{n}$ if and only if there exists a $k \in \mathbb{N}$ such that $m_j = n_j$ for all $j > k$ and $m_k < n_k$. This defines a linear ordering in $\widehat{G}_{\boldsymbol{p}}$, with smallest element $\boldsymbol{0} = (0, 0, \ldots)$. From now on we shall consider the system $\{\psi_{\boldsymbol{n}} : \boldsymbol{n} \in \widehat{G}_{\boldsymbol{p}}\}$ with the enumeration induced by this ordering in $\widehat{G}_{\boldsymbol{p}}$.

REMARK 4.1. (a) Of course, it is also possible to use $\mathbb{N}$ as the index set for the characters $\{\psi_{\boldsymbol{n}} : \boldsymbol{n} \in \widehat{G}_{\boldsymbol{p}}\}$, preserving the enumeration introduced above as follows. Define

$$M_k = \begin{cases} 1 & \text{if } k = 1, \\ \prod_{j=1}^{k-1} p_j & \text{if } k \geq 2. \end{cases}$$

To each $\boldsymbol{n} \in \widehat{G}_{\boldsymbol{p}}$ we assign the natural number $n = \sum_{j=1}^{k} n_j M_j$. This defines an order preserving bijection between $\widehat{G}_{\boldsymbol{p}}$ and $\mathbb{N}$. Denoting the character $\psi_{\boldsymbol{n}}$ by ψ_n, where n corresponds to $\boldsymbol{n}$ as above, we may write the Vilenkin system as $\{\psi_n\}_{n=0}^{\infty}$.

(b) Although we work with the groups $G_{\boldsymbol{p}}$, we could also have chosen to work with the interval $[0, 1]$. There is a natural measure preserving identification between the groups $G_{\boldsymbol{p}}$ and the interval $[0, 1]$. As in the dyadic case, this identification is given by the mapping $\boldsymbol{\theta} \mapsto x(\boldsymbol{\theta}) \in [0, 1]$, where

$$x(\boldsymbol{\theta}) = \sum_{k=1}^{\infty} \frac{\theta_k}{M_{k+1}} \quad \text{for all } \boldsymbol{\theta} = (\theta_1, \theta_2, \ldots) \in G_{\boldsymbol{p}}.$$

We denote the Borel σ-algebra in $G_{\boldsymbol{p}}$ by $\mathcal{B}$. For $k = 1, 2, \ldots$ we define

$$\mathcal{B}_k = \Big\{ A \times \prod_{j>k} \mathbb{Z}_{p_j} : A \subset \prod_{j=1}^{k} \mathbb{Z}_{p_j} \Big\}, \tag{17}$$

and $\mathcal{B}_0 = \{\emptyset, G_{\boldsymbol{p}}\}$. Then $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \ldots$ are sub-σ-algebras of $\mathcal{B}$ and the σ-algebra generated by $\bigcup_{k=0}^{\infty} \mathcal{B}_k$ is equal to $\mathcal{B}$. For a fixed complex Banach space X and fixed $1 < p < \infty$ we consider the Bochner space $E = L_X^p(G_{\boldsymbol{p}})$. We denote by $\mathbb{E}_k$ the conditional expectation projection in $L_X^p(G_{\boldsymbol{p}})$ with respect to $\mathcal{B}_k$ $(k \in \mathbb{N})$. Note that

$$(\mathbb{E}_k f)(\boldsymbol{\theta}) = \int f(\theta_1, \ldots, \theta_k, \theta'_{k+1}, \theta'_{k+2}, \ldots) \, d(\theta'_{k+1}, \theta'_{k+2}, \ldots) \tag{18}$$

for all $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots) \in G_{\boldsymbol{p}}$, where the integration is taken over $\prod_{j>k} \mathbb{Z}_{p_j}$ with respect to the (normalized) Haar measure.

As is well known (see e.g. [DU77], Theorem V.2.1),

$$\mathbb{E}_k f \to f \quad \text{as } k \to \infty \tag{19}$$

in norm for all $f \in L_X^p(G_{\boldsymbol{p}})$. Defining

$$\Delta_0 = \mathbb{E}_0, \quad \Delta_k = \mathbb{E}_k - \mathbb{E}_{k-1} \quad (k \geq 1), \tag{20}$$

it follows from (19) that

$$f = \sum_{k=0}^{\infty} \Delta_k f \quad \text{for all } f \in L_X^p(G_{\boldsymbol{p}}) \tag{21}$$

(norm convergent series in $L_X^p(G_{\boldsymbol{p}})$). Hence $\{\Delta_k\}_{k=0}^{\infty}$ is a Schauder decomposition of $L_X^p(G_{\boldsymbol{p}})$. If we assume in addition that X is a UMD-space (see e.g. [Bur83], [Bou83]), then (21) converges unconditionally for all $L_X^p(G_{\boldsymbol{p}})$. So $\{\Delta_k\}_{k=0}^{\infty}$ is an unconditional Schauder decomposition of $L_X^p(G_{\boldsymbol{p}})$.

