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On operator ideals related to (p,σ)-absolutely continuous operators

by

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Abstract. We study tensor norms and operator ideals related to the ideal $\mathcal{P}_{p,\sigma}$, $1<p<\infty$, $0<\sigma<1$, of (p,σ)-absolutely continuous operators of Matter. If α is the tensor norm associated with $\mathcal{P}_{p,\sigma}$ (in the sense of Defant and Floret), we characterize the $(\alpha')^t$-nuclear and $(\alpha')^t$- integral operators by factorizations by means of the composition of the inclusion map $L^r(\mu)\to L^1(\mu)+L^p(\mu)$ with a diagonal operator $B_w : L^\infty(\mu)\to L^r(\mu)$, where r is the conjugate exponent of $p'/(1-\sigma)$. As an application we study the reflexivity of the components of the ideal $\mathcal{P}_{p,\sigma}$.

1. Introduction. The ideal $\mathcal{P}_{p,\sigma}$ of (p,σ)-absolutely continuous operators was introduced by Matter [8] in order to get a classification of the absolutely continuous operators previously defined by Niculescu [10]. Since $\mathcal{P}_{p,\sigma}$ is a maximal ideal, it is interesting to study the tensor norm α (or the transposed α^t) associated with $\mathcal{P}_{p,\sigma}$ and the properties of the operator ideals naturally related to α. The results obtained could be applied to study the metric properties of α as well as some topological properties of the components of $\mathcal{P}_{p,\sigma}$. As far as we know, this work has not been done yet. Concretely, the main questions can be reduced to the following:

1. Find the tensor norm α such that $(E\hat{\otimes}_\alpha F)'=\mathcal{P}_{p,\sigma}(F,E')$ for every pair of Banach spaces E and F.

2. Characterize the α-nuclear and α-integral operators.

In this spirit, we have characterized in [6] the tensor norm $g_{p',\sigma}$ which solves question 1. *In the present paper we give a full answer to problem* 2. Although this can be done without any reference to tensor products (Definitions 2 and 4 below have a meaning in the context of purely operator ideals), we have chosen the tensorial approach for two reasons. The first one

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is to motivate the complicated Definition 2. The second one is to get the applications we have in mind and which will be developed in Section 4.

It turns out that the $g_{p,\sigma}$-nuclear operators can be characterized (Section 2) by factorizations which contain the composition of an inclusion of type $\ell^r(\mu) \to \ell^1(\mu) + \ell^p(\mu)$ with a diagonal operator $B : \ell^\infty(\mu) \to \ell^r(\mu)$, where r, p, σ satisfy a certain relation (see (3)). *The main result of our paper* is that an analogous factorization with "continuous" spaces $L^\infty(\mu) \to L^r(\mu) \to L^1(\mu) + L^p(\mu)$ characterizes the $g_{p,\sigma}$-integral operators, i.e. the operators of the ideal associated with the tensor norm $g_{p,\sigma}$ (Section 3).

Section 4 is devoted to applications. We begin by studying the coincidence of $g_{p,\sigma}$-nuclear and $g_{p,\sigma}$-integral operators. This result is applied to study some metric properties of $g_{p,\sigma}$ and its dual tensor norm $g'_{p,\sigma}$. Finally, we consider some topological properties of the spaces $\mathcal{P}_{p,\sigma}(E,F)$ (density of the space of finite rank operators and reflexivity).

In order to reduce the length of the paper, we assume the reader is familiar with the theories which we need. For instance, we refer to [3] for results about tensor norms and to [11] for operator ideals. In general, we follow the standard notation. We point out the following special symbols and conventions:

We denote by BAN the class of all Banach spaces. If $E \in \mathrm{BAN}$, B_E will be the closed unit ball of E, E' the topological dual Banach space, J_E the natural embedding of E into its bidual E'' and $\mathrm{FIN}(E)$ the set of all finite-dimensional subspaces of E. Sometimes, to call attention to the norm of the Banach space E involved, we write $\|\cdot\|_E$. Given $p \in [1,\infty]$, a measure space $(\Omega, \mathcal{M}, \mu)$ and a measurable, not everywhere null real function g, we denote by $L^p(\Omega, \mathcal{M}, g, \mu)$ (or simply $L^p(g,\mu)$ if there is no risk of confusion) the Banach space of classes of functions f such that fg belongs to the Lebesgue space $L^p(\Omega, \mathcal{M}, \mu)$, provided with the norm $\|f\| = \|fg\|_{L^p(\mu)}$.

In the whole paper, σ will be a parameter such that $0 < \sigma < 1$. Given $(x_i) \in E^{\mathbb{N}}$ and $1 \le p < \infty$ we put

$$\varepsilon_p((x_i)) = \sup_{a \in B_{E'}} \Big(\sum_{i=1}^{\infty} |\langle x_i, a\rangle|^p\Big)^{1/p}, \qquad \pi_p((x_i)) = \Big(\sum_{i=1}^{\infty} \|x_i\|^p\Big)^{1/p};$$

$$\delta_{p,\sigma}((x_i)) = \sup_{a \in B_{E'}} \Big(\sum_{i=1}^{\infty} (|\langle x_i, a\rangle|^{1-\sigma} \|x_i\|^{\sigma})^{p/(1-\sigma)}\Big)^{(1-\sigma)/p}.$$

If $p = \infty$, the definitions are the same with $\sum_{i=1}^{\infty}$ replaced by $\sup_{i\in\mathbb{N}}$, hence $\varepsilon_\infty((x_i)) = \pi_\infty((x_i)) = \delta_{\infty,\sigma}((x_i)) = \sup_{i\in\mathbb{N}} \|x_i\|$.

The word *operator* (or *map*) will always be used to denote a continuous linear map between normed spaces. If $E, F \in \mathrm{BAN}$ and $1 \le p < \infty$, we say that $T \in \mathcal{P}_p(E,F)$, the set of *p-absolutely summing operators* from E into F, if there is $C \ge 0$ such that $\pi_p((Tx_i)) \le C\varepsilon_p((x_i))$ for all sequences $(x_i) \in E^{\mathbb{N}}$. We then put $\Pi_p(T) = \inf C$ for which the above inequality holds. $(\mathcal{P}_p, \Pi_p)$ is a normed ideal in the sense of Pietsch (see [11]). In connection with $\mathcal{P}_p$, Matter, in his study of absolutely continuous operators, introduces for every σ the ideal $\mathcal{P}_{p,\sigma}$ of (p,σ)-*absolutely continuous operators* setting $T \in \mathcal{P}_{p,\sigma}(E,F)$ if there is $C \ge 0$ such that $\pi_{p/(1-\sigma)}((Tx_i)) \le C\delta_{p,\sigma}((x_i))$ for all $((x_i)) \in E^{\mathbb{N}}$. Then we put $\Pi_{p,\sigma}(T) = \inf C$ for all such constants C. Another equivalent formulation is to say that $T \in \mathcal{P}_{p,\sigma}(E,F)$ if there are $G \in \mathrm{BAN}$ and $S \in \mathcal{P}_p(E,G)$ such that

$$(1) \qquad \forall x \in E \quad \|T(x)\| \le \|x\|^{\sigma} \|S(x)\|^{1-\sigma}.$$

Then

$$(2) \qquad \Pi_{p,\sigma}(T) = \inf\,(\Pi_p(S))^{1-\sigma},$$

where the infimum is taken over the maps S such that (1) holds (see [8], Theorem 4.1). $(\mathcal{P}_{p,\sigma}, \Pi_{p,\sigma})$ turns out to be a maximal normed operator ideal in BAN. Clearly $\mathcal{P}_p \subset \mathcal{P}_{p,\sigma}$. In [8] there are given some relations of the ideal $\mathcal{P}_{p,\sigma}$ to other classical operator ideals. Since we shall not use them in this paper, we simply refer the interested reader to the quoted reference.

