We conclude that $M \le 2+2b^3 \le 2+2(1/2)^3 = 9/4$. Our considerations also show that equality holds if and only if $b_1 = b_2 = b_3 = b = 1/2$ (condition for equality in the inequality between arithmetic and geometric means) and $-c^2 = |c|^2 = 4b^3 = 1/2$, i.e. $c = \pm i/\sqrt{2}$. This proves that the maximum of M (under the constraint $F = 1$) is $9/4$ and that the maximum is attained at the points

$$w = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \pm\frac{i}{\sqrt{2}}\right)$$

and only there, and finally this yields $d(2,\mathbb{C}) = \sqrt{9/4} = 3/2$. ■

REMARK 8. Theorem 7 exhibits a significant difference between real and complex trilinear forms. This is surprising in so far as the corresponding constants for bilinear forms are $d(n,\mathbb{K}) = \sqrt{n}$ regardless of whether $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

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Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad Complutense de Madrid
E-28040 Madrid, Spain
E-mail: cobos@eucmax.sim.ucm.es

Matematiska institutionen
Lunds universitet
Box 118
S-22100 Lund, Sweden
E-mail: jaak@maths.lth.se

Fakultät für Mathematik und Informatik
Mathematisches Institut
Universität Leipzig
Augustusplatz 10/11
D-04109 Leipzig, Germany
E-mail: kuehn@mathematik.uni-leipzig.de

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Applying the density theorem for derivations to range inclusion problems

by

K. I. BEIDAR (Tainan) and MATEJ BREŠAR (Maribor)

Abstract. The problem of when derivations (and their powers) have the range in the Jacobson radical is considered. The proofs are based on the density theorem for derivations.

1. Introduction. In both ring theory and the theory of Banach algebras, there are a number of results showing that under certain conditions a derivation (or its power) of a ring (algebra) must be zero or must map into the Jacobson radical. The ring-theoretic results are often proved by combining Kharchenko's theory of differential identities with some elementary (but clever) algebraic manipulations (see [3] for details about background and numerous references). Many of the analytic results in this vein were obtained as attempts to get noncommutative versions of the classical Singer–Wermer theorem [23]. Their proofs usually combine analytic and algebraic tools. For a more detailed discussion on this topic and bibliography we refer the reader to the survey articles [16, 19] and our recent paper [2].

It is our aim here to present a new possible approach to these problems, which works in both algebraic and analytic setting. It is based on an extension of the Jacobson density theorem, recently obtained in [2]. In order to state this result we have to introduce some notation and terminology. Let $\mathcal{A}$ be any ring and $\mathcal{M}$ be a simple left $\mathcal{A}$-module. Recall that $\mathcal{D} = \mathrm{End}({}_{\mathcal{A}}\mathcal{M})$ is a division ring by Schur's lemma. Let d be a derivation of $\mathcal{A}$. We say that d is *$\mathcal{M}$-inner* if there exists an additive map $T : \mathcal{M} \to \mathcal{M}$ such that $a^d x = T(ax) - a(Tx)$ for all $a \in \mathcal{A}$, $x \in \mathcal{M}$ (we shall always write derivations as exponents). The concept of $\mathcal{M}$-innerness obviously extends the concept of (ordinary) innerness. Moreover, in case $\mathcal{A}$ is a primitive ring and $\mathcal{M}$ is a faithful simple module, every X-inner derivation (cf. [3]) is also $\mathcal{M}$-inner,

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but not vice versa [2]. A derivation which is not $\mathcal{M}$-inner is called $\mathcal{M}$-*outer*. The main result of [2] considers compositions of $\mathcal{M}$-outer derivations and automorphisms [2, Theorem 5.3]. We now state only its special case, sufficient for the purposes of the present paper.

THEOREM 1.1. *Suppose that d is $\mathcal{M}$-outer. Let m be a positive integer and assume that either* $\mathrm{char}(D) = 0$ *or* $\mathrm{char}(D) > m$. *Let $x_1, \ldots, x_n$ be elements in $\mathcal{M}$ linearly independent over $\mathcal{D}$ and let $y_1, \ldots, y_n, z_1, \ldots, z_n$ be any elements in $\mathcal{M}$. Then there exists $a \in \mathcal{A}$ such that $ax_i = y_i$ and $a^{d^m} x_i = z_i$ for all $i = 1, \ldots, n$.*

The simplest, but nevertheless a very important one, is the case when $m = 1$. For this case the theorem was proved somewhat earlier [6], but only for dense algebras of linear operators. We also mention the papers [5, 22, 25] which already indicated that $\mathcal{M}$-outer derivations behave nicely with respect to the action on $\mathcal{M}$.

