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Received July 14, 1998
Revised version October 13, 1999 (4149)

STUDIA MATHEMATICA 138 (1) (2000)

Tauberian theorems for Cesàro summable double integrals over $\mathbb{R}^2_+$

by

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Abstract. Given $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$, denote by $s(w,z)$ its integral over the rectangle $[0,w]\times[0,z]$ and by $\sigma(u,v)$ its $(C,1,1)$ mean, that is, the average value of $s(w,z)$ over $[0,u]\times[0,v]$, where $u,v,w,z>0$. Our permanent assumption is that $(*)$ $\sigma(u,v)\to A$ as $u,v\to\infty$, where A is a finite number.

First, we consider real-valued functions f and give one-sided Tauberian conditions which are necessary and sufficient in order that the convergence $(**)$ $s(u,v)\to A$ as $u,v\to\infty$ follow from $(*)$. Corollaries allow these Tauberian conditions to be replaced either by Schmidt type slow decrease (or increase) conditions, or by Landau type one-sided Tauberian conditions.

Second, we consider complex-valued functions and give a two-sided Tauberian condition which is necessary and sufficient in order that $(**)$ follow from $(*)$. In particular, this condition is satisfied if $s(u,v)$ is slowly oscillating, or if $f(x,y)$ obeys Landau type two-sided Tauberian conditions.

At the end, we extend these results to the mixed case, where the $(C,1,0)$ mean, that is, the average value of $s(w,v)$ with respect to the first variable over the interval $[0,u]$, is considered instead of $\sigma_{11}(u,v) := \sigma(u,v)$.

1. Summability $(C,1,1)$ of double integrals over $\mathbb{R}^2_+$. We remind the reader that a complex-valued function $f(x,y)$ is said to be *locally integrable* over $\mathbb{R}^2_+ := [0,\infty)\times[0,\infty)$, in symbols $f\in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$, if for all $0<u,v<\infty$ the integral

$$s(u,v) := \int_0^u\int_0^v f(x,y)\,dx\,dy$$

2000 *Mathematics Subject Classification*: 40B05, 40E05.
Key words and phrases: improper double integral, convergence in Pringsheim's sense, Cesàro summability $(C,1,1)$, $(C,1,0)$, and $(C,0,1)$, one-sided and two-sided Tauberian conditions, Schmidt type slow decrease (or increase), slow oscillation, Landau type Tauberian conditions.
This research was partially supported by the Hungarian National Foundation for Scientific Research under Grant T 029 094.

exists in Lebesgue's sense. We set

$$\sigma(u,v) := \sigma_{11}(u,v) := \frac{1}{uv}\int_0^u\int_0^v s(w,z)\,dw\,dz, \quad u,v>0.$$

We recall (for single integrals, see e.g. [1, p. 11] or [6, p. 26]) that the (formal) integral

$$\int_0^\infty\int_0^\infty f(x,y)\,dx\,dy \tag{1.1}$$

is said to be (*Cesàro*) *summable* $(C,1,1)$ to a finite number A if

$$\lim_{u,v\to\infty}\sigma(u,v)=A. \tag{1.2}$$

We use the notion of convergence *in Pringsheim's sense*, that is, in (1.2) and (1.3) below, both u and v tend to ∞ independently of each other.

It is plain that if the limit

$$\lim_{u,v\to\infty} s(u,v)=A \tag{1.3}$$

exists (in other words: the double improper integral $\int_0^{\to\infty}\int_0^{\to\infty} f(x,y)\,dx\,dy$ converges) and if the function $s(u,v)$ is bounded on $\mathbb{R}^2_+$, then the limit (1.2) also exists. The converse statement is not true in general, even if $s(u,v)$ is bounded on $\mathbb{R}^2_+$.

However, if the function $f(x,y)$ is of constant sign on $\mathbb{R}^2_+$, then the limit (1.2) exists if and only if the limit (1.3) exists, which in turn is equivalent to the boundedness of $s(u,v)$ on $\mathbb{R}^2_+$. This follows immediately by Fubini's theorem:

$$\begin{aligned}\sigma(u,v) &= \frac{1}{uv}\int_0^u\int_0^v dw\,dz\int_0^w\int_0^z f(x,y)\,dx\,dy\\ &= \int_0^u\int_0^v (1-x/u)(1-y/v)f(x,y)\,dx\,dy.\end{aligned}$$

It is also plain that summability $(C,1,1)$ of the integral (1.1) does not depend on the values of $f(x,y)$ assumed on any finite rectangle $(0,u_0)\times(0,v_0)$, where $u_0,v_0>0$ are fixed.

