Fundamental solution, eigenvalue asymptotics and eigenfunctions of degenerate elliptic operators with positive potentials

by

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Abstract. We show a weighted version of Fefferman–Phong’s inequality and apply it to give an estimate of fundamental solutions, eigenvalue asymptotics and exponential decay of eigenfunctions for certain degenerate elliptic operators of second order with positive potentials.

1. Introduction and main results. In [Sh1] Shen studied $L^p$ boundedness of the operators $VL^{-1}, V^{1/2}L^{-1/2}, V^{3/2}L^{-1}$ for the Schrödinger operator $L = -\Delta + V$ with certain positive potentials $V$. Here $L^{-1}$ is an integral operator with the minimal Green function (or minimal fundamental function) for $L$ as its integral kernel (see e.g. [Mu], [Smi]). In [KS] we extended Shen’s results to uniformly elliptic operators $L$ and gave a simple proof of some part of his results. In particular our estimates imply boundedness of several operators on weighted $L^p$ spaces and Morrey spaces. Furthermore, in [Su] Sugano investigated several estimates for the operators $V^\alpha (-\Delta + V)^{-\beta}$ and $V^\alpha \nabla (-\Delta + V)^{-\beta}$ by using arguments as in [Sh1] and [KS]. In [Sh2] Shen also studied eigenvalue asymptotics and exponential decay of eigenfunctions. The main ingredients of his work are the function $m(x, V)$ introduced to control the fundamental solution of $L$ and Fefferman–Phong’s inequality associated with this weight $m(x, V)$.

The purpose of this paper is to show a weighted version of Fefferman–Phong’s inequality and its applications. The first application is to give an estimate of fundamental solutions of the following degenerate elliptic operators with positive potentials:

$$L = -\nabla (A(x)\nabla) + V(x),$$

where $V(x) \geq 0$, $A(x) = (a_{ij}(x))_{i,j=1}^n$ and $a_{ij}(x)$ is a measurable function.
satisfying
\[(1) \quad \mu w(x)|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq \mu^{-1} w(x)|\xi|^2, \quad \xi \in \mathbb{R}^n,\]
for some positive constant $\mu \in (0, 1]$ and a non-negative measurable function $w$. The second application is to show the eigenvalue asymptotics and exponential decay of eigenfunctions of the operator
\[H = -\frac{1}{w} \text{div}(A(x) \nabla U(x)) + U(x)\]
on $L^2(w \, dx)$, where $U \geq 0$. Throughout this paper we assume that $w$ satisfies the so-called $A_2$ condition of Muckenhoupt.

**Definition 1.** We say $w \in A_2$ if there exists a constant $C$ such that
\[\left( \frac{1}{|B|} \int_B w(x) \, dx \right)^{1/2} \left( \frac{1}{|B|} \int_B w(x)^{-1} \, dx \right) \leq C\]
for every ball $B \subset \mathbb{R}^n$.

We show the following estimate for the fundamental solution $\Gamma_w(x, y; V)$ of $L$ under certain conditions on $V$: for every $k > 0$ there exists a constant $C_k$ such that
\[\Gamma_w(x, y; V) \leq \frac{C_k}{(1 + m_w(x, V/w)|x - y|^2)} \frac{|x - y|^2}{w(B(x, |x - y|))} \]
where $B(x, r) = \{ y \in \mathbb{R}^n : |x - y| < r \}$, $w(B) = \int_B w(x) \, dx$ and $m_w(x, U)$ is defined by
\[\frac{1}{m_w(x, U)} = \sup \left\{ r > 0 : \frac{r^2}{w(B(x, r))} \int_{B(r)} U(y)w(y) \, dy \leq 1 \right\}.\]

This type of estimate was first established by Shen [Sh1] for $L = -\Delta + V$ (see also [Zh]). By using this estimate, we can obtain the boundedness of the operator $\sqrt{L}$ on various spaces.

We also assume $U(x) \geq 0$ throughout this paper. We define the potential class $(\text{RH})_q(w)$ which will be considered in this paper.

**Definition 2.** (1) For $1 < q < \infty$ we say $U \in (\text{RH})_q(w)$ if there exists a constant $C$ such that
\[(2) \quad \left( \frac{1}{w(B)} \int_B U(x)^q w(x) \, dx \right)^{1/q} \leq C \frac{1}{w(B)} \int_B U(x)w(x) \, dx\]
for every ball $B \subset \mathbb{R}^n$. If the inequality only holds for balls with radius less than or equal to 1, then we say $U \in (\text{RH})_{q, \text{loc}}(w)$.

We say $U \in (\text{RH})_q(w)$ if there exists a constant $C$ such that
\[\sup_{x \in B} |U(x)| \leq C \frac{1}{w(B)} \int_B U(x)w(x) \, dx\]
for every ball $B \subset \mathbb{R}^n$.

Note that Hölder’s inequality yields $(\text{RH})_q(w) \subset (\text{RH})_p(w)$ for $1 < q < \infty$. It is well known that if $U \in (\text{RH})_q(w)$, then $U \in (\text{RH})_{q + \epsilon}(w)$ for some $\epsilon > 0$ (see Lemma 1) and satisfies the doubling condition: there exists a constant $C_0$ such that
\[\frac{1}{w(B/2, 2)} \int_{B(2, r)} U(y)w(y) \, dy \leq C_0 \frac{1}{w(B(x, r))} \int_{B(x, r)} U(y)w(y) \, dy\]
for every $x \in \mathbb{R}^n$ and $r > 0$. We also note that almost all properties of $(\text{RH})_q(w)$ hold for $(\text{RH})_{q, \text{loc}}(w)$ (see [Sh2]).