Some rather straightforward applications of Theorem 1.1 (for $m = 1$) were obtained already in [2]. The goal of the present paper is to give some further evidence of its applicability by generalizing certain results existing in the literature, in particular, those in [25] and [4]. As we shall see, by using Theorem 1.1 some problems on derivations can be reduced quite easily to $\mathcal{M}$-inner derivations. Arriving at an $\mathcal{M}$-inner derivation d of a ring (algebra) $\mathcal{A}$, we have two options: either to study the operator inducing this derivation or to reduce the problem to primitive rings and then apply the methods and results of the theory of derivations on primitive (and prime) rings.

2. The results. The first, algebraic part of an interesting paper by Turovskiĭ and Shul'man [25] shows that a derivation whose powers satisfy some special conditions must be, in our terminology, $\mathcal{M}$-inner (in particular, the transitivity (but not density) of ranges of powers of $\mathcal{M}$-outer derivations was discovered). As an application they proved that for a derivation d of a Banach algebra $\mathcal{A}$ the following five conditions are equivalent: (1) $\mathcal{A}^{d^{2n}} \subseteq J(\mathcal{A})$; (2) $\mathcal{A}^{d^{2n-1}} \subseteq J(\mathcal{A})$; (3) $(\mathcal{A}^{d^n})^n \subseteq J(\mathcal{A})$; (4) $\mathcal{A}^{d^n} \subseteq Q(\mathcal{A})$; (5) $\mathcal{A}^{d^n}$ is a thin set [25, Proposition 2.2]. Here, $J(\mathcal{A})$ denotes the Jacobson radical of $\mathcal{A}$ and $Q(\mathcal{A})$ denotes the set of quasi-nilpotent elements in $\mathcal{A}$. A subset $\mathcal{T}$ of a ring $\mathcal{A}$ is said to be *thin* if $\mathcal{T}x \neq \mathcal{M}$ for every simple left $\mathcal{A}$-module $\mathcal{M}$ and every element $x \in \mathcal{M}$. For instance, the set of left quasi-regular elements in $\mathcal{A}$ is thin. Namely, if a is any left quasi-regular element, i.e. such that $a + a' + a'a = 0$ for some $a' \in \mathcal{A}$, then we clearly have $ax \neq -x$ for any nonzero x in a left $\mathcal{A}$-module $\mathcal{M}$. In particular, the set of nilpotent elements in $\mathcal{A}$ is therefore thin, and, in case $\mathcal{A}$ is a Banach algebra, the set $Q(\mathcal{A})$ is thin.

The basic example illuminating the Turovskiĭ–Shul'man result is the following: let $a \in \mathcal{A}$ be such that $a^n = 0$ and $a^{n-1}\mathcal{A}a^{n-1} \neq 0$, and let d be the inner derivation induced by a. Then $d^{2n-1} = 0$ while $d^{2n-2} \neq 0$. It is easy to see that in this case $(\mathcal{A}^{d^n})^n = 0$ (see the proof of Theorem 2.1 below). In particular, every element in $\mathcal{A}^{d^n}$ is nilpotent.

This example suggests that it is natural to expect that the index of nilpotence of a nilpotent derivation d (i.e., the least integer n such that $d^n = 0$) is an odd number. Indeed, this is true for any nilpotent derivation on a 2-torsionfree semiprime ring [7]. In general, however, this does not hold. Consider the following example. Let $\mathcal{A}$ be the ring of $m \times m$ upper triangular matrices over a field. Set $a = e_{12} + e_{23} + \ldots + e_{2n-1,2n}$ where $2n \leq m$ (here, e_{ij}'s denote the matrix units). Note that $a^k = e_{1,k+1} + e_{2,k+2} + \ldots + e_{2n-k,2n}$, $k = 1, \ldots, 2n-1$, and $a^{2n} = 0$. Hence we see that $a^k \mathcal{A} a^{2n-k} = 0$, $k = 0, 1, \ldots, 2n$, which shows that $d^{2n} = 0$ where d is the inner derivation induced by a. However, $d^{2n-1} \neq 0$ (for instance, $e_{11}^{d^{2n-1}} \neq 0$). Of course, $\mathcal{A}^{d^{2n-1}} \subseteq J(\mathcal{A})$.