We refer to [4], where the reader can find the basic notions and results on the interrelation between convergence and Cesàro summability $(C,1)$ of integrals over $\mathbb{R}_+$.

2. Main results. First, we consider the special case where the function f assumes only real values. In Theorem 1, we give one-sided Tauberian conditions which are necessary and sufficient in order that convergence follow from summability $(C,1,1)$.

THEOREM 1. *If a real-valued function $f\in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ is such that the integral (1.1) is summable $(C,1,1)$ to a finite number A, then we have (1.3) if and only if*

$$\sup_{\lambda>1}\liminf_{u,v\to\infty}\frac{1}{(\lambda u-u)(\lambda v-v)}\int_u^{\lambda u}\int_v^{\lambda v}[s(x,y)-s(u,v)]\,dx\,dy\ge 0 \tag{2.1}$$

and

$$\sup_{0<\lambda<1}\liminf_{u,v\to\infty}\frac{1}{(u-\lambda u)(v-\lambda v)}\int_{\lambda u}^{u}\int_{\lambda v}^{v}[s(u,v)-s(x,y)]\,dx\,dy\ge 0. \tag{2.2}$$

If conditions (2.1) *and* (2.2) *are satisfied, then we necessarily have*

$$\lim_{u,v\to\infty}\frac{1}{(\lambda u-u)(\lambda v-v)}\int_u^{\lambda u}\int_v^{\lambda v}[s(x,y)-s(u,v)]\,dx\,dy=0 \tag{2.3}$$

for every $\lambda>1$, and

$$\lim_{u,v\to\infty}\frac{1}{(u-\lambda u)(v-\lambda v)}\int_{\lambda u}^{u}\int_{\lambda v}^{v}[s(u,v)-s(x,y)]\,dx\,dy=0 \tag{2.4}$$

for every $0<\lambda<1$.

A few comments are appropriate here.

(i) A real-valued function $s(u,v)$ defined on $\mathbb{R}^2_+$ is said to be *slowly decreasing with respect to the first variable* if

$$\lim_{\lambda\to 1+0}\liminf_{u,v\to\infty}\min_{u\le x\le\lambda u}[s(x,v)-s(u,v)]\ge 0. \tag{2.5}$$

In other words, this means that for every $\varepsilon>0$ there exist $u_0\ge 0$ and $\lambda>1$ such that

$$s(x,v)-s(u,v)\ge-\varepsilon \quad \text{whenever} \quad u_0<u<x\le\lambda u \text{ and } u_0<v.$$

The term "slowly decreasing" was introduced by Schmidt [5] for single sequences of real numbers. (See also [1, pp. 124–125].)

Analogously, a function $s(u,v)$ defined on $\mathbb{R}^2_+$ is said to be *slowly decreasing with respect to the second variable* if

$$\lim_{\lambda\to 1+0}\liminf_{u,v\to\infty}\min_{v\le y\le\lambda v}[s(u,y)-s(u,v)]\ge 0. \tag{2.6}$$

(ii) We prove that condition (2.5) is equivalent to the following:

$$\lim_{\lambda\to 1-0}\liminf_{u,v\to\infty}\min_{\lambda u\le x\le u}[s(u,v)-s(x,v)]\ge 0. \tag{2.7}$$

In fact, if for some $\lambda>1$ we have

$$\liminf_{u,v\to\infty}\min_{u\le x\le\lambda u}[s(x,v)-s(u,v)]=L,$$

where L is a finite number, then

$$\liminf_{x,v\to\infty} \min_{\lambda^{-1}x\le u\le x} [s(x,v)-s(u,v)] \le L.$$

Conversely, it is also true that if the latter lower limit equals some L, then the former cannot exceed L.

Condition (2.6) can be reformulated in an analogous manner.

In Corollary 1, we give Tauberian conditions, in terms of Schmidt type slow decrease, sufficient for convergence to follow from summability $(C,1,1)$.

