[L1] I. Labuda, *Denumerability conditions and Orlicz–Pettis type theorems*, Comment. Math. 18 (1974), 45–49.

[L2] —, *Submeasures and locally solid topologies on Riesz spaces*, Math. Z. 195 (1987), 179–196.

[L3] —, *Spaces of measurable functions*, Comment. Math., Tomus spec. in honorem Ladislai Orlicz II 1979, 217–249.

[MO] W. Matuszewska and W. Orlicz, *A note on modular spaces. IX*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 16 (1968), 801–808.

[Mu] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math. 1034, Springer, Berlin, 1983.

[NS] D. Noll and W. Stadler, *Abstract sliding hump technique and characterization of barrelled spaces*, Studia Math. 94 (1989), 103–120.

[O] W. Orlicz, *On perfectly convergent series in certain function spaces*, Prace Mat. 1 (1955), 393–414 (in Polish); English transl. in: W. Orlicz, Collected Papers, Part I, Polish Sci. Publ., Warszawa, 1988, 830–850.

[P] M. Paštéka, *Convergence of series and submeasures on the set of positive integers*, Math. Slovaca 40 (1990), 273–278.

[R] S. Rolewicz, *Metric Linear Spaces*, Polish Sci. Publ. & Reidel, Warszawa & Dordrecht, 1984.

[SF] J. J. Sember and A. R. Freedman, *On summing sequences of 0's and 1's*, Rocky Mountain J. Math. 11 (1981), 419–425.

[T] P. Turpin, *Convexités dans les espaces vectoriels topologiques généraux*, Dissertationes Math. 131 (1976).

[W] W. Wnuk, *Representations of Orlicz lattices*, ibid. 235 (1984).

Faculty of Mathematics and Computer Sciences
A. Mickiewicz University
Matejki 48/49
60-769 Poznań, Poland
E-mail: drewlech@amu.edu.pl

Department of Mathematics
University of Mississippi
University, MS 38677, U.S.A.
E-mail: mmlabuda@hilbert.math.olemiss.edu

Received February 1, 1999
Revised version October 13, 1999 (4260)

STUDIA MATHEMATICA 138 (1) (2000)

Extreme points of the complex binary trilinear ball

by

FERNANDO COBOS (Madrid), THOMAS KÜHN (Leipzig)
and JAAK PEETRE (Lund)

Abstract. We characterize all the extreme points of the unit ball in the space of trilinear forms on the Hilbert space $\mathbb{C}^2$. This answers a question posed by R. Grząślewicz and K. John [7], who solved the corresponding problem for the real Hilbert space $\mathbb{R}^2$. As an application we determine the best constant in the inequality between the Hilbert–Schmidt norm and the norm of trilinear forms.

It is well known that the extreme points of the unit ball in the space $\mathcal{L}(H)$ of all bounded linear operators on a Hilbert space H are just isometries or coisometries (see [8]). For real Hilbert spaces H, $\mathcal{L}(H)$ can be identified with the space $\mathcal{B}(H,H)$ of all bounded bilinear forms on H. This leads in a natural way to the problem of characterizing extreme points of the unit ball of multilinear forms. In the case of trilinear forms on $H=\mathbb{R}^2$ this question was solved by R. Grząślewicz and K. John [7]. The complex case, where $H=\mathbb{C}^2$, was left there as an open problem (see [7], Remark 5). Accordingly, we prove here such a result. As an application we compute the exact value of the best constant d in the inequality $\|T\|_2 \le d\|T\|$ between the Hilbert–Schmidt norm and the norm of a trilinear form T in the binary case, thus complementing our previous results in [3] for the n-ary case, where the asymptotic behaviour of these constants was investigated. For more background material about trilinear forms we refer to [4].

Let $\mathcal{B}(H,H,H)$ be the space of all trilinear forms $T : H\times H\times H \to \mathbb{C}$ equipped with the norm

$$\|T\| = \sup\{|T(x,y,z)| : \|x\| = \|y\| = \|z\| = 1\}.$$

Our main results are the following.

THEOREM 1. *For a trilinear form $T : H\times H\times H \to \mathbb{C}$ on the Hilbert space $H=\mathbb{C}^2$ one has $\|T\|=1$ if and only if there are three orthonormal bases*

2000 *Mathematics Subject Classification*: 46B20, 47A30, 15A69.

The first- and second-named authors supported in part by DGES (PB97-0254).

$\{e_1, e_2\}, \{f_1, f_2\}, \{g_1, g_2\}$ *of* H *and complex numbers* b_1, b_2, b_3, c *such that, if* $T_{jkl} \equiv T(e_j, f_k, g_l)$, *then* $T_{111} = 1, T_{112} = T_{121} = T_{211} = 0, T_{122} = b_1$, $T_{212} = b_2$, $T_{221} = b_3$, $T_{222} = c$ *and* $F \equiv |b_1|^2 + |b_2|^2 + |b_3|^2 + |c|^2/2 + |X| \leq 1$, *where we have set* $X \equiv 2b_1b_2b_3 + c^2/2$.