For $k = 1, 2, \ldots$ define $\boldsymbol{d}_k = (\delta_{jk})_{j=1}^{\infty} \in \widehat{G}_{\boldsymbol{p}}$ and $\boldsymbol{d}_0 = \boldsymbol{0}$ (note that $\boldsymbol{d}_k$ corresponds to the function ϕ_k). Since

$$\operatorname{span}\{\psi_{\boldsymbol{n}} : \boldsymbol{n} \in \widehat{G}_{\boldsymbol{p}},\ \boldsymbol{n} < \boldsymbol{d}_{k+1}\} = L^p(\mathbb{Z}_{p_1} \times \ldots \times \mathbb{Z}_{p_k}) \subset L^p(G_{\boldsymbol{p}}), \tag{22}$$

it is easy to see that

$$\mathrm{R}(\mathbb{E}_k) = \operatorname{span}\{\psi_{\boldsymbol{n}} \otimes x : \boldsymbol{n} \in \widehat{G}_{\boldsymbol{p}},\ \boldsymbol{n} < \boldsymbol{d}_{k+1} \text{ and } x \in X\} \tag{23}$$

for all $k \in \mathbb{N}$. Moreover,

$$\mathbb{E}_k f = \sum_{\boldsymbol{n} < \boldsymbol{d}_{k+1}} \psi_{\boldsymbol{n}} \otimes c_{\boldsymbol{n}}(f) \quad \text{for all } f \in L_X^p(G_{\boldsymbol{p}}), \tag{24}$$

where

$$c_{\boldsymbol{n}}(f) = \int_{G_{\boldsymbol{p}}} \overline{\psi_{\boldsymbol{n}}(\boldsymbol{\theta})} f(\boldsymbol{\theta}) \, d\boldsymbol{\theta}. \tag{25}$$

It follows in particular that

$$\Delta_k f = \sum_{\boldsymbol{d}_k \leq \boldsymbol{n} < \boldsymbol{d}_{k+1}} \psi_{\boldsymbol{n}} \otimes c_{\boldsymbol{n}}(f) \tag{26}$$

for all $f \in L_X^p(G_{\boldsymbol{p}})$ and all $k \in \mathbb{N}$. Note that

$$\begin{aligned} &\{\boldsymbol{n} \in \widehat{G}_{\boldsymbol{p}} : \boldsymbol{d}_k \leq \boldsymbol{n} < \boldsymbol{d}_{k+1}\} \\ &\quad = \{\boldsymbol{n} = (n_1, n_2, \ldots) \in \widehat{G}_{\boldsymbol{p}} : n_k \neq 0 \text{ and } n_j = 0 \text{ for all } j > k\}. \end{aligned} \tag{27}$$

For $k=1,2,\dots$ and $1\le j\le p_k-1$ we define

$$\boldsymbol{d}_{(k,j)}=(j\delta_{ik})_{i=1}^{\infty}\in\widehat{G}_{\boldsymbol{p}}.$$

These $\boldsymbol{d}_{(k,j)}$ can be associated with ϕ_k^j. It will also be convenient to define $\boldsymbol{d}_{(k,p_k)}=\boldsymbol{d}_{k+1}$ and $\boldsymbol{d}_{(0,0)}=\boldsymbol{0}$. Note that $\boldsymbol{d}_{(k,1)}=\boldsymbol{d}_k$ for $k=1,2,\dots$ The set

$$\Lambda=\{(k,j):k=1,2,\dots,\ 1\le j\le p_k-1\}\cup\{(0,0)\}\tag{28}$$

is linearly ordered by the lexicographical ordering. Note that $(k,j)<(l,i)$ in Λ implies that $\boldsymbol{d}_{(k,j)}<\boldsymbol{d}_{(l,i)}$ in $\widehat{G}_{\boldsymbol{p}}$. For $(k,j)\in\Lambda$, $k\ge1$, we define

$$\Delta_{(k,j)}f=\sum_{\boldsymbol{d}_{(k,j)}\le\boldsymbol{n}<\boldsymbol{d}_{(k,j+1)}}\psi_{\boldsymbol{n}}\otimes c_{\boldsymbol{n}}(f)\tag{29}$$

for all $f\in L_X^p(G_{\boldsymbol{p}})$ and $\Delta_{(0,0)}=\Delta_0$. Note that

$$\begin{aligned}\{\boldsymbol{n}\in\widehat{G}_{\boldsymbol{p}}:\boldsymbol{d}_{(k,j)}\le\boldsymbol{n}<\boldsymbol{d}_{(k,j+1)}\}\\ =\{\boldsymbol{n}=(n_1,n_2,\dots):n_k=j\text{ and }n_i=0\text{ for all }i>k\}\end{aligned}\tag{30}$$

for all $(k,j)\in\Lambda$, $k\ge1$. Furthermore,

$$\Delta_k=\sum_{j=1}^{p_k-1}\Delta_{(k,j)},\qquad \mathbb{E}_k=\sum_{(l,i)<(k+1,1)}\Delta_{(l,i)}\tag{31}$$

for all $k=1,2,\dots$

Lemma 4.2. *For $k=1,2,\dots$ and $1\le j\le p_{k-1}$ we have*

$$\Delta_{(k,j)}f=\phi_k^j\mathbb{E}_{k-1}(\bar\phi_k^jf)\quad\text{for all } f\in L_X^p(G_{\boldsymbol{p}}).\tag{32}$$

