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Weighted spaces of holomorphic functions on Banach spaces

by

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Abstract. We deal with weighted spaces $\mathcal{H}V_0(U)$ and $\mathcal{H}V(U)$ of holomorphic functions defined on a balanced open subset U of a Banach space X. We give conditions on the weights to ensure that the weighted spaces of m-homogeneous polynomials constitute a Schauder decomposition for them. As an application, we study their reflexivity. We also study the existence of a predual. Several examples are provided.

Weighted spaces of holomorphic functions on a balanced open subset of $\mathbb{C}^n$ arise naturally in partial differential equations, complex analysis and spectral theory. The one variable case has been extensively studied by Rubel and Shields [28], Williams [34], Bierstedt and Summers [10] and Bierstedt and Bonet [6]. Bierstedt, Meise and Summers [9] studied a projective description of weighted inductive limits of holomorphic functions on open subsets of $\mathbb{C}^n$. Recently Bierstedt, Bonet and Galbis [7] achieved significant advances in the knowledge of these spaces on balanced domains U in $\mathbb{C}^n$.

Weighted spaces of holomorphic functions on $\mathbb{C}^n$, $n \in \mathbb{N}$, appear in a natural way as the Fourier–Laplace transform of spaces of ultradistributions. In the infinite-dimensional case they constitute a generalization of the algebra $\mathcal{H}_b(X)$ of holomorphic functions of bounded type on a Banach space X, which has recently received much attention; see e.g. [4, 5]. The spaces considered here are relevant to Schauder decompositions of spaces of holomorphic functions [13, 18, 27], their locally convex properties [3, 13, 15] and linearization and biduality [6, 10, 18, 27], among other topics.

The first section of [7] has been our starting point in order to extend their results to the setting of Banach spaces. Although several of our results are based on the ones proved there, when moving to infinite dimensions we lose local compactness and, consequently, the fact that the elements of the topological dual of $\mathcal{H}V_0(U)$ have an integral representation ([10], Theorem 1.1b, or [33]). Actually, this is one of the main tools used in Theorem 1.5(d) of

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[7] to obtain the biduality $(\mathcal{H}V_0(U)'_b)'_b = \mathcal{H}V(U)$. The proof also makes use of general theorems about biduality [6], which require building a topological predual of $\mathcal{H}V(U)$. So, we follow a different approach: in Section 1 we assume "reasonable" conditions on the families of weights V in order that the families of weighted m-homogeneous polynomials $(\mathcal{P}V(^mX))_{m=0}^\infty$ and $(\mathcal{P}V_0(^mX))_{m=0}^\infty$ are Schauder decompositions of the Fréchet spaces $\mathcal{H}V(U)$ and $\mathcal{H}V_0(U)$ respectively. As an application we deduce that $\mathcal{H}V(U)$ (resp. $\mathcal{H}V_0(U)$) is a reflexive Fréchet space if and only if $(\mathcal{P}V(^mX))_{m=0}^\infty$ (resp. $(\mathcal{P}V_0(^mX))_{m=0}^\infty$) is a sequence of reflexive spaces. Moreover, by connecting the topologies τ_V and τ_b we prove that if V is a sequence of rapidly decreasing weights then the Fréchet space $\mathcal{H}V(X)$ is reflexive if and only if each element of the sequence $(\mathcal{P}(^mX), \|\cdot\|)_{m=0}^\infty$ is reflexive. This approach was used by Prieto [27] for the space $\mathcal{H}_b(X)$, and so our results can also be considered as a generalization to weighted spaces (see Corollary 13).

Section 2 is devoted to examples. These are chosen in order to emphasize the importance of dealing with the spaces $\mathcal{P}V(^mX)$ and $\mathcal{P}V_0(^mX)$ instead of the space $\mathcal{P}(^mX)$ of all continuous m-homogeneous polynomials, even if X is a finite-dimensional space of dimension greater than one. Examples 18 and 19 show that, when X is an infinite-dimensional complex Banach space, one faces situations that are meaningless in the finite-dimensional setting.