**Definition 3.** (1) We say $w \in D_q$ if there exists a constant $C > 0$ such that $w(B(x, t)) \leq C t^q w(B(x, r))$ for every $t > 1$.

(2) We say $w \in (\text{RD})_q$ if there exists a constant $C > 0$ such that $w(B(x, t)) \geq Ct^q w(B(x, r))$ for every $t > 1$.

We now state the main results of this paper. Let $\Gamma_w(x, y; V)$ be the fundamental solution to the operator $L = -\nabla (A(x) \nabla) + V(x)$ with $A(x)$ satisfying (1) and $V(x) \geq 0$.

**Theorem 1.** Assume $w \in A_2 \cap (\text{RD})_q \cap D_q$ with $2 < \nu \leq q$ and $V/w \in (\text{RH})_q(w)$ for some $q > q/2$. Then for each $k > 0$ there exists a constant $C_k$ such that
\[0 \leq \Gamma_w(x, y; V) \leq \frac{C_k}{(1 + m_w(x, V/w)|x - y|^2)} \frac{|x - y|^2}{w(B(x, |x - y|))}.\]

**Theorem 2.** Let $S = \{\text{meas}(w(x))^{1/2} L^{-1} \}$. Under the same assumptions as in Theorem 1, there exists a constant $C > 0$ such that
\[|f(x)| \leq CM_w(f/w)(x),\]
where $M_w$ is the Hardy–Littlewood maximal function with respect to the measure $w(x) \, dx$, i.e., $(M_w f)(x) = \sup_{x \in B} w(B)^{-1} \int_B |f(x)|w(x) \, dx$. Hence we also have
\[||f||_p \leq C ||f||_p \, w(x) \, dx\]
for every $1 < p < \infty$.

**Theorem 2** is an extension of a result of J. Zhong [Zh, Lemma 3.2] in which only the non-degenerate case $A(x) \equiv I$ and non-negative polynomials $V$ were considered.
THEOREM 3. Under the same assumptions as in Theorem 1, there exists a constant $C > 0$ such that

$$|T^* f(x)| \leq C (M_w(|f|^{q'})(x))^{1/q'},$$

where $1/q + 1/q = 1$ and $T^*$ is the conjugate operator to $T = VL^{-1}$.

REMARK 1. It is known that $w \in A_2$ implies $w \in D_{2n}$. Hence Theorems 1 and 2 hold for $w \in A_2 \cap (RD)_v$ with $2 < v$ and $V/w \in (RH)^q_q(w)$ for some $q > n$.

COROLLARY 1. (a) Let $V/w \in (RH)^q_q(w)$ and $T = VL^{-1}$. Then, under the same assumptions as in Theorem 1, we have

$$\|Tf\|_{L^p(wdx)} \leq C \|f\|_{L^p(wdx)}$$

for every $1 < p < \infty$, where $\sigma(x) = w(x)^{1-p}$.

(b) Suppose $V/w \in (RH)^q_q(w)$ with $q > \gamma/2$. Then

$$\|Tf\|_{L^p(wdx)} \leq C \|f\|_{L^p(wdx)}$$

for every $1 < p < q$, where $\sigma(x) = w(x)^{1-p}$.

Next, we consider the operator

$$H = -\frac{1}{w} \text{div}(A(x)\nabla) + U(x)$$

on $L^2(udw)$, where $U \geq 0, U \in (RH)^q_q(w)$ for some $q > \gamma/2$. Here we assume that $A(x)$ satisfies (1) with a weight $w \in A_2 \cap D_\gamma$ with $\gamma > 2$. Then the operator $H$ can be realized as a self-adjoint operator on $L^2(udw)$ by the Friedrichs extension (see e.g. [Da, Theorem 1.2.8]). For $\lambda > 0$, we denote by $N(\lambda, H)$ the number of eigenvalues to $H$ less than or equal to $\lambda$. Then we have the following estimate for $N(\lambda, H)$.

THEOREM 4. Assume $w \in A_2 \cap D_\gamma \cap (RD)_v$ with $\gamma > 2$ and $U \geq 0, U + 1 \in (RH)^q_q(w)$ with $q > \gamma/2$. Assume also that there exist constants $d_j, j = 1, 2$ such that

$$0 < d_1 \leq w(B(x,1)) \leq d_2 < \infty$$

for every $x \in \mathbb{R}^n$. Then there exist positive constants $C_j, j = 1, 2, 3, 4$, such that for $\lambda \geq 1$,

$$C_1 \lambda^{\gamma/2} w(\{x : m_w(x, U + 1) \leq C_1 \lambda\}) \leq N(\lambda, H) \leq C_3 \lambda^{\gamma/2} w(\{x : m_w(x, U + 1) \leq C_4 \lambda\}).$$

Theorem 4 easily implies the following.