Proof. From (19) and (23) it follows that

$$\operatorname{span}\{\psi_{\boldsymbol{n}}\otimes x:\boldsymbol{n}\in\widehat{G}_{\boldsymbol{p}},\ x\in X\}\tag{33}$$

is dense in $L_X^p(G_{\boldsymbol{p}})$. Therefore it is sufficient to prove (32) for $f=\psi_{\boldsymbol{n}}\otimes x$ with $\boldsymbol{n}\in\widehat{G}_{\boldsymbol{p}}$, $x\in X$. For $\boldsymbol{n}=\boldsymbol{0}$ this is clear. Assuming that $\boldsymbol{n}\ne\boldsymbol{0}$ take l such that $\psi_{\boldsymbol{n}}\otimes x\in\mathrm{R}(\Delta_l)$, i.e., $\boldsymbol{d}_l\le\boldsymbol{n}<\boldsymbol{d}_{l+1}$. Now we consider three cases:

- $l>k$. Then $\Delta_{(k,j)}(\psi_{\boldsymbol{n}}\otimes x)=0$ and

$$\begin{aligned}\phi_k^j\mathbb{E}_{k-1}(\bar\phi_k^j\cdot\psi_{\boldsymbol{n}}\otimes x)&=\phi_k^j\mathbb{E}_{k-1}(\mathbb{E}_k(\bar\phi_k^j\cdot\psi_{\boldsymbol{n}}\otimes x))\\&=\phi_k^j\mathbb{E}_{k-1}(\bar\phi_k^j\cdot\mathbb{E}_k(\psi_{\boldsymbol{n}}\otimes x))=0,\end{aligned}$$

since $\mathbb{E}_k(\psi_{\boldsymbol{n}}\otimes x)=\mathbb{E}_k\Delta_l(\psi_{\boldsymbol{n}}\otimes x)=0$.

- $l<k$. Then $\Delta_{(k,j)}(\psi_{\boldsymbol{n}}\otimes x)=0$ and

$$\phi_k^j\mathbb{E}_{k-1}(\bar\phi_k^j\cdot\psi_{\boldsymbol{n}}\otimes x)=\phi_k^j\mathbb{E}_{k-1}(\bar\phi_k^j)\cdot\psi_{\boldsymbol{n}}\otimes x=0,$$

since $\mathbb{E}_{k-1}(\bar\phi_k^j)=0$ for $1\le j\le p_k-1$.

- $l=k$. Then $\psi_{\boldsymbol{n}}=\prod_{i=1}^k\phi_i^{n_i}$ and

$$\phi_k^j\mathbb{E}_{k-1}(\bar\phi_k^j\cdot\psi_{\boldsymbol{n}}\otimes x)=\phi_k^j\prod_{i=1}^{k}\phi_i^{n_i}\mathbb{E}_{k-1}(\bar\phi_k^j\phi_k^{n_k})\otimes x=\begin{cases}\psi_{\boldsymbol{n}}\otimes x&\text{if } j=n_k,\\0&\text{otherwise,}\end{cases}$$

since $\mathbb{E}_{k-1}(\bar\phi_k^j\phi_k^{n_k})=\delta_{jn_k}\mathbb{I}$.

Hence the lemma is proved. ∎

We consider again the collection $\{\Delta_{(k,j)}:(k,j)\in\Lambda\}$ of projections in $L_X^p(G_{\boldsymbol{p}})$, where Λ is linearly ordered by the lexicographical ordering. The corresponding partial sum projections are given by

$$S_{(k,j)}=\sum_{(l,i)<(k,j)}\Delta_{(l,i)}\tag{34}$$

if $(k,j)\ne(0,0)$ and $S_{(0,0)}=\Delta_{(0,0)}=\Delta_0$. Note that it follows from (31) and (32) that for $k\ge1$ we have

$$\begin{aligned}S_{(k,j)}&=\sum_{(l,i)<(k,1)}\Delta_{(l,i)}+\sum_{i=1}^{j-1}\Delta_{(k,i)}\\&=\mathbb{E}_{k-1}+\sum_{i=1}^{j-1}\phi_k^i\mathbb{E}_{k-1}\bar\phi_k^i=\sum_{i=0}^{j-1}\phi_k^i\mathbb{E}_{k-1}\bar\phi_k^i\end{aligned}\tag{35}$$

for all $1\le j\le p_k-1$.

Lemma 4.3. *Suppose that X is a UMD-space and that $\max_k p_k-1=m<\infty$ (i.e., the Vilenkin system $\{\psi_{\boldsymbol{n}}:\boldsymbol{n}\in\widehat{G}_{\boldsymbol{p}}\}$ is bounded). Then the collection $\{S_{(k,j)}:(k,j)\in\Lambda\}$ of operators in $E=L_X^p(G_{\boldsymbol{p}})$, $1<p<\infty$, is R-bounded.*

Proof. Since each of the operators $S_{(k,j)}$ is a sum of at most m operators of the form $\phi_k^i\mathbb{E}_{k-1}\bar\phi_k^i$, it is enough to show that the collection $\{\phi_k^i\mathbb{E}_{k-1}\bar\phi_k^i: 1\le i\le p_k-1,\ k\in\mathbb{N}\}$ is R-bounded.