In Section 3 we obtain a predual of the space $\mathcal{H}V(X)$ which has the structure of an (LB)-space. One of the reasons for this study is that building a "good" predual of a given space of holomorphic functions is a process of linearization (see for example [24], [25], [30], [16]). This allows one to see those spaces as some kind of very big duals (like tensor products linearize bilinear mappings) and it has revealed to be a very useful tool in the theory (see especially S. Dineen's book [14]). Another reason, connected with the former, was to try to obtain a theorem of biduality of spaces of weighted holomorphic functions like the one given in [7], [6] or [10] described above.

1. The spaces $\mathcal{H}V_0(U)$ and $\mathcal{H}V(U)$. Throughout this paper X denotes a complex Banach space and U a balanced open subset of X. Also, we always consider countable families of continuous non-negative weights $V = \{v : U \to [0,\infty[\,:$ for each $x \in U$ there exists $v \in V$ such that $v(x) > 0\}$. We define the space $\mathcal{H}V(U)$ as the space of all holomorphic functions f on U such that

$$p_v(f) := \sup\{v(x)|f(x)| : x \in U\} < \infty \quad \text{for all } v \in V.$$

We recall that a subset A of U is *U-bounded* if it is bounded and its distance to $X \setminus U$ is greater than 0. A function $g : U \to [0,\infty[$ is said to *vanish at infinity outside U-bounded sets* if for each $\varepsilon > 0$ there exists a U-bounded set A satisfying $g(x) < \varepsilon$ for all $x \in U \setminus A$. We define

$$\mathcal{H}V_0(U) := \{f \in \mathcal{H}V(U) : \text{for each } v \in V,\ v|f| \text{ vanishes at infinity outside } U\text{-bounded sets}\}.$$

Both spaces are endowed with the weighted topology τ_V generated by the family $(p_v)_{v\in V}$ of seminorms. Let us remark that if X is a Banach space of finite dimension the elements A in the definition of $\mathcal{H}V_0(U)$ are considered to be compact, but any compact subset of an infinite-dimensional Banach space has empty interior, hence every $g : U \to [0,\infty[$ continuous on U and vanishing at infinity outside compact subsets of U is identically 0.

The space $\mathcal{H}_b(U)$ (see [4], [16], [17]) is the space of holomorphic functions on U which are bounded on U-bounded subsets of U, endowed with the locally convex topology τ_b of uniform convergence on U-bounded subsets of U. As usual, we denote by $\mathcal{P}(^mX)$ the space of continuous m-homogeneous polynomials endowed with the norm $\|P\| := \sup\{|P(x)| : x \in X,\ \|x\| \le 1\}$. Given $f \in \mathcal{H}_b(U)$, $\sum_{m=0}^\infty P_m f$ denotes the Taylor series expansion of f at the origin. For each $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ we define $\mathcal{P}V(^mX) := \mathcal{P}(^mX) \cap \mathcal{H}V(U)$ and $\mathcal{P}V_0(^mX) := \mathcal{P}(^mX) \cap \mathcal{H}V_0(U)$ endowed with the restriction of the topology τ_V. Since the space $\mathcal{P}_a(^mX)$ of all m-homogeneous polynomials (not necessarily continuous) is complete when endowed with the topology of pointwise convergence on X, the spaces $\mathcal{P}V(^mX)$ and $\mathcal{P}V_0(^mX)$ are closed subspaces of $\mathcal{H}V(U)$ and $\mathcal{H}V_0(U)$ respectively. It is well known that the restriction of τ_b to $\mathcal{P}(^mX)$ coincides with the norm topology for all $m = 0, 1, \ldots$ If $A \subset U$ and $f : U \to \mathbb{C}$ we write