It is our aim now to examine the conditions in the Turovskiĭ–Shul'man result from the ring-theoretic point of view. It is immediate that Theorem 1.1 enables the reduction to $\mathcal{M}$-inner derivations when considering these conditions. Basically, such reduction, though using a different approach, has been done in [25] as well. What we would like to point out here is the utility of the property that $\mathcal{M}$-inner derivations obviously have: they leave the primitive ideal $\mathrm{ann}(\mathcal{M}) = \{a \in \mathcal{A} \mid a\mathcal{M} = 0\}$ invariant. Therefore, when arriving at an $\mathcal{M}$-inner derivation it is not always necessary to carefully study properties of the element inducing this derivation (as in [25]), but one can consider the induced derivation on the primitive ring $\mathcal{A}/\mathrm{ann}(\mathcal{M})$.

First we show that the first three conditions remain equivalent in a more general setting.

THEOREM 2.1. *Let n be a positive integer and $\mathcal{A}$ be an algebra over a field K such that either* $\mathrm{char}(K) = 0$ *or* $\mathrm{char}(K) > 2n$. *Let d be a derivation of $\mathcal{A}$. The following conditions are equivalent:*

(1) $\mathcal{A}^{d^{2n}} \subseteq J(\mathcal{A})$;
(2) $\mathcal{A}^{d^{2n-1}} \subseteq J(\mathcal{A})$;
(3) $(\mathcal{A}^{d^n})^n \subseteq J(\mathcal{A})$.

Proof. Let $\mathcal{M}$ be any simple left $\mathcal{A}$-module. Theorem 1.1 clearly shows that d must be $\mathcal{M}$-inner whenever any of the three conditions is satisfied. Therefore, $\mathrm{ann}(\mathcal{M})$ is invariant under d and so d induces a derivation on the primitive algebra $\mathcal{A}/\mathrm{ann}(\mathcal{M})$ satisfying the same condition as d. Thus, with no loss of generality we may assume that $\mathcal{A}$ is a primitive; in particular, $J(\mathcal{A}) = 0$.

Assume first that $d^{2n} = 0$. Then, since $\mathrm{char}(K) \neq 2$ and $\mathcal{A}$ is prime, [7] tells us that $d^{2n-1} = 0$. Thus (1) and (2) are equivalent.

Assume that (2) holds. Then there is $q \in Q_s(\mathcal{A})$, the symmetric ring of quotients of $\mathcal{A}$, such that $q^n = 0$ and $a^d = [q, a]$ for all $a \in \mathcal{A}$; this could be deduced from Kharchenko's characterization of algebraic derivations of prime rings [11], but more explicitly we have [12, Theorem 1]. Now, by induction on k one shows easily that $(\mathcal{A}^{d^n})^k \in \mathcal{A}q^k$ for any positive integer k. Consequently, (3) holds.

Finally, we show that (3) implies (2). Assuming (3) we see that there is a nonzero element $a_0 \in \mathcal{A}$ such that $a_0\mathcal{A}^{d^n} = 0$. For any $b, c \in \mathcal{A}$ we thus have $a_0(bc)^{d^n} = 0$, that is,

$$(1) \quad \binom{n}{1} a_0 b^{d^{n-1}} c^d + \binom{n}{2} a_0 b^{d^{n-2}} c^{d^2} + \ldots + \binom{n}{n-1} a_0 b^d c^{d^{n-1}} + a_0 b c^{d^n} = 0.$$

We claim that $a_0 b^{d^{n-k}} c^{d^{n+k-1}} = 0$ for $k = 0, 1, \ldots, n$. For $k = 0$ this holds by the initial assumption. Assuming that our claim is true for some nonnegative integer $\le k$, we deduce by replacing b by $b^{d^{n-k-1}}$ and c by c^{d^k} in (1) that it is also true for $k+1$. Thus, our claim is true indeed, and for $k = n$ we get $a_0\mathcal{A}\mathcal{A}^{d^{2n-1}} = 0$. The primeness of $\mathcal{A}$ yields $\mathcal{A}^{d^{2n-1}} = 0$. The proof is complete.