Since X is a UMD-space, it follows from Proposition 3.8 that the collection $\{\mathbb{E}_k\}_{k=0}^{\infty}$ is R-bounded. Since $\|\phi_k\|_\infty\le1$ for all $k\in\mathbb{N}$, the result follows from Corollary 3.18. ∎

Corollary 4.4. *Suppose that X is a UMD-space, $\sup_k p_k<\infty$ and let $E=L_X^p(G_{\boldsymbol{p}})$. Then $\{\Delta_{(k,j)}:(k,j)\in\Lambda\}$ is an unconditional Schauder decomposition of E.*

Proof. It follows immediately from (35) that if $\sup_k p_k<\infty$, then the $S_{(k,j)}$ are uniformly bounded. Since these $S_{(k,j)}$ are the partial sum projections corresponding to $\{\Delta_{(k,j)}:(k,j)\in\Lambda\}$ and since $\operatorname{span}\{\mathrm{R}(\Delta_{(k,j)}):(k,j)\in\Lambda\}$ is dense in E, it follows that $\{\Delta_{(k,j)}:(k,j)\in\Lambda\}$ is a Schauder decomposition of E.

Since $\{\Delta_k\}_{k=0}^\infty$ is an unconditional blocking of $\{\Delta_{(k,j)} : (k,j) \in \Lambda\}$, since the partial sum projections $\{S_{(k,j)} : (k,j) \in \Lambda\}$ are R-bounded and since $\sup_k p_k < \infty$, it is an immediate consequence of Corollary 3.6 that the decomposition $\{\Delta_{(k,j)} : (k,j) \in \Lambda\}$ is unconditional (see Remark 3.7). ■

For $\mathbf{0} < \boldsymbol{n} \in \widehat{G}_p$ and $f \in L^p_X(G_p)$ we define

$$P_n f = \sum_{m<n} \psi_m \otimes c_m(f), \tag{36}$$

and $P_0 = \Delta_0 = S_{(0,0)}$.

We will now formulate a Paley identity for a Vilenkin system. To this end, for every $\boldsymbol{n} \in \widehat{G}_p$, define the disjoint subsets A_n and B_n of Λ by

$$A_n = \{0,0\} \cup \bigcup_{k=1}^{\infty} \{(k,j) : 1 \le j \le p_k - n_k - 1\},$$

$$B_n = \bigcup_{k=1}^{\infty} \{(k,j) : p_k - n_k \le j \le p_k - 1\} \tag{37}$$

(see also [DS97]). Note that $\Lambda = A_n \cup B_n$ and $A_n \cap B_n = \emptyset$ for every $\boldsymbol{n} \in \widehat{G}_p$. For a subset $A \subset \Lambda$ we define

$$P_A = \sum_{(k,j)\in A} \Delta_{(k,j)}. \tag{38}$$

Then we have

$$P_n = \psi_n P_{B_n} \bar{\psi}_n. \tag{39}$$

Indeed, if $\boldsymbol{m} < \boldsymbol{n}$, then $\bar{\psi}_n \psi_m \in \mathrm{R}(P_{B_n})$, whereas for $\boldsymbol{m} \ge \boldsymbol{n}$ we have $\bar{\psi}_n \psi_m \in \mathrm{R}(P_{A_n})$. For the Paley–Walsh system, (39) is called the *Paley identity* (see also R. E. A. C. Paley [Pal32]).

Define for $\boldsymbol{n} \in \widehat{G}_p$ the projection D_n in $L^p_X(\widehat{G}_p)$ by

$$D_n f = \psi_n \otimes c_n(f), \qquad f \in L^p_X(\widehat{G}_p). \tag{40}$$

Note that $P_n = \sum_{m<n} D_m$ for all $\mathbf{0} < \boldsymbol{n} \in \widehat{G}_p$ and $P_0 = D_0$. It is clear that $\mathrm{span}\{\mathrm{R}(D_n) : \boldsymbol{n} \in \widehat{G}_p\}$ is dense in $L^p_X(G_p)$.

THEOREM 4.5. *Suppose that X is a UMD-space and that the Vilenkin system $\{\psi_n : \boldsymbol{n} \in \widehat{G}_p\}$ is bounded. Then $\{D_n : \boldsymbol{n} \in \widehat{G}_p\}$ is a Schauder decomposition of $L^p_X(G_p)$, $1 < p < \infty$.*

Proof. Since the fine decomposition $\{\Delta_{(k,j)} : (k,j) \in \Lambda\}$ is unconditional and since the collection $\{\psi_n : \boldsymbol{n} \in \widehat{G}_p\}$ is uniformly bounded, it follows from (39) that the collection of projections $\{P_n : \boldsymbol{n} \in \widehat{G}_p\}$ is uniformly bounded. Since $\{P_n : \boldsymbol{n} \in \widehat{G}_p\}$ are the partial sums corresponding to $\{D_n : \boldsymbol{n} \in \widehat{G}_p\}$ and since $\mathrm{span}\{\mathrm{R}(D_n) : \boldsymbol{n} \in \widehat{G}_p\}$ is dense in $L^p_X(G_p)$, the result now follows. ■