Given $p \in [1,\infty]$, p' is the conjugate of p: $1/p + 1/p' = 1$. *Throughout the paper, given $1 \le p \le \infty$ and $0 < \sigma < 1$, r will always represent the number satisfying*

$$(3) \qquad \frac{1}{r} + \frac{1}{\frac{p'}{1-\sigma}} = 1.$$

Inspired by the notation used by Saphar [12], we denote by $g_{p,\sigma}$ the tensor norm defined on every tensor product $E \otimes F$ of Banach spaces E and F by

$$g_{p,\sigma}(z; E\otimes F) = \inf\Big\{\pi_r((x_i))\delta_{p',\sigma}((y_i)) \,\Big|\, z = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes F\Big\}.$$

In [6] we have proved that the tensor norm associated to the ideal $\mathcal{P}_{p',\sigma}$, $1 \le p < \infty$, is the dual tensor norm of the transposed of $g_{p,\sigma}$. Moreover $g_p \le g_{p,\sigma}$ and $\pi = g_{1\sigma}$.

Given a compatible couple (A_0, A_1) of Banach spaces (i.e. two Banach spaces which are continuously embedded in a larger Hausdorff topological vector space E), the spaces $A_0 + A_1$ and $A_0 \cap A_1$ will always be endowed with their canonical norms

$$\|x\|_{A_0+A_1} = \inf\{\|a\|_{A_0} + \|b\|_{A_1} \mid x = a + b,\ a \in A_0,\ b \in A_1\}$$

and

$$\|x\|_{A_0\cap A_1} = \max\{\|x\|_{A_0}, \|x\|_{A_1}\}.$$

It is easy to check that, given a measure space (Ω, μ) and $1 < p < \infty$, there is always a continuous inclusion map

$$L^{\infty}(\mu) \cap L^{p'}(\mu) \subset L^{p'/(1-\sigma)}(\mu).$$

Hence we have a continuous inclusion map $J_r^{\mu} : L^r(\mu) \subset L^1(\mu) + L^p(\mu)$.

On the other hand, it is clear that every continuous linear map $T \in \mathcal{L}(E,F)$, $E, F \in \mathrm{BAN}$, defines canonically a linear map T_M from E into the dual M' of every subspace $M \subset F'$ by

$$\forall x \in E,\ y \in M, \qquad \langle T_M(x), y\rangle = \langle T(x), y\rangle.$$

In [7] we have shown the following theorem which will be used in Section 3. Since its proof widely uses results of Banach lattice theory, *we shall always deal with vector spaces over the field of real numbers.* We refer the reader to [1] for all questions concerning Banach lattices.

THEOREM 1. *Let $1 < p < \infty$ and $0 < \sigma < 1$. Let $T \in \mathcal{L}(E,F)$ be such that for every $M \in \mathrm{FIN}(F')$, the restriction of T_M to every $N \in \mathrm{FIN}(E)$ factorizes as*

$$N \to \ell^{\infty}(\Omega_N, d_N) \xrightarrow{D_N} \ell^r(\Omega_N, \mu_N) \xrightarrow{J_r^{\mu_N}} \ell^1(\Omega_N, \mu_N) + \ell^p(\Omega_N, \mu_N) \to M'$$

where every (Ω_N, μ_N) is a discrete measure space with a finite number of atoms and every D_N is a positive diagonal operator. Then there is a σ-finite measure space (Ω, μ) such that $J_F T$ factorizes as

$$E \to L^{\infty}(\Omega, \mu) \xrightarrow{B_w} L^r(\Omega, \mu) \xrightarrow{J_r^{\mu}} L^1(\Omega, \mu) + L^p(\Omega, \mu) \to F''$$

where B_w is a diagonal operator.

2. $g_{p,\sigma}$-nuclear operators. Using a standard argument (see for example 12.6 in [3]), it can be proved that given $E, F \in \mathrm{BAN}$, every element of the completion $E' \hat{\otimes}_{g_{p,\sigma}} F$ can be written as $z = \sum_{i=1}^{\infty} x_i' \otimes y_i$ where $\pi_r((x_i')) < \infty$ and $\delta_{p',\sigma}((y_i)) < \infty$. Let $\Phi_{E,F} : E' \hat{\otimes}_{g_{p,\sigma}} F \to \mathcal{L}(E,F)$ be the canonical map. In accordance with the general theory of tensor norms and operator ideals, we set

DEFINITION 2. Let $E, F \in \mathrm{BAN}$. An operator $S : E \to F$ is said to be *$g_{p,\sigma}$-nuclear* if $S = \Phi_{E,F}(z)$ for some $z \in E' \hat{\otimes}_{g_{p,\sigma}} F$.

In that case we write $S \in \mathcal{N}_{p,\sigma}(E,F)$ and define

$$\mathbf{N}_{p,\sigma}(S) = \inf\Big\{\pi_r((x_i'))\delta_{p',\sigma}((y_i)) \mid S = \Phi_{E,F}(z),\ z = \sum_{i=1}^{\infty} x_i' \otimes y_i \in E' \hat{\otimes}_{g_{p,\sigma}} F\Big\}.$$

It is easy to see that $(\mathcal{N}_{p,\sigma}, \mathbf{N}_{p,\sigma})$ is a normed operator ideal. It is clear that if $p = 1$, we have $\mathcal{N}_{1\sigma}(E,F) = \mathcal{N}_1(E,F)$, the space of nuclear operators from E into F. Hence, *from now on, we always assume that* $1 < p < \infty$. Next we characterize the (p,σ)-nuclear operators by means of special factorizations. We shall need non-canonical measures μ on $\mathbb{N}$. Then we shall put $\mu_i = \mu(\{e_i\})$ and denote the corresponding $L^u(\mathbb{N}, \mu)$ spaces by $\ell^u(\mu)$, $1 \le u \le \infty$.

THEOREM 3. *Let $E, F \in \mathrm{BAN}$ and let $T \in \mathcal{L}(E,F)$. The following are equivalent:*

(1) *T is $g_{p,\sigma}$-nuclear.*

(2) *There is a measure μ on $\mathbb{N}$ such that T can be factorized as*

$$\begin{array}{ccccc} E & \xrightarrow{\quad\quad T \quad\quad} & & & F \\ \downarrow A & & & & \uparrow D \\ \ell^{\infty}(\mu) & \xrightarrow{B} & \ell^r(\mu) & \xrightarrow{J_r^{\mu}} & \ell^1(\mu) + \ell^p(\mu) \end{array}$$

where B is a diagonal operator generated by a sequence $(b_i) \in \ell^r(\mu)$. Furthermore $\mathbf{N}_{p,\sigma}(T) = \inf\{\|D\| \cdot \|B\| \cdot \|A\|\}$, where the infimum is taken over all such factorizations.