In the ring-theoretic setting, the condition (5) of the Turovskiĭ–Shul'man result is not equivalent to the first three conditions, not even for $n = 1$. The simplest way to realize this is to consider the ring of real quaternions. Every commutator in this ring is a quasi-regular element, and so the range of any nonzero inner derivation d is a thin set. However, none of the conditions (i), (ii) and (iii) is fulfilled. So, what can one expect in general if the range of a derivation is a thin set? The next result is related to [2, Theorem 7.3] and considers a more general problem. Following [2] we denote by $J_m(\mathcal{A})$, where m is a positive integer, the ideal of $\mathcal{A}$ consisting of those elements $a \in \mathcal{A}$ such that $a\mathcal{M} = 0$ for every simple left $\mathcal{A}$-module $\mathcal{M}$ with $\dim(\mathcal{M}_{\mathcal{D}}) \ge m$ where $\mathcal{D} = \mathrm{End}({}_{\mathcal{A}}\mathcal{M})$. Of course, $J_1(\mathcal{A}) = J(\mathcal{A})$.

THEOREM 2.2. *Let $\mathcal{A}$ be a ring and d be a derivation of $\mathcal{A}$. Suppose that there exist a nonnegative integer n and a thin subset $\mathcal{T}$ of $\mathcal{A}$ such that $a^n a^d \in \mathcal{A}a + \mathcal{T}$ for every $a \in \mathcal{A}$. Then $\mathcal{A}^d \subseteq J_{n+2}(\mathcal{A})$.*

Proof. Pick any simple left $\mathcal{A}$-module $\mathcal{M}$ such that $\dim(\mathcal{M}_{\mathcal{D}}) \ge n+2$ where $\mathcal{D} = \mathrm{End}({}_{\mathcal{A}}\mathcal{M})$; we show that $\mathcal{A}^d\mathcal{M} = 0$.

Assume first that d is $\mathcal{M}$-outer. Choose any $y \in \mathcal{M}$ and any $\mathcal{D}$-independent elements $x_1, \ldots, x_{n+1} \in \mathcal{M}$. If $n \ge 1$, then applying Theorem 1.1 we see that there is $a \in \mathcal{A}$ such that $a^d x_1 = x_2$, $ax_1 = 0$, $ax_i = x_{i+1}$, $i = 2, \ldots, n$, $ax_{n+1} = y$. In the case $n = 0$, just pick $a \in \mathcal{A}$ so that $ax_1 = 0$ and $a^d x_1 = y$. In any case, since $a^n a^d \in \mathcal{A}a + \mathcal{T}$, we see that there is $t \in \mathcal{T}$ such that $tx_1 = y$. But this means that $\mathcal{T}x_1 = \mathcal{M}$, a contradiction. Thus d is $\mathcal{M}$-inner, i.e., there is an additive map $T : \mathcal{M} \to \mathcal{M}$ such that $a^d x = T(ax) - a(Tx)$ for all $a \in \mathcal{A}$, $x \in \mathcal{M}$. Repeating the argument given in the proof of [2, Theorem 7.2] we see that $\mathcal{A}^d\mathcal{M} \ne 0$ implies that there is $x_1 \in \mathcal{M}$ such that x_1 and $x_2 = Tx_1$ are $\mathcal{D}$-independent. Picking any $y \in \mathcal{M}$ and any elements x_i, $i = 3, \ldots, n+2$, so that $x_1, \ldots, x_{n+2}$ are $\mathcal{D}$-independent, and then applying the Jacobson density theorem to get $a \in \mathcal{A}$ satisfying $ax_1 = 0$, $ax_i = x_{i+1}$, $i = 2, \ldots, n+1$, $ax_{n+2} = -y$, we again arrive at the contradiction $\mathcal{T}x_1 = \mathcal{M}$. The proof is thus complete.

In particular, Theorem 2.2 shows that if the range of a derivation is thin, then it is contained in $J_2(\mathcal{A})$. Hence it follows at once that if the range consists of nilpotent elements, then the derivation maps into $J(\mathcal{A})$. This fact is not new [8, Theorem 7], but the proof is. Another simple consequence is that a derivation of a Banach algebra whose range consists of quasi-nilpotent elements maps into $J(\mathcal{A})$ (cf. [14, 17, 20, 25]).

We continue with another, less obvious application of Theorem 2.2, concerned with a certain Engel type condition. Define $[b, a]_n$ where n is a positive integer as follows: $[b, a]_1 = [b, a]$ and $[b, a]_n = [[b, a]_{n-1}, a]$ for $n > 1$.