It is well known that a Banach space X is a UMD-space if and only if the trigonometric system generates a Schauder decomposition in $L^p_X(\mathbb{T})$, $1 < p < \infty$. For the bounded Vilenkin systems we get a similar result. We have shown that if X is a UMD-space, then the bounded Vilenkin system generates a Schauder decomposition in L^p_X. The converse of this implication is also true. Using Vilenkin systems, we get the following characterization of UMD-spaces:

THEOREM 4.6. *Let $\{\psi_n : \boldsymbol{n} \in \widehat{G}_p\}$ be a bounded Vilenkin system and $1 < p < \infty$. The following statements for a Banach space X are equivalent:*

(i) *X is a UMD-space.*
(ii) *The coarse blocking $\{\Delta_k\}_{k=0}^\infty$ is unconditional in $L^p_X(G_p)$.*
(iii) *The fine blocking $\{\Delta_{(k,j)} : (k,j) \in \Lambda\}$ is unconditional in $L^p_X(G_p)$.*
(iv) *$\{D_n : \boldsymbol{n} \in \widehat{G}_p\}$ is a Schauder decomposition of $L^p_X(G_p)$.*

Proof. The implication (i)⇒(ii) is clear, since the coarse blocking is associated with a martingale. The converse implication can be obtained via approximate embeddings of the Paley–Walsh martingale into the martingale associated with the coarse blocking of the Vilenkin system. This can be achieved using the identification of G_p with $[0,1]$ (see Remark 4.1(b)) in combination with the technique used by B. Maurey in [Mau75]. We leave the details to the interested reader.

The implication (ii)⇒(iii) follows from the proof of Corollary 4.4, while the converse is obvious, since the coarse blocking is a blocking of the fine blocking.

The implication (iii)⇒(iv) is given in Theorem 4.5. Now we shall prove the implication (iv)⇒(iii). It is sufficient to show that $\{P_A : A \subset \Lambda\}$ is uniformly bounded (here Λ is defined by (28) and P_A via (38)). Let $A \subset \Lambda$ be given. For $0 \le j \le m = \max_k p_k - 1$ define

$$F_j = \{k \in \mathbb{N} : (k,j) \in A\},$$

and let $A_j = \{(k,j) : k \in F_j\}$. Then $A = \bigcup_{j=0}^m A_j$, a disjoint union. For $1 \le j \le m$ define $\boldsymbol{n}(j) \in G_p$ by

$$[\boldsymbol{n}(j)]_k = \begin{cases} p_k - j - 1 & \text{if } k \in F_j,\ 1 \le j \le p_k - 1, \\ 0 & \text{otherwise,} \end{cases}$$

and define $\boldsymbol{m}(j) \in G_p$ by

$$[\boldsymbol{m}(j)]_k = \begin{cases} p_k - j & \text{if } k \in F_j,\ 1 \le j \le p_k - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$P_{A_j} = P_{B_{m(j)}} P_{A_{n(j)}}, \quad 1 \le j \le m.$$

By hypothesis, $\{D_n : n \in \widehat{G}_p\}$ is a Schauder decomposition of $L^p_X(G_p)$ and so the partial sum operators $\{P_n : n \in \widehat{G}_p\}$ are uniformly bounded, i.e., $\|P_n\| \le K$ for all $n \in \widehat{G}_p$ and some $K > 0$. Now it follows from (39) that

$$\|P_{A_j}\| \le K(K+1), \quad 1 \le j \le m,$$

and so $\|P_A\| \le mK(K+1)+1$. ∎

In the special case of the Paley–Walsh system, the implication (i)⇒(iv) of Theorem 4.6 is given in [SF94] and [SF95]. The equivalence (i)⇔(iv) for the Paley–Walsh system is given in [Wen93]. The implication (i)⇒(ii), in a more general setting, is given in [SF94].

5. Multiplier theorems. The following theorem is a vector-valued analogue of the Marcinkiewicz multiplier theorem for the bounded Vilenkin system. It should be noted that its scalar-valued specialization yields an extension of a multiplier theorem given by G. I. Sunouchi in [Sun51].

We will use the notation $l+1$ to denote the successor of the element $l \in \widehat{G}_p$.

THEOREM 5.1. *Let X be a UMD-space with property (α) and $1 < p < \infty$. Let $\{\psi_n : n \in \widehat{G}_p\}$ be a bounded Vilenkin system. Suppose $\{\lambda_n : n \in \widehat{G}_p\} \subset \mathbb{C}$ is such that*

- $|\lambda_l| \le K$ *for all* $l \in \widehat{G}_p$,
- $\sum_{d_k \le l < d_{k+1}} |\lambda_{l+1} - \lambda_l| \le K$ *for all* $k \ge 0$,

for some constant $K > 0$. Then there exists a (unique) bounded linear operator T_λ in $L^p_X(G_p)$ satisfying $T_\lambda(\psi_n \otimes x) = (\lambda_n \psi_n) \otimes x$ for all $n \in \widehat{G}_p$ and all $x \in X$. Moreover, $\|T_\lambda\| \le CK$, where the constant C depends on p, X and the Vilenkin system.

Proof. By Corollary 3.6 it is sufficient to prove that the collection $\{P_n : n \in \widehat{G}_p\}$ of partial sum projections, as defined by (36), is R-bounded.