THEOREM 2. *A trilinear form* T *as in Theorem 1 is an extreme point of the unit ball of* $\mathcal{B}(H,H,H)$ *if and only if* $F = 1$, *and in addition either the equality* $X = 0$ *or the strict inequality* $|X - c^2/2| < |X| + |c|^2/2$ *holds.*

Proof of Theorem 1. If $\|T\| = 1$, then by compactness of the unit sphere of H there are vectors e_1, f_1, g_1 of norm one with $T(e_1, f_1, g_1) = 1$. Choose now vectors e_2, f_2, g_2 in the unit sphere of H such that $e_2 \perp e_1$, $f_2 \perp f_1$ and $g_2 \perp g_1$. Then $\|T\| = 1$ yields

$$|T(xe_1 + e_2, f_1, g_1)|^2 = |x + T_{211}|^2 = |x|^2 + 2\operatorname{Re} x\overline{T}_{211} + |T_{211}|^2$$
$$\leq |x|^2 + 1 \quad \forall x \in \mathbb{C},$$

whence $T_{211} = 0$, and similarly we get $T_{121} = T_{112} = 0$. Observe that the choice of the second vectors in the three bases is only unique up to multiplication by arbitrary unimodular complex numbers. Setting $\varphi_1 = \arg T_{122}$, $\varphi_2 = \arg T_{212}$, $\varphi_3 = \arg T_{221}$ and replacing e_2, f_2, g_2 by $e_2' = e^{i\alpha_1}e_2$, $f_2' = e^{i\alpha_2}f_2$, $g_2' = e^{i\alpha_3}g_2$ with $\alpha_j = \varphi_j - \frac{1}{2}\sum_{k=1}^{3}\varphi_k$ we get $T_{122}' \equiv T(e_1, f_2', g_2') = e^{i(\alpha_2+\alpha_3)}T_{122} = |T_{122}|$, and similarly $T_{212}' = |T_{212}|$, $T_{221}' = |T_{221}|$. So we can and do assume that the three complex numbers b_j are even real numbers with $0 \leq b_j \leq 1$. This will simplify some of our further arguments.

Clearly, every trilinear form T with $T_{111} = 1$ is of norm $\|T\| \geq 1$. So it remains to show that the condition $F \leq 1$ is equivalent to $\|T\| \leq 1$. Note that the latter inequality means that

$$|xyz + b_1x + b_2y + b_3z + c|^2 \leq (1 + |x|^2)(1 + |y|^2)(1 + |z|^2) \quad \forall x, y, z \in \mathbb{C},$$

which in turn is equivalent, by Cauchy's inequality, to

$$(1) \qquad |yz + b_1|^2 + |b_2y + b_3z + c|^2 \leq (1 + |y|^2)(1 + |z|^2) \quad \forall y, z \in \mathbb{C}.$$

First we show the equivalence of (1) and $F \leq 1$ in the special cases where one of the b_j's, say b_1, equals either 0 or 1.

CASE $b_1 = 0$. Here (1) simplifies to

$$|b_2y + b_3z + c|^2 \leq 1 + |y|^2 + |z|^2 \quad \forall y, z \in \mathbb{C},$$

which is, again by Cauchy's inequality, equivalent to

$$|b_2|^2 + |b_3|^2 + |c|^2 \leq 1,$$

but this is just $F \leq 1$ for $b_1 = 0$.

CASE $b_1 = 1$. Upon writing $|yz + 1|^2 = |yz|^2 + 2\operatorname{Re} yz + 1$, expanding the right-hand side of (1) and collecting terms, inequality (1) reads

$$(2) \qquad |b_2y + b_3z + c|^2 \leq |y|^2 - 2\operatorname{Re} yz + |z|^2 = |y - \overline{z}|^2 \quad \forall y, z \in \mathbb{C},$$

which is equivalent to $b_2 = b_3 = c = 0$. (One implication is trivial, the other follows by taking the special values $y = \overline{z} = 0, 1, i$ in (2).) But, for $b_1 = 1$, the condition $F \leq 1$ is equivalent to $b_2 = b_3 = c = 0$.

Next we show that in the remaining case, where $0 < b_j < 1$ for $j = 1, 2, 3$, inequality (1) is equivalent to the non-negative definiteness of a certain real (3×3)-matrix.