Proof. (1)⇒(2). Suppose that T is a $g_{p,\sigma}$-nuclear operator. Given $\varepsilon > 0$, write $T = \Phi_{E,F}(\sum_{i=1}^{\infty} x_i' \otimes y_i)$ where $\pi_r((x_i'))\delta_{p',\sigma}((y_i)) \le \mathbf{N}_{p,\sigma}(T) + \varepsilon$ and $x_i' \neq 0$, $y_i \neq 0$ for each $i \in \mathbb{N}$. If we put $z_i = y_i/\delta_{p',\sigma}((y_i))$, we have $T = \Phi_{E,F}(\delta_{p',\sigma}((y_i)) \sum_{i=1}^{\infty} x_i' \otimes z_i)$ and $\delta_{p',\sigma}((z_i)) = 1$. Now, we consider the measure μ on $\mathbb{N}$ such that $\mu_i = \|z_i\|^{p'/1-\sigma}$, $i \in \mathbb{N}$, and define $A \in \mathcal{L}(E, \ell^{\infty}(\mu))$ and $B \in \mathcal{L}(\ell^{\infty}(\mu), \ell^r(\mu))$ as

$$A(x) = \delta_{p',\sigma}((y_i))\left(\frac{\langle x, x_i'\rangle}{\|x_i'\|}\right)_{i=1}^{\infty} \qquad \forall x \in E$$

and

$$B((\alpha_i)) = (\alpha_i \|x_i'\| \mu_i^{-1/r})_{i=1}^{\infty} \qquad \forall (\alpha_i) \in \ell^{\infty}(\mu).$$

We easily obtain $\|A\| \le \delta_{p',\sigma}((y_i))$ and $\|B\| \le \pi_r((x_i'))$.

Finally, we define

$$D : \ell^1(\mu) + \ell^p(\mu) \to F$$

by

$$\forall (\lambda_i) \in \ell^1(\mu) + \ell^p(\mu) \qquad D((\lambda_i)) = \sum_{i=1}^{\infty} \lambda_i \mu_i^{1/r} z_i.$$

D is well defined and continuous: in fact, first remark that

$$\sup_{j\in\mathbb{N}} \|z_j\| = \sup_{j\in N}\ \sup_{y' \in B_{F'}} |\langle z_j, y'\rangle|^{1-\sigma}\|z_j\|^{\sigma} \le \delta_{p',\sigma}((z_i)) = 1.$$

On the other hand, given $\eta > 0$ and $(\lambda_i) \in \ell^1(\mu) + \ell^p(\mu)$ there is a decomposition $(\lambda_i) = (\alpha_i) + (\beta_i)$ such that $(\alpha_i) \in \ell^1(\mu)$, $(\beta_i) \in \ell^p(\mu)$ and

$\|(\alpha_i)\|_{\ell^1(\mu)} + \|(\beta_i)\|_{\ell^p(\mu)} \le \|(\lambda_i)\| + \eta$. Noting that we have

$$\forall i \in \mathbb{N} \qquad \sup_{y' \in B_{F'}} |\langle z_i, y'\rangle| \mu_i^{1/r-1} \le \|z_i\|^{(1/r-1)p'/(1-\sigma)+1} = 1,$$

for each $(\lambda_i) \in \ell^1(\mu) + \ell^p(\mu)$ we get

$$\begin{aligned}
&\|D((\lambda_i))\| \\
&\quad \le \sup_{y' \in B_{F'}} \Big| \sum_{i=1}^{\infty} \alpha_i \mu_i \mu_i^{1/r-1} \langle z_i, y'\rangle \Big| + \sup_{y' \in B_{F'}} \Big| \sum_{i=1}^{\infty} \beta_i \mu_i^{1/p} \mu_i^{1/r-1/p} \langle z_i, y'\rangle \Big| \\
&\quad \le \sum_{i=1}^{\infty} |\alpha_i| \mu_i + \sup_{y' \in B_{F'}} \Big(\sum_{i=1}^{\infty} |\beta_i|^p \mu_i \Big)^{1/p} \Big(\sum_{i=1}^{\infty} \|z_i\|^{p'\sigma/(1-\sigma)} |\langle z_i, y'\rangle|^{p'} \Big)^{1/p'} \\
&\quad = \|(\alpha_i)\|_{\ell^1(\mu)} + \|(\beta_i)\|_{\ell^p(\mu)} \delta_{p',\sigma}((z_i))^{1/(1-\sigma)} \\
&\quad \le \|(\alpha_i)\|_{\ell^1(\mu)} + \|(\beta_i)\|_{\ell^p(\mu)} \le \|(\lambda_i)\| + \eta
\end{aligned}$$

and hence $\|D((\lambda_i))\| \le \|(\lambda_i)\|$ and $\|D\| \le 1$. Obviously $T = DJ_r^\mu BA$ and

$$\|D\| \cdot \|B\| \cdot \|A\| \le \pi_r((x_i'))\delta_{p',\sigma}((y_i)) \le \mathbf{N}_{p,\sigma}(T) + \varepsilon.$$

(2)⇒(1). Take a factorization of T as in the previous diagram. Let $P_i : \ell^\infty(\mu) \to K$, $i \in \mathbb{N}$, be the canonical projection on the ith axis. Then $x_i' := P_i A \in E'$ and $A(x) = (\langle x_i', x\rangle)_{i=1}^\infty$. If the diagonal operator B is given by the sequence (b_i), we have

$$\forall x \in E \qquad T(x) = \sum_{i=1}^{\infty} b_i \langle x_i', x\rangle DJ_r^\mu(e_i) = \sum_{i=1}^{\infty} \langle y_i', x\rangle z_i$$

where we have defined $y_i' := b_i \mu_i^{1/r} x_i'$ and $z_i := \mu_i^{-1/r} DJ_r^\mu(e_i)$.

Let us see that $\pi_r((y_i')) < \infty$. Given $\varepsilon > 0$, for every $n \in \mathbb{N}$, there is $x_n \in E$ such that $\|x_n\| \le 1+\varepsilon$ and $\langle x_n, x_n'\rangle = \|x_n'\|$. Then $\|(\langle x_n, x_n'\rangle)\|_{\ell^\infty(\mu)} \le (1+\varepsilon)\|A\|$ and

$$\begin{aligned}
\pi_r((y_i')) &= \sup \Big\{ \Big| \sum_{i=1}^{\infty} \alpha_i b_i \mu_i^{1/r} \langle x_i, x_i'\rangle \Big| \;\Big|\; \pi_{p'/(1-\sigma)}((\alpha_i)) \le 1 \Big\} \\
&= \sup\{ |\langle (\alpha_i \mu_i^{-(1-\sigma)/p'}), B((\langle x_i, x_i'\rangle))\rangle| \mid \pi_{p'/(1-\sigma)}((\alpha_i)) \le 1\} \\
&\le \|B\|(1+\varepsilon)\|A\|
\end{aligned}$$

since $(\alpha_i \mu_i^{-(1-\sigma)/p'}) \in \ell^{p'/(1-\sigma)}(\mu)$. In consequence $\pi_r((y_i')) \le \|B\| \cdot \|A\|$.

On the other hand, as $(\ell^1(\mu) + \ell^p(\mu))' = \ell^\infty(\mu) \cap \ell^{p'}(\mu)$ (see [4], Theorem 2.7.1), for every $w' \in F'$ we have $D'(w') = (\langle e_i/\mu_i, D'(w')\rangle)_{i=1}^\infty$. Hence, putting $H := DJ_r^\mu$ we get

$$\begin{aligned}
\delta_{p',\sigma}((z_i)) &= \sup_{y' \in B_{F'}} \Big(\sum_{i=1}^{\infty} \Big| \frac{\langle J_r^\mu(e_i), D'(y')\rangle}{\mu_i} \Big|^{p'} \mu_i \Big\| \frac{H(e_i)}{\mu_i} \Big\|^{p'\sigma/(1-\sigma)} \Big)^{(1-\sigma)/p'} \\
&\le (\sup_{i \in \mathbb{N}} \sup_{z' \in B_{F'}} |\langle J_r^\mu(e_i)/\mu_i, D'(z')\rangle|^{p'\sigma/(1-\sigma)})^{(1-\sigma)/p'} \\
&\qquad \times \sup_{y' \in B_{F'}} \Big(\sum_{i=1}^{\infty} \Big| \frac{\langle J_r^\mu(e_i), D'(y')\rangle}{\mu_i} \Big|^{p'} \mu_i \Big)^{(1-\sigma)/p'} \\
&= \sup_{z' \in B_{F'}} \|D'(z')\|_{\ell^\infty(\mu)}^\sigma \sup_{y' \in B_{F'}} \|D'(y')\|_{\ell^{p'}(\mu)}^{1-\sigma} \\
&\le \sup_{z' \in B_{F'}} \|D'(z')\|_{\ell^\infty(\mu) \cap \ell^{p'}(\mu)} \le \|D'\| = \|D\|.
\end{aligned}$$