Since, by Corollary 4.4, the fine blocking $\{\Delta_{(k,j)} : (k,j) \in \Lambda\}$ is an unconditional decomposition of $L^p_X(G_p)$, it follows from Corollary 3.15 that $\{P_{B_n} : n \in \widehat{G}_p\}$ is R-bounded. Now Corollary 3.18 combined with (39) yields that $\{P_n : n \in \widehat{G}_p\}$ is R-bounded as well, and we are done. ∎

Let X be a UMD-space, let $\mathbb{D}$ be the dyadic group and $\{\psi_k\}_{k=0}^\infty$ the Paley–Walsh system enumerated as in Remark 4.1(a) (i.e., the Paley–Walsh enumeration). Let D and Δ denote the Schauder decomposition and the dyadic blocking in $L^2_X(\mathbb{D})$ corresponding to the Paley–Walsh system, respectively. So $D_k f = \psi_k \otimes c_k(f)$, with $c_k(\cdot)$ given by (25) and $\Delta_k = \sum_{l=2^{k-1}}^{2^k-1} D_l$. As before let we let $P_k = \sum_{l=0}^{k} D_l$. It will be convenient to have the following terminology available.

DEFINITION 5.2. We say that (MPW) *holds* for X if statement (i) of the multiplier Theorem 3.5 is true in $L^2_X(\mathbb{D})$ for the Paley–Walsh system with respect to the dyadic blocking.

Note that if X is not UMD, then the dyadic blocking is not unconditional, so the multiplier theorem cannot hold. By Theorem 5.1 every UMD-space with property (α) has (MPW). Finally, we shall characterize those UMD-spaces X for which (MPW) holds.

We will begin with the following lemma.

LEMMA 5.3. *Let X be a UMD-space for which* (MPW) *holds. Define*

$$Q_m f = \sum_{l=2^{k-1}}^{m} \psi_l \otimes c_l(f), \quad f \in L^2_X(\mathbb{D}),$$

for $2^{k-1} \le m < 2^k$ and $k \in \mathbb{N}$. Then, for any sequence $\varepsilon_k = \pm 1$, for any collection $\{m_k\}$ in $\mathbb{N}$ satisfying $2^{k-1} \le m < 2^k$ $(k = 1, 2, \ldots)$ and for all $M \in \mathbb{N}$ we have

$$\Big\| \sum_{k=1}^{M} \varepsilon_k Q_{m_k} \Big\| \le C_2(X)K.$$

Proof. Let the sequences $\{\varepsilon_k\}$ and $\{m_k\}$ and $M \in \mathbb{N}$ be given. Define a sequence $\{\lambda_n\}$ by

$$\lambda_n = \begin{cases} \varepsilon_k & \text{if } 2^{k-1} \le n \le m_k \text{ and } k \le M, \\ 0 & \text{otherwise.} \end{cases}$$

This sequence satisfies the conditions of Theorem 3.5, with constant $K = 1$, and so (MPW) implies that

$$T_\lambda f = \sum_{n=1}^{\infty} \lambda_n c_n(f) \psi_n$$

is a bounded operator on $L^2_X(\mathbb{D})$. Since $T_\lambda = \sum_{k=1}^{M} \varepsilon_k Q_{m_k}$, this gives the result. ∎

For $m \in \mathbb{N}$, define $F_m = \{l_0, l_1, \ldots, l_k\}$ where $l_0 < l_1 < \ldots < l_k$ and $m = 2^{l_0} + 2^{l_1} + \ldots + 2^{l_k}$. Also define $\Delta_{F_m} = \sum_{l \in F_m} \Delta_l$.

LEMMA 5.4. *Let X be a UMD-space with* (MPW). *For any choice of $\{m_k\}$ as in Lemma 5.3, the collection $\{\Delta_{F_{m_k}}\}_{k=1}^\infty$ is R-bounded with a uniform R-bound.*

Proof. The Paley identity (39) states that $P_m = \psi_m \Delta_{F_m} \psi_m$. So, by Lemma 3.3 and Corollary 3.18, it is enough to show that

$$\Big\|\sum_{k=1}^{M} \varepsilon_k P_{m_k} f_k\Big\|_{L^2_E(\Omega)} \le C \Big\|\sum_{k=1}^{M} \varepsilon_k f_k\Big\|_{L^2_E(\Omega)}$$

for $f_1,\dots,f_M \in E = L^2_X(\mathbb{D})$ and all $M\in\mathbb{N}$, for some $C>0$.

Since $P_{m_k} = \mathbb{E}^X_{k-1} + Q_{m_k}$, it follows from Proposition 3.8 that it suffices to show that

$$\Big\|\sum_{k=1}^{M} \varepsilon_k Q_{m_k} f_k\Big\|_{L^2_E(\Omega)} \le C' \Big\|\sum_{k=1}^{M} \varepsilon_k f_k\Big\|_{L^2_E(\Omega)}$$

for some $C'>0$.