COROLLARY 1. *If a real-valued function $f \in L^1_{\rm loc}(\mathbb{R}^2_+)$ is such that the integral* (1.1) *is summable $(C,1,1)$ to a finite number A, and the function $s(u,v)$ is slowly decreasing in each variable, then we have* (1.3).

(iii) In particular, condition (2.5) is satisfied if there exist constants $H>0$ and $x_0\ge 0$ such that

$$x\int_0^v f(x,z)\,dz \ge -H \quad \text{for almost every } (x,v)\in\mathbb{R}^2_+ \text{ with } x,v>x_0. \tag{2.8}$$

Similarly, condition (2.6) is satisfied if

$$y\int_0^u f(w,y)\,dw \ge -H \quad \text{for almost every } (u,y)\in\mathbb{R}^2_+ \text{ with } u,y>x_0. \tag{2.9}$$

For single sequences of real numbers, an analogous condition was introduced by Landau [2].

In Corollary 2, we give Landau type one-sided Tauberian conditions sufficient for convergence to follow from summability $(C,1,1)$.

COROLLARY 2. *If a real-valued function $f \in L^1_{\rm loc}(\mathbb{R}^2_+)$ is such that the integral* (1.1) *is summable $(C,1,1)$ to a finite number A, and conditions* (2.8) *and* (2.9) *are satisfied, then we have* (1.3).

(iv) The symmetric counterparts of conditions (2.1) and (2.2) read as follows:

$$\inf_{\lambda>1}\limsup_{u,v\to\infty}\frac{1}{(\lambda u-u)(\lambda v-v)}\int_u^{\lambda u}\int_v^{\lambda v}[s(x,y)-s(u,v)\,dx\,dy \le 0 \tag{2.10}$$

and

$$\inf_{0<\lambda<1}\limsup_{u,v\to\infty}\frac{1}{(u-\lambda u)(v-\lambda v)}\int_{\lambda u}^{u}\int_{\lambda v}^{v}[s(u,v)-s(x,y)]\,dx\,dy \le 0. \tag{2.11}$$

Now, Theorem 1 remains true if conditions (2.10) and (2.11) are substituted for (2.1) and (2.2). The proof runs along the same lines as that of Theorem 1. In particular, it follows that if the integral (1.1) is summable

$(C,1,1)$ to a finite number and conditions (2.10) and (2.11) are satisfied, then conditions (2.1) and (2.2) are also satisfied; and vice versa.

(v) Accordingly, we may say that a function $s(u,v)$ defined on $\mathbb{R}^2_+$ is *slowly increasing with respect to the first variable* if

$$\lim_{\lambda\to 1+0}\limsup_{u,v\to\infty}\max_{u\le x\le\lambda u}[s(x,v)-s(u,v)]\le 0 \tag{2.12}$$

(cf. (2.5)), or equivalently, if for every $\varepsilon>0$ there exist $u_0\ge 0$ and $\lambda>1$ such that

$$s(x,v)-s(u,v)\le\varepsilon \quad \text{whenever} \quad u_0<u<x\le\lambda u \text{ and } u_0<v.$$

(vi) Analogously to (2.7), condition (2.12) can be reformulated as follows:

$$\lim_{\lambda\to 1-0}\limsup_{u,v\to\infty}\max_{\lambda u\le x\le u}[s(u,v)-s(x,v)]\le 0$$

(cf. (2.7)). Furthermore, the existence of constants $H>0$ and $x_0\ge 0$ such that

$$x\int_0^v f(x,z)\,dz\le H \quad \text{for almost every } (x,v)\in\mathbb{R}^2_+ \text{ with } x,v>x_0$$

is a Landau type condition sufficient for the fulfillment of (2.12).

The slowly increasing property as well as a Landau type condition with respect to the second variable are defined in an analogous manner.

In this way, one can formulate the symmetric counterparts of Corollaries 1 and 2 by substituting "slowly increasing" for "slowly decreasing" and conditions (2.12) and its symmetric counterpart with respect to v for (2.8) and (2.9), respectively.

Second, we consider the general case where the function f assumes complex values. In Theorem 2, we give a two-sided Tauberian condition which is necessary and sufficient in order that convergence follow from summability $(C,1,1)$.