$$\|f\|_A := \sup\{|f(x)| : x \in A\} \in [0,\infty].$$

If the family of continuous weights has a unique element $V = \{v\}$ such that $v(x) > 0$ for all $x \in U$, then $\mathcal{H}V(U)$ and $\mathcal{H}V_0(U)$ endowed with the norm $\|\cdot\|_v := p_v$ are Banach spaces (and are denoted by $\mathcal{H}v(U)$ and $\mathcal{H}v_0(U)$ respectively). These spaces have been studied in [29]. Moreover, if $v \equiv 1$, then $\mathcal{H}v(U) = \mathcal{H}^\infty(U)$. On the other hand, V can be chosen so that $\mathcal{H}V(U) = \mathcal{H}_b(U)$ (see Example 14). Accordingly, weighted spaces of holomorphic functions can be considered as a generalization of the space $\mathcal{H}^\infty(U)$ of holomorphic bounded functions or as a generalization of the space $\mathcal{H}_b(U)$ of holomorphic functions of bounded type. In this paper we adopt the second point of view (see Remark 9).

Definition 1. A family V of non-negative continuous weights defined on U satisfies *Condition I* if for each U-bounded subset A of U there exists v in V such that $\inf\{v(x) : x \in A\} > 0$.

Proposition 2. *Let U be a balanced open subset of a Banach space X. The topology τ_V is stronger than the restriction of the norm topology to*

$\mathcal{P}V(^mX)$ and to $\mathcal{P}V_0(^mX)$. If the family V satisfies Condition I then the space $\mathcal{H}V(U)$ is a subset of $\mathcal{H}_b(U)$ and the topology τ_V is stronger than the restriction of τ_b to $\mathcal{H}V(U)$, and moreover, the spaces $\mathcal{H}V(U)$, $\mathcal{H}V_0(U)$, $\mathcal{P}V(^mX)$ and $\mathcal{P}V_0(^mX)$ $(m = 0, 1, \ldots)$ are Fréchet spaces.

Proof. The first statement follows from [13], Lemma 1.13. The rest is easily checked. ∎

Let us remark that $\mathcal{P}V(^mX)$ and $\mathcal{P}V_0(^mX)$, $m = 0, 1, \ldots$, are Fréchet spaces even if Condition I does not hold. To prove this it is enough to use the first part of Proposition 2.

A weight v defined on a balanced open subset U of a Banach space X is said to be *radial* if $v(tx) = v(x)$ for all $x \in X$ and all $t \in \mathbb{C}$ such that $|t| = 1$.

PROPOSITION 3. *If U is a balanced open subset of a Banach space X, V is a family of radial weights and $f \in \mathcal{H}V(U)$ (resp. $\mathcal{H}V_0(U)$), then $P_m f \in \mathcal{H}V(X)$ (resp. $P_m f \in \mathcal{H}V_0(X)$) for every $m \in \mathbb{N}$. Moreover, $p_v(P_m(f)) \le p_v(f)$ for each $v \in V$ and each $m \in \mathbb{N}_0$.*

Proof. Consider $f \in \mathcal{H}V(U)$. Since each $v \in V$ is a radial weight and U is balanced, we apply the Cauchy inequalities and the maximum modulus principle, to conclude

$$\begin{aligned}
\sup_{x\in U} v(x)|P_m(f)(x)| &\le \sup_{x\in U} v(x) \sup_{|t|\le 1} |P_m(f)(tx)| \\
&\le \sup_{x\in U} v(x) \sup_{|t|\le 1} |f(tx)| \\
&= \sup_{x\in U} v(x) \sup_{|t|=1} |f(tx)| = \sup_{x\in U} \sup_{|t|=1} v(tx)|f(tx)| \\
&= \sup_{x\in U} v(x)|f(x)| = p_v(f) \qquad \text{for all } m = 0, 1, \ldots
\end{aligned}$$

Moreover, given a balanced U-bounded set A, if $x \in U \setminus A$ and $t \in \mathbb{C}$ with $|t| = 1$, we have $tx \in U \setminus A$, so that

$$\sup_{x\in U\setminus A} v(x)|P_m(f)(x)| \le \sup_{x\in U\setminus A} v(x) \sup_{|t|=1} |f(tx)| = \sup_{y\in U\setminus A} v(y)|f(y)|.$$

Hence if $f \in \mathcal{H}V_0(X)$, then $P_m f \in \mathcal{H}V_0(X)$ for all $m = 0, 1, \ldots$ ∎