Since $Q_{m_k} = Q_{m_k}\Delta_k = \Delta_k Q_{m_k}$, we can write

$$\sum_{k=1}^{M} \varepsilon_k(\omega) Q_{m_k} f_k = \Big(\sum_{k=1}^{M} \varepsilon_k(\omega) Q_{m_k}\Big) f \quad \text{for all } \omega\in\Omega,$$

with $f=\sum_{k=1}^{M}\Delta_k f_k$. For every ω fixed it follows from Lemma 5.3 that

$$\Big\|\sum_{k=1}^{M} \varepsilon_k(\omega) Q_{m_k} f_k\Big\|_{L^2_X(\mathbb{D})} \le C_2(X)K\|f\|_{L^2_X(\mathbb{D})}.$$

Integration over Ω yields

$$\begin{aligned}\Big\|\sum_{k=1}^{M} \varepsilon_k Q_{m_k} f_k\Big\|_{L^2_E(\Omega)} &\le C_2(X)K\|f\|_{L^2_X(\mathbb{D})} = C_2(X)K\Big\|\sum_{k=1}^{M}\Delta_k f\Big\|_{L^2_X(\mathbb{D})}\\ &\overset{(1)}{\le} C''\Big\|\sum_{k=1}^{M}\varepsilon_k\Delta_k f_k\Big\|_{L^2_E(\Omega)}\\ &\overset{(2)}{\le} C'\Big\|\sum_{k=1}^{M}\varepsilon_k f_k\Big\|_{L^2_E(\Omega)},\end{aligned}$$

where (1) follows from the unconditionality of the decomposition $\{\Delta_k\}$ and (2) follows from Proposition 3.8, as $\Delta_k = \mathbb{E}^X_k - \mathbb{E}^X_{k-1}$. ∎

Note that $\max\{F_{m_k}\} = k-1$ for all $k\in\mathbb{N}$.

LEMMA 5.5. *Let X be a UMD-space for which* (MPW) *holds. Let $\{\mathfrak{D}_k\}_{k=0}^\infty$ be an unconditional decomposition of X. For $F\subset\mathbb{N}$ finite define $\mathfrak{D}_F=\sum_{j\in F}\mathfrak{D}_j$. Then $\{\mathfrak{D}_F : F\subset\mathbb{N}$, finite$\}$ is R-bounded.*

Proof. Define $\Psi : X\to L^2_X(\mathbb{D})$ by

$$\Psi(x)=\sum_{k=0}^{\infty} r_k\otimes\mathfrak{D}_k(x),$$

where the r_k denotes the kth Rademacher function on $\mathbb{D}$, i.e., $r_0\equiv 0$ and $r_k=\psi_{2^{k-1}}$ for $k\ge 1$. This defines an isomorphism (cf. the proof of Theorem 3.9), and moreover we have

$$\Delta_F(\Psi(x)) = \Psi(\mathfrak{D}_F(x))$$

for any finite $F\subset\mathbb{N}$, where $\Delta_F=\sum_{k\in F}\Delta_k$. So it is enough to show that $\{\Delta_F : F\subset\mathbb{N}, \text{ finite}\}$ is R-bounded.

Let $\{F_k\}_{k=1}^M$ be an arbitrary collection of subsets of $\mathbb{N}$. Now define a new collection $\{G_k\}_{k=1}^M$ as follows: $G_1=F_1$ and $G_k=F_k\cup\alpha(k)$, with $\alpha(k)\in\mathbb{N}$ given by

$$\alpha(k) = 1+\max\{\alpha(k-1),\max(F_k)\},\qquad k\in\mathbb{N}$$

(with $\alpha(0)=0$). With this choice the sequence $\{\max(G_k)\}_{k=1}^M$ is strictly increasing. Now observe that the sequence $\{n_k\}_{k=1}^M$ defined by $n_k=\sum_{j\in G_k}2^j$ is a subsequence of some $\{m_k\}$ as considered in Lemmas 5.3 and 5.4. Hence by the previous lemma, there exists a constant $C>0$ independent of the choice of the F_k's and such that

$$\Big\|\sum_{k=1}^{M}\varepsilon_k\Delta_{G_k}f_k\Big\|_{L^2_X(\Omega)} \le C\Big\|\sum_{k=1}^{M}\varepsilon_k f_k\Big\|_{L^2_X(\Omega)}$$

for all $f_1,\dots,f_M\in L^2_X(\Omega)$. Since $\Delta_{\alpha(k)}=\mathbb{E}^X_{\alpha(k)}-\mathbb{E}^X_{\alpha(k-1)}$, it follows from Proposition 3.8 that the collection $\{\Delta_{\alpha(k)}\}_{k=0}^\infty$ is R-bounded. Since

$$\Delta_{F_k}=\Delta_{G_k}-\Delta_{\alpha(k)},$$

it now follows easily that $\{\Delta_F : F\subset\mathbb{N}, \text{ finite}\}$ is R-bounded. ∎

Now we are in a position to prove the final result of this paper.