Expanding both sides of (1) we can rewrite it as

$$(3) \qquad \underbrace{(1 - b_2^2)}_{P}|y|^2 - 2\operatorname{Re}\underbrace{(b_1z + b_2\overline{(b_3z + c)})}_{Q}y + \underbrace{(1 + |z|^2 - b_1^2 - |b_3z + c|^2)}_{R} \geq 0 \quad \forall y, z \in \mathbb{C}.$$

Since $P = 1 - b_2^2 > 0$, (3) is equivalent to $PR \geq |Q|^2$, that is to say, to the inequality

$$(4) \qquad (1 - b_2)^2(1 + |z|^2 - b_1^2 - |b_3z + c|^2) \geq |b_1z + b_2\overline{(b_3z + c)}| \quad \forall z \in \mathbb{C}.$$

After expanding and collecting terms, (4) takes the form

$$(5) \qquad A|z|^2 - B\operatorname{Re} z^2 - 2\operatorname{Re} Cz + D \geq 0 \quad \forall z \in \mathbb{C},$$

where we have set

$$A \equiv 1 - b_1^2 - b_2^2 - b_3^2,$$
$$B \equiv 2b_1b_2b_3,$$
$$C \equiv b_1b_2c + b_3\overline{c},$$
$$D \equiv (1 - b_1^2)(1 - b_2^2) - |c|^2.$$

Thanks to our particular choice of the three bases the quantities A, B, D are *real* numbers. Splitting z and C in real and imaginary parts,

$$z = x + iy, \qquad C = C_1 + iC_2,$$

and using the relations $\operatorname{Re} z^2 = x^2 - y^2$, $\operatorname{Re} Cz = C_1x - C_2y$ we can transform (5) into

$$(A - B)x^2 + (A + B)y^2 - 2C_1x + 2C_2y + D \geq 0 \quad \forall x, y \in \mathbb{R},$$

which means that the real (3×3)-matrix

$$M \equiv \begin{pmatrix} A - B & 0 & -C_1 \\ 0 & A + B & C_2 \\ -C_1 & C_2 & D \end{pmatrix}$$

is non-negative definite ($M \geq 0$ for short), meaning that $\langle Mu, u\rangle \geq 0$ $\forall u \in \mathbb{R}^3$, where $\langle\cdot,\cdot\rangle$ denotes the standard inner product. So it remains to prove that *the conditions* $M \geq 0$ *and* $F \leq 1$ *are equivalent*. For the proof of this equivalence we can assume that we additionally have

$$A \geq B\ (>0) \quad \text{and} \quad A \geq |c|^2/2, \tag{6}$$

since either of the conditions $M \geq 0$ and $F \leq 1$ implies (6). Indeed, if $F \leq 1$, then clearly $A \geq |c|^2/2$. Moreover, by the triangle inequality, we obtain

$$A = 1 - \sum_{j=1}^{3} b_j^2 \geq \frac{|c|^2}{2} + |X| = \frac{|c|^2}{2} + \left|B + \frac{c^2}{2}\right| \geq B,$$

where equality holds if and only if

$$-c^2 = |c|^2 \leq 2B. \tag{7}$$

On the other hand, if $M \geq 0$, then all principal minors of M are ≥ 0, in particular $A - B \geq 0$, whence $A \geq B$. For the second-order minors we get

$$(A-B)D - C_1^2 \geq 0, \qquad (A+B)D - C_2^2 \geq 0,$$

and consequently

$$2AD \geq C_1^2 + C_2^2 = |C|^2. \tag{8}$$

Let us introduce the quantity $E \equiv b_1^2 b_2^2 + b_3^2$. Since $D + |c|^2 = A + E = (1-b_1^2)(1-b_2^2) > 0$, inequality (8) implies

$$\begin{aligned} 2A(A+E) &= 2A(D+|c|^2) \geq |C|^2 + 2A|c|^2 \\ &= E|c|^2 + \underbrace{B\operatorname{Re} c^2}_{\geq -A|c|^2} + 2A|c|^2 \geq (A+E)|c|^2, \end{aligned}$$

therefore $A \geq |c|^2/2$.

Next we prove that $\det M \geq 0$ *is equivalent to* $F \leq 1$. We have

$$\begin{aligned} \det M &= (A^2 - B^2)D - C_1^2(A+B) - C_2^2(A-B) \\ &= (A^2-B^2)D - A|C|^2 - B\operatorname{Re} C^2. \end{aligned}$$

Taking into account that

$$|C|^2 = E|c|^2 + B\operatorname{Re} c^2 \quad \text{and} \quad \operatorname{Re} C^2 = B|c|^2 + E\operatorname{Re} c^2$$

we see that $\det M \geq 0$ is equivalent to

$$(A^2-B^2)D \geq A(E|c|^2 + B\operatorname{Re} c^2) + B(B|c|^2 + E\operatorname{Re} c^2).$$

Adding now $(A^2-B^2)|c|^2$ to both sides, we get

$$(A^2-B^2)(A+E) = (A^2-B^2)(D+|c|^2) \geq (A|c|^2 + B\operatorname{Re} c^2)(A+E).$$

Dividing by $A+E$ (>0) and adding $|c|^4/4$ implies

$$\begin{aligned} (A - |c|^2/2)^2 &= A^2 - A|c|^2 + |c|^4/4 \\ &\geq B^2 + B\operatorname{Re} c^2 + |c|^4/4 = |B + c^2/2|^2 = |X|^2. \end{aligned}$$

Taking square roots (observe that we have $A \geq |c|^2/2$) we arrive at $A - |c|^2/2 \geq |X|$, but this is just $F \leq 1$. Since all operations can be reversed, we have shown that

$$\det M \geq 0 \quad \text{if and only if} \quad F \leq 1.$$

Finally, we prove the *equivalence of* $M \geq 0$ *and* $F \leq 1$. Note that without loss of generality we can do this under the additional assumption that the inequalities (6) hold. If $M \geq 0$, then clearly $\det M \geq 0$, but this implies $F \leq 1$, as we have already shown. Now let $F \leq 1$; this gives $\det M \geq 0$. We distinguish two cases.