THEOREM 2. *If a complex-valued function $f\in L^1_{\rm loc}(\mathbb{R}^2_+)$ is such that the integral* (1.1) *is summable $(C,1,1)$ to a finite number A, then we have* (1.3) *if and only if*

$$\inf_{\substack{0<\lambda<\infty\\ \lambda\ne 1}}\limsup_{u,v\to\infty}\left|\frac{1}{(\lambda u-u)(\lambda v-v)}\int_u^{\lambda u}\int_v^{\lambda v}[s(x,y)-s(u,v)]\,dx\,dy\right|=0, \tag{2.13}$$

in which case we necessarily have (2.3) *for every $\lambda>1$, and* (2.4) *for every $0<\lambda<1$.*

Again, one can make similar comments as in the real case above. In addition, Theorem 2 can be extended to functions with values in ordered linear spaces over the real numbers. We do not enter into details. Instead, we formulate two corollaries of Theorem 2.

In Corollary 3, we give Tauberian conditions, in terms of Schmidt type slow oscillation, sufficient for convergence to follow from summability $(C,1,1)$.

COROLLARY 3. *If a complex-valued function* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ *is such that the integral* (1.1) *is summable* $(C,1,1)$ *to a finite number* A,

$$\lim_{\lambda\to 1+0} \limsup_{u,v\to\infty} \max_{u\le x\le \lambda u} |s(x,v)-s(u,v)| = 0, \tag{2.14}$$

and

$$\lim_{\lambda\to 1+0} \limsup_{u,v\to\infty} \max_{v\le y\le \lambda v} |s(u,y)-s(u,v)| = 0, \tag{2.15}$$

then we have (1.3).

We recall that a complex-valued function $s(u,v)$ defined on $\mathbb{R}^2_+$ is said to be *slowly oscillating with respect to the first variable* if condition (2.14) is satisfied. In other words, this means that for every $\varepsilon>0$ there exist $u_0\ge 0$ and $\lambda>1$ such that

$$|s(x,v)-s(u,v)|\le\varepsilon \quad \text{whenever} \quad u_0<u<x\le\lambda u \text{ and } u_0<v.$$

Analogously, $s(u,v)$ is said to be *slowly oscillating with respect to the second variable* if condition (2.15) is satisfied.

Again, it is not difficult to check that (2.14) is equivalent to the following condition:

$$\lim_{\lambda\to 1-0} \limsup_{u,v\to\infty} \max_{\lambda u\le x\le u} |s(u,v)-s(x,v)| = 0.$$

Condition (2.15) can be reformulated in an analogous manner.

In Corollary 4, we give Landau type two-sided Tauberian conditions sufficient for convergence to follow from summability $(C,1,1)$.

COROLLARY 4. *If a complex-valued function* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ *is such that the integral* (1.1) *is summable* $(C,1,1)$ *to a finite number* A, *and there exist constants* $H>0$ *and* $x_0\ge 0$ *such that*

$$\Big|x\int_0^v f(x,z)\,dz\Big| \le H \quad \textit{for a.e. } (x,v)\in\mathbb{R}^2_+ \textit{ with } x,v>x_0, \tag{2.16}$$

and

$$\Big|y\int_0^u f(w,y)\,dw\Big| \le H \quad \textit{for a.e. } (u,y)\in\mathbb{R}^2_+ \textit{ with } u,y>x_0, \tag{2.17}$$

then we have (1.3).

3. Proofs. We begin with the following auxiliary result, which is interesting in itself.

LEMMA 1. *If the integral* (1.1) *is summable* $(C,1,1)$ *to a finite number* A, *then for every* $0<\lambda<\infty$, $\lambda\ne 1$, *we have*

$$\lim_{u,v\to\infty} \frac{1}{(\lambda u-u)(\lambda v-v)} \int_u^{\lambda u}\int_v^{\lambda v} s(x,y)\,dx\,dy = A. \tag{3.1}$$