A consequence of Proposition 3 is that if the weights are radial, the projections from $\mathcal{H}V(U)$ (resp. $\mathcal{H}V_0(U)$) on $\mathcal{P}V(^mX)$ (resp. $\mathcal{P}V_0(^mX)$) are continuous for all $m = 0, 1, \ldots$ If moreover for each function $f \in \mathcal{H}V(U)$ (resp. $f \in \mathcal{H}V_0(U)$) the Taylor series expansion at 0 of f converges to f for the τ_V topology, then $(\mathcal{P}V(^mX))_{m=0}^\infty$ (resp. $(\mathcal{P}V_0(^mX))_{m=0}^\infty$) is said to be a *Schauder decomposition* of $\mathcal{H}V(U)$ (resp. $\mathcal{H}V_0(U)$) (see [13], Definition 3.7). However, this last condition is not satisfied in general. For example, if we consider $V = \{v\}$, where $v : \mathbb{C} \to]0, \infty[$, $v(z) = e^{-|z|^p}$ $(p \in \mathbb{N})$, then it is obvious that the series $\sum_{m=0}^\infty z^{pm}/m!$ does not converge to e^{z^p} in $\mathcal{H}\{e^{-|z|^p}\}(\mathbb{C})$. Moreover, for every $p \in \mathbb{N}$ there is a function in $\mathcal{H}\{e^{-|z|^p}\}_0(\mathbb{C})$ such that its Taylor series expansion does not converge for the τ_V topology (see [22], Theorem 2.3). On the other hand, in [11], Lemma 1.2(b), the authors prove that given a sequence (b_n) of complex numbers such that $\lim_{n\to\infty} |b_n|\Gamma(n/p + 1) = 0$, we have $g(z) := \sum_{m=0}^\infty b_m z^{mp} \in \mathcal{H}\{e^{-|z|^p}\}_0(\mathbb{C})$ and from their proof one can easily deduce that the series $\sum_{m=0}^\infty b_m z^{mp}$ converges to g in $\mathcal{H}\{e^{-|z|^p}\}_0(\mathbb{C})$. Nevertheless, we can prove the following proposition.

PROPOSITION 4. *Let U be a balanced open subset of a Banach space X and V be a family of radial weights on U that are bounded on U-bounded subsets of U. Suppose V satisfies Condition I. Then the polynomials that belong to $\mathcal{H}V_0(U)$ are τ_V-dense in $\mathcal{H}V_0(U)$.*

Proof. If $f \in \mathcal{H}V_0(U)$, then $f \in \mathcal{H}_b(U)$ by Proposition 2. Since U is balanced, the series $\sum_{m=0}^\infty P_m f$ converges to f uniformly on each U-bounded subset of U, i.e., if we define $S_l := \sum_{k=0}^l P_k f$ for all $l = 0, 1, \ldots$, then the sequence $(\|S_l - f\|_A)_l$ converges to 0 in $\mathbb{R}$ for every U-bounded set A. Hence the sequence $\left(\frac{1}{n+1}\sum_{l=0}^n \|S_l - f\|_A\right)_n$ of arithmetic means also converges to 0. Consider the Cesàro means

$$C_n f(x) = \frac{1}{n+1}\sum_{l=0}^n \Big(\sum_{k=0}^l P_k f(x)\Big), \qquad x \in U,\ n \in \mathbb{N}.$$

By Proposition 3, $(C_n f)_{n=1}^\infty \subset \mathcal{H}V_0(U)$. Moreover,

$$|C_n f(x)| \le \max_{|\lambda|=1} |f(\lambda x)|, \qquad n = 1, 2, \ldots,\ x \in U$$

([24], Proposition 5.2 or [7], Lemma 1.1). Now each $v \in V$ is radial and $\|v\|_A < \infty$, thus

$$\begin{aligned}
\|v(C_n f - f)\|_U &= \max\{\|v(C_n f - f)\|_{U\setminus A}, \|v(C_n f - f)\|_A\} \\
&\le \max\{2\|vf\|_{U\setminus A}, \|v\|_A\|(C_n f - f)\|_A\} \\
&\le \max\Big\{2\|vf\|_{U\setminus A}, \|v\|_A\Big(\frac{1}{n+1}\sum_{l=0}^n \|S_l - f\|_A\Big)\Big\}.
\end{aligned}$$