COROLLARY 2. Under the same assumptions as in Theorem 4, there exist positive constants $C_j, j = 1, 2, 3, 4$, such that for $\lambda \geq 1$,

$$(w(x)dx \times dp)(\{|p|^{2n/\nu} + C_1 m_w(x, U + 1)^2 < \lambda\}) \leq C_2 N(\lambda, H),$$

$$C_3 N(\lambda, H) \leq (w(x)dx \times dp)(\{|p|^{2n/\nu} + C_4 m_w(x, U + 1)^2 < \lambda\}).$$

This generalizes Theorem 0.9 of [Sh2] which deals with the non-degenerate case $A(x) = I$. Next, we show the exponential decay of eigenfunctions $u \in L^2(udw)$ with eigenvalue $\lambda \geq 1$ of $H$, that is, $Hu = \lambda u$ in $L^2(udw)$. To state the result, we define the Agmon type distance $d(x, y)$ associated with the potential $U$ and a weight $w$:

$$d(x, y) = \inf \left\{ \int_{0}^{1} \left(\int_{\gamma(0)}^{\gamma(t)} |u|^2 \right)^{1/2} \right\}.$$

Let $E_\lambda = \{x \in \mathbb{R}^n : m_w(x, U + 1) < \sqrt{\lambda}\}$ and define $d_\lambda(x) = \inf\{d(x, y) : y \in E_\lambda\}$.

THEOREM 5. Under the same assumption as in Theorem 4, let $u \in L^2(udw)$ satisfy $Hu = \lambda u$ in $L^2(udw)$ for some $\lambda \geq 1$. Then for sufficiently small $\varepsilon > 0$ there exist constants $C_\varepsilon, C_\varepsilon'$ such that

$$|u(x)| \leq C_\varepsilon e^{-\varepsilon d_\lambda(x)} \|u\|_{L^2(udw)}.$$

COROLLARY 3. Let $u \in L^2(udw)$ be an eigenfunction of $H$ with eigenvalue $\lambda \geq 1$. If $m_w(x, U + 1) \to \infty$, then there exist constants $C_{\lambda, \lambda}'$ such that

$$|u(x)| \leq C_{\lambda, \lambda}' e^{-d_\lambda(0)} \|u\|_{L^2(udw)} \quad \text{and} \quad |u(x)| \leq C_{\lambda, \lambda}' e^{-\varepsilon d_\lambda} \|u\|_{L^2(udw)}.$$

We use the following notation throughout this paper: $B(x, r) = B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$ for $x \in \mathbb{R}^n$ and $r > 0$; $f \sim g$ means that there exist positive constants $C_1$ and $C_2$ such that $C_1 f \leq g \leq C_2 f$; $\nabla = (\nabla_1, \ldots, \nabla_n)$, $\nabla f = \partial f/\partial x_j$.

2. Preliminaries. We collect some properties of the class $(RH)_q^q(w)$.

LEMMA 1. (i) If $U \in (RH)_q^q(w)$, then $U \in A_\infty(w)$.

(ii) If $U_1, U_2 \in (RH)_q^q(w)$, then $U = U_1 + U_2 \in (RH)_q^q(w)$ for every $\alpha, \beta > 0$.

(iii) If $U_1, U_2 \in (RH)_q^q(w)$, then $U_1 = U_2 \in (RH)_q^q(w)$.

(iv) If $U \in (RH)_q^q(w)$, then $W = U^q \in (RH)_q^q(w)$ for every $\alpha > 0$.

(v) If $U \in (RH)_q^q(w)$, then $U \in (RH)_q^{q+\varepsilon}(w)$ for some $\varepsilon > 0$.

Proof. For (i) see [CoF]. (ii) is easy. To show (iii), first by using Hölder’s inequality and the assumption $U_j \in (RH)_q^q(w), j = 1, 2$, we have
The following lemma makes the quantity \( m_w(x, U) \) well defined and 0 < \( m_w(x, U) \) < \( \infty \) for \( U \in (RH)_q(w) \) with \( q > \gamma/2 \).

**Lemma 2.** Assume \( w \in D_\gamma \) with \( \gamma > 0 \) and \( U \in (RH)_q(w) \) with \( q > \gamma/2 \). Then

\[
\frac{r^2}{w(B_r(x))} \int_{B_r(x)} U(y)w(y) dy \leq C \frac{R^2}{w(B_r)} \int_{B_r(x)} U(y)w(y) dy
\]

for some positive constant \( C_0 \) and for every \( x \in \mathbb{R}^n \) and 0 < \( r < R < \infty \), where \( \alpha = 2 - \gamma/q > 0 \).

**Proof.** We write \( B_r = B_r(x) \) for simplicity. By using Hölder’s inequality and (2), we have

\[
\frac{r^2}{w(B_r)} \int_{B_r} U(y)w(y) dy \leq C \left( \frac{r}{R} \right)^{2} \frac{w(B_r)}{w(B_r)} \int_{B_r} U(y)w(y) dy
\]

for every 0 < \( r < R < \infty \). Since \( w \in D_\gamma \), it follows that \( w(B_r)/w(B_r) \leq C(R/r)^{\gamma} \). Hence we arrive at the conclusion. \( \blacksquare \)

**Lemma 3.** Under the same assumptions as in Lemma 2, if

\[
r^2 \frac{w(B_r(x))}{w(B_r)} \int_{B_r(x)} U(y)w(y) dy \sim 1,
\]

then \( r \sim m_w(x, U)^{-1} \).