THEOREM 2.3. *Let d be a continuous derivation of a Banach algebra $\mathcal{A}$. Suppose there is a positive integer n such that $[a^d, a]_n$ is quasi-nilpotent for every $a \in \mathcal{A}$. Then $\mathcal{A}^d \subseteq J(\mathcal{A})$.*

Proof. Our assumption yields that $a^n a^d \in \mathcal{A}a + Q(\mathcal{A})$, $a \in \mathcal{A}$. In view of Theorem 2.2 it now suffices to show that $\mathcal{A}^d$ annihilates modules of dimension at most $n+1$. Let $\mathcal{M}$ be such a module. Since d leaves $\mathrm{ann}(\mathcal{M})$ invariant by Sinclair's theorem [22], d induces a derivation on the quotient algebra $\mathcal{A}/\mathrm{ann}(\mathcal{M})$ which also satisfies the condition of the theorem. As $\mathcal{A}/\mathrm{ann}(\mathcal{M})$ is a finite-dimensional primitive (complex) algebra, it is isomorphic to a matrix algebra $M_k(\mathbb{C})$ with $1 \le k \le n+1$. We have to show that the induced derivation is zero.

Thus, there is no loss of generality in proving the theorem for the special case when $\mathcal{A} = M_k(\mathbb{C})$. Of course, we may assume that $k > 1$. It is well known that every derivation of $\mathcal{A}$ is inner, so that $a^d = [a_0, a]$ for some $a_0 \in \mathcal{A}$. Thus, the initial assumption can be written as $r([[\ldots[[a_0, a], a]\ldots], a]) = 0$ for every $a \in \mathcal{A}$, where $r(\cdot)$ denotes the spectral radius. Following [4] we replace a by $\lambda a + a_0$ in this relation, where $\lambda \in \mathbb{C}$, and arrive at

$$|\lambda| r([[\ldots[[a_0, a], a_0]\ldots], a_0] + \ldots + \lambda^{n-1}[[\ldots[[a_0, a], a]\ldots], a])) = 0.$$

Thus, the function

$$\lambda \mapsto r([[\ldots[[a_0, a], a_0]\ldots], a_0] + \ldots + \lambda^{n-1}[[\ldots[[a_0, a], a]\ldots], a]))$$

equals 0 for every $\lambda \ne 0$. However, since this function is subharmonic by Vesentini's theorem [1, Theorem 3.4.7], it must equal 0 at $\lambda = 0$ as well.

This means that

$$r([[\ldots[[a_0,a],a_0]\ldots],a_0]) = 0 \quad \text{for every } a \in \mathcal{A},$$

i.e., $\mathcal{A}^{d^{n+1}} \subseteq Q(\mathcal{A})$ (of course, quasi-nilpotent matrices are just nilpotent matrices). Now, the result of Turovskiĭ and Shul'man [25] mentioned above implies that $d^{2n+1}=0$. As in the proof of Theorem 2.1 we now apply a result on nilpotent derivations [12] (or more directly, [10] or [15]) to conclude that there is a scalar matrix λ such that the matrix $a_0-\lambda$ is nilpotent. Since $a^d=[a_0,a]=[a_0-\lambda,a]$ there is no loss of generality in assuming that a_0 is nilpotent. Also, we may assume that a_0 is in its Jordan canonical form. In particular, it is then a strictly upper triangular matrix. Suppose that $a_0 \neq 0$. Permuting Jordan blocks, if necessary, we may then assume that its upper left 2×2 block is $\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$. Let a be the matrix whose upper left 2×2 block is $\left[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right]$ and has zero entries elsewhere. By assumption, the matrix $[a^d,a]_n$ is nilpotent. Since a is an idempotent matrix, we see that either $[a^d,a]_n=[a_0,a]$ (if n is even) or $[a^d,a]_n=[[a_0,a],a]$ (if n is odd). In any case, $[a^d,a]_n$ has a nonzero upper left 2×2 block and has zero entries elsewhere. The upper left 2×2 block is either $\left[\begin{smallmatrix}1&-1\\0&-1\end{smallmatrix}\right]$ (if n is even) or $\left[\begin{smallmatrix}-1&1\\-2&1\end{smallmatrix}\right]$ (if n is odd). But then $[a^d,a]_n$ is not nilpotent. This contradiction shows that $a_0=0$.

For $n=1$, Theorem 2.3 was proved in [4]. We also mention here a ring-theoretic result of Lanski [13] which considers the condition $[a^d,a]_n=0$.

