STUDIA MATHEMATICA 138 (2) (2000)

Absolutely continuous dynamics and real coboundary cocycles in L^p-spaces, $0 < p < \infty$

by

ANA I. ALONSO (Valladolid), JIALIN HONG (Beijing) and RAFAEL OBAYA (Valladolid)

Abstract. Conditions for the existence of measurable and integrable solutions of the cohomology equation on a measure space are deduced. They follow from the study of the ergodic structure corresponding to some families of bidimensional linear difference equations. Results valid for the non-measure-preserving case are also obtained.

1. Introduction. Let $(\Omega, \mathcal{A}, m_0)$ be a measure space. For each $0 < p < \infty$, we set

$$L^p(\Omega, m_0) = \Big\{ f : \Omega \to \mathbb{R} \;\Big|\; \int_\Omega |f(\xi)|^p \, dm_0 < \infty \Big\}.$$

If $0 < p < 1$, the map $f \mapsto \int_\Omega |f(\xi)|^p \, dm_0$ defines a quasi-norm in $L^p(\Omega, m_0)$. If $1 \le p < \infty$, the map $f \mapsto \|f\|_p = (\int_\Omega |f(\xi)|^p \, dm_0)^{1/p}$ defines a norm which makes $L^p(\Omega, m_0)$ a Banach space. We denote by $L^\infty(\Omega, m_0)$ the set of essentially bounded functions on Ω; by the essential supremum of $f \in L^\infty(\Omega, m_0)$, in symbols $\|f\|_\infty$, we mean the greatest lower bound of all the essential upper bounds of f. It is well known that $(L^\infty(\Omega, m_0), \| \ \|_\infty)$ is also a Banach space.

Throughout this paper $T : (\Omega, \mathcal{A}, m_0) \to (\Omega, \mathcal{A}, m_0)$ is an ergodic automorphism. Each measurable function $f : \Omega \to \mathbb{R}$ will be called a *cocycle*. We define $S_n f = \sum_{j=0}^{n-1} f \circ T^j$ for $n \ge 1$, $S_0 f = f$ and $S_n f = \sum_{j=n}^{-1} f \circ T^j$ for $n \le -1$, which provides the cocycle identity $S_{n+m} f = S_n f \circ T^m + S_m f$. We say that a cocycle f is a *coboundary cocycle* if there exists a measurable solution $h : \Omega \to \mathbb{R}$ of the cohomology equation $f = h \circ T - h$. The function h is called a *transfer function*. It is easy to check that two transfer functions

2000 *Mathematics Subject Classification*: Primary 28D05.

Research of the first two authors partially supported by Junta de Castilla y Leon, under project VA01/97.

Research of the third author supported by the Spanish Ministry of Education and Sciences.

h_1 and h_2 of the same cocycle f differ by a constant, i.e., $h_1 = h_2 + c$ almost everywhere on Ω. For $0 < p \leq \infty$, we say that f is an *L^p-coboundary* if it is a coboundary and the transfer function h belongs to $L^p(\Omega, m_0)$.

We now consider the case where Ω is a compact metric space and $T : \Omega \to \Omega$ is a minimal homeomorphism. Gottschalk and Hedlund [7] prove that a continuous cocycle f is a coboundary with transfer function $h \in C(\Omega)$ if and only if $\sup_{n\in\mathbb{N}} \|S_n f\|_\infty < \infty$. The study of L^p-solutions, for $1 \leq p \leq \infty$, can be formulated as a particular case of the following more general problem: solve the functional equation $(\mathrm{Id} - T)x = y$ for a linear contraction T on a Banach space X. The usual method to deal with this problem is to study the convergence of the averages $x_n = (1/n)\sum_{k=1}^{n}\sum_{j=0}^{k-1} T^j y$. With this point of view, Browder [5] proves that if X is a reflexive Banach space then $y \in \mathrm{Rg}(\mathrm{Id} - T)$ if and only if $\sup_{n\geq 1} \|\sum_{j=1}^{n} T^j y\| < \infty$. This result was extended by Lin and Sine in [10] for the adjoint operator $T^* : X^* \to X^*$ of the linear contraction T on X, which permits us to apply the above arguments in L^1-spaces (see also Assani [2]). The paper of Krengel and Lin [8] contains a continuous version of the previous results.

In the present paper we obtain necessary and sufficient conditions for the existence of L^p-solutions of the cohomology equation given by a cocycle f on Ω, with $0 < p < \infty$. Such conditions are based on the boundedness of the ergodic averages of the sequence $\{S_n f\}_{n\in\mathbb{N}}$. To this end we introduce a family of bidimensional linear difference equations which define a projective flow whose equations are defined precisely by the cohomology equation. In this situation the existence of a measurable transfer function is equivalent to the existence of a density function for an invariant measure on the projective bundle which is absolutely continuous with respect to the product measure. Moreover, the orders of integrability of both functions are related. These facts enable us to apply the results of Novo and Obaya [12, 13].

We show that if $\Theta \subset \Omega$ is a measurable subset with $m_0(\Theta) > 0$ and $\{n_k\}_{k\in\mathbb{N}}$ is an increasing sequence of positive integers with

$$\sup_{k\in\mathbb{N}} (1/n_k) \sum_{j=1}^{n_k} \int_{\Theta} |S_j f|^p \, dm_0 < \infty,$$

then f is an L^p-coboundary, which in particular extends the previously known results to the range $0 < p < 1$. We also obtain the corresponding version of this same statement for the non-measure-preserving case, assuming that m_0 is not invariant but equivalent to an invariant measure (and hence the map $L^p(\Omega, m_0) \to L^p(\Omega, m_0)$, $f \mapsto f \circ T$, is not necessarily a contraction even if it is continuous). From Assani and Woś [3] and Sato [14], we deduce that a cocycle is a coboundary if and only if the corresponding projective flow satisfies the pointwise ergodic theorem with respect to the product measure. The pointwise ergodic theorem is also studied by Martín-Reyes and de la Torre [11].

Notation and preliminaries are stated in Section 2; we formulate and prove the results in Section 3.

2. Preliminaries. In what follows we consider a one-dimensional cocycle $f : \Omega \to \mathbb{R}$. We introduce the family of linear difference equations

$$\mathbf{x}(n+1) = \begin{pmatrix} 1 & 0 \\ f(\xi \cdot n) & 1 \end{pmatrix} \mathbf{x}(n) = M(\xi \cdot n)\mathbf{x}(n), \qquad \xi \in \Omega, \tag{2.1}$$

where $\xi \cdot n = T^n(\xi)$ for $\xi \in \Omega$, $n \in \mathbb{Z}$ and $\mathbf{x} = (u, v)^T$.