**Proof.** We use the notation

\[
\Phi(x, r) = \frac{r^2}{w(B_r(x))} \int_{B_r(x)} U(y)w(y) dy.
\]

By assumption there exist positive constants \( C_1, C_2 \) such that

\[
C_1 \leq \frac{r^2}{w(B_r(x))} \int_{B_r(x)} U(y)w(y) dy \leq C_2.
\]

We may assume 0 < \( C_1 < 1 < C_2 \) and that the constant \( C_0 \) in Lemma 2 satisfies \( C_0 > 1 \). Then Lemma 2 and the definition of \( m_w(x, U) \) imply

\[
(C_0/C_2)^{-1/\alpha} \leq \frac{1}{m_w(x, U)} \leq (C_0/C_1)^{1/\alpha}.
\]

Indeed, let \( R = (C_0/C_2)^{-1/\alpha} \). Then \( R < r \) and it follows from Lemma 2 that

\[
\Phi(x, R) \leq \frac{1}{C_2} \Phi(x, r) \leq \frac{1}{C_0} \Phi(x, r) \leq 1.
\]

Hence \( m_w(x, U)^{-1} \geq (C_0/C_2)^{-1/\alpha} \). On the other hand, for every \( R > (C_0/C_1)^{1/\alpha} \), we have \( R > r \) and \( C_1 \leq \Phi(x, r) \leq C_0(R/r)^{\alpha} \Phi(x, R) \leq C_1 \Phi(x, R) \) by Lemma 2. Hence \( m_w(x, U)^{-1} \leq (C_0/C_1)^{1/\alpha} \). \( \blacksquare \)
LEMMA 4. Under the same assumptions as in Lemma 2, we have the following properties.

(i) For any constant $C > 0$, we have $m_w(x, U) \sim m_w(y, U)$ if $|x - y| \leq C/m_w(x, U)$.

(ii) There exist positive constants $C_1, C_2, k_0$ such that

$$m_w(y, U) \leq C_1(1 + m_w(x, U)|x - y|)^{k_0}m_w(x, U),$$

$$m_w(y, U) \geq C_2(m_w(x, U)|x - y|)^{k_0/(k_0 + 1)}.$$ 

Proof. Recall that $U(x)w(x)dx$ is a doubling measure for $U \in (RH)_q(w)$ with $q > 1$. Since the proof is similar to that in [Sh1, Lemma 1.4] by using Lemmas 2 and 3, we omit the details. \hfill \blacksquare

3. A weighted Fefferman–Phong inequality. In this section we show a weighted version of Fefferman–Phong’s inequality. We also recall some estimates for the fundamental solution of $L_0 = -\nabla(A(x)\nabla)$. First we note the following Poincaré-type inequality.

LEMMA 5. Let $w \in A_2$. Then there exists a constant $C > 0$ such that

$$\int_{B_r(x_0)} \int_{B_r(y_0)} |u(x) - u(y)|^2w(x)w(y)\,dx\,dy$$

$$\leq Cr^2 \int_{B_r(x_0)} |\nabla u(x)|^2w(x)\,dx$$

for $u \in C^1(B_r(x_0)).$

Proof. For simplicity, we write $B_r = B_r(x_0)$ in the proof. The assertion is an easy consequence of the following well known Poincaré inequality: under the assumption $w \in A_2$, we have

$$\int_{B_r} |u(x) - (u)_{B_r}|^2w(x)\,dx \leq Cr^2 \int_{B_r} |\nabla u(x)|^2\,dx,$$

where $(u)_{B_r} = w(B_r)^{-1}\int_{B_r} u(x)w(x)\,dx$. For the proof of (4) see [FKS]. Actually, for every $y \in B_r$, we have

$$\int_{B_r} |u(x) - u(y)|^2w(x)\,dx$$

$$\leq 2 \int_{B_r} ((u(x) - (u)_{B_r})^2 + |u(y) - (u)_{B_r}|^2)w(x)\,dx$$

$$\leq Cr^2 \int_{B_r} w(x)|\nabla u(x)|^2\,dx + 2w(B_r)|u(y) - (u)_{B_r}|^2.$$

Hence by multiplying with $w(y)$, integrating on $y \in B_r$ and using the inequality (4) again, we arrive at the desired estimate. \hfill \blacksquare

Next we show a weighted version of Fefferman–Phong’s inequality. See [Sh1] for its original version [see also [Fe] and [Sm]].

LEMMA 6. Assume $w \in A_2 \cap D_+$ with $\gamma > 0$ and $U \in (RH)_q(w)$ with $q > \gamma/2$. Then

$$\int |u(x)|^2m_w(x, U)^2w(x)\,dx$$

$$\leq C \left( \int |\nabla u(x)|^2w(x) + |u(x)|^2U(x)w(x)\,dx \right).$$

Proof. Let $x_0 \in \mathbb{R}^n$ and $r_0 = m_w(x_0, U)^{-1}$. Since $U$ is an $A_{\infty}(w)$-weight, there exists a constant $\epsilon > 0$ such that

$$w \left( \left\{ x \in B_{r_0}(x_0) : U(x) \geq \frac{\epsilon}{w(B_{r_0}(x_0))} \int_{B_{r_0}(x_0)} U(y)w(y)\,dy \right\} \right) \geq \frac{1}{2}w(B_{r_0}(x_0)).$$

Put

$$\bar{U} \equiv w(B_{r_0}(x_0))^{-1} \int_{B_{r_0}(x_0)} U(y)w(y)\,dy$$

and $A = \{ y \in B_{r_0}(x_0) : U(x) \geq \epsilon \bar{U} \}$. Then, by the definition of $r_0$, we have

$$\bar{U} = r_0^{-2}$$

and $w(A) \geq \frac{1}{2}w(B_{r_0})$ with $B_{r_0} = B_{r_0}(x_0)$. Hence we obtain

$$\int_{B_{r_0}} \int_{B_{r_0}} \min(\epsilon/r_0^2, U(y)) |u(x)|^2w(x)w(y)\,dx\,dy$$

$$\geq \int_{B_{r_0}} |u(x)|^2w(x) \left( \int_{A} (\epsilon/r_0^2)w(y)\,dy \right)\,dx$$

$$\geq \frac{\epsilon}{2r_0^2}w(B_{r_0}) \int_{B_{r_0}} |u(x)|^2w(x)\,dx.$$ 

On the other hand, we will show

$$\int_{B_{r_0}} \int_{B_{r_0}} \min(\epsilon/r_0^2, U(y)) |u(x)|^2w(x)w(y)\,dx\,dy$$

$$\leq Cw(B_{r_0}) \left( \int_{B_{r_0}} |u(x)|^2w(x)\,dx + \int_{B_{r_0}} |u(x)|^2U(x)w(x)\,dx \right).$$