Strict inequality $A > B$. Then we have the following chain of principal minors of the matrix M:

$$\begin{aligned} A - B &> 0 \quad \text{(first order minor)}, \\ A^2 - B^2 &> 0 \quad \text{(second order minor)}, \\ \det M &\geq 0 \quad \text{(third order minor)}, \end{aligned}$$

whence $M \geq 0$.

Equality $A = B$. As observed earlier (see (7)), we have equality in the inequality $A \geq B$ if and only if $-c^2 = |c|^2$, whence $\operatorname{Re} c = 0$ and $C_1 = \operatorname{Re} C = \operatorname{Re}(b_1 b_2 c + b_3 \overline{c}) = 0$ as well, so that in this case

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2A & C_2 \\ 0 & C_2 & D \end{pmatrix}$$

A sufficient condition for $M \geq 0$ is then

$$2A > 0 \quad \text{and} \quad 2AD - C_2^2 \geq 0.$$

The first inequality follows from

$$2A = 2B = 4b_1 b_2 b_3 > 0 \quad (\text{since all } b_j > 0).$$

On the other hand, $\operatorname{Re} C = 0$ implies

$$C_2^2 = |C|^2 = E|c|^2 + B\operatorname{Re} c^2 = E|c|^2 - A|c|^2,$$

which gives

$$\begin{aligned} 2AD - C_2^2 &= 2A(A + E - |c|^2) - E|c|^2 + A|c|^2 \\ &= (A+E)(2A - |c|^2) \geq 0, \end{aligned}$$

because $A + E = (1-b_1^2)(1-b_2^2) > 0$, and $2A - |c|^2 \geq 0$ by (6). The proof of Theorem 1 is finished. ∎

Proof of Theorem 2. Let T be in the unit ball of $\mathcal{B}(H,H,H)$, and let $w \equiv (b_1, b_2, b_3, c) \in \mathbb{C}^4$ be the vector of its coefficients with respect to the three orthonormal bases. Again we assume without loss of generality that the b_j's are *real* numbers with $0 \le b_j \le 1$. Then $F(w) \le 1$ by Theorem 1. If we even have strict inequality, then the same holds for a certain neighbourhood of w in $\mathbb{C}^4$, in particular w could be moved along a straight line without violating $F \le 1$, whence T cannot be an extreme point.

Assume now that $F(w) = 1$ and that T is not extremal, so it can be represented as a non-trivial convex linear combination

$$T = (1-\theta)T' + \theta T''$$

with $0 < \theta < 1$ and $\|T'\| \le 1$, $\|T''\| \le 1$. This implies

$$1 = T(e_1, f_1, g_1) = (1-\theta)T'(e_1, f_1, g_1) + \theta T''(e_1, f_1, g_1)$$

and therefore $T'_{111} = T''_{111} = 1$ as well. The same argument as in the proof of Theorem 1 yields now $T'_{211} = T'_{121} = T'_{112} = 0$, and similarly for T''. Let w' and w'' be the coefficient vectors of T' and T'' in $\mathbb{C}^4$. This means that w is an inner point of the segment with endpoints w' and w'' in

$$G \equiv \{u \in \mathbb{C}^4 : F(u) \le 1\},$$

so w is not an extreme point of the subset G of $\mathbb{C}^4$. Conversely, it is obvious that a non-extremal point $w \in G$ cannot correspond to an extremal trilinear form T of the unit ball of $\mathcal{B}(H,H,H)$. Thus we have established that T is extremal if and only if its coefficient vector w (with respect to our three orthonormal bases) is an extreme point of the set G. So it only remains to determine all these extreme points. Note that G is convex, since it can be identified with the intersection of two convex sets, namely the unit ball of $\mathcal{B}(H,H,H)$ and the affine manifold of all trilinear forms R with coefficients $R_{111} = 1$, $R_{211} = R_{121} = R_{112} = 0$ with respect to the three bases.

Given any $w \in G$ with $F(w) = 1$, we must now decide whether or not it is an extreme point of G. We distinguish three cases.

CASE 1: $X = 0$. Suppose that $w = (1-\theta)w' + \theta w''$ for some $0 < \theta < 1$ and $w', w'' \in G$. By convexity we get

$$\begin{aligned} 1 = F(w) &= \sum_{j=1}^{3} |b_j|^2 + \frac{|c|^2}{2} \\ &\le (1-\theta)\Big(\sum_{j=1}^{3} |b'_j|^2 + \frac{|c'|^2}{2}\Big) + \theta\Big(\sum_{j=1}^{3} |b''_j|^2 + \frac{|c''|^2}{2}\Big) \\ &\le (1-\theta)F(w') + \theta F(w'') \le 1. \end{aligned}$$

But the convexity is even *strict*, therefore $w = w' = w''$, and w is an extreme point.