Let $P^1(\mathbb{R})$ be the space of real lines through the origin in $\mathbb{R}^2$, identified with $\mathbb{R}/(\pi\mathbb{Z})$. Assume that $\{\mathbf{x}(n)\}_{n\in\mathbb{Z}}$ satisfies (2.1) and take $\varphi(n) = \cot^{-1}(v(n)/u(n))$. This provides the relation

$$\varphi(n+1) = \cot^{-1}(f(\xi \cdot n) + \cot \varphi(n)). \tag{2.2}$$

We denote by $\varphi(n, \xi, \varphi_0)$ the solution of (2.2) with initial data $\varphi(0, \xi, \varphi_0) = \varphi_0$. Then $F(\xi, \varphi) = (\xi \cdot 1, \varphi(1, \xi, \varphi))$ and in general $F^n(\xi, \varphi) = (\xi \cdot n, \varphi(n, \xi, \varphi))$ defines the equation of the discrete skew-product flow induced by (2.1) on $K_\mathbb{R}$. We also denote by $\mathbf{x}(n, \xi, \varphi)$ the solution of (2.1) with $\mathbf{x}(0, \xi, \varphi) = (\sin \varphi, \cos \varphi)$. The symbol l will represent the Lebesgue measure on $P^1(\mathbb{R})$ and $m_1 = m_0 \otimes l$ the completed product measure on $K_\mathbb{R}$.

For each $\varrho \in \mathbb{R}$ we consider the change of variables defined on $P^1(\mathbb{R})$ by the relation $\cot \psi = \cot \varphi + \varrho$. Since

$$\frac{\partial \psi}{\partial \varphi}(\varphi) = \frac{1 + \cot^2 \varphi}{1 + (\cot \varphi + \varrho)^2} = \frac{1}{\sin^2 \varphi + (\cos \varphi + \varrho \sin \varphi)^2},$$

we have

$$\int_{P^1(\mathbb{R})} \frac{1}{\sin^2 \varphi + (\cos \varphi + \varrho \sin \varphi)^2} \, d\varphi = 1. \tag{2.3}$$

It makes sense to introduce the function $G(\xi, \varphi) = \sin^2 \varphi + (\cos \varphi + f(\xi) \sin \varphi)^2$ and, in general, $G_j(\xi, \varphi) = \sin^2 \varphi + (\cos \varphi + S_j f(\xi) \sin \varphi)^2$ for $j \geq 1$; this allows us to characterize the F-invariant measures absolutely continuous with respect to m_1. Observe that

$$G_j^{-1}(\xi, \varphi) \leq 2(1 + (S_j f(\xi))^2) \tag{2.4}$$

for every $(\xi, \varphi) \in K_\mathbb{R}$. From relation (2.4) we deduce that

$$\begin{aligned} \int_{K_\mathbb{R}} (G_j^{-1}(\xi, \varphi))^{1+p/2} \, dm_1 &\leq \int_{\Omega} \int_{P^1(\mathbb{R})} (2(1 + (S_j f(\xi))^2))^{p/2} G_j^{-1}(\xi, \varphi) \, d\varphi \, dm_0 \\ &= \int_{\Omega} (2(1 + (S_j f(\xi)))^2)^{p/2} \, dm_0 \end{aligned}$$

for each $p > 0$. Thus, if $f \in L^p(\Omega, m_0)$ then $G_j \in L^{p/2}(K_{\mathbb{R}}, m_1)$ and $G_j^{-1} \in L^{1+p/2}(K_{\mathbb{R}}, m_1)$.

The results collected in this section are due to Novo and Obaya [12, 13].

PROPOSITION 2.1. *Let $P \in L^1(K_{\mathbb{R}}, m_1)$ be a positive function. The following statements are equivalent*:

(1) *the measure $d\mu = P dm_1$ is F-invariant*;

(2) *P satisfies the functional equation $P \circ F = P \cdot G$ almost everywhere.*

Let μ be an absolutely continuous F-invariant measure with $d\mu = P dm_1$. We define $D = \{(\xi, \varphi) \in K_{\mathbb{R}} \mid P(\xi, \varphi) > 0\}$ and $Q = (1/P)\chi_D$. Then Q is a non-negative solution of the functional equation $Q \circ F = Q \cdot G^{-1}$.

We introduce the family $\{\mu_n\}_{n\in\mathbb{N}}$ of measures on $K_{\mathbb{R}}$ defined by $\mu_n(A) = (1/n)\sum_{j=1}^n m_1(F^j(A))$ for every measurable subset $A \subset K_{\mathbb{R}}$. Set

$$P_n(\xi, \varphi) = \frac{1}{n}\sum_{j=1}^n \frac{1}{|\mathbf{x}(j, \xi, \varphi)|^2} = \frac{1}{n}\sum_{j=1}^n G_j^{-1}(\xi, \varphi) = \frac{1}{n}\sum_{j=1}^n \frac{1}{\sin^2\varphi + (\cos\varphi + S_j f(\xi)\sin\varphi)^2}$$

for each $n \in \mathbb{N}$. Then

$$\mu_n(A) = \frac{1}{n}\sum_{j=1}^n m_1(F^j(A)) = \int_A P_n(\xi, \varphi)\, dm_1$$

for every measurable subset $A \subset K_{\mathbb{R}}$; that is, the measure μ_n is absolutely continuous with respect to m_1 with density function P_n.

Let $0 < p < \infty$ be a fixed real constant. We define

$$Q_n(\xi, \varphi) = \frac{1}{n}\sum_{j=1}^n |\mathbf{x}(j, \xi, \varphi)|^{2p} = \frac{1}{n}\sum_{j=1}^n G_j(\xi, \varphi)^p = \frac{1}{n}\sum_{j=1}^n (\sin^2\varphi + (\cos\varphi + S_j f(\xi)\sin\varphi)^2)^p$$

for each $n \in \mathbb{N}$. The existence of absolutely continuous F-invariant measures and the integrability exponents of their density functions can be characterized in terms of the behavior of the families $\{P_n\}_{n\in\mathbb{N}}$ and $\{Q_n\}_{n\in\mathbb{N}}$ respectively, as the following theorems summarize.