Thus $T \in \mathcal{N}_{p,\sigma}(E,F)$ and $\mathbf{N}_{p,\sigma}(T) \le \|A\| \cdot \|B\| \cdot \|D\|$. With the bound of the necessary condition, we obtain the final result. ∎

3. $g_{p,\sigma}$-integral operators. According to the general theory of tensor norms and operator ideals (see [5]) we make the next definition:

DEFINITION 4. Let $E, F \in$ BAN. An operator $S : E \to F$ is called *$g_{p,\sigma}$-integral* if it belongs to the maximal normed operator ideal $(\mathcal{I}_{p,\sigma}, I_{p,\sigma})$ associated with the tensor norm $g_{p,\sigma}$, i.e. to the associated maximal ideal $(\mathcal{N}_{p,\sigma}^{\max}, \mathbf{N}_{p,\sigma}^{\max})$ in the sense of Pietsch [11].

In particular, an operator $T \in \mathcal{L}(E, F')$ is $g_{p,\sigma}$-integral if and only if $T \in (E \otimes_{g'_{p,\sigma}} F)'$. The aim of this section is to characterize the $g_{p,\sigma}$-integral operators by means of suitable factorizations. The basic idea is to apply the ultraproduct technique to the diagram obtained in the above section for $g_{p,\sigma}$-nuclear operators.

THEOREM 5. *Let $E, F \in$ BAN and $T \in \mathcal{L}(E,F)$. The following conditions are equivalent*:

(1) *T is $g_{p,\sigma}$-integral.*

(2) *There is a σ-finite measure space $(\Omega, \mathcal{M}, \mu)$ such that $J_F T$ can be factorized as*

$$\begin{array}{ccccc}
E & \xrightarrow{\ T\ } & F & \xrightarrow{\ J_F\ } & F'' \\
\downarrow{\scriptstyle A} & & & & \uparrow{\scriptstyle D} \\
L^\infty(\mu) & \xrightarrow[B_w]{} & L^r(\mu) & \xrightarrow[J_r^\mu]{} & L^1(\mu) + L^p(\mu)
\end{array}$$

where B_w is the multiplication operator by a measurable function $w \in L^r(\mu)$.

(3) *The same as in* (2) *with an arbitrary measure space $(\Omega, \mathcal{M}, \mu)$.*

Furthermore, in (2) *and* (3), *the canonical* $g_{p,\sigma}$*-integral norm* $\mathbf{I}_{p,\sigma}(T)$ *of* T *satisfies*

$$\mathbf{I}_{p,\sigma}(T) = \inf\{\|D\| \cdot \|B_w\| \cdot \|A\|\},$$

where the infimum is taken over all such factorizations.

Proof. (1)⇒(2). The proof goes along the same lines as in the classical case of p-integral operators, although using, of course, our main results of Theorems 1 and 3 (see for example [3], Chapter 18, or [4], Theorem 9.3, where more details can be found).

(2)⇒(3). Trivial.

(3)⇒(1). Suppose that $T \in \mathcal{L}(E,F)$ is such that $J_F T$ factorizes as in the above diagram. Let us see that $T \in (E \otimes_{g'_{p,\sigma}} F')'$ and $\mathbf{I}_{p,\sigma}(T) \le \|D\| \cdot \|B_w\| \cdot \|A\|$. For every $x_i \in E$ and $y'_i \in F'$, $i = 1, \ldots, n$, we have

$$\Big\langle J_F T, \sum_{i=1}^n x_i \otimes y'_i \Big\rangle = \Big\langle J_r^\mu B_w, \sum_{i=1}^n A(x_i) \otimes D'(y'_i) \Big\rangle.$$

Hence, by the metric mapping property of tensor norms, it is enough to see that $H := J_r^\mu B_w$ belongs to $(L^\infty(\mu) \hat{\otimes}_{g'_{p,\sigma}} (L^\infty(\mu) \cap L^{p'}(\mu)))'$ and $\mathbf{I}_{p,\sigma}(H) \le \|B_w\|$. Since the set $\mathcal{T}$ of step functions on Ω with support of finite measure is dense in $L^\infty(\mu) \cap L^{p'}(\mu)$ and $L^1(\mu) + L^p(\mu) \subset (L^\infty(\mu) \cap L^{p'}(\mu))'$, by the density Lemma (see Lemma 13.4 in [3]) we only have to show that H is in $(L^\infty(\mu) \hat{\otimes}_{g'_{p,\sigma}} \mathcal{T})'$.

Let $z \in L^\infty(\mu) \otimes_{g'_{p,\sigma}} \mathcal{T}$ and let M_0 and M_1 be finite-dimensional subspaces of $L^\infty(\mu)$ and $\mathcal{T}$ respectively such that $z \in M_0 \otimes M_1$. Let $\{\chi_{B_j}\}_{j=1}^m$ be a basis of M_1 such that $\{B_j\}_{j=1}^m$ is a finite sequence of pairwise disjoint sets in Ω and let $\{f_i\}_{i=1}^n$ be a basis in M_0. For $1 \le j \le m$, let $\xi_j = \chi_{B_j} \in (L^\infty(\mu) \cap L^{p'}(\mu))'$ and $\phi_j \in (L^\infty(\mu))'$ be such that

$$\langle h, \phi_j \rangle = \int_\Omega \chi_{B_j} w h \, d\mu \qquad \forall h \in L^\infty(\mu).$$

Then ϕ_j is well defined. In fact, since $\chi_\Omega \in L^\infty(\mu)$, we have $w \in L^r(\mu)$ and hence $w\chi_{B_j} \in L^1(\mu) + L^p(\mu)$ can be written as $w\chi_{B_j} = a_j + b_j$ with $a_j \in L^1(\mu)$ and $b_j \in L^p(\mu)$. Then, by Hölder's inequality,

$$\forall h \in L^\infty(\mu) \qquad |\langle h, \phi_j \rangle| \le \max(1, \mu(B_j)^{1/p'}) \|h\|_{L^\infty(\mu)} (\|a_j\|_{L^1(\mu)} + \|b_j\|_{L^p(\mu)})$$

and hence

$$|\langle h, \phi_j \rangle| \le \max(1, \mu(B_j)^{1/p'}) \|h\|_{L^\infty(\mu)} \|w\chi_{B_j}\|_{L^1(\mu)+L^p(\mu)}.$$

Now, given $f \otimes h \in M_0 \otimes M_1$, there are finite scalar sequences $\{\alpha_i\}_{i=1}^n$ and $\{\beta_j\}_{j=1}^m$ such that $f = \sum_{i=1}^n \alpha_i f_i$ and $h = \sum_{j=1}^m \beta_j \chi_{B_j}$. Then

$$\langle f \otimes h, H \rangle = \sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j \int_{B_j} w f_i \, d\mu = \sum_{j=1}^m \frac{\langle \xi_j, h \rangle}{\mu(B_j)} \sum_{i=1}^n \alpha_i \int_{B_j} w f_i \, d\mu$$

$$= \sum_{j=1}^m \Big(\mu(B_j)^{1/r-1} \int_{B_j} w f \, d\mu\Big)\big(\mu(B_j)^{-1/r} \langle \xi_j, h \rangle\big)$$
$$= \Big\langle \sum_{j=1}^m (\mu(B_j)^{1/r-1} \phi_j) \otimes (\mu(B_j)^{-1/r} \xi_j), f \otimes h \Big\rangle.$$

Hence $\langle z, H \rangle = \langle z, U \rangle$ for every $z \in M_0 \otimes M_1$ where

$$U := \sum_{j=1}^m (\mu(B_j)^{1/r-1} \widehat{\phi}_j) \otimes (\mu(B_j)^{-1/r} \widehat{\xi}_j) \in M'_0 \otimes_{g_{p,\sigma}} M'_1 = (M_0 \otimes_{g'_{p,\sigma}} M_1)'$$

and $\widehat{\phi}_j$ and $\widehat{\xi}_j$ denote the canonical images of ϕ_j and ξ_j in M'_0 and M'_1 respectively.