CASE 2: $|X - c^2/2| = |X| + |c|^2/2$ and $X \ne 0$. The first condition implies that the three complex numbers $X - c^2/2 = 2b_1b_2b_3$, X and $-c^2/2$ have the same argument, so they are all non-negative real numbers. The second condition gives then $0 < X = 2b_1b_2b_3 + c^2/2$, whence $c = it$ for some real t with $|t| < r \equiv 2\sqrt{b_1b_2b_3}$, so w is a non-trivial convex linear combination of the points $w_\pm \equiv (b_1, b_2, b_3, \pm ir)$. Moreover, we have

$$\begin{aligned} F(w_\pm) &= \sum_{j=1}^{3} b_j^2 + \frac{r^2}{2} + \Big|2b_1b_2b_3 - \frac{r^2}{2}\Big| = \sum_{j=1}^{3} b_j^2 + 2b_1b_2b_3 \\ &= \sum_{j=1}^{3} b_j^2 + |X| + \frac{|c|^2}{2} = F(w) = 1, \end{aligned}$$

whence $w_\pm \in G$, and w is not extremal.

CASE 3: $|X - c^2/2| < |X| + |c|^2/2$. First we remark that this condition implies that $X \ne 0, c \ne 0$ and $b_1 < 1$. Indeed, $X = 0$ or $c = 0$ obviously contradict the condition, and $b_1 = 1$ (together with $F(w) = 1$) would again yield $c = 0$.

Assume now that w belongs to a segment which is contained in G, without loss of generality let w be the midpoint. Since $F(w) = 1$, the point w belongs to the boundary $\partial G = \{u \in \mathbb{C}^4 : F(u) = 1\}$ of the convex set G, so the whole segment is contained in ∂G. Let $w_t \equiv w + at$ be this segment, where $a = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$ and t is a real parameter, say $t \in [-1, 1]$. So we have

$$(9) \qquad 1 = F(w_t) = \sum_{j=1}^{3} |b_j + a_j t|^2 + \frac{|c + a_4 t|^2}{2} + |Y(t)| \quad \forall t \in [-1, 1]$$

where we have set

$$(10) \qquad Y(t) \equiv 2(b_1 + a_1t)(b_2 + a_2t)(b_3 + a_3t) + (c + a_4t)^2/2.$$

In order to prove the extremality of w for G we have to show that $a = 0$. Solving (9) for $|Y(t)|$ we see that this is a quadratic polynomial in t, so $|Y(t)|^2$ is a polynomial of degree ≤ 4. From (10) we obtain

$$|Y(t)|^2 = Y(t)\overline{Y(t)} = (2a_1a_2a_3)^2t^6 + \text{lower order terms},$$

and we conclude that $a_1a_2a_3 = 0$. So at least one of the factors must vanish, say $a_1 = 0$. Letting $Y_\pm \equiv Y(\pm 1)$, the triangle inequality gives the following estimate:

$$(11) \qquad \begin{aligned} |Y_+| + |Y_-| &\ge |Y_+ + Y_-| = |2X + 4b_1a_2a_3 + a_4^2| \\ &\ge 2|X| - 4b_1|a_2a_3| - |a_4|^2. \end{aligned}$$

Taking now the sum of equations (9) for $t = 1$ and $t = -1$, inequality (11) yields

$$(12)\qquad 2 = F(w_-) + F(w_+) = 2\sum_{j=1}^{3} b_j^2 + |c|^2 + 2|a_2|^2 + 2|a_3|^2 + |a_4|^2 + |Y_+| + |Y_-| \ge 2F(w) + 2(1-b_1)(|a_2|^2 + |a_3|^2),$$

where we have used moreover the trivial estimate $2|a_2a_3| \le |a_2|^2 + |a_3|^2$.

Since $F(w) = 1$, inequality (12) implies that $a_2 = a_3 = 0$ (remember that in our case $b_1 < 1$) and that, in addition, we even have equality in (11). Since $a_2 = a_3 = 0$, this holds if and only if the four complex numbers

$$Y_+ = 2X + ca_4 + \frac{a_4^2}{2}, \quad Y_- = 2X - ca_4 + \frac{a_4^2}{2}, \quad -a_4^2, \quad X\ (\neq 0)$$

have the same argument, say $Y_+ = \alpha X$, $Y_- = \beta X, -a_4^2 = \gamma X$ for certain real numbers $\alpha, \beta, \gamma \ge 0$. This gives

$$(13)\qquad ca_4 = \frac{1}{2}(Y_+ - Y_-) = \frac{\alpha - \beta}{2}X.$$

If $\alpha \neq \beta$, then we obtain

$$-c^2\gamma X = c^2 a_4^2 = \left(\frac{\alpha-\beta}{2}\right)^2 X^2,$$

and from $c \neq 0$, $X \neq 0$ we conclude that also $\gamma \neq 0$ and that

$$-\frac{c^2}{2} = \delta X, \quad \text{where} \quad \delta = \frac{(\alpha-\beta)^2}{8\gamma} > 0.$$