From the above chain of inequalities we conclude that $(C_n f)_{n=1}^\infty$ τ_V-converges to f. ∎

Proposition 4 is a generalization and a slight improvement of ([7], Proposition 1.2, parts (b)–(d)) for balanced open subsets of $\mathbb{C}^N$. The argument in our proof is clearly an adaptation of the one given there.

COROLLARY 5. *If U is a balanced open subset of $\mathbb{C}^N$, $N \ge 1$, and V is a family of continuous radial weights, then $\mathcal{H}V_0(U)$ has the approximation property.*

Proof. This is immediate from the proof of Proposition 4 since U-bounded sets are relatively compact subsets of U and weights are continuous on U. ∎

This corollary is Theorem 1.5(a) of [7] without the additional hypothesis on $\mathcal{H}V_0(U)$ of containing all the polynomials on $\mathbb{C}^N$. When $U=\mathbb{C}$ this hypothesis is quite natural. In fact, in [6] it is pointed out in a remark after Proposition 10 that if for a given $k\in\mathbb{N}$, $z^k\notin\mathcal{H}V(\mathbb{C})$ (resp. $\mathcal{H}V_0(\mathbb{C})$) then $\mathcal{H}V(\mathbb{C})\subset\operatorname{span}\{1,z,\ldots,z^{k-1}\}$ (resp. $\mathcal{H}V_0(\mathbb{C})\subset\operatorname{span}\{1,z,\ldots,z^{k-1}\}$). But the situation for a general Banach space is different, as shown by Examples 16 and 19 below.

In [7] it is stated in a remark after Lemma 1.1 that the hypothesis of all polynomials being contained in $\mathcal{H}V_0(\mathbb{C}^N)$ means exactly that each $v\in V$ is rapidly decreasing (that is, $v(x)\|x\|^m$ is a bounded function on X for each $m\in\mathbb{N}_0$). This can be generalized to Banach spaces.

PROPOSITION 6. *Let V be a family of radial weights on a Banach space X.*

(a) *If for a given $m\in\mathbb{N}$, $\mathcal{P}(^mX)$ is contained in $\mathcal{H}V(X)$, then $\mathcal{P}(^{m-1}X)$ is contained in $\mathcal{H}V_0(X)$.*

(b) *Each element of V is rapidly decreasing if and only if $\mathcal{P}(^mX)$ is contained in $\mathcal{H}V(X)$ for all $m\in\mathbb{N}_0$. In that case the topology τ_V coincides with the norm topology on $\mathcal{P}V(^mX)$ for all $m\in\mathbb{N}_0$.*

Proof. (a) Let $m\in\mathbb{N}$ be such that $\mathcal{P}(^mX)=\mathcal{P}V(^mX)$ algebraically. By Proposition 2 the norm topology $\tau_{\|\cdot\|}$ is weaker than $\tau_V|_{\mathcal{P}(^mX)}$, but $(\mathcal{P}(^mX),\|\cdot\|)$ is a Banach space and by Proposition 2, $(\mathcal{P}(^mX),\tau_V|_{\mathcal{P}(^mX)})$ is a Fréchet space, thus the open mapping theorem implies that both topologies coincide. Given $y\in X$, $y\neq0$, by the Hahn–Banach theorem we can find a continuous linear map φ_y on X such that $\varphi_y(y)=\|y\|$ and $\|\varphi_y\|=1$. If $Q\in\mathcal{P}(^{m-1}X)$, then $P_y(x):=\varphi_y(x)Q(x)$, $x\in X$, belongs to $\mathcal{P}(^mX)$, and $\|P_y\|\le\|\varphi_y\|\|Q\|=\|Q\|$ for all $y\in X$. Hence $\{P_y: y\in X\}$ is a $\tau_{\|\cdot\|}$-bounded set. Since $\tau_{\|\cdot\|}=\tau_V|_{\mathcal{P}(^mX)}$, for each $v\in V$ we have