THEOREM 2.2. *The following facts are equivalent*:

(i) *$(K_{\mathbb{R}}, F)$ admits an invariant measure absolutely continuous with respect to m_1*;

(ii) *there is an F-invariant subset $D \subset K_{\mathbb{R}}$ with $m_1(D) = 1$ such that the limit $P(\xi, \varphi) = \lim_{n\to\infty} P_n(\xi, \varphi)$ exists and is positive for every $(\xi, \varphi) \in D$.*

Moreover, under these conditions the measure $d\mu = P dm_1$ is F-invariant and normalized.

THEOREM 2.3. *The following facts are equivalent*:

(i) *$(K_{\mathbb{R}}, F)$ admits an absolutely continuous F-invariant measure $d\mu = P dm_1$ with $P \in L^{1+p}(K_{\mathbb{R}}, m_1)$*;

(ii) *there is an F-invariant subset $D \subset K_{\mathbb{R}}$ with $m_1(D) = 1$ such that the limit $Q(\xi, \varphi) = \lim_{n\to\infty} Q_n(\xi, \varphi)$ exists and is positive for every $(\xi, \varphi) \in D$.*

Moreover, under these conditions, $(1/Q)^{1/p} \in L^{1+p}(K_{\mathbb{R}}, m_1)$ and if we take $\lambda = (\int_{K_{\mathbb{R}}} (1/Q)^{1/p}\, dm_1)^{-1}$, then the measure $d\mu = \lambda(1/Q)^{1/p} dm_1$ is normalized and F-invariant.

We say that a function Q is *fiber-quadratic* if there exist measurable functions $a, b, c : \Omega \to \mathbb{R}$ and a T-invariant subset $\Omega_0 \subset \Omega$ with $m_0(\Omega_0) = 1$ such that

$$Q(\xi, \varphi) = a(\xi)\cos^2\varphi + b(\xi)\sin^2\varphi + 2c(\xi)\sin\varphi\cos\varphi$$

for every $(\xi, \varphi) \in \Omega_0 \times P^1(\mathbb{R})$. We take $\mathbf{X}(\xi) = (a(\xi), b(\xi), c(\xi))^T$; it is easy to check that $Q(F(\xi, \varphi)) = Q(\xi, \varphi)G^{-1}(\xi, \varphi)$ in $K_{\mathbb{R}}$ if and only if

$$\mathbf{X}(T(\xi)) = \begin{pmatrix} 1 & 0 & 0 \\ f^2(\xi) & 1 & -2f(\xi) \\ -f(\xi) & 0 & 1 \end{pmatrix} \mathbf{X}(\xi). \tag{2.5}$$

An F-invariant measure μ with $d\mu = (1/Q)dm_1$ is said to be a *linear F-invariant measure* if Q is a fiber-quadratic solution of $Q \circ F = Q \cdot G^{-1}$. Linear invariant measures are directly associated with the ergodic structure of the projective flow.

We finally recall that if F admits an absolutely continuous invariant measure $d\nu = P dm_1$ with $P \in L^{1+p}(K_{\mathbb{R}}, m_1)$ then it admits a linear invariant measure with the same order of integrability. A qualitative study of the projective flow in the case of singular dynamics can be found in Alonso and Obaya [1].

3. Real coboundary cocycles.

We point out a direct relation between the solutions of the cohomology equation and the coefficients of the density function of a linear invariant measure.

PROPOSITION 3.1. *Let $0 < p < \infty$. The following statements are equivalent*:

(i) *the cohomology equation $f = h \circ T - h$ has a solution $h \in L^p(\Omega, m_0)$*;

(ii) *F admits an absolutely continuous invariant measure $d\mu = P dm_1$ with density function $P \in L^{1+p/2}(K_{\mathbb{R}}, m_1)$.*

Proof. We first show that (i) implies (ii). If h is a measurable solution of the cohomology equation, i.e. $h\circ T - h = f$, then the function

$$Q(\xi,\varphi) = \cos^2\varphi + (1+h^2(\xi))\sin^2\varphi - 2h(\xi)\sin\varphi\cos\varphi = 1 + h^2(\xi)\sin^2\varphi - 2h(\xi)\sin\varphi\cos\varphi$$

satisfies $Q\circ F = Q\cdot G^{-1}$. Then $d\nu = P\,dm_1$, with $P = 1/Q$, is a linear F-invariant measure.

Let us introduce the matrix $S(\xi) = \begin{pmatrix} 1+h^2(\xi) & -h(\xi) \\ -h(\xi) & 1\end{pmatrix}$. Its eigenvalues $0 \le \gamma_1(\xi) \le 1 \le \gamma_2(\xi)$ satisfy $\gamma_1(\xi)\cdot\gamma_2(\xi) = 1$ and $\gamma_1(\xi)+\gamma_2(\xi) = 2 + h^2(\xi)$, which yields

$$\gamma_1(\xi) \le Q(\xi,\varphi) \le \gamma_2(\xi) \le 2 + h^2(\xi),$$
$$\gamma_1(\xi) \le P(\xi,\varphi) \le \gamma_2(\xi) \le 2 + h^2(\xi).$$

Therefore, we obtain a constant $C_1 > 1$ such that

$$\int_{K_{\mathbb{R}}} P^{1+p/2}(\xi,\varphi)\,dm_1 \le \int_{K_{\mathbb{R}}} (2+h^2(\xi))^{p/2} P(\xi,\varphi)\,d\varphi\,dm_0 \le C_1\int_{\Omega}(1+h^p(\xi))\,dm_0 < \infty,$$

which shows that $P \in L^{1+p/2}(K_{\mathbb{R}})$.

Now we consider the converse implication. Assume that F admits a linear invariant measure $d\mu = P\,dm_1$ with $P \in L^{1+p/2}(K_{\mathbb{R}}, m_1)$. Set $Q(\xi,\varphi) = 1/P(\xi,\varphi) = a(\xi)\cos^2\varphi + b(\xi)\sin^2\varphi + 2c(\xi)\sin\varphi\cos\varphi$. Note that $a(T(\xi)) = a(\xi)$ and hence, by ergodicity, we have $a(\xi) = a$ almost everywhere with respect to m_0. The function $h(\xi) = -c(\xi)/a$ is a measurable solution of the cohomology equation.

It follows from Theorem 2.3 that there exists an F-invariant subset $D \subset K_{\mathbb{R}}$ with $m_1(D) > 0$ where the limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}(u^2(j,\xi,\varphi)+v^2(j,\xi,\varphi))^{p/2}$$

is a positive number for every $(\xi,\varphi)\in D$.