Since by Hölder's inequality

$$\|\phi_j\| = \sup\Big\{\Big| \int_{B_j} wh \, d\mu \Big| \;\Big|\; \|h\|_{L^\infty(\mu)} \le 1\Big\}$$
$$\le \Big(\int_{B_j} |w|^r \, d\mu\Big)^{1/r} \sup\Big\{\Big(\int_{B_j} |h|^{p'/(1-\sigma)} \, d\mu\Big)^{(1-\sigma)/p'} \;\Big|\; \|h\|_{L^\infty(\mu)} \le 1\Big\}$$
$$\le \Big(\int_{B_j} |w|^r \, d\mu\Big)^{1/r} \mu(B_j)^{(1-\sigma)/p'},$$

we obtain

$$\pi_r((\mu(B_j)^{1/r-1} \widehat{\phi}_j)) \le \Big(\sum_{j=1}^m \mu(B_j)^{1-r} \mu(B_j)^{r(1-\sigma)/p'} \int_{B_j} |w|^r d\mu\Big)^{1/r}$$
$$= \|w\|_{L^r(\mu)} = \|B_w\|.$$

On the other hand, if

$$K = \{f \in L^\infty(\mu) \cap L^{p'}(\mu) \mid \sup(\|f\|_{L^\infty(\mu)}, \|f\|_{L^{p'}(\mu)}) \le 1\},$$

we get

$$\delta_{p',\sigma}((\mu(B_j)^{-1/r} \widehat{\xi}_j)) \le \sup_{f \in K} \Big(\sum_{j=1}^m |\langle \mu(B_j)^{-1/r-(1-\sigma)/p'} \xi_j, f \rangle|^{p'}$$
$$\times \mu(B_j)^{1-\sigma} \|\mu(B_j)^{-1/r} \xi_j\|^{p'\sigma/(1-\sigma)}\Big)^{(1-\sigma)/p'}$$

and since $\|\xi_j\| \le \|\chi_{B_j}\|_{L^1(\mu)} = \mu(B_j)$, using Hölder's inequality for the evaluation of $\langle \xi_j, f \rangle$, we continue the inequality as

$$\le \sup_{f \in K} \Big(\sum_{j=1}^m \mu(B_j)^{p'/p-p'+1-\sigma+\sigma} \int_{B_j} |f|^{p'} \, d\mu\Big)^{(1-\sigma)/p'} = \sup_{f \in K} \|f\|_{L^{p'}(\mu)}^{1-\sigma} \le 1.$$

Hence

$$|\langle H,z\rangle| = |\langle U,z\rangle| \le g_{p,\sigma}(U) g'_{p,\sigma}(z; M_0 \otimes M_1) \le \|B_w\| g'_{p,\sigma}(z; M_0 \otimes M_1)$$

and $\mathbf{I}_{p,\sigma}(H) \le \|B_w\|$. ∎

REMARK 6. We have $\mathcal{I}_{p,\sigma} \subset \mathcal{I}_r \subset \mathcal{I}_p \subset \mathcal{P}_p \subset \mathcal{P}_{p/(1-\sigma)} \subset \mathcal{P}_{p,\sigma}$.

This is a direct consequence of Theorem 5, Theorem 19.2.6 and Propositions 19.2.10 and 17.3.9 in [11] and Proposition 4.2 in [8] combined with the fact that $g_{p'} \ge (g_p^t)'$ in BAN. ∎

4. Applications. In this last section we state some results which can be derived from the previous factorization theorems of Sections 3 and 4. The first one concerns the equality $\mathcal{N}_{p,\sigma}(E,F) = \mathcal{I}_{p,\sigma}(E,F)$. We make the following

DEFINITION 7. If $E, F \in$ BAN, we say that $T \in \mathcal{L}(E,F)$ is a *strictly $g_{p,\sigma}$-integral operator* if it can be factored as

$$E \xrightarrow{A} L^\infty(\mu) \xrightarrow{C_w} L^r(\mu) \xrightarrow{J_r^\mu} L^1(\mu) + L^p(\mu) \xrightarrow{D} F$$

where C_w is the multiplication operator by a function $w \in L^r(\mu)$. In that case, we define $\mathbf{SI}_{p,\sigma}(T) = \inf\{\|A\| \cdot \|C_w\| \cdot \|D\|\}$, where the infimum is taken over all such factorizations.

The ideal of strictly $g_{p,\sigma}$-integral operators will be denoted by $\mathcal{SI}_{p,\sigma}$. Clearly, if F is a dual space (or F is 1-complemented in its bidual), we have $\mathcal{SI}_{p,\sigma}(E,F) = \mathcal{I}_{p,\sigma}(E,F)$ and $\mathbf{SI}_{p,\sigma}(T) = \mathbf{I}_{p,\sigma}(T)$.

Given a measure space $(\Omega, \mathcal{A}, \mu)$ and a Banach space E, we denote by $L^p(\mu, E)$ the Lebesgue–Bochner space of order p.

LEMMA 8. *Suppose that $\mu(\Omega) < \infty$, $\phi \in L^r(\mu, E')$, $g_0 \in L^p(\Omega,\mu)$ is μ-everywhere positive, $\|g_0\|_{L^p(\mu)} = 1$ and T_h is a diagonal positive operator from $L^r(\Omega,\mu)$ into $L^1(\Omega,\mu) + L^p(\Omega, g_0, \mu)$. Then $T_\phi : x \mapsto h(t)\langle x, \phi(t)\rangle$ is a $g_{p,\sigma}$-nuclear operator from E into $L^1(\Omega,\mu) + L^p(\Omega, g_0, \mu)$.*

Proof. Suppose that $\phi(t) = \sum_{i=1}^n x_i' \chi_{A_i(t)}$ is a simple function with pairwise disjoint sets $\{A_i\}_{i=1}^n$. Then

$$T_\phi(x) = \Big\langle \sum_{i=1}^n (\mu(A_i)^{1/r} x_i') \otimes (\mu(A_i)^{-1/r} h\chi_{A_i}), x\Big\rangle.$$

We have

$$\pi_r((\mu(A_i)^{1/r} x_i')_{i=1}^n) = \Big(\sum_{i=1}^n \mu(A_i)\|x_i'\|^r\Big)^{1/r} = \|\phi\|.$$