Proof. CASE $\lambda>1$. By definition,

$$\frac{1}{(\lambda u-u)(\lambda v-v)} \int_u^{\lambda u}\int_v^{\lambda v} s(x,y)\,dx\,dy$$

$$= \frac{1}{(\lambda-1)^2uv}\Big\{\int_0^{\lambda u}\int_0^{\lambda v} - \int_0^{\lambda u}\int_0^{v} - \int_0^{u}\int_0^{\lambda v} + \int_0^{u}\int_0^{v}\Big\} s(x,y)\,dx\,dy$$

$$= \Big(1+\frac{1}{\lambda-1}\Big)^2 \frac{1}{\lambda u\lambda v}\int_0^{\lambda u}\int_0^{\lambda v} s(x,y)\,dx\,dy$$

$$- \frac{1}{\lambda-1}\Big(1+\frac{1}{\lambda-1}\Big)\frac{1}{\lambda uv}\int_0^{\lambda u}\int_0^{v} s(x,y)\,dx\,dy$$

$$- \frac{1}{\lambda-1}\Big(1+\frac{1}{\lambda-1}\Big)\frac{1}{u\lambda v}\int_0^{u}\int_0^{\lambda v} s(x,y)\,dx\,dy$$

$$+ \frac{1}{(\lambda-1)^2}\frac{1}{uv}\int_0^{u}\int_0^{v} s(x,y)\,dx\,dy$$

$$= \Big(1+\frac{1}{\lambda-1}\Big)^2\sigma(\lambda u,\lambda v) - \frac{1}{\lambda-1}\Big(1+\frac{1}{\lambda-1}\Big)\sigma(\lambda u,v)$$

$$- \frac{1}{\lambda-1}\Big(1+\frac{1}{\lambda-1}\Big)\sigma(u,\lambda v) + \frac{1}{(\lambda-1)^2}\sigma(u,v)$$

$$= \sigma(\lambda u,\lambda v) + \frac{1}{\lambda-1}[\sigma(\lambda u,\lambda v)-\sigma(\lambda u,v)] + \frac{1}{\lambda-1}[\sigma(\lambda u,\lambda v)-\sigma(u,\lambda v)]$$

$$+ \frac{1}{(\lambda-1)^2}[\sigma(\lambda u,\lambda v)-\sigma(\lambda u,v)-\sigma(u,\lambda v)+\sigma(u,v)].$$

Now, (3.1) follows from (1.2).

Case $0 < \lambda < 1$. This time, we have

$$\frac{1}{(u-\lambda u)(v-\lambda v)}\int\limits_{\lambda u}^{u}\int\limits_{\lambda v}^{v} s(x,y)\,dx\,dy$$
$$= \frac{1}{(1-\lambda)^2 uv}\Big\{\int\limits_0^u\int\limits_0^v - \int\limits_0^{\lambda u}\int\limits_0^{v} - \int\limits_0^{u}\int\limits_0^{\lambda v} + \int\limits_0^{\lambda u}\int\limits_0^{\lambda v}\Big\} s(x,y)\,dx\,dy$$
$$= \frac{1}{(1-\lambda)^2}\frac{1}{uv}\int\limits_0^u\int\limits_0^v s(x,y)\,dx\,dy$$
$$+\frac{1}{1-\lambda}\Big(1-\frac{1}{1-\lambda}\Big)\frac{1}{\lambda uv}\int\limits_0^{\lambda u}\int\limits_0^{v} s(x,y)\,dx\,dy$$
$$+\frac{1}{1-\lambda}\Big(1-\frac{1}{1-\lambda}\Big)\frac{1}{u\lambda v}\int\limits_0^{u}\int\limits_0^{\lambda v} s(x,y)\,dx\,dy$$
$$+\Big(1-\frac{1}{1-\lambda}\Big)^2\frac{1}{\lambda u\lambda v}\int\limits_0^{\lambda u}\int\limits_0^{\lambda v} s(x,y)\,dx\,dy$$
$$= \frac{1}{(1-\lambda)^2}[\sigma(u,v)-\sigma(\lambda u,v)-\sigma(u,\lambda v)+\sigma(\lambda u,\lambda v)]$$
$$+\frac{1}{1-\lambda}[\sigma(\lambda u,v)-\sigma(\lambda u,\lambda v)]+\frac{1}{1-\lambda}[\sigma(u,\lambda v)-\sigma(\lambda u,\lambda v)]+\sigma(\lambda u,\lambda v).$$