For almost every $\xi\in\Omega$ we can find an element $\varphi = \varphi(\xi) \in P^1(\mathbb{R})$ such that $(\xi,\varphi(\xi)), (\xi,\varphi(\xi)+\pi/2) \in D$ and introduce the fundamental matrix $V(n,\xi) = (\mathbf{x}(n,\xi,\varphi(\xi)), \mathbf{x}(n,\xi,\varphi(\xi)+\pi/2))$. We define $u(n,\xi) = 1$ and $v(n,\xi) = h(T^n(\xi))$ for every $n\in\mathbb{Z}$ and $\xi\in\Omega$. Then $\mathbf{x}(n,\xi) = (u(n,\xi), v(n,\xi))^T$ satisfies (2.1) and

$$\mathbf{x}(n,\xi) = V(n,\xi)V^T(0,\xi)(1,h(\xi))^T.$$

We denote by $\|\,\|_S$ the Schur norm, defined by $\|[v_{i,j}]\|_S = (\sum_{i,j=1}^n |v_{i,j}^2|)^{1/2}$.

A straightforward calculation shows that

$$|\mathbf{x}(n,\xi)|^2 \le \|V(n,\xi)\|_S^2\|V^T(0,\xi)(1,h(\xi))^T\|_2^2 \le (1+h^2(\xi))(|\mathbf{x}(n,\xi,\varphi(\xi))|^2 + |\mathbf{x}(n,\xi,\varphi(\xi)+\pi/2)|^2)$$

and, consequently,

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n}|h(T^j(\xi))|^p \le \limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n}|\mathbf{x}(j,\xi)|^p < \infty.$$

Birkhoff's ergodic theorem allows us to conclude that $h\in L^p(\Omega, m_0)$, which completes the proof of the theorem. ∎

We now state the main result of this paper concerning coboundary cocycles.

THEOREM 3.2. *Let $0<p<\infty$ and $f\in L^p(\Omega,m_0)$. Assume that there exists a measurable subset $\Theta\subset\Omega$ with $m_0(\Theta)>0$ and a sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers with*

$$\tau_0 = \lim_{k\to\infty}\frac{1}{n_k}\sum_{j=1}^{n_k}\int_{\Theta}|S_jf|^p\,dm_0 < \infty.$$

Then f is an L^p-coboundary.

We deduce some preliminary results from the hypotheses of Theorem 3.2 which will be used later in its proof. Set $q = \min(1/2, p/2)$. It is easy to verify that

$$\int_{\Omega}|S_nf(\xi)|^q\,dm_0 \le \Big(\int_{\Omega}|S_nf(\xi)|^{2q}\,dm_0\Big)^{1/2} \le \Big(\int_{\Omega}\sum_{j=1}^{n}|f(T^j(\xi))|^{2q}\,dm_0\Big)^{1/2} \le n^{1/2}\Big(\int_{\Omega}|f(\xi)|^{2q}\,dm_0\Big)^{1/2}.$$

Therefore, $\lim_{n\to\infty}(1/n)\int_\Omega|S_nf(\xi)|^q\,dm_0 = 0$. For each $m\in\mathbb{N}$ we define $\Theta_m = \bigcup_{j=-m}^{m}T^j(\Theta)$. Since $S_jf(T(\xi)) = S_{j+1}f(\xi) - f(\xi)$, we also have

$$\tau_m = \sup\Big\{\frac{1}{n_k}\sum_{j=1}^{n_k}\int_{\Theta_m}|S_jf(\xi)|^q\,dm_0 \;\Big|\; k\in\mathbb{N}\Big\} < \infty.$$

LEMMA 3.3. *There exists a function $P\in L^1(K_{\mathbb{R}},m_1)$ and a sequence $\{n_l\}_{l\in\mathbb{N}}$ of positive integers such that*

(i) *$P\chi_{\Theta_m}\in L^{1+q/2}(K_{\mathbb{R}},m_1)$ for every $m\in\mathbb{N}$,*

(ii) *$\{P_{n_l}\chi_{\Theta_m}\}_{l\in\mathbb{N}}$ converges to $P\chi_{\Theta_m}$ in the $\sigma(L^{1+q/2}(K_{\mathbb{R}},m_1), L^{1+2/q}(K_{\mathbb{R}},m_1))$-topology for every $m\in\mathbb{N}$.*

(iii) *$\{(1/n_l)P_{n_l}\}_{l\in\mathbb{N}}$ converges pointwise to zero almost everywhere.*

Proof. On account of the definition of the family $\{P_n\}_{n\in\mathbb{N}}$, we have

$$P_n^{1+q/2}(\xi,\varphi)=\left(\frac{1}{n}\sum_{j=1}^{n}G_j^{-1}(\xi,\varphi)\right)^{1+q/2}\le\frac{1}{n}\sum_{j=1}^{n}(G_j^{-1}(\xi,\varphi))^{1+q/2}$$
$$\le\frac{1}{n}\sum_{j=1}^{n}2^{q/2}(1+(S_jf(\xi))^2)^{q/2}G_j^{-1}(\xi,\varphi).$$

Consequently, for every $k,m\in\mathbb{N}$ we obtain $C_1>1$ such that

$$\int_{K_{\mathbb{R}}}P_{n_k}^{1+q/2}(\xi,\varphi)\chi_{\Theta_m}(\xi)\,dm_1 \tag{3.1}$$
$$\le\frac{C_1}{n_k}\sum_{j=1}^{n_k}\int_{K_{\mathbb{R}}}(1+|S_jf(\xi)|^q)\chi_{\Theta_m}(\xi)G_j^{-1}(\xi,\varphi)\,d\varphi\,dm_0$$
$$\le\frac{C_1}{n_k}\sum_{j=1}^{n_k}\int_{\Theta_m}(1+|S_jf(\xi)|^q)\,dm_0\le C_1(2+\tau_m).$$