Now, put $F = L^1(\Omega,\mu) + L^p(\Omega, g_0, \mu)$ to simplify the notation. As $g_0 \in L^p(\mu)$, we have $F' = L^\infty(\mu) \cap L^{p'}(\Omega, g_0^{p-1}, \mu)$ and $\chi_\Omega \in F'$. Put $h_0 = T_h'(\chi_\Omega) \in L^{r'}(\mu)$. Since T_h' is also a positive operator, we have $\|h_0\|_{L^{r'}(\mu)} = \|T_h'\| \le \|T_h\|$. By Hölder's inequality, if $f_i \in F'$ is such that $\|f_i\|_{F'} \le 1$ and $\|h\chi_{A_i}\|_F = |\langle h\chi_{A_i}, f_i\rangle|$, we have

$$\begin{aligned}\|h\chi_{A_i}\|_F = |\langle \chi_{A_i}, T_h'(f_i)\rangle| &\le \Big|\int_{A_i} T_h'(f_i)\,d\mu\Big| \le \mu(A_i)^{1/r}\Big(\int_{A_i} |T_h'(f_i)|^{r'} d\mu\Big)^{1/r'} \\ &\le \|f_i\|_{L^\infty(\mu)} \mu(A_i)^{1/r}\Big(\int_{A_i} |h_0|^{r'} d\mu\Big)^{1/r'} \\ &\le \mu(A_i)^{1/r}\Big(\int_{A_i} |h_0|^{r'} d\mu\Big)^{1/r'}.\end{aligned}$$

In consequence, using again Hölder's inequality we get

$$\begin{aligned}&\delta_{p',\sigma}((\mu(A_i)^{-1/r} h\chi_{A_i})_{i=1}^n) \\ &= \sup_{\|f\|_{F'} \le 1} \Big(\sum_{i=1}^n \|\mu(A_i)^{-1/r} h\chi_{A_i}\|_F^{p'\sigma/(1-\sigma)} |\langle \mu(A_i)^{-1/r} T_h(\chi_{A_i}), f\rangle|^{p'}\Big)^{(1-\sigma)/p'} \\ &\le \sup_{\|f\|_{F'} \le 1} \Big(\sum_{i=1}^n \Big(\int_{A_i} |h_0|^{r'} d\mu\Big)^{p'\sigma/(r'(1-\sigma))} \mu(A_i)^{-p'/r} |\langle \chi_{A_i}, T_h'(f)\rangle|^{p'}\Big)^{(1-\sigma)/p'} \\ &= \sup_{\|f\|_{F'} \le 1} \Big(\sum_{i=1}^n \Big(\int_{A_i} |h_0|^{r'} d\mu\Big)^{\sigma} \mu(A_i)^{-p'/r} \Big(\int_{A_i} |T_h'(f)|\,d\mu\Big)^{p'}\Big)^{(1-\sigma)/p'} \\ &\le \sup_{\|f\|_{F'} \le 1} \|f\|_{L^\infty(\mu)}^{1-\sigma} \Big(\sum_{i=1}^n \Big(\int_{A_i} |h_0|^{r'} d\mu\Big)^{\sigma} \mu(A_i)^{(-p')/r} \Big(\int_{A_i} |h_0|\,d\mu\Big)^{p'}\Big)^{(1-\sigma)/p'} \\ &\le \Big(\sum_{i=1}^n \Big(\int_{A_i} |h_0|^{r'} d\mu\Big)^{\sigma} \Big(\int_{A_i} |h_0|^{r'} d\mu\Big)^{1-\sigma}\Big)^{(1-\sigma)/p'} \le \|h_0\|_{L^{r'}(\mu)} = \|T_h\|.\end{aligned}$$

Hence $T_\phi \in \mathcal{N}_{p,\sigma}(E, L^1(\Omega,\mu) + L^p(\Omega, g_0, \mu))$ and $\mathbf{N}_{p,\sigma}(T_\phi) \le \|T_h\| \cdot \|\phi\|$. This means that $V : \phi \mapsto T_\phi$ is a continuous linear map from the vector space $\mathcal{H}$ of simple functions on $L^r(\mu, E')$ into $\mathcal{N}_{p,\sigma}(E, L^1(\Omega,\mu) + L^p(\Omega, g_0, \mu))$ and $\|V(\phi)\| \le \|T_h\| \cdot \|\phi\|$. As $\mathcal{H}$ is dense in $L^r(\mu, E')$, V can be continuously extended to $L^r(\mu, E')$ and hence $T_\phi = V(\phi)$ is $g_{p,\sigma}$-nuclear for all $\phi \in L^r(\mu, E')$. Moreover, $\mathbf{N}_{p,\sigma}(T_\phi) \le \|T_h\| \cdot \|\phi\|$. ∎

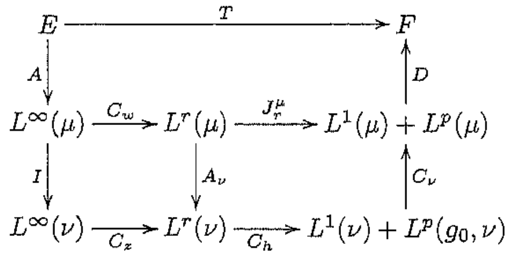
It is interesting to know sufficient conditions as general as possible for the equality $\mathcal{N}_{p,\sigma}(E,F) = \mathcal{SI}_{p,\sigma}(E,F)$:

THEOREM 9. *Given $E, F \in$ BAN, the equality $\mathcal{N}_{p,\sigma}(E,F) = \mathcal{SI}_{p,\sigma}(E,F)$ holds if E' has the Radon–Nikodym property. In that case, $\mathbf{N}_{p,\sigma}(T) = \mathbf{SI}_{p,\sigma}(T)$ for every $T \in \mathcal{SI}_{p,\sigma}(E,F)$.*

Proof. Suppose that E' has the Radon–Nikodym property. Fix $T \in \mathcal{SI}_{p,\sigma}(E,F)$. Given $\varepsilon > 0$, there is a measure space $(\Omega, \mathcal{M}, \mu)$ and a factorization as in Definition 7 such that $\mathbf{SI}_{p,\sigma}(T) + \varepsilon \ge \|A\| \cdot \|C_w\| \cdot \|D\|$. Since $w = C_w(\chi_\Omega) \in L^r(\mu)$, w has σ-finite support $S(w)$. Then we can suppose that $(\Omega, \mathcal{M}, \mu)$ is a σ-finite measure space by projecting onto the sectional subspace relative to $S(w)$ if necessary. Now, let h_0 be an everywhere positive function in Ω such that $h_0 \in L^1(\mu) \cap L^p(\mu)$ and $\|h_0\|_{L^p(\mu)} = 1$. We consider the finite measure ν on $\mathcal{M}$ such that

$$\nu(A) = \int_A h_0 \, d\mu \qquad \forall A \in \mathcal{M}.$$

Put $g_0 := h_0^{(p-1)/p}$. The maps $A_\nu : f \mapsto f h_0^{-1/r}$ and $C_\nu : f \mapsto f h_0$ are isometries from $L^r(\mu)$ onto $L^r(\nu)$ and from $L^1(\nu) + L^p(g_0, \nu)$ onto $L^1(\mu) + L^p(\mu)$ respectively. Since h_0 is everywhere positive and ν is absolutely continuous with respect to μ, the identity $I : L^\infty(\mu) \to L^\infty(\nu)$ is also an isometry. Then there are multiplication operators C_z and C_h which make the diagram



commutative. Since E' has the Radon–Nikodym property, there is a function ϕ in $L^\infty(\nu, E')$ such that for all $x \in E$ we have $IA(x)(t) = \langle x, \phi(t) \rangle$ ν-almost everywhere on Ω. Clearly $z\phi \in L^r(\nu, E')$ and $\|g_0\|_{L^p(\nu)} = 1$ by the construction of h_0. Now, the conclusion is obtained by direct application of Lemma 8 since

$$\mathbf{N}_{p,\sigma}(T) \le \|D\| \cdot \|C_h\| \cdot \|z\phi\| = \|D\| \cdot \|J_r^\mu\| \cdot \|A_\nu C_w A\| \le \mathbf{SI}_{p,\sigma}(T) + \varepsilon.$$

By Theorem 5, as ε is arbitrary we get $\mathbf{SI}_{p,\sigma}(T) = \mathbf{N}_{p,\sigma}(T)$. ∎

The second application we consider concerns metric properties of the tensor norms $g'_{p,\sigma}$ and $g_{p,\sigma}$, $1 < p < \infty$, $0 < \sigma < 1$.