Again, (3.1) follows from (1.2). ■

Proof of Theorem 1. Necessity. Assume (1.2) and (1.3). By Lemma 1,

$$\lim_{u,v\to\infty}\frac{1}{(\lambda u-u)(\lambda v-v)}\int\limits_{u}^{\lambda u}\int\limits_{v}^{\lambda v}[s(x,y)-s(u,v)]\,dx\,dy$$
$$=\lim_{u,v\to\infty}\frac{1}{(\lambda u-u)(\lambda v-v)}\int\limits_{u}^{\lambda u}\int\limits_{v}^{\lambda v} s(x,y)\,dx\,dy$$
$$-\lim_{u,v\to\infty} s(u,v) = A-A=0.$$

This proves (2.3) in case $\lambda > 1$, and (2.4) in case $0<\lambda<1$.

Sufficiency. Assume (1.2), (2.1), and (2.2). We have to prove that (1.3) is also satisfied. To this end, let $\varepsilon > 0$ be given. By (2.1) there exists $\lambda_1 > 1$ such that

$$\liminf_{u,v\to\infty}\frac{1}{(\lambda_1 u-u)(\lambda_1 v-v)}\int\limits_{u}^{\lambda_1 u}\int\limits_{v}^{\lambda_1 v}[s(x,y)-s(u,v)]\,dx\,dy \ge -\varepsilon, \tag{3.2}$$

and by (2.2) there exists $0<\lambda_2<1$ such that

$$\liminf_{u,v\to\infty}\frac{1}{(u-\lambda_2 u)(v-\lambda_2 v)}\int\limits_{\lambda_2 u}^{u}\int\limits_{\lambda_2 v}^{v}[s(u,v)-s(x,y)]\,dx\,dy \ge -\varepsilon. \tag{3.3}$$

By (1.2) and Lemma 1, for every $\lambda > 1$ we have

$$\liminf_{u,v\to\infty}\frac{1}{(\lambda u-u)(\lambda v-v)}\int\limits_{u}^{\lambda u}\int\limits_{v}^{\lambda v}[s(x,y)-s(u,v)]\,dx\,dy = A-\limsup_{u,v\to\infty} s(u,v);$$

while for every $0<\lambda<1$,

$$\liminf_{u,v\to\infty}\frac{1}{(u-\lambda u)(v-\lambda v)}\int\limits_{\lambda u}^{u}\int\limits_{\lambda v}^{v}[s(u,v)-s(x,y)]\,dx\,dy = \liminf_{u,v\to\infty} s(u,v)-A.$$

Thus, (3.2) and (3.3) are equivalent to the following:

$$A-\varepsilon\le\liminf_{u,v\to\infty}s(u,v)\le\limsup_{u,v\to\infty}s(u,v)\le A+\varepsilon.$$

As $\varepsilon>0$ can be arbitrarily small, (1.3) follows. ■

Proof of Corollary 1. We show that conditions (2.5) and (2.6) imply (2.1) and (2.2) in Theorem 1. In fact, by the obvious estimate

$$\frac{1}{(\lambda u-u)(\lambda v-v)}\int\limits_{u}^{\lambda u}\int\limits_{v}^{\lambda v}[s(x,y)-s(u,v)]\,dx\,dy$$
$$\ge \min_{\substack{u\le x\le\lambda u\\ v\le y\le \lambda v}}[s(x,y)-s(u,y)]+\min_{v\le y\le\lambda v}[s(u,y)-s(u,v)],$$

we find that (2.5) and (2.6) are sufficient for (2.1) to hold.

In an analogous way, one can deduce (2.2) from (2.5) and (2.6). ■

Proof of Corollary 2. It is plain that conditions (2.8) and (2.9) imply (2.5) and (2.6), respectively, and Corollary 1 applies. ■

Proof of Theorem 2. It also relies on representations (3.1) and (3.2), and is modelled after the proof of Theorem 1. We leave the details to the reader. ■

Proof of Corollary 3. We show that conditions (2.14) and (2.15) imply condition (2.13) in Theorem 2. Indeed, it is plain that

$$\Big|\frac{1}{(\lambda u-u)(\lambda v-v)}\int\limits_{u}^{\lambda u}\int\limits_{v}^{\lambda v}[s(x,y)-s(u,v)]\,dx\,dy\Big|$$
$$\le \max_{\substack{u\le x\le\lambda u\\ v\le y\le \lambda v}}|s(x,y)-s(u,y)|+\max_{v\le y\le\lambda v}|s(u,y)-s(u,v)|.$$