THEOREM 5.6. *For a UMD-space X the following statements are equivalent:*

(i) (MPW) *holds for X.*

(ii) *The partial sum projections $P_n=\sum_{k\le n}D_n$ $(n\in\mathbb{N})$ of the Paley–Walsh system are R-bounded in $L^2_X(\mathbb{D})$.*

(iii) *For any unconditional decomposition $\mathfrak{D}=\{\mathfrak{D}_k\}_{k=0}^\infty$ in $L^2_X(\mathbb{D})$, the collection $\{\sum_{k\in F}\mathfrak{D}_k : F\subset\mathbb{N}$, finite$\}$ is R-bounded.*

Moreover, if (MPW) *holds for X, then for any unconditional decomposition $\mathfrak{D}=\{\mathfrak{D}_k\}_{k=0}^\infty$ in X, the collection $\{\sum_{k\in F}\mathfrak{D}_k : F\subset\mathbb{N}$, finite$\}$ is R-bounded.*

Proof. It is well known that if X is a UMD-space, then so is $L^2_X(\mathbb{D})$. Similarly, from Fubini's theorem, it follows that $L^2_X(\mathbb{D})$ satisfies (MPW) whenever X has this property. Now Lemma 5.5 gives the implication (i)⇒(iii). An application of the Paley identity (39) shows (iii)⇒(ii). The unconditionality of the blocking Δ together with Theorem 3.5(ii) gives (ii)⇒(i). Finally, Lemma 5.5 yields the last statement of the theorem. ∎

The row decomposition in Example 3.10 is an unconditional decomposition of $\mathcal{C}_p$, whereas the corresponding collection $\mathcal{R}$ is not R-bounded. Hence the last part of Theorem 5.6 combined with considerations presented in Example 3.10 implies the following corollary.

COROLLARY 5.7. *The multiplier theorem with respect to the Paley–Walsh system fails in any* $L^2_{\mathcal{C}_p}(\mathbb{D})$, $1 < p < \infty$ *and* $p \neq 2$.

References

[BG94] E. Berkson and T. A. Gillespie, *Spectral decompositions and harmonic analysis on UMD spaces*, Studia Math. 112 (1994), 13–49.

[Bou83] J. Bourgain, *Some remarks on Banach spaces in which martingale differences are unconditional*, Ark. Mat. 21 (1983), 163–168.

[Bou85] —, *Vector-valued singular integrals and the* H^1*-BMO duality*, in: Probability Theory and Harmonic Analysis, Dekker, New York, 1985, 1–19.

[Bur83] D. Burkholder, *A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions*, in: Proc. Conf. on Harmonic Analysis in Honor of Antoni Zygmund (Chicago, 1981), Wadsworth, Belmont, 1983, 270–286.

[DJT95] J. Diestel, H. Jarchow and A. Tonge, *Absolutely Summing Operators*, Cambridge Univ. Press, 1995.

[DU77] J. Diestel and J. J. Uhl, *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., Providence, RI, 1977.

[DS97] P. Dodds and F. Sukochev, *Non-commutative bounded Vilenkin systems*, preprint, 1997.

[EG77] R. E. Edwards and G. I. Gaudry, *Littlewood-Paley and Multiplier Theory*, Ergeb. Math. Grenzgeb. 90, Springer, Berlin, 1977.

[GK70] I. C. Gohberg and M. G. Kreĭn, *Theory and Applications of Volterra Operators in Hilbert Space*, Transl. Math. Monogr. 24, Amer. Math. Soc., Providence, RI, 1970.

[KP79] N. J. Kalton and N. T. Peck, *Twisted sums of sequence spaces and the three space problem*, Trans. Amer. Math. Soc. 255 (1979), 1–30.

[KP70] S. Kwapień and A. Pełczyński, *The main triangle projection in matrix spaces and its applications*, Studia Math. 34 (1970), 43–68.

[LT77] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces. I*, Ergeb. Math. Grenzgeb. 92, Springer, Berlin, 1977.

[Mar39] J. Marcinkiewicz, *Sur les multiplicateurs des séries de Fourier*, Studia Math. 8 (1939), 78–91.

[Mau75] B. Maurey, *Système de Haar*, in: Séminaire Maurey–Schwartz 1974–1975: Espaces L_p, applications radonifiantes et géométrie des espaces de Banach, Exp. Nos. I et II, Centre Math., École Polytech., Paris, 1975, p. 26.

[Pal32] R. E. A. C. Paley, *A remarkable series of orthogonal functions*, Proc. London Math. Soc. 34 (1932), 241–279.

[Pis78] G. Pisier, *Some results on Banach spaces without local unconditional structure*, Composito Math. 37 (1978), 3–19.

[SWS90] F. Schipp, W. R. Wade and P. Simon, *Walsh Series*, Adam Hilger, Bristol, 1990.

[Ste70] E. M. Stein, *Topics in Harmonic Analysis Related to the Littlewood–Paley Theory*, Ann. of Math. Stud. 63, Princeton Univ. Press, Princeton, NJ, 1970.

[SF94] F. A. Sukochev and S. V. Ferleger, *Harmonic analysis in symmetric spaces of measurable operators*, Dokl. Akad. Nauk 339 (1994), 307–310 (in Russian); English transl.: Russian. Acad. Sci. Dokl. Math. 50 (1995), 432–437.

[SF95] —, —, *Harmonic analysis in (UMD)-spaces: Applications to the theory of bases*, Mat. Zametki 58 (1995), 890–905 (in Russian); English transl.: Math. Notes 58 (1995), 1315–1326.

[Sun51] G. I. Sunouchi, *On the Walsh-Kaczmarz series*, Proc. Amer. Math. Soc. 2 (1951), 5–11.

[Wat58] C. Watari, *On generalized Walsh Fourier series*, Tôhoku Math. J. (2) 10 (1958), 211–241.

[Wen93] J. Wenzel, *Mean convergence of vector-valued Walsh series*, Math. Nachr. 162 (1993), 117–124.

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