THEOREM 10. *$g'_{p,\sigma}$ is a totally accessible tensor norm, i.e. for all $E, F \in$ BAN, $F \otimes_{g'_{p,\sigma}} E$ is a subspace of $\mathcal{P}_{p',\sigma}(E', F)$.*

Proof. Let $z = \sum_{i=1}^n y_i \otimes x_i \in F \otimes E$ and let $H_z \in \mathcal{L}(E', F)$ be the canonical linear map associated with z. By Proposition 12.4 and the duality theorem 15.5 of [3], we have $\Pi_{p'\sigma}(H_z) \le g'_{p,\sigma}(z; F \otimes E)$. On the other hand, given $N \in \mathrm{FIN}(F)$ such that $z \in N \otimes E$, there is $V \in \mathcal{I}_{p,\sigma}(N, E') = \mathcal{SI}_{p,\sigma}(N, E')$ such that $\mathbf{I}_{p,\sigma}(V) = \mathbf{SI}_{p,\sigma}(V) \le 1$ and $g'_{p,\sigma}(z; N \otimes E) = \langle z, V \rangle$.

By Theorem 9, as N' has the Radon–Nikodym property, $V \in \mathcal{N}_{p,\sigma}(N, E')$ and given $\varepsilon > 0$, there is a factorization $V = DBA$ as in the diagram of Theorem 3 such that

$$\|D\| \cdot \|B\| \cdot \|A\| \le \mathbf{N}_{p,\sigma}(V) + \varepsilon = \mathbf{I}_{p,\sigma}(V) + \varepsilon \le 1 + \varepsilon.$$

As ℓ^∞ has the extension property, A can be extended to a map $\overline{A} \in \mathcal{L}(F, \ell^\infty)$ such that $\|\overline{A}\| \le \|A\|$. By Theorem 3, $W := DB\overline{A}$ is in $\mathcal{N}_{p,\sigma}(F, E')$ and there is a representation $W = \Phi_{E,F}(w)$ where $w = \sum_{j=1}^\infty y'_j \otimes x'_j \in F' \hat{\otimes}_{g_{p,\sigma}} E'$ such that

$$\pi_r((y'_j))\delta_{p',\sigma}((x'_i)) \le \mathbf{N}_{p,\sigma}(W) + \varepsilon \le \|D\| \cdot \|B\| \cdot \|\overline{A}\| + \varepsilon \le 1 + 2\varepsilon.$$

Clearly $\langle z, V \rangle = \langle z, W \rangle$. Then

$$\begin{aligned} g'_{p,\sigma}(z; F \otimes E) &\le g'_{p,\sigma}(z; N \otimes E) = \langle z, V \rangle = \langle z, W \rangle \\ &\le g_{p,\sigma}(w)\Pi_{p'\sigma}(H_z) \le (1 + 2\varepsilon)\Pi_{p'\sigma}(H_z). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get $g'_{p,\sigma}(z; F \otimes E) \le \Pi_{p'\sigma}(H_z)$. ∎

COROLLARY 11. *$g_{p,\sigma}$ is an accessible tensor norm, i.e. $M \otimes_{g_{p,\sigma}} F$ and $F \otimes_{g_{p,\sigma}} M$ are subspaces of $\mathcal{I}_{p,\sigma}(M', F'')$ and $\mathcal{I}_{p,\sigma}(F', M)$ respectively if $F \in$ BAN and M is a finite-dimensional space.*

Proof. This is immediate by Theorem 10 and Proposition 15.6 of [3]. ∎

In the last application we study some topological vector space properties of $E \otimes_{g_{p,\sigma}} F$ and $\mathcal{P}_{p,\sigma}(E, F)$. In the next theorem we need a theorem of Matter [9] which uses interpolation spaces $(F_0, F_1)_{1-\sigma,1}$. We refer the reader to [2] for this topic.

THEOREM 12. *Let $1 \le p < \infty$, $0 < \sigma < 1$. Suppose $E \in$ BAN does not contain ℓ^1, and E' has the approximation property. Then for every $F \in$ BAN, the set of finite rank operators is dense in $\mathcal{P}_{p,\sigma}(E, F)$.*

Proof. For each $x \in E$, denote by f_x the scalar function on E' given by $f_x(x') = \langle x, x' \rangle$ if $x' \in E'$. Given a probability measure μ on $B_{E'}$, let $I_\mu : E \to L^p(B_{E'}, \mu)$ send every $x \in E$ to the class of f_x in $L^p(B_{E'}, \mu)$. Let E_μ be the quotient space of E by the kernel of I_μ, $K_\mu \in \mathcal{L}(E, E_\mu)$ the canonical quotient map and $J_{p,\sigma}$ the inclusion $E_\mu \subset (E_\mu, L^p(\mu))_{1-\sigma,1}$. Clearly, $I_\mu = J_p J_{p,\sigma} K_\mu$ where J_p is the inclusion $(E_\mu, L^p(\mu))_{1-\sigma,1} \subset L^p(\mu)$.

Fix $T \in \mathcal{P}_{p,\sigma}(E, F)$. By a theorem of Matter (corollary of Theorem B in [9]), there are a probability measure μ on $B_{E'}$ and $R \in \mathcal{L}((E_\mu, L^p(\mu))_{1-\sigma,1}, F)$ such that

$$T = RJ_{p,\sigma}K_\mu. \tag{4}$$

By the theorem of Grothendieck–Pietsch (see Theorem 11.3 in [3]), I_μ is p-absolutely summing. Moreover, by interpolation space properties, there is

$C > 0$ such that

$$\begin{aligned}(5)\qquad \|J_{p,\sigma}K_\mu(x)\| &\le \|K_\mu(x)\|^{\sigma}_{E_\mu}\|C^{1/(1-\sigma)}I_\mu(x)\|^{1-\sigma}_{L^p(\mu)}\\ &\le \|x\|^{\sigma}\|C^{1/(1-\sigma)}I_\mu(x)\|^{1-\sigma}_{L^p(\mu)} \qquad \forall x\in E.\end{aligned}$$

Then, by (1), $J_{p,\sigma}K_\mu \in \mathcal{P}_{p,\sigma}(E,(E_\mu,L^p(\mu))_{1-\sigma,1})$. As E does not contain ℓ^1, by Theorem 2.2 of Niculescu [10], $J_{p,\sigma}K_\mu$ is compact. Since E' has the approximation property, given $\varepsilon > 0$, by 43.1(6) of [5], there is $S \in E'\otimes (E_\mu, L^p(\mu))_{1-\sigma,1}$ such that

$$(6)\qquad \sup_{x\in B_E}\|(J_{p,\sigma}K_\mu - S)(x)\| \le \varepsilon/2.$$

By definition of the elements of $(E_\mu, L^p(\mu))_{1-\sigma,1}$, S can be written as

$$S = \sum_{n=1}^{k} x'_n \otimes \Big(\sum_{j=1}^{\infty} f_{x_j^n}\Big)$$

where the series $\sum_{j=1}^{\infty} f_{x_j^n}$ is convergent in $(E_\mu, L^p(\mu))_{1-\sigma,1}$ (Proposition 3, Section 1, Chapter II in [2]) and hence in $L^p(\mu)$. Then there is $j_0\in\mathbb{N}$ such that

$$(7)\qquad \Big\|\sum_{j=j_0+1}^{\infty} f_{x_j^n}\Big\| \le \frac{\varepsilon}{2k\|x'_n\|} \qquad \forall n=1,\ldots,k.$$