Thus, (2.14) and (2.15) are sufficient for (2.13) to hold. ■

Proof of Corollary 4. It is obvious that conditions (2.16) and (2.17) imply (2.14) and (2.15), respectively, and Corollary 3 applies. ∎

4. Summability $(C,1,0)$ of double integrals over $\mathbb{R}^2_+$. Given a complex-valued function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$, we set

$$\sigma_{10}(u,v) := \frac{1}{u}\int_0^u s(w,v)\,dw = \int_0^u\int_0^v \left(1-\frac{x}{u}\right) f(x,y)\,dx\,dy, \quad u,v>0.$$

We recall that the integral (1.1) is said to be (*Cesàro*) *summable* $(C,1,0)$ to a finite number A if

$$\lim_{u,v\to\infty} \sigma_{10}(u,v) = A.$$

Analogously to summability $(C,1,1)$, one can develop the theory of summability $(C,1,0)$ of double integrals over $\mathbb{R}^2_+$. Because of similarity, we only sketch it without detailed proofs.

First, we consider the real case. In Theorem 3, we give one-sided Tauberian conditions which are necessary and sufficient in order that convergence follow from summability $(C,1,0)$.

THEOREM 3. *If a real-valued function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ is such that the integral* (1.1) *is summable $(C,1,0)$ to a finite number A, then we have* (1.3) *if and only if*

$$\sup_{\lambda>1} \liminf_{u,v\to\infty} \frac{1}{\lambda u - u}\int_u^{\lambda u} [s(x,v)-s(u,v)]\,dx \ge 0 \tag{4.1}$$

and

$$\sup_{0<\lambda<1} \liminf_{u,v\to\infty} \frac{1}{(u-\lambda u)}\int_{\lambda u}^{u} [s(u,v)-s(x,v)]\,dx \ge 0. \tag{4.2}$$

If conditions (4.1) *and* (4.2) *are satisfied, then we necessarily have*

$$\lim_{u,v\to\infty} \frac{1}{\lambda u - u}\int_u^{\lambda u} [s(x,v)-s(u,v)]\,dx = 0$$

for every $\lambda>1$, and

$$\lim_{u,v\to\infty} \frac{1}{u-\lambda u}\int_{\lambda u}^{u} [s(u,v)-s(x,v)]\,dx = 0$$

for every $0<\lambda<1$.

Corollaries 5 and 6 allow these Tauberian conditions to be replaced either by a Schmidt type slow decrease condition, or a Landau type one-sided Tauberian condition.

COROLLARY 5. *If a real-valued function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ is such that the integral* (1.1) *is summable $(C,1,0)$ to a finite number A and the function $s(u,v)$ is slowly decreasing with respect to u, then we have* (1.3).

COROLLARY 6. *If a real-valued function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ is such that the integral* (1.1) *is summable $(C,1,0)$ to a finite number A and condition* (2.8) *is satisfied, then we have* (1.3).

Second, we consider the complex case. In Theorem 4, we give a two-sided Tauberian condition which is necessary and sufficient in order that convergence follow from summability $(C,1,0)$.

THEOREM 4. *If a complex-valued function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ is such that the integral* (1.1) *is summable $(C,1,1)$ to a finite number A, then we have* (1.3) *if and only if*

$$\inf_{\substack{0<\lambda<1\\ \lambda\ne 1}} \limsup_{u,v\to\infty} \left|\frac{1}{\lambda u - u}\int_u^{\lambda u} [s(x,v)-s(u,v)]\,dx\right| = 0,$$

in which case we necessarily have (4.1) *for every $\lambda>1$, and* (4.2) *for every $0<\lambda<1$.*

Corollaries 7 and 8 allow these Tauberian conditions to be replaced either by a Schmidt type slow oscillation condition, or by a Landau type two-sided Tauberian condition.

COROLLARY 7. *If a complex-valued function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ is such that the integral* (1.1) *is summable $(C,1,0)$ to a finite number A and condition* (2.14) *is satisfied, then we have* (1.3).

COROLLARY 8. *If a complex-valued function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ is such that the integral* (1.1) *is summable $(C,1,0)$ to a finite number A and condition* (2.16) *is satisfied, then we have* (1.3).