Indeed, since $|u(x)|^2 \leq 2(|u(x) - u(y)|^2 + |u(y)|^2)$, we have
\[
\int_{B_{r_0}} \int_{B_{r_0}} \min(\epsilon/r_0^2, U(y))|u(x)|^2 w(x)w(y) \, dx \, dy \\
\leq 2 \int_{B_{r_0}} \int_{r_0}^{r_0} |u(x) - u(y)|^2 w(x)w(y) \, dx \, dy \\
+ 2w(B_{r_0}) \int_{B_{r_0}} |u(y)|^2 U(y)w(y) \, dy.
\]

By Lemma 5, we arrive at the inequality (7). Therefore, by (6) and (7) we obtain
\[
\left( |\nabla u|^2 + |u|^2 U(x)\right) w(x) \, dx \geq \frac{C}{r_0^6} \int_{B_{r_0}} |u|^2 w \, dx.
\]

By Lemma 4(i), we have \(m_w(x, U) \sim 1/r_0\) for \(x \in B_{r_0}\). Hence
\[
\int_{B_{r_0}} m_w(x, U)^n (|\nabla u|^2 + |u|^2 U) w \, dx \geq C \int_{B_{r_0}} |u(x)|^2 m_w(x, U)^{n+2} w(x) \, dx.
\]

Integrating this over \(x_0 \in \mathbb{R}^n\) and changing the order of integration, we obtain the desired estimate (5). Here we use
\[
|z-x_0| < 1/m_w(x_0, U) \Rightarrow \int_{|z-x_0| < 1/m_w(x_0, U)} \, dz_0 \sim m_w(x_0, U)^{-n}.
\]

**Lemma 7.** Assume \(w \in A_2 \cap (RD)_\nu\) with \(\nu > 2\). Let \(\Gamma_0(x, y)\) be the fundamental solution of \(L_0\). Then
\[
(0 \leq) \Gamma_0(x, y) \leq C \frac{|x-y|^2}{w(B(x, |x-y|)).}
\]

**Proof.** By the estimate in [FKJ], we know
\[
(0 \leq) \Gamma_0(x, y) \leq C \int_{|x-y|}^{\infty} s^2 \frac{ds}{w(B(x, s))} \leq C \frac{r^2}{w(B(x, r))}.
\]

The assumption \(w \in (RD)_\nu\) with \(\nu > 2\) implies that
\[
\int_{r}^{\infty} s^2 \frac{ds}{w(B(x, s))} \leq C \frac{r^2}{w(B(x, r))}.
\]

In fact, since \(w \in (RD)_\nu\) with \(\nu > 2\), we obtain
\[
\int_{r}^{\infty} s^2 \frac{ds}{w(B(x, s))} = \int_{1}^{r^2 s_0^2} \frac{ds}{w(B(x, rs_0))} \leq C \frac{r^2}{w(B(x, r))}.
\]

The observation (8) can also be found in [CSW, p. 316].

**4. An estimate of fundamental solutions.** First we note the following subsolution estimate.

**Lemma 8.** Assume \(w \in A_2\). Let \(u\) be a non-negative subsolution of \(L_0\) on \(B_{2R} = B_{2R}(y)\). Then for all \(\sigma \in (0, 1)\) there exists a constant \(C_\sigma\) such that
\[
\sup_{x \in B_{\sigma R}(x_0)} u(x) \leq \frac{C_\sigma}{w(B_{R}(x_0))} \int_{B_{R}(x_0)} u(x)w(x) \, dx.
\]

**Proof.** We write \(B_r = B_r(y)\) for simplicity. The result in [FKS] yields the following estimate:
\[
\sup_{x \in B_{\sigma R}} u(x) \leq C_{\sigma} \left( \int_{B_{R}} u(x)^2 w(x) \, dx \right)^{1/2}.
\]

We conclude by using the reverse Hölder type estimate:
\[
\left( \frac{1}{w(B_{R})} \int_{B_{R}} u(x)^2 w(x) \, dx \right)^{1/2} \leq C \frac{1}{w(B_{2R})} \int_{B_{2R}} u(x)w(x) \, dx.
\]

This inequality can be shown by using the argument as in [Gu] (see also [CFG], [Ku1,2]).