This implies the contradiction

$$|X - c^2/2| = (1+\delta)|X| = |X| + |c|^2/2,$$

so we must have $\alpha = \beta$. Then (13) gives $ca_4 = 0$, and $c \neq 0$ yields $a_4 = 0$. This shows $a = 0$, and the proof is finished. ∎

REMARK 3: *The real case.* The analogous results for trilinear forms on the real Hilbert space $\mathbb{R}^2$, found in [7], are the following:

(i) $\|T\| = 1$ if and only if the (real) numbers b_j and c satisfy $|b_j| \le 1$, $|c| \le 1$ and $L \equiv b_1^2 + b_2^2 + b_3^2 + 2b_1b_2b_3 + c^2 \le 1$.

(ii) T is an extreme point if and only if $L = 1$, and either $|b_j| < 1$ for all j or $|b_j| = 1$ for all j.

REMARK 4: *The rôle of the quantity X.* In order to clarify the significance of the quantity $X = 2b_1b_2b_3 + c^2/2$ we now show that X has an invariant-theoretic meaning. Every trilinear form

$$T = T(x,y,z) = \sum_{j,k,l=1}^{2} a_{jkl}x_jy_kz_l$$

on $\mathbb{C}^2$ can be viewed as a bilinear form T_x whose coefficients depend on x:

$$T_x = T_x(y,z) = \sum_{k,l=1}^{2}\Big(\sum_{j=1}^{2} a_{jkl}x_j\Big)y_kz_l.$$

Let $A(x)$ denote the corresponding (2×2)-matrix of its coefficients. Then the determinant $\Delta(x) \equiv \det A(x)$ is an invariant of T_x (with respect to non-singular linear transformations of the variables y and z). Since $\Delta(x)$ is a 2-homogeneous polynomial in x, say $\Delta(x) = Ax_1^2 + Bx_1x_2 + Cx_2^2$, its discriminant $D \equiv B^2 - 4AC$ is an invariant (with respect to non-singular linear transformations of the variable x). Consequently, $J \equiv D(\Delta(x))$ is an invariant of the binary trilinear form T. It was shown by E. Schwartz [9] in 1922 that there are no other invariants. Actually, the invariant Δ had already been introduced by Cayley [1], who named it the *hyperdeterminant*, as early as 1843; it has been rediscovered many times (see [6]). According to the first part of the proof of Theorem 1, every complex binary trilinear form T with $\|T\| = 1$ can be represented in the special shape

$$T = x_1y_1z_1 + b_1x_1y_2z_2 + b_2x_2y_1z_2 + b_3x_2y_2z_1 + cx_2y_2z_2.$$

Therefore

$$A(x) = \begin{pmatrix} x_1 & b_3x_2 \\ b_2x_2 & b_1x_1 + cx_2 \end{pmatrix}, \qquad \Delta(x) = b_1x_1^2 + cx_1x_2 - b_2b_3x_2^2,$$

and the invariant is $J = c^2 + 4b_1b_2b_3$, which is X up to a factor 2: $J = 2X$.

For an elementary introduction to invariant theory the reader is referred to [5].

REMARK 5: *Normal shape of trilinear forms.* Let us return to the first step of the proof of Theorem 1, that is to say, to the fact that for any given trilinear form T on $\mathbb{C}^2$ there are three orthonormal bases of $\mathbb{C}^2$ such that the corresponding coefficients of T satisfy $T_{111} = \|T\|$ and $T_{112} = T_{121} = T_{211} = 0$.

More generally, when considering a trilinear form T on the product $H_1 \times H_2 \times H_3$ of three arbitrary Hilbert spaces (over the same field) one can iterate our arguments from the complex binary case obtaining, in a quasi-unique way, three orthonormal bases of H_1, H_2 and H_3 and a special representation of T, called the normal shape. This procedure works even for infinite-dimensional Hilbert spaces, provided that the form T is compact. For more details we refer to [2], where the normal shape was introduced.

REMARK 6: *Comparison with the Hilbert–Schmidt norm.* Besides $\|T\|$ there are other natural norms to be considered in $\mathcal{B}(H,H,H)$ as the Hilbert–Schmidt norm. For a trilinear form T on an arbitrary Hilbert space H with $\dim H<\infty$ we set

$$\|T\|_2=\Big(\sum_{j,k,l}|T(e_j,f_k,g_l)|^2\Big)^{1/2},$$

where $\{e_j\},\{f_k\}$ and $\{g_l\}$ are fixed orthonormal bases of H. It turns out that this quantity does not depend on the special choice of the bases and defines a norm on $\mathcal{B}(H,H,H)$, namely the Hilbert–Schmidt norm. Clearly, we always have $\|T\|\le\|T\|_2$.