Since $\{P_{n_k}\chi_{\Theta_m}\mid k\in\mathbb{N}\}$ is a bounded subset of $L^{1+q/2}(K_{\mathbb{R}},m_1)$ for every $m\in\mathbb{N}$, following a diagonal Cantor process we can find a sequence $\{n_l\}_{l\in\mathbb{N}}$ of positive integers and functions $P_m^*\in L^{1+q/2}(K_{\mathbb{R}},m_1)$ such that $\{(1/n_l)P_{n_l}\chi_{\Theta_m}\}_{l\in\mathbb{N}}$ converges pointwise to zero and $\{P_{n_l}\chi_{\Theta_m}\}_{l\in\mathbb{N}}$ converges to P_m^* in the $\sigma(L^{1+q/2},L^{1+2/q})$-topology for every $m\in\mathbb{N}$. This shows that $\{(1/n_l)P_{n_l}\}_{l\in\mathbb{N}}$ converges to zero almost everywhere in $K_{\mathbb{R}}$. Moreover, it is obvious that $P_{m+1}^*|_{\Theta_m}=P_m^*$ and the above process defines a measurable function P on $K_{\mathbb{R}}$ with $P|_{\Theta_m}=P_m^*$ for every $m\in\mathbb{N}$. On the other hand, it follows from the above convergence and relation (2.3) that

$$\int_{K_{\mathbb{R}}}P(\xi,\varphi)\chi_{\Theta_m}(\xi)\,dm_1=\lim_{l\to\infty}\int_{K_{\mathbb{R}}}P_{n_l}(\varphi,\xi)\chi_{\Theta_m}(\xi)\,dm_1=m_0(\Theta_m),$$

which shows that $P\in L^1(K_{\mathbb{R}},m_1)$ and completes the proof of the statement. ■

Lemma 3.4. *The sequence $\{P_{n_l+1}\chi_{\Theta_m}\}_{l\in\mathbb{N}}$ of functions converges to $P\chi_{\Theta_m}$ in the $\sigma(L^1(K_{\mathbb{R}},m_1),L^\infty(K_{\mathbb{R}},m_1))$-topology for every $m\in\mathbb{N}$.*

Proof. We can write

$$|P_{n_l+1}(\xi,\varphi)-P_{n_l}(\xi,\varphi)|\le\frac{1}{n_l+1}P_{n_l}(\xi,\varphi)+\frac{2}{n_l+1}G_{n_l+1}^{-1}(\xi,\varphi).$$

Note that $G_j^{-1}(\xi,0)=1$ and $G_j^{-1}(\xi,\varphi)\le1+\cot^2\varphi$ for every $(\xi,\varphi)\in K_{\mathbb{R}}$ with $\sin\varphi\ne0$. Moreover, using Lemma 3.3 we deduce that $\{P_{n_l+1}-P_{n_l}\}_{l\in\mathbb{N}}$ converges pointwise to zero almost everywhere. Take $h\in L^\infty(K_{\mathbb{R}},m_1)$. For each measurable subset $A\subset K_{\mathbb{R}}$ one has

$$\int_A|(P_{n_l+1}(\xi,\varphi)-P_{n_l}(\xi,\varphi))\chi_{\Theta_m}(\xi)h(\xi,\varphi)|\,dm_1$$
$$\le\Big(\int_A|h(\xi,\varphi)|^{1+2/q}\,dm_1\Big)^{q/(q+2)}$$
$$\times\Big(\int_A|P_{n_l+1}(\xi,\varphi)-P_{n_l}(\xi,\varphi)|^{1+q/2}\,dm_1\Big)^{2/(q+2)}$$
$$\le\|h\|_\infty m_1(A)^{q/(q+2)}2C_1(2+\tau_m)$$

according to (3.1). This shows that the sequence $\{(P_{n_l+1}-P_{n_l})\chi_{\Theta_m}h\}_{l\in\mathbb{N}}$ is uniformly integrable. Under these conditions we deduce from Vitali's theorem that

$$\lim_{l\to\infty}\int_{K_{\mathbb{R}}}|(P_{n_l+1}(\xi,\varphi)-P_{n_l}(\xi,\varphi))\chi_{\Theta_m}(\xi)h(\xi,\varphi)|\,dm_1=0,$$

which proves that $\{P_{n_l+1}\chi_{\Theta_m}\}_{l\in\mathbb{N}}$ also converges to $P\chi_{\Theta_m}$ in the $\sigma(L^1,L^\infty)$-topology for every $m\in\mathbb{N}$. ■

Lemma 3.5. *The sequence $\{((P_{n_l}\chi_{\Theta_m})\circ F)\cdot G^{-1}\}_{l\in\mathbb{N}}$ of functions converges to $((P\chi_{\Theta_m})\circ F)G^{-1}$ in the $\sigma(L^1(K_{\mathbb{R}},m_1),L^\infty(K_{\mathbb{R}},m_1))$-topology for every $m\in\mathbb{N}$.*

Proof. If $P\in L^1(K_{\mathbb{R}},m_1)$ then the function $((P\chi_{\Theta_m})\circ F)G^{-1}$ is measurable and

$$\int_{K_{\mathbb{R}}}P(F(\xi,\varphi))\chi_{\Theta_m}(F(\xi,\varphi))G^{-1}(\xi,\varphi)\,dm_1=\int_{K_{\mathbb{R}}}P(\xi,\varphi)\chi_{\Theta_m}(\xi)\,dm_1,$$

which shows that $((P\chi_{\Theta_m})\circ F)\cdot G^{-1}\in L^1(K_{\mathbb{R}},m_1)$.

Take $h\in L^\infty(K_{\mathbb{R}},m_1)$. Then $h\circ F^{-1}\in L^\infty(K_{\mathbb{R}},m_1)$ and

$$\int_{K_{\mathbb{R}}}(P_{n_l}(F(\xi,\varphi))-P(F(\xi,\varphi)))\chi_{\Theta_m}(F(\xi,\varphi))G^{-1}(\xi,\varphi)h(\xi,\varphi)\,dm_1$$
$$=\int_{K_{\mathbb{R}}}(P_{n_l}(\xi,\varphi)-P(\xi,\varphi))\chi_{\Theta_m}(\xi,\varphi)h(F^{-1}(\xi,\varphi))\,dm_1.$$

Consequently, it follows from Lemma 3.3 that $\{((P_{n_l}\chi_{\Theta_m})\circ F)\cdot G^{-1}\}_{l\in\mathbb{N}}$ converges to $((P\chi_{\Theta_m})\circ F)\cdot G^{-1}$ in the $\sigma(L^1,L^\infty)$-topology. ■

Lemma 3.6. *The measure $d\mu=Pdm_1$ is invariant under F.*

Proof. It is easy to verify that $G_j^{-1}(F(\xi,\varphi))=G(\xi,\varphi)G_{j+1}^{-1}(\xi,\varphi)$ for every $j\ge1$. Then

$$P_{n_l}(F(\xi,\varphi))G^{-1}(\xi,\varphi)=\frac{1}{n_l}\sum_{j=0}^{n_l}G_{j+1}^{-1}(\xi,\varphi).$$