**Lemma 9.** Assume \(w \in A_2 \cap D_\gamma \) with \(\gamma > 0\) and \(V/w \in (RD)_{\nu}(w)\) with \(q > \gamma/2\). Let \(u\) be a solution of \(L_0u = 0\) on \(B_{4R}(x_0)\). Then for every \(k > 0\) there exists a constant \(C_k\) such that
\[
\sup_{x \in B_{R}(x_0)} |u(x)| \leq \frac{C_k}{(1 + Rm_w(x_0, V/w))^k} \left( \int_{B_{2R}(x_0)} u(x)^2 w(x) \, dx \right)^{1/2}.
\]

**Proof.** We write \(B_r = B_r(x_0)\) for simplicity. Since
\[
\frac{1}{2} \nabla(A(x)\nabla|u|^2) = V|u|^2 + \sum_{i,j=1}^n a_{ij}(x) \nabla_i u \nabla_j u \geq 0,
\]

\(v = |u|^2\) is a subsolution to \(L_0v = 0\). Hence Lemma 8 shows that
\[
\sup_{x \in B_{R}} |u| \leq \frac{1}{w(B_{R/2})} \int_{B_{R/2}} |u|^2 w \, dx \leq \frac{1}{w(B_{R/2})} \int_{B_{R/2}} |u|^2 w \, dx \leq \frac{C}{R} \int_{B_{R/2}} |u|^2 \, dx.
\]

By Caccioppoli’s inequality, we have
\[
\int_{B_{R/2}} |\nabla u|^2 \, dx \leq \frac{C}{R^2} \int_{B_{R/2}} |u|^2 \, dx.
\]
Hence (10) and Lemma 6 yield
\[ \int_{B_R} m_w(x, V/w)^2 |u(x)|^2 w(x) \, dx \leq \frac{C}{R^2} \int_{B_{2R}} |u|^2 w \, dx. \]
By Lemma 4(ii),
\[ m_w(x, V/w) \geq C m_w(x_0, V/w)/(1 + R m_w(x_0, V/w))^{k_0/(k_0+1)} \]
for every \( x \in B_R \). Thus
\[ \int_{B_R} |u(x)|^2 w(x) \, dx \leq \frac{C}{(1 + R m_w(x_0, V/w))^{2/(k_0+1)}} \int_{B_{2R}} |u|^2 w \, dx. \]
By repeating this argument and using (9) and \( w \in D_\gamma \), we arrive at the desired estimate. \( \blacksquare \)

**Proof of Theorem 1.** By the maximum principle (see e.g. [CW]), we have
\[ 0 \leq \Gamma_w(x, y; V) \leq \Gamma^0(x, y). \]
Let \( x_0 \neq y \in \mathbb{R}^n \) and put \( R = |x_0 - y| \). Since \( u(x) = \Gamma_w(x, y; V) \) satisfies \( Lu = 0 \) on \( B_{R/2}(x_0) \), Lemma 9 shows that
\[ \sup_{x \in B_{R/4}(x_0)} |u(x)| \leq \frac{C}{(1 + R m_w(x_0, V/w))^{k_0}} \left( \int_{B_{R/2}(x_0)} |u(x)|^2 w(x) \, dx \right)^{1/2}. \]
Note that \( \Gamma^0(x, y) \leq C \Gamma^0(x_0, y) \) for \( x \in B_{R/2}(x_0) \) by Harnack's inequality (see [FKS, Lemma 2.3.5]). Hence by Lemma 7, we have
\[ \left( \frac{1}{w(B_{R/2}(x_0))} \int_{B_{R/2}(x_0)} |u(x)|^2 w(x) \, dx \right)^{1/2} \leq C \frac{R^2}{w(B(x_0, R))}. \]
Therefore we obtain
\[ \Gamma_w(x_0, y; V) = u(x_0) \leq \sup_{x \in B_{R/4}(x_0)} |u(x)| \leq \frac{C}{(1 + R m_w(x_0, V/w))^{k_0}} \frac{R^2}{w(B(x_0, R))}. \]
Since \( R = |x_0 - y| \), we get the desired estimate. \( \blacksquare \)

Once we obtain Theorem 1, we can prove Theorems 2 and 3 in a similar way to [KS].

**Proof of Theorem 2.** Let \( f \in C_0^\infty \). By Theorem 1, we have
\[ |Df(x)| \leq \int_{B_{2R}} m_w(x, V/w)^2 C_k|x - y|^2 \frac{w(B(x, |x - y|))}{w(B(x_0, |x - y|))} |f(y)| \, dy. \]
Put \( r = 1/m_w(x, V/w) \). Then, by the doubling property of \( w(x) \), the right hand side of (11) is dominated by
\[ \sum_{j = -\infty}^{\infty} \left\{ \frac{\Gamma(u, y, x; V)}{(1 + 2^{-j - 1} k w(B(x, 2^{-j - 1})))} \int_{B_{x_0, 2^{-j - 1}}} f(y) \, dy \right\} \]
if we take \( k \geq 3 \). \( \blacksquare \)