$$M:=\sup_{y\in X}\sup_{x\in X}v(x)|\varphi_y(x)Q(x)|<\infty.$$

In particular, $\sup_{x\in X}v(x)\|x\||Q(x)|\le M$. Then, for each $\varepsilon>0$, $v(x)|Q(x)|<\varepsilon$ for all $x\in X$ such that $\|x\|>M\varepsilon^{-1}$. Thus $Q\in\mathcal{P}V_0(^{m-1}X)$.

(b) If we assume that every $v\in V$ is rapidly decreasing, then given $P\in\mathcal{P}(^mX)$ we have

$$v(x)|P(x)|=v(x)|P(x/\|x\|)|\,\|x\|^m\le\|P\|v(x)\|x\|^m\quad\text{for all } x\in X,$$

hence $P\in\mathcal{P}V(^mX)$.

Conversely, if $\mathcal{P}(^mX)$ is contained in $\mathcal{H}V(X)$ for all $m\in\mathbb{N}_0$, we consider the family $\{\varphi_y: y\in X\}$ defined in part (a). Then $\{\varphi_y^m: y\in X\}\subset\mathcal{P}(^mX)$

and $\|\varphi_y^m\|=1$, for all $y\in Y$. This set is $\tau_{\|\cdot\|}$-bounded and hence τ_V-bounded. Thus

$$\sup_{x\in X}v(x)\|x\|^m=\sup_{x\in X}v(x)|\varphi_x^m(x)|\le\sup_{y\in X}\sup_{x\in X}v(x)|\varphi_y^m(x)|<\infty$$

for all $m\in\mathbb{N}$. ∎

We are going to introduce an additional condition on the weights to ensure that the spaces of weighted m-homogeneous polynomials are a Schauder decomposition of the space of weighted holomorphic functions.

DEFINITION 7. Let U be an open balanced set in a Banach space X, and V a family of continuous radial weights. We say that V satisfies *Condition II* if for each v in V there exist $R>1$ and w in V such that

$$p_v(P_mf)\le\frac{1}{R^m}p_w(f)\quad\text{for all } f\in\mathcal{H}V(U),\ m=0,1,\ldots$$

Since the weights are radial, by Proposition 3 Condition II is equivalent to

$$p_v(P_mf)\le\frac{1}{R^m}p_w(P_mf)\quad\text{for all } f\in\mathcal{H}V(U),\ m=0,1,\ldots$$

When dealing with entire functions, Proposition 8 below gives a condition that implies Condition II and that is easier to check.

PROPOSITION 8. *Let X be a Banach space and V be a family of continuous radial weights. If for each v in V there exist $R>1$ and w in V such that*

$$(\textit{Condition II}')\qquad v(x)\le w(Rx)\quad\textit{for all } x\in X,$$

then the family V satisfies Condition II.

Proof. Given f in $\mathcal{H}V(X)$, since all the elements of V are radial, by Proposition 3, $P_mf\in\mathcal{P}V(^mX)$, $m=0,1,\ldots$, and

$$\begin{aligned}R^mp_v(P_mf)&=R^m\sup_{x\in X}v(x)|P_mf(x)|\\&=\sup_{x\in X}v(x)|P_mf(Rx)|\le\sup_{x\in X}w(Rx)|P_mf(Rx)|\\&=\sup_{y\in X}w(y)|P_mf(y)|=p_w(P_mf)\le p_w(f).\ ∎\end{aligned}$$

REMARK 9. If V is a family of radial weights defined on an open balanced subset U of a Banach space X and satisfies Conditions I and II, then the Fréchet space $\mathcal{H}V(U)$ is a Banach space if and only if there exists a natural number m_0 such that $\mathcal{P}V(^mX)=\{0\}$, $m\ge m_0$, and each $(\mathcal{P}V(^mX),\tau_V)$ is a Banach space. Indeed, suppose that there exists a continuous norm $|||\cdot|||$ on $\mathcal{H}V(U)$ generating the τ_V-topology and a strictly increasing sequence $(m_k)_{k=1}^\infty$ of natural numbers such that $\mathcal{P}V(^{m_k}X)\neq\{0\}$, $k=1,2,\ldots$ Choose