The continuity of the derivation in Theorem 2.3 has been used in the proof at one point only, when using Sinclair's result on invariance of primitive ideals under continuous derivations. The problem whether the continuity in this result can be removed seems to be extremely difficult and is known as the noncommutative Singer–Wermer conjecture. It is known to be true in commutative Banach algebras, as shown by Thomas [24]. Actually, he proved that every derivation of a commutative Banach algebra $\mathcal{A}$ maps into $J(\mathcal{A})$, thereby extending the Singer–Wermer theorem to discontinuous derivations and answering the classical Singer–Wermer conjecture. The solution of the noncommutative Singer–Wermer conjecture would be the most natural and certainly the most desirable extension of Thomas' theorem. But there are other ways to get noncommutative versions of this result [18, 21]. We conclude this paper with a result of that type, dealing with (not necessarily continuous) derivations of (not necessarily commutative) Banach algebras whose range is a commutative set. We remark that derivations having commutative range have also been considered in a pure ring-theoretic setting by Herstein [9]. He showed, in particular, that the existence of such a derivation on a prime ring $\mathcal{A}$ with $\operatorname{char}(\mathcal{A})\neq 2$ forces $\mathcal{A}$ to be commutative (actually, it takes just a few lines to verify this by an elementary computation). Let us now consider the case when $\mathcal{A}$ is an arbitrary ring.

LEMMA 2.4. *If the range of a derivation d of a ring $\mathcal{A}$ is a commutative subset of $\mathcal{A}$, then $2\mathcal{A}^d \subseteq J_2(\mathcal{A})$.*

Proof. Let $\mathcal{M}$ be any simple left $\mathcal{A}$-module $\mathcal{M}$ with $\dim(\mathcal{M}_\mathcal{D}) \geq 2$ where $\mathcal{D}=\operatorname{End}({}_\mathcal{A}\mathcal{M})$. We have to show that $2\mathcal{A}^d\mathcal{M}=0$. Suppose first that d is $\mathcal{M}$-outer. Pick any $\mathcal{D}$-independent elements $x_1,x_2\in\mathcal{M}$. Then, by Theorem 1.1, there are $a,b\in\mathcal{A}$ such that $a^dx_1=x_2$, $b^dx_1=0$ and $b^dx_2\neq 0$. But then $b^da^dx_1\neq 0$ while $a^db^dx_1=0$, which clearly contradicts our assumption that $a^db^d=b^da^d$. Thus, d is $\mathcal{M}$-inner, which implies that $\operatorname{ann}(\mathcal{M})$ is invariant under d and so d induces a derivation $\widehat{d}$ on a primitive noncommutative (namely, $\dim(\mathcal{M}_\mathcal{D})\geq 2$) ring $\mathcal{A}/\operatorname{ann}(\mathcal{M})$. Clearly, the range of $\widehat{d}$ is commutative. But then Herstein's result mentioned above implies that $2\widehat{d}=0$, that is, $2\mathcal{A}^d\mathcal{M}=0$.

THEOREM 2.5. *If the range of a derivation d of a Banach algebra $\mathcal{A}$ is a commutative subset of $\mathcal{A}$, then $\mathcal{A}^d\subseteq J(\mathcal{A})$.*

Proof. In view of Lemma 2.4 it suffices to show that $\mathcal{A}^d$ annihilates one-dimensional modules, or equivalently, that $\mathcal{A}^d$ lies in the kernel of each multiplicative functional of $\mathcal{A}$. Let ϕ be a multiplicative functional on $\mathcal{A}$.

Let $\mathcal{C}$ be the closed subalgebra of $\mathcal{A}$ generated by $\mathcal{A}^d$. Since $\mathcal{A}^d\subseteq\mathcal{C}$, the restriction of d to $\mathcal{C}$ is a derivation of $\mathcal{C}$. As $\mathcal{C}$ is a commutative Banach algebra, $\mathcal{C}^d\subseteq J(\mathcal{C})$ by Thomas' theorem [24]. In particular, $\mathcal{A}^{d^2}=(\mathcal{A}^d)^d\subseteq J(\mathcal{C})$, which implies that $\mathcal{A}^{d^2}$ consists of quasi-nilpotent elements. Consequently, $\phi(d^2(\mathcal{A}))=0$ and so $\phi(a^d)^2=\phi((a^d)^2)=\frac{1}{2}\phi((a^2)^{d^2}-a^{d^2}a-aa^{d^2})=0$. Hence $\phi(a^d)=0$ for any $a\in\mathcal{A}$ and the proof is complete.

It should be mentioned that one can also prove Theorem 2.5 using a different approach as developed in [18, 21]. The main goal of the present paper, however, is to illustrate how the new techniques can be applied.

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Department of Mathematics
National Cheng-Kung University
Tainan, Taiwan
E-mail: beidar@mail.ncku.edu.tw

Department of Mathematics
University of Maribor
Maribor, Slovenia
E-mail: bresar@uni-mb.si

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