On the other hand, a straightforward computation shows that

$$P_{n_l}(F(\xi,\varphi))G^{-1}(\xi,\varphi) = \frac{1}{n_l}\sum_{j=0}^{n_l} G_{j+1}^{-1}(\xi,\varphi) = \frac{n_l+1}{n_l}P_{n_l+1}(\xi,\varphi) - \frac{1}{n_l}P_1(\xi,\varphi),$$

and hence

$$P_{n_l}(F(\xi,\varphi))G^{-1}(\xi,\varphi))\chi_{\Theta_m}(\xi) = \left(\frac{n_l+1}{n_l}P_{n_l+1}(\xi,\varphi) - \frac{1}{n_l}P_1(\xi,\varphi)\right)\chi_{\Theta_m}(\xi)$$

for every $m\in\mathbb{N}$. Moreover

$$P_{n_l}(F(\xi,\varphi))\chi_{\Theta_m}(\xi) = P_{n_l}(F(\xi,\varphi))\chi_{\Theta_{m+1}}(F(\xi,\varphi))\chi_{\Theta_m}(\xi).$$

Taking limits as $l\to\infty$ in the $\sigma(L^1,L^\infty)$-topology and using Lemma 3.5, we deduce the equality $((P\chi_{\Theta_{m+1}})\circ F)\cdot G^{-1}\cdot\chi_{\Theta_m} = P\chi_{\Theta_m}$ for every $m\in\mathbb{N}$; therefore $P\circ F = P\cdot G^{-1}$ almost everywhere on $K_{\mathbb{R}}$. This shows that the measure $d\mu = Pdm_1$ is invariant under F. ■

Proof of Theorem 3.2. We argue by contradiction and assume that there is no linear invariant measure with density function in $L^{1+p/2}$. Consider the functions

$$Q_n(\xi,\varphi) = \frac{1}{n}\sum_{j=1}^{n} G_j(\xi,\varphi)^{p/2} = \frac{1}{n}\sum_{j=1}^{n}(\sin^2\varphi + (\cos\varphi + S_jf(\xi)\sin\varphi)^2)^{p/2}.$$

Lemma 3.5 provides the invariant measure $d\mu = Pdm_1$, absolutely continuous with respect to m_1. Birkhoff's ergodic theorem assures the existence of the pointwise limit of the sequence $\{Q_n\}_{n\in\mathbb{N}}$, and from Theorem 2.3 we deduce that $\lim_{n\to\infty} Q_n(\xi,\varphi) = \infty$ almost everywhere on $K_{\mathbb{R}}$ (with respect to μ). Consequently, $\lim_{n\to\infty}(1/n)\sum_{j=1}^n |S_jf(\xi)|^p = \infty$ for almost every $\xi\in\Omega$, and Fatou's lemma yields $\lim_{n\to\infty}(1/n)\sum_{j=1}^n\int_\Theta |S_jf(\xi)|^p\,dm_0 = \infty$. This contradicts the hypotheses of the theorem and proves the existence of L^p-solutions of the cohomology equation. ■

PROPOSITION 3.7. *Let $0<p<\infty$. The following statements are equivalent*:

(i) *the cohomology equation $f = h\circ T - h$ has a solution $h\in L^p(\Omega,m_0)$*;
(ii) *F satisfies the pointwise ergodic theorem in $L^{1+2/p}(K_{\mathbb{R}},m_1)$.*

Proof. We first assume that the cocycle f is an L^p-coboundary. From Proposition 3.1 we deduce that F admits an absolutely continuous invariant measure with density function in $L^{1+p/2}(K_{\mathbb{R}},m_1)$. Consequently, Theorem 2.2 assures the existence of the pointwise limit $P = \lim_{n\to\infty} P_n$ almost everywhere in $K_{\mathbb{R}}$. More precisely, it follows from Theorem 5.4 of [13] that the sequence $\{P_n\}_{n\in\mathbb{N}}$ is uniformly integrable and converges to P in the norm topology of $L^1(K_{\mathbb{R}},m_1)$. Moreover, Birkhoff's ergodic theorem yields

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} m_1(F^{-j}(A)) = \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} m_1(F^j(A)) = \int_A P(\xi,\varphi)\,dm_1 = \mu(A).$$

We now repeat the arguments used in Lemma 3.3 to conclude that $P\in L^{1+p/2}(K_{\mathbb{R}},m_1)$ and $\{P_n\}_{n\in N}$ converges to P in the $\sigma(L^{1+p/2}(K_{\mathbb{R}},m_1), L^{1+2/p}(K_{\mathbb{R}},m_1))$-topology. Finally, Theorem 4 of Assani [3] shows that F satisfies the pointwise ergodic theorem in $L^{1+2/p}(K_{\mathbb{R}},m_1)$.

The proof of the converse is also immediate. It follows from [3] that if F satisfies the pointwise ergodic theorem in $L^{1+2/p}(K_{\mathbb{R}},m_1)$ then it admits an absolutely continuous invariant measure $d\mu = Pdm_1$ with $P\in L^{1+p/2}(K_{\mathbb{R}},m_1)$. Proposition 3.1 shows that f is an L^p-coboundary. ■

In the general case, the expression of the transfer function h for a real coboundary f is not easy to calculate (see Bradley [4]). The sequences $\{P_n\}_{n\in\mathbb{N}}$ and $\{Q_n\}_{n\in\mathbb{N}}$ provide an alternative way to obtain it: if f is a coboundary, then

$$P(\xi,\varphi) = \int_{K_{\mathbb{R}}}\frac{1}{\sin^2\varphi + (\cos\varphi + (h(\eta)-h(\xi))\sin\varphi)^2}\,dm_0(\eta),$$

and if f is an L^p-coboundary, then

$$Q(\xi,\varphi) = \int_{K_{\mathbb{R}}}(\sin^2\varphi + (\cos\varphi + (h(\eta)-h(\xi))\sin\varphi)^2)^{p/2}\,dm_0(\eta).$$

In particular, when f is an L^1-coboundary and $\int_\Omega h\,dm_0 = 0$, the previous expressions lead to the well known relation $h = \lim_{n\to\infty}(1/n)\sum_{j=1}^n S_jf$ almost everywhere on Ω.

The calculation of h can also be simplified when $\exp(h)\in L^1(\Omega,m_0)$, as shown in the next result.

PROPOSITION 3.8. *Let f be a cocycle on Ω. The following statements are equivalent*:

(i) *the cohomology equation $f = h\circ T - h$ has a solution h with $\exp(h)\in L^1(\Omega,m_0)$*;

(ii) *there exists an invariant subset $\Omega_0\subset\Omega$ with $m_0(\Omega_0)=1$ such that the limit*

$$R^*(\xi) = \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\exp(S_jf(\xi))$$

exists and is positive for every $\xi\in\Omega_0$.