In [3] we investigated the best constant d in the inequality $\|T\|_2\le d\|T\|$, where T is any trilinear form on a given finite-dimensional Hilbert space H. Clearly, for $H=\mathbb{K}^n$ with $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, we have

$$d=d(n,\mathbb{K})=\sup\{\|T\|_2:\|T:\mathbb{K}^n\times\mathbb{K}^n\times\mathbb{K}^n\to\mathbb{K}\|=1\}.$$

By probabilistic arguments we showed in [3] that

$$\frac{\sqrt{2}\,n}{3\sqrt{\pi}}\le d(n,\mathbb{R})\le n\quad\text{and}\quad\frac{n}{3\sqrt{\pi}}\le d(n,\mathbb{C})\le n.$$

As an application of our results we now determine the exact value of d in the binary case.

THEOREM 7. *We have*

$$d(2,\mathbb{R})=2\quad\textit{and}\quad d(2,\mathbb{C})=3/2.$$

Proof. Let T be any trilinear form on $\mathbb{R}^2$ or $\mathbb{C}^2$ with $\|T\|=1$ and coefficient vector $w=(b_1,b_2,b_3,c)$. Then

$$M\equiv 1+\sum_{j=1}^{3}|b_j|^2+|c|^2=\|T\|_2^2.$$

Due to the description of all extremal trilinear forms of the unit ball in terms of the vector w (see Theorem 2 and Remark 3) we only have to solve the maximum problem $M=\max$ under certain constraints; then $d(2,\mathbb{K})=\sqrt{\max}$. In the *real* case the constraints are

$$|b_j|\le 1,|c|\le 1\quad\text{and}\quad L\equiv\sum_{j=1}^{3}b_j^2+c^2+2b_1b_2b_3=1,$$

which gives $M=2-2b_1b_2b_3\le 4$. Equality holds if and only if

$$w=(-1,1,1,0),\ (1,-1,1,0),\ (1,1,-1,0),\ \text{or}\ (-1,-1,-1,0).$$

This shows $d(2,\mathbb{R})=2$.

In the *complex* case the constraint is

$$F\equiv\sum_{j=1}^{3}|b_j|^2+\frac{|c|^2}{2}+|X|=1.$$

Again we assume without loss of generality that the numbers b_j are real with $0\le b_j\le 1$.

If $b_1b_2b_3=0$, then $F=\sum_{j=1}^3|b_j|^2+|c|^2$, whence $M=2$. Next we show that in the remaining case $0<b_1b_2b_3\le 1$, M cannot attain its maximum unless $c=\pm i|c|$. Indeed, let $w\ne u\equiv(b_1,b_2,b_3,\pm i|c|)$. Then trivially $M(u)=M(w)$, while $|X(u)|<|X(w)|$, and consequently $F(u)<F(w)=1$. (The inequality $|X(u)|<|X(w)|$ is geometrically obvious: Fix $b_1b_2b_3>0$ and $r\ge 0$ and consider the circle $K\equiv\{2b_1b_2b_3+c^2/2:c\in\mathbb{C},\ |c|=r\}$ in $\mathbb{C}$. Since the centre of K is on the positive real half-axis, the only complex number on K with minimal absolute value is the "left" intersection point with the real axis, that means, $c^2=-|c|^2$ or $c=\pm i|c|$.) By continuity of F and $\lim_{t\to\infty}F(tu)=\infty$ there is some real $t>1$ with $F(tu)=1$. This gives $M(tu)>M(u)=M(w)$, so M is not maximal at w.

Using the notation $b=\sqrt[3]{b_1b_2b_3}$ and the inequality between the arithmetic and the geometric means we get

$$\sum_{j=1}^{3}b_j^2\ge 3(b_1b_2b_3)^{2/3}=3b^2.$$

Therefore our constraint and the triangle inequality imply

$$1=F\ge\sum_{j=1}^{3}b_j^2+2b_1b_2b_3\ge 3b^2+2b^3,$$

in other words

$$2b^3+3b^2-1=(2b-1)(b+1)^2\le 0,$$

which gives $b\le 1/2$.