**Proof of Theorem 3.** Let \( r = 1/m_w(x, V/w) \). Then
\[ |T^* f(x)| \leq \int_{\mathbb{R}^n} \frac{C_k|x - y|^2}{w(B(x, |x - y|))} \frac{w(B(x, |x - y|))}{w(B(x_0, |x - y|))} |f(y)| \, dy \]
\[ \leq \int_{\mathbb{R}^n} \frac{C_k|x - y|^2}{w(B(x, |x - y|))} \frac{w(B(x, |x - y|))}{w(B(x_0, |x - y|))} |f(y)| \, dy \]
\[ \leq C \sum_{j = -\infty}^{\infty} \left\{ \frac{(2^{-j})^2}{(1 + 2^{-j - 1} k w(B(x_0, 2^{-j - 1})))} \right\} \frac{w(B(x, |x - y|))}{w(B(x_0, |x - y|))} \int_{B_{x_0, 2^{-j - 1}}} f(y) \, dy \]
\[ \times \left( \frac{1}{w(B(x, 2^{-j - 1})))} \left\{ \frac{1}{w(B(x, 2^{-j - 1})))} \int_{B_{x_0, 2^{-j - 1}}} f(y) \, dy \right\}^{1/q} \right\}^{1/p} \]
\[ \times \left( \frac{1}{w(B(x_0, 2^{-j - 1})))} \int_{B_{x_0, 2^{-j - 1}}} f(y) \, dy \right)^{1/q} \]
\[ \leq C_k \int_{\mathbb{R}^n} \frac{C_k|f(x)|^{1/q}}{w(B(x_0, 2^{-j - 1})))} \int_{B_{x_0, 2^{-j - 1}}} f(y) \, dy \]
Note that \( V dx \) is a doubling measure because \( V/w \in A_\infty(w) \).
For the case \( j \geq 1 \), since \( w \in (RD)_w \), we obtain
\[ \frac{(2^{-j})^2}{w(B(x, 2^{-j - 1})))} \int_{B_{x_0, 2^{-j - 1}}} V(y) \, dy \]
\[ \leq 2^{2j} C_k^{-1} C_2^{-1} \int_{B_{x_0, 2^{-j - 1}}} V(y) \, dy = 4 C_k^{-1} 2^{-j - 1}, \]
where \( k_0 = 2 - \nu + \log_2 C_0 \). For the case \( j \leq 0 \), by Lemma 2 we obtain
Choose a function \( \eta \in C_0^\infty(Q(0,1)) \) such that \( \eta(x) = 1 \) on \( x \in Q(0,1/2) \) and put \( \eta_{k,l}(x) = \lambda^{\nu/4} \eta(\sqrt{\lambda}(x-x_l)) \), \( l = 1, \ldots, m \). Let \( \mathcal{H} \) be the subspace spanned by \( \{ \eta_{k,l} \}_{k,l=1}^m \). Since \( Q_k \)s are disjoint, \( \mathcal{H} \) is an \( m \)-dimensional subspace of \( L^2(w dx) \). Let \( L = \text{sup}(|\nabla \eta(x)| + |\eta(x)|) \) and \( r_k = 1/m |\eta_{k,l}(x_k, U + 1) \). Then

\[
\sum_{i,j=1}^n a_{ij} \nabla_i \eta_{k,l} \nabla_j \eta_{k,l} dx \leq \mu^{-1} L^2 \lambda \left[ \lambda^{\nu/2} w(Q(x_k, 1/\sqrt{\lambda})) \right] \text{div} \mathcal{H}
\]

and

\[
\int U \eta_{k,l}^2 w dx \leq L^2 \lambda^{\nu/2} \left[ \int U w dx \right] \text{div} \mathcal{H}
\]

Part (a) follows from this. Let \( 1/p + 1/p' = 1 \). For \( q' < p' \) it is known that \( M_w \) is bounded on \( L^{p'/q'}(wdx) \). Hence \( T^* \) is bounded on \( L^{p'/q'}(wdx) \) for \( 1 < p < q \). By duality we obtain the boundedness of \( T \) on \( L^p(w^{-p} dx) \). This gives (b). □

5. Eigenvalue asymptotics and decay of eigenfunctions. In this section we show Theorems 4 and 5. We denote by \( Q(x, r) \) the cube with sidelength \( r \) and center \( x \in \mathbb{R}^n \).

Proof of Theorem 4. We follow the argument of [Sh2].

Step 1. First, we show the lower bound. Let \( E_k = \{ x : m_w(x, U + 1) \leq \sqrt{\lambda} \} \) for \( \lambda \geq 1 \) and divide \( \mathbb{R}^n \) into disjoint cubes \( \{ Q_l \} \) whose sides are parallel to the coordinate axes. Let \( m \) be the number of cubes \( Q_l \) such that \( Q_l \cap E_k \neq \emptyset \) and let \( Q_k = Q(x_k, 1/\sqrt{\lambda}) \) be such that \( Q_k \cap E_k \neq \emptyset \). Then

\[
w(E_k) = \sum_{l=1}^m w(Q_l \cap E_k) \leq \sum_{k=1}^m w(Q_k) \leq C_d \lambda^{-\nu/2}.
\]

Here we have used

\[
w(Q(x, 1/\sqrt{\lambda})) \leq C(1/\sqrt{\lambda})^\nu w(Q(x, 1))
\]

since \( w \in (RD)_w \) and \( \sup_w w(Q(x, 1)) \leq d_2 \) by assumption. Next we show that there exists an \( m \)-dimensional subspace \( \mathcal{H} \) of \( L^2(w dx) \) such that

\[
\sum_{i,j=1}^n a_{ij}(x) \nabla_i u \nabla_j u dx + \int U u^2 w dx \leq C \left[ \lambda \right]^\nu u^2 w dx \quad (u \in \mathcal{H}).
\]

Now (12) and the min-max principle imply that

\[
N(C\lambda, \mathcal{H}) \geq m \geq C \frac{1}{d_2} \lambda^{\nu/2} w(E_k).
\]
Here the constant $C$ does not depend on $l$.