Proof. We first show that (i) implies (ii). Let h be a measurable solution of the cohomology equation $f = h\circ T - h$ and $g = \exp(h)$. We have

$g(T^n(\xi)) = g(\xi)\exp(S_n f(\xi))$ for every $n \in \mathbb{N}$; thus

$$R_n^*(\xi) = \frac{1}{n}\sum_{j=1}^{n}\exp(S_j f(\xi)) = \frac{1}{g(\xi)}\frac{1}{n}\sum_{j=1}^{n} g(T^j(\xi)).$$

Birkhoff's ergodic theorem assures the existence of an invariant subset $\Omega_0 \subset \Omega$ with $m_0(\Omega_0) = 1$ such that $R^*(\xi) = (1/g(\xi))\int_\Omega g(\xi)\, dm_0$ for every $\xi \in \Omega_0$.

Now we consider the converse implication. It is easy to check the relation $R_{n+1}^*(\xi) = \exp(f(\xi))(1/(n+1) + nR_n^*(T(\xi))/(n+1))$ for every $n \in \mathbb{N}$. Taking limits as $n \to \infty$ we obtain $R^*(T(\xi)) = R^*(\xi)\exp(-f(\xi))$ for every $\xi \in \Omega$, hence defining $g(\xi) = 1/R^*(\xi)$ we get a positive solution of $g(T(\xi)) = g(\xi)\exp(f(\xi))$. Moreover, we also have

$$R^*(\xi) = \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\frac{g(T^j(\xi))}{g(\xi)} = \lim_{n\to\infty} R^*(\xi)\frac{1}{n}\sum_{j=0}^{n-1} g(T^j(\xi)).$$

Therefore, there exists a T-invariant subset $\Omega_0 \subset \Omega$ with $m_0(\Omega_0) = 1$ such that $\lim_{n\to\infty}(1/n)\sum_{j=0}^{n-1} g(T^j(\xi)) = 1$ for every $\xi \in \Omega_0$. From Birkhoff's ergodic theorem for positive functions we deduce that $g \in L^1(\Omega, m_0)$, which proves the claim. ∎

We complete this section with some results and examples concerning the non-measure-preserving case. Let $(\Omega, \mathcal{A}, m_0)$ be a normalized measure space and $T : \Omega \to \Omega$ an invertible non-singular transformation. In all what follows, we assume there exists an invariant measure equivalent to m_0, which permits us to apply the previous theory.

PROPOSITION 3.9. *Let $r > 0$ and $p, q \ge 1$ be real numbers with $1/p + 1/q = 1$. Assume that the following conditions hold:*

(i) *m_0 is equivalent to an F-invariant measure $\widetilde{m}_0$ with $dm_0 = w\,d\widetilde{m}_0$ and density function $w \in L^q(\Omega, \widetilde{m}_0)$;*

(ii) *there exists a measurable subset $\Theta \subset \Omega$ with $m_0(\Theta) > 0$ and a sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers with*

$$\tau_0' = \lim_{k\to\infty}\frac{1}{n_k}\sum_{j=1}^{n_k}\int_\Theta |S_j f|^r\, dm_0 < \infty.$$

Then f is an $L^{p/r}$-coboundary, i.e. there exists $h \in L^{p/r}(\Omega, m_0)$ such that $f = h\circ T - h$ almost everywhere in $K_{\mathbb{R}}$.

Proof. Set $\Theta_l = \{\xi \in \Theta \mid 1/l < w(\xi) < l\}$. We can find an index $l \in \mathbb{N}$ with $m_0(\Theta_l) > 0$; then

$$\limsup_{k\to\infty}\frac{1}{n_k}\sum_{j=1}^{n_k}\int_{\Theta_l}|S_j f(\xi)|^r\, d\widetilde{m}_0 = \limsup_{k\to\infty}\frac{1}{n_k}\sum_{j=1}^{n_k}\int_{\Theta_l}|S_j f(\xi)|^r\frac{1}{w(\xi)}\, dm_0 < \infty.$$

Theorem 3.2 assures the existence of a function $h \in L^p(\Omega, \widetilde{m}_0)$ such that $f = h\circ T - h$ almost everywhere in $K_{\mathbb{R}}$. Moreover,

$$\int_\Omega |h(\xi)|^{p/r}\, dm_0 \le \Big(\int_\Omega |h(\xi)|^r\, d\widetilde{m}_0\Big)^{1/p}\Big(\int_\Omega w(\xi)^q\, d\widetilde{m}_0\Big)^{1/q} < \infty,$$

which implies that $h \in L^{p/r}(\Omega, m_0)$ and completes the proof. ∎

EXAMPLE 3.10. The following example, based on a construction given by Furstenberg [6], shows that the conclusions of Proposition 3.9 are optimal.

Let $\{\nu_k\}_{k\in\mathbb{N}}$ be the sequence of positive numbers defined by $\nu_{k+1} = 2^{\nu_k} + \nu_k + 1$ for $k \ge 1$ and consider the irrational number $\alpha = \sum_{k=1}^{\infty} 2^{-\nu_k}$. The map $T : S^1 \to S^1$, $\xi \mapsto \xi + \alpha$, represents the rotation of angle α on S^1. In this case m_0 is the Lebesgue measure on the circle.

We introduce the functions

$$f(\xi) = \sum_{|k|\ne 0}\frac{1}{|k|}(e^{2\pi i n_k\alpha} - 1)e^{2\pi i n_k\xi} \quad\text{and}\quad h(\xi) = \sum_{|k|\ne 0}\frac{1}{|k|}e^{2\pi i n_k\xi}.$$

It is easy to check that $f \in C^\infty(S^1)$. Furthermore, $h \in L^p(S^1, m_0)$ for every $0 < p < \infty$ and $h(\xi + \alpha) - h(\xi) = f(\xi)$ almost everywhere on S^1 (see [12]).

Now we take the base $\Omega = S^1$ and denote by F the discrete transformation induced by (2.1) on $K_{\mathbb{R}} = \Omega \times P^1(\mathbb{R})$. Fix $1 \le p \le \infty$. The map $\tau : L^p(K_{\mathbb{R}}, m_1) \to L^p(K_{\mathbb{R}}, m_1)$, $B \mapsto B\circ F$, is continuous with $\|\tau\| = \|G\|_\infty^{1/p} > 1$.