Now we show that, under the assumption $c^2=-|c|^2$, we have $M\le 2+2b^3$. If $|c|^2\ge 4b^3$ $(=4b_1b_2b_3)$ then

$$F=\sum_{j=1}^{3}b_j^2+\frac{|c|^2}{2}+\left|2b^3-\frac{|c|^2}{2}\right|=\sum_{j=1}^{3}b_j^2+|c|^2-2b^3=1;$$

consequently,

$$M=1+\sum b_j^2+|c|^2=2+2b^3.$$

For $|c|^2\le 4b^3$ we have $F=\sum_{j=1}^3 b_j^2+2b^3=1$, whence

$$M=2-2b^3+|c|^2\le 2+2b^3.$$

We conclude that $M \le 2+2b^3 \le 2+2(1/2)^3 = 9/4$. Our considerations also show that equality holds if and only if $b_1 = b_2 = b_3 = b = 1/2$ (condition for equality in the inequality between arithmetic and geometric means) and $-c^2 = |c|^2 = 4b^3 = 1/2$, i.e. $c = \pm i/\sqrt{2}$. This proves that the maximum of M (under the constraint $F = 1$) is $9/4$ and that the maximum is attained at the points

$$w = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \pm\frac{i}{\sqrt{2}}\right)$$

and only there, and finally this yields $d(2,\mathbb{C}) = \sqrt{9/4} = 3/2$. ∎

REMARK 8. Theorem 7 exhibits a significant difference between real and complex trilinear forms. This is surprising in so far as the corresponding constants for bilinear forms are $d(n,\mathbb{K}) = \sqrt{n}$ regardless of whether $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

References

[1] A. Cayley, *On a theory of determinants*, Cambridge Philos. Soc. Trans. 8 (1843), 1–16.
[2] F. Cobos, T. Kühn and J. Peetre, *Schatten–von Neumann classes of multilinear forms*, Duke Math. J. 65 (1992), 121–156.
[3] —, —, —, *On S_p-classes of trilinear forms*, J. London Math. Soc. (2) 59 (1999), 1003–1022.
[4] —, —, —, *On the structure of bounded trilinear forms*, http:/www.maths.lth.se/matematiklu/personal/jaak.
[5] J. A. Dieudonné and J. B. Carrell, *Invariant Theory, Old and New*, Academic Press, New York and London, 1971.
[6] I. M. Gel'fand, M. M. Kapranov and A. V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Birkhäuser, Boston, MA, 1994.
[7] R. Grząślewicz and K. John, *Extreme elements of the unit ball of bilinear operators on ℓ_2^2*, Arch. Math. (Basel) 50 (1988), 264–269.
[8] R. V. Kadison, *Isometries of operator algebras*, Ann. of Math. 54 (1951), 325–338.
[9] E. Schwartz, *Über binäre trilineare Formen*, Math. Z. 12 (1922), 18–35.

Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad Complutense de Madrid
E-28040 Madrid, Spain
E-mail: cobos@eucmax.sim.ucm.es

Matematiska institutionen
Lunds universitet
Box 118
S-22100 Lund, Sweden
E-mail: jaak@maths.lth.se

Fakultät für Mathematik und Informatik
Mathematisches Institut
Universität Leipzig
Augustusplatz 10/11
D-04109 Leipzig, Germany
E-mail: kuehn@mathematik.uni-leipzig.de

Received April 23, 1999
Revised version November 3, 1999 (4307)

STUDIA MATHEMATICA 138 (1) (2000)

Applying the density theorem for derivations to range inclusion problems

by

K. I. BEIDAR (Tainan) and MATEJ BREŠAR (Maribor)

Abstract. The problem of when derivations (and their powers) have the range in the Jacobson radical is considered. The proofs are based on the density theorem for derivations.

1. Introduction.

In both ring theory and the theory of Banach algebras, there are a number of results showing that under certain conditions a derivation (or its power) of a ring (algebra) must be zero or must map into the Jacobson radical. The ring-theoretic results are often proved by combining Kharchenko's theory of differential identities with some elementary (but clever) algebraic manipulations (see [3] for details about background and numerous references). Many of the analytic results in this vein were obtained as attempts to get noncommutative versions of the classical Singer–Wermer theorem [23]. Their proofs usually combine analytic and algebraic tools. For a more detailed discussion on this topic and bibliography we refer the reader to the survey articles [16, 19] and our recent paper [2].

It is our aim here to present a new possible approach to these problems, which works in both algebraic and analytic setting. It is based on an extension of the Jacobson density theorem, recently obtained in [2]. In order to state this result we have to introduce some notation and terminology. Let $\mathcal{A}$ be any ring and $\mathcal{M}$ be a simple left $\mathcal{A}$-module. Recall that $\mathcal{D} = \mathrm{End}({}_{\mathcal{A}}\mathcal{M})$ is a division ring by Schur's lemma. Let d be a derivation of $\mathcal{A}$. We say that d is *$\mathcal{M}$-inner* if there exists an additive map $T : \mathcal{M} \to \mathcal{M}$ such that $a^d x = T(ax) - a(Tx)$ for all $a \in \mathcal{A}$, $x \in \mathcal{M}$ (we shall always write derivations as exponents). The concept of $\mathcal{M}$-innerness obviously extends the concept of (ordinary) innerness. Moreover, in case $\mathcal{A}$ is a primitive ring and $\mathcal{M}$ is a faithful simple module, every X-inner derivation (cf. [3]) is also $\mathcal{M}$-inner,

2000 *Mathematics Subject Classification*: Primary 16W25, 47B47; Secondary 16N20, 46H99.
The second author was partially supported by a grant from the Ministry of Science of Slovenia.