First, for each $l$ we have

$$
\int B_l \left( \left| \nabla (u\phi_l) \right|^2 w + 2 \left| \nabla u \right|^2 \phi_l^2 w + C \left( m_{w}(x, U + 1) \right)^2 u^2 w dx \right.
$$

It follows from Lemma 10(d) that

$$
\sum_{l=1}^{\infty} \int B_l \left( \left| \nabla (u\phi_l) \right|^2 w + (U + 1)(u\phi_l)^2 w \right) dx
$$

$$
\leq CC \sum_{l=1}^{\infty} \int B_l \left( \left| \nabla u \right|^2 \phi_l^2 w + m_w(x, U + 1)^2 u^2 w + (U + 1)u^2 \phi_l^2 w \right) dx
$$

$$
\leq C \int B_l \left( \left| \nabla u \right|^2 w dx + C \int m_w(x, U + 1)^2 u^2 w dx + C \int (U + 1)u^2 w dx \right).
$$

Therefore, Lemma 6 implies

$$
\sum_{l=1}^{\infty} \int B_l \left( \left| \nabla (u\phi_l) \right|^2 w + (U + 1)(u\phi_l)^2 w \right) dx
$$

$$
\leq C \int B_l \left( \left| \nabla u \right|^2 w + (U + 1)u^2 w \right) dx.
$$

Now, we consider two cases. If $B_l \cap E_{\lambda} \neq \emptyset$, then by the definition of $E_{\lambda}$ we have $x \in B_l \cap E_{\lambda}$, for every $l \leq \lambda^2/2$. Hence, Lemma 4(i) implies $\inf_{x \in B_l} m_w(x, U + 1) \geq C\lambda^{1/2}$. Using this and Lemma 6 we obtain

$$
\int B_l \left( \left| \nabla (u\phi_l) \right|^2 + (U + 1)(u\phi_l)^2 \right) w dx
$$

$$
\geq C \int B_l \left( m_w(x, U + 1)^2 (u\phi_l)^2 w dx \right.
$$

$$
\geq C \int B_l \left( m_w(x, U + 1)^2 (u\phi_l)^2 w dx \right.
$$

Next, suppose $B_l = B(x_l, r_l) \subset E_{\lambda}$, where $r_l = m_w(x_l, U + 1)^{-1}$. Let $Q_l = Q(x_l, 2r_l)$ denote the cube with sidelength $2r_l$ and center $x_l$. Then $B_l \subset Q_l$. Divide $Q_l$ into disjoint subcubes $Q_l^D$ with sidelength comparable to $\lambda^{-1/2}$. Note that $r_l \geq \lambda^{-1/2}$, since $x_l \in E_{\lambda}$. Let $\mathcal{H}$ be the subspace generated by $\{\phi_l(x)1_{Q_l^D} : B_l \subset E_{\lambda}\}$. If $\int_{Q_l^D} u\phi_l w dx = 0$ for each $Q_l^D$, that is, $u \in \mathcal{H}^D$, then (4) yields

$$
\lambda \int (u\phi_l)^2 w dx = \lambda \sum_{\beta} \int_{Q_l^D} (u\phi_l)^2 w dx
$$

$$
\leq C\lambda \int \left| \nabla (u\phi_l) \right|^2 w dx
$$

$$
\leq C \int \left| \nabla (u\phi_l) \right|^2 w dx + \lambda \int (U + 1)(u\phi_l)^2 w dx.
$$

Thus, (16) and (17) imply

$$
\int B_l \left( \left| \nabla (u\phi_l) \right|^2 w + (U + 1)(u\phi_l)^2 w \right) dx \geq C\lambda \int (u\phi_l)^2 w dx
$$

for every $u \in \mathcal{H}^D$. From (15) and (18), there exists a constant $C_0 > 0$ such that for every $\lambda \geq C_0$ we obtain

$$
C\lambda \int B_l \left( u^2 w dx \right. \left. + \left| \nabla u \right|^2 + U w^2 \right) dx
$$

for $u \in \mathcal{H}^D$. Let $m_l$ be the number of $Q_l^D$ associated with $B_l \subset E_{\lambda}$. Then there exists a positive constant $C$ such that $m_l \leq C\lambda^{1/2} w(B_l)$. Indeed, assumptions (3) and $w \in D_{\gamma}$ imply

$$
w(B_l) \geq C(n)m_l w(Q_l^D) \geq C(n)m_l \left( 1/\lambda \right)^\gamma w(Q(x_l, 1)) \geq Cm_l \lambda^{-\gamma/2}.
$$

Hence by Lemma 10(d) we obtain

$$
\dim \mathcal{H} \leq C \sum_{B_l \subset E_{\lambda}} \lambda^{1/2} w(B_l) \leq C' \lambda^{1/2} w(E_{\lambda})
$$

Thus we have proved the upper bound. ■

Proof of Theorem 5. Since we reason as in [Sh2], we just mention the key estimates of the proof. For an eigenfunction $u \in L^2(\omega dx)$ with eigenvalue $\lambda \geq 1$, we obtain the subsolution estimate: there exists a constant $C$ such that

$$
|u(x)| \leq C \left( \frac{1}{w(B_r(x))} \int_{B_r(x)} |u(y)|^2 w(y) dy \right)^{1/2}
$$

for $\lambda r^2 \leq 1$. Second, we obtain the estimate

$$
\|e^{d(x)} u\|_{L^2(\omega dx)} \leq C' \|u\|_{L^1(\omega dx)}
$$

for small $\delta > 0$ and some constants $C, C'$. Now, by the assumption (7) and $w \in D_{\gamma}$, we have $C\gamma \leq w(B(x, r))$ for some positive constant $C$. Combining these estimates, we conclude that

$$
|u(x)| \leq C w(B(x, U + 1)^{1/2} e^{-\delta d} \omega(x) |u|_{L^2(\omega dx)}.
$$

Once we obtain this estimate we can prove Theorem 5 in the same way as in Theorem 0.20 of [Sh2]. ■
Corollary 3 follows from Theorem 5 as in [Sh2].

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References