We introduce the function $Q(\xi, \varphi) = 1 + h^2(\xi)\sin^2\varphi - 2h(\xi)\sin\varphi\cos\varphi$ and $P = 1/Q$. The measure $d\mu = P\,dm_1$ is invariant under F; moreover, $dm_1 = Q\,d\mu$ and $Q \in L^p(K_{\mathbb{R}}, \mu)$ for every $0 < p < \infty$.

We fix an exponent $p > 0$ and take a measurable function H such that $H \in L^p(K_{\mathbb{R}}, \mu) - L^p(K_{\mathbb{R}}, m_1)$. In particular this implies that $H \in L^{p'}(K_{\mathbb{R}}, m_1)$ for every $0 < p' < p$. If we define $B = H\circ F - H$, it is obvious that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\int_{K_{\mathbb{R}}}|S_j B(\xi,\varphi)|^p\, d\mu < \infty.$$

There exists $l \in \mathbb{N}$ such that, if $K_l = \{(\xi,\varphi) \in K_{\mathbb{R}} \mid 1/l < Q(\xi,\varphi) < l\}$, then $m_1(K_l) > 0$ and

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\int_{K_l}|S_j B(\xi,\varphi)|^p\, dm_1 < \infty.$$

Notice that B is an $L^{p'}$-coboundary for every $0 < p' < p$ (with respect to m_1), but it is not an L^p-coboundary.

Acknowledgements. The third author would like to thank the members of the "Departamento de Matemática Aplicada a la Ingeniería", University of Valladolid, for their hospitality during the preparation of this paper.

References

[1] A. I. Alonso and R. Obaya, *Dynamical description of bidimensional linear systems with a measurable 2-sheet*, J. Math. Anal. Appl. 212 (1997), 154–175.

[2] I. Assani, *Note on the equation $y = (I - T)x$*, preprint, 1997.

[3] I. Assani and J. Woś, *An equivalent measure for some nonsingular transformations and application*, Studia Math. 97 (1990), 1–12.

[4] R. C. Bradley, *On a theorem of K. Schmidt*, Statist. Probab. Lett. 24 (1995), 9–12.

[5] F. E. Browder, *On the iteration of transformations in non-compact minimal dynamical systems*, Proc. Amer. Math. Soc. 9 (1958), 773–780.

[6] H. Furstenberg, *Strict ergodicity and transformation of the torus*, Amer. J. Math. 83 (1961), 573–601.

[7] W. Gottschalk and G. Hedlund, *Topological Dynamics*, Amer. Math. Soc. Colloq. Publ. 36, Providence, RI, 1955.

[8] U. Krengel and M. Lin, *On the range of the generator of a markovian semigroup*, Math. Z. 185 (1984), 553–565.

[9] V. P. Leonov, *On the dispersion of the time averages of a stationary stochastic process*, Teor. Veroyatnost. i Primenen. 6 (1961), 93–101 (in Russian); English transl. in Theory Probab. Appl. 6 (1961).

[10] M. Lin and R. Sine, *Ergodic theory and the functional equation $(I - T)x = y$*, J. Operator Theory 10 (1983), 153–166.

[11] F. J. Martín-Reyes and A. de la Torre, *On the pointwise ergodic theorem*, Studia Math. 108 (1994), 1–4.

[12] S. Novo and R. Obaya, *An ergodic classification of bidimensional linear systems*, J. Dynam. Differential Equations 8 (1996), 373–406.

[13] —, —, *An ergodic and topological approach to almost periodic bidimensional linear systems*, in: Contemp. Math. 215, Amer. Math. Soc., 1998, 299–323.

[14] R. Sato, *Pointwise ergodic theorems in Lorentz spaces $L(p,q)$ for null preserving transformations*, Studia Math. 114 (1995), 227–236.

Departamento de Matemática Aplicada
a la Ingeniería
E.T.S. de Ingenieros Industriales
Universidad de Valladolid
47011 Valladolid, Spain
E-mail: anaalo@wmatem.eis.uva.es
rafoba@wmatem.eis.uva.es

Institute of Computational Mathematics and
Scientific/Engineering Computing
Chinese Academy of Sciences
Beijing 100080, P.R. China
E-mail: hjl@lsec.cc.ac.cn

Received June 29, 1998
Revised version August 13, 1999
(4136)

STUDIA MATHEMATICA 138 (2) (2000)

Schauder decompositions and multiplier theorems

by

P. CLÉMENT (Delft), B. DE PAGTER (Delft),
F. A. SUKOCHEV (Adelaide, S.A.) and H. WITVLIET (Delft)

Abstract. We study the interplay between unconditional decompositions and the R-boundedness of collections of operators. In particular, we get several multiplier results of Marcinkiewicz type for L^p-spaces of functions with values in a Banach space X. Furthermore, we show connections between the above-mentioned properties and geometric properties of the Banach space X.

1. Introduction. A number of important operators in analysis may be represented as multiplier operators with respect to a given Schauder decomposition $\{D_n\}_{n=1}^{\infty}$ of a Banach space X, i.e.,

$$T_\lambda(x) = \sum \lambda_k D_k x, \quad x \in X, \tag{1}$$

where $\lambda = \{\lambda_k\} \in \mathbb{C}$. The characterization of the sequences λ for which (1) defines a bounded operator T_λ on X for a given decomposition $\{D_n\}_{n=1}^{\infty}$ is an interesting problem. The study of this problem for the Schauder decomposition defined by the trigonometric system in $L^p(0,1)$ led J. Marcinkiewicz [Mar39] (see also [EG77]) to his famous multiplier theorem.

A similar description to that of Marcinkiewicz was obtained by G. I. Sunouchi [Sun51] for the Schauder decomposition defined by the Paley–Walsh system in $L^p(0,1)$. Vector-valued extensions of the Marcinkiewicz theorem are given in [Bou83] (see also [BG94]).

In all results mentioned above the descriptions of the sequences λ for which T_λ is bounded are given in terms of certain blockings $\Delta = \{\Delta_k\}_{k=1}^{\infty}$ of the Schauder decomposition $\{D_n\}_{n=1}^{\infty}$ (the dyadic blocking for both trigonometric and Paley–Walsh systems), which turns out to be an unconditional decomposition of X. In fact, the study of the operators given by (1) naturally

2000 *Mathematics Subject Classification*: 46B15, 46E40, 42A45.
Research of the third author partially supported by A.R.C. and the Netherlands Organization for Scientific Research (N.W.O.).