If we put $y_n = \sum_{j=1}^{j_0} x_j^n$ and consider the canonical operator from E into $(E_\mu, L^p(\mu))_{1-\sigma,1}$ defined by the tensor $Z = \sum_{n=1}^{k} x'_n \otimes (\sum_{j=1}^{j_0} f_{x_j^n}) = \sum_{n=1}^{k} x'_n\otimes f_{y_n}$, by (6) and (7) we have

$$\begin{aligned}(8)\qquad \|J_{p,\sigma}K_\mu - Z\| &= \sup_{x\in B_E}\|(J_{p,\sigma}K_\mu - Z)(x)\|\\ &\le \sup_{x\in B_E}\|(J_{p,\sigma}K_\mu - S)(x)\| + \sup_{x\in B_E}\Big\|\sum_{n=1}^{k}\langle x'_n, x\rangle \sum_{j=j_0+1}^{\infty} f_{x_j^n}\Big\| \le \varepsilon.\end{aligned}$$

Furthermore, for every $x\in E$ and $x'\in E'$,

$$\langle Z(x), x'\rangle = \sum_{n=1}^{k}\langle x, x'_n\rangle\langle y_n, x'\rangle = \Big\langle \sum_{n=1}^{k}\langle x, x'_n\rangle y_n, x'\Big\rangle.$$

Consequently, $J_pZ(x) = f_{W(x)}$ where $W\in\mathcal{L}(E,E)$ is the map $W(x) = \sum_{n=1}^{k}\langle x, x'_n\rangle y_n$. Setting $H := J_p(J_{p,\sigma}K_\mu - Z)\in\mathcal{L}(E, L^p(\mu))$, since $Z(E)\subset E_\mu$, we have

$$(9)\qquad \forall x\in E\quad \|(J_{p,\sigma}K_\mu - Z)(x)\| \le C\|x\|^{\sigma}\|H(x)\|^{1-\sigma}.$$

Moreover, $H\in\mathcal{P}_p(E, L^p(\mu))$ since J_pZ is a finite rank map and $J_pJ_{p,\sigma}K_\mu = I_\mu \in \mathcal{P}_p(E, L^p(\mu))$. Then, for every weakly p-absolutely summable sequence $(x_i)_{i=1}^{\infty}$ in E, using (8) we have

$$\begin{aligned}(10)\qquad &\Big(\sum_{i=1}^{\infty}\|H(x_i)\|^p\Big)^{1/p}\\ &= \Big(\int_{B_{E'}}\sum_{i=1}^{\infty}|\langle J_p(J_{p,\sigma}K_\mu - Z)(x_i), x'\rangle|^p\,d\mu(x')\Big)^{1/p}\\ &= \bigg(\|(J_{p,\sigma}K_\mu - Z)'\|\int_{B_{E'}}\bigg(\sum_{i=1}^{\infty}\bigg|\bigg\langle x_i, \frac{(J_p(J_{p,\sigma}K_\mu - Z))'(x')}{\|(J_{p,\sigma}K_\mu - Z)'\|}\bigg\rangle\bigg|^p\bigg)\,d\mu(x')\bigg)^{1/p}\\ &\le \varepsilon_p((x_i))\varepsilon^{1/p}.\end{aligned}$$

Hence, by (9) and (10) we see that $J_{p,\sigma}K_\mu - Z\in\mathcal{P}_{p,\sigma}(E,(E_\mu, L^p(\mu))_{1-\sigma,1})$ and by (2), $\Pi_{p,\sigma}(J_{p,\sigma}K_\mu - Z)\le C\varepsilon^{(1-\sigma)/p}$. Then, by (4),

$$\Pi_{p,\sigma}(T - RZ) \le \|R\|\Pi_{p,\sigma}(J_{p,\sigma}K_\mu - Z) \le C\|R\|\varepsilon^{(1-\sigma)/p}$$

and $E'\otimes F$ is dense in $\mathcal{P}_{p,\sigma}(E,F)$. ■

THEOREM 13. *Suppose that F has the approximation property, $0<\sigma<1$ and $1<p<\infty$. Then $E\,\hat{\otimes}_{g_{p,\sigma}} F$, $\mathcal{N}_{p,\sigma}(E,F)$, $E\,\hat{\otimes}_{g'_{p,\sigma}} F$, $\mathcal{P}_{p,\sigma}(E,F)$ and $\mathcal{I}_{p,\sigma}(E,F)$ are reflexive if and only if E and F are.*

Proof. Let E and F be reflexive. Then F' also has the approximation property. By Theorems 10, 12 and 9 and [6],

$$(E\,\hat{\otimes}_{g_{p,\sigma}} F)' = \mathcal{P}_{p'\sigma}(F,E') = E'\,\hat{\otimes}_{g'_{p,\sigma}} F'$$

and

$$(E'\,\hat{\otimes}_{g'_{p,\sigma}} F')' = \mathcal{I}_{p,\sigma}(E',F) = \mathcal{SI}_{p,\sigma}(E',F) = \mathcal{N}_{p,\sigma}(E',F).$$

By a result of Grothendieck (see Proposition 21.7 in [3]), we have $\mathcal{N}_{p,\sigma}(E',F) = E\,\hat{\otimes}_{g_{p,\sigma}} F$. Hence $E\,\hat{\otimes}_{g_{p,\sigma}} F$ (and in consequence every space of the statement) is reflexive. ■

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Tauberian theorems for Cesàro summable double integrals over $\mathbb{R}_+^2$

by

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Abstract. Given $f \in L^1_{\text{loc}}(\mathbb{R}_+^2)$, denote by $s(w,z)$ its integral over the rectangle $[0,w]\times[0,z]$ and by $\sigma(u,v)$ its $(C,1,1)$ mean, that is, the average value of $s(w,z)$ over $[0,u]\times[0,v]$, where $u,v,w,z>0$. Our permanent assumption is that $(*)$ $\sigma(u,v)\to A$ as $u,v\to\infty$, where A is a finite number.

First, we consider real-valued functions f and give one-sided Tauberian conditions which are necessary and sufficient in order that the convergence $(**)$ $s(u,v)\to A$ as $u,v\to\infty$ follow from $(*)$. Corollaries allow these Tauberian conditions to be replaced either by Schmidt type slow decrease (or increase) conditions, or by Landau type one-sided Tauberian conditions.

Second, we consider complex-valued functions and give a two-sided Tauberian condition which is necessary and sufficient in order that $(**)$ follow from $(*)$. In particular, this condition is satisfied if $s(u,v)$ is slowly oscillating, or if $f(x,y)$ obeys Landau type two-sided Tauberian conditions.

At the end, we extend these results to the mixed case, where the $(C,1,0)$ mean, that is, the average value of $s(w,v)$ with respect to the first variable over the interval $[0,u]$, is considered instead of $\sigma_{11}(u,v) := \sigma(u,v)$.

1. Summability $(C,1,1)$ of double integrals over $\mathbb{R}_+^2$. We remind the reader that a complex-valued function $f(x,y)$ is said to be *locally integrable* over $\mathbb{R}_+^2 := [0,\infty)\times[0,\infty)$, in symbols $f\in L^1_{\text{loc}}(\mathbb{R}_+^2)$, if for all $0<u,v<\infty$ the integral

$$s(u,v) := \int_0^u\int_0^v f(x,y)\,dx\,dy$$

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