The proofs of Theorems 3 and 4 are based on the following auxiliary result, which is a partial counterpart of Lemma 1.

LEMMA 2. *If the integral* (1.1) *is summable $(C,1,0)$ to a finite number A, then for every $0<\lambda<\infty$, $\lambda\ne 1$, we have*

$$\lim_{u,v\to\infty} \frac{1}{\lambda u - u}\int_u^{\lambda u} s(x,v)\,dx = A.$$

We close our paper with two remarks.

REMARK 1. The (Cesàro) summability $(C,0,1)$ of the integral (1.1) is defined by the convergence of the mean

$$\sigma_{01}(u,v) := \frac{1}{v}\int_0^v s(u,z)\,dz = \int_0^u\int_0^v \left(1-\frac{y}{v}\right) f(x,y)\,dx\,dy \quad \text{as } u,v\to\infty.$$

Each of Theorems 3, 4 and Corollaries 5–8 has a symmetric counterpart when summability $(C,0,1)$ is considered in place of summability $(C,1,0)$ of the integral (1.1).

REMARK 2. Analogous results were proved in [3] for double numerical series with rectangular partial sums s_{jk}, $j,k=0,1,2,\ldots$ Making use of the method of this paper, we are now able to improve some of those results. For example, [3, Corollary 1] remains valid if we drop the condition of slow decrease in $(1,1)$ sense. Likewise, condition (2.3) in [3, Corollary 2] and condition (5.1) in [3, Corollary 5] are superfluous.

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Received August 10, 1998 (4163)
Revised version November 3, 1999

STUDIA MATHEMATICA 138 (1) (2000)

Vector series whose lacunary subseries converge

by

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Abstract. The area of research of this paper goes back to a 1930 result of H. Auerbach showing that a scalar series is (absolutely) convergent if all its zero-density subseries converge. A series $\sum_n x_n$ in a topological vector space X is called $\mathcal{L}$-convergent if each of its lacunary subseries $\sum_k x_{n_k}$ (i.e. those with $n_{k+1}-n_k\to\infty$) converges. The space X is said to have the Lacunary Convergence Property, or LCP, if every $\mathcal{L}$-convergent series in X is convergent; in fact, it is then subseries convergent. The Zero-Density Convergence Property, or ZCP, is defined similarly though of lesser importance here. It is shown that for every $\mathcal{L}$-convergent series the set of all its finite sums is metrically bounded; however, it need not be topologically bounded. Next, a space with the LCP contains no copy of the space c_0. The converse holds for Banach spaces and, more generally, sequentially complete locally pseudoconvex spaces. However, an F-lattice of measurable functions is constructed that has both the Lebesgue and Levi properties, and thus contains no copy of c_0, and, nonetheless, lacks the LCP. The main (and most difficult) result of the paper is that if a Banach space E contains no copy of c_0 and λ is a finite measure, then the Bochner space $L_0(\lambda,E)$ has the LCP. From this, with the help of some Orlicz–Pettis type theorems proved earlier by the authors, the LCP is deduced for a vast class of spaces of (scalar and vector) measurable functions that have the Lebesgue type property and are "metrically-boundedly sequentially closed" in the containing L_0 space. Analogous results about the convergence of $\mathcal{L}$-convergent positive series in topological Riesz spaces are also obtained. Finally, while the LCP implies the ZCP trivially, an example is given that the converse is false, in general.

1. Introduction. We first recall a few more or less standard definitions and facts. As usual, a series in a topological vector space X is said to be *con-*

2000 *Mathematics Subject Classification*: Primary 40A05, 40A30, 46A16, 46A40, 46E30; Secondary 46B03, 46G10.

Key words and phrases: subseries convergence, lacunary subseries, zero-density subseries, lacunary convergence property, topological Riesz space of measurable functions, topological vector space of Bochner measurable functions, Lebesgue property, Levi property, copy of c_0.

A substantial part of the research presented here was done while the first author held a visiting position in the Department of Mathematics, University of Mississippi, in the Spring and Fall Semesters of 1996, and in the Spring Semester of 1998. The first author also acknowledges a support from the State Committee for Scientific Research (Poland), Grant no. 2 P03A 05115, in the final preparation of the paper.