$P_k \in \mathcal{P}V(^{m_k}X)$ with $|||P_k||| = 1$, $k = 1, 2, \ldots$ Given $v \in V$, by Condition II, we can find $R > 1$ and $w \in V$ such that

$$p_v(P_k) \le R^{-m_k} p_w(P_k), \quad k = 1, 2, \ldots$$

Since $(P_k)_{k=1}^{\infty}$ is a bounded set, this inequality implies that $(P_k)_{k=1}^{\infty}$ τ_V-converges to 0, a contradiction. Conversely, if $\mathcal{P}V(^mX) = \{0\}$, $m \ge m_0$, then $(\mathcal{H}V(U), \tau_V) = \bigoplus_{m=0}^{m_0} (\mathcal{P}V(^mX), \tau_V)$, and the latter space has a Banach space structure if and only if $(\mathcal{P}V(^mX), \tau_V)$, $m = 0, \ldots, m_0$, are Banach spaces.

LEMMA 10. *If V is a family of radial weights defined on an open balanced subset U of a Banach space X and satisfies Conditions I and II, then the topology τ_V on $\mathcal{H}V(U)$ is generated by the family of seminorms such that $p(f) = \sum_{m=0}^{\infty} p(P_m(f))$ for all $f \in \mathcal{H}V(U)$, and $p|_{\mathcal{P}V(^mX)}$ is τ_V-continuous on $\mathcal{P}V(^mX)$, for all $m = 0, 1, \ldots$*

Proof. Consider a seminorm p as above. Since $\{f \in \mathcal{H}V(U) : p(f) \le 1\} = \bigcap_{k=0}^{\infty} \{f \in \mathcal{H}V(U) : \sum_{m=0}^{k} p(P_m f) \le 1\}$ is a closed absolutely convex absorbing set in the Fréchet space $(\mathcal{H}V(U), \tau_V)$, it is a neighbourhood of 0, and p is τ_V-continuous. Conversely, given $v \in V$, we apply Condition II to get $R > 1$ and $w \in V$ such that

$$p(P_m(f)) \le \frac{1}{R^m} p_w(f) \quad \text{for all } f \in \mathcal{H}V(U),\ m = 0, 1, \ldots$$

Hence

$$\sum_{m=0}^{\infty} p_v(P_m f) \le \sum_{m=0}^{\infty} \frac{1}{R^m} p_w(f) = \frac{R}{R-1} p_w(f).$$

Thus if we define q_v on $\mathcal{H}V(U)$ as

$$q_v(f) := \sum_{m=0}^{\infty} p_v(P_m f) \quad \text{for all } f \in \mathcal{H}V(U)$$

we find that q_v is τ_V-continuous and $p_v \le q_v$. ∎

It is clear that a similar description of the topology τ_V can be stated for the space $\mathcal{H}V_0(U)$.

The $\mathcal{S}$-absolute γ-complete decompositions are a useful tool in Infinite-Dimensional Holomorphy in order to lift properties from spaces of polynomials to spaces of holomorphic functions (see for instance [13], Chapter 3, [27], [18]). In particular, the fact that $(\mathcal{P}V(^mX))_m$ is such a decomposition of $\mathcal{H}V(U)$, under some assumptions on the weights (Theorem 11 below), leads to characterizing the reflexivity of $\mathcal{H}V(U)$ in terms of the reflexivity of $\mathcal{P}V(^mX)$, $m \in \mathbb{N}$ (Corollary 13).

A Schauder decomposition (E_n) of a locally convex space E is said to be *γ-complete* if given $x_n \in E_n$, $n = 0, 1, \ldots$, such that $(\sum_{n=0}^{m} x_n)_{m=0}^{\infty}$ is bounded, the series $\sum_{n=0}^{\infty} x_n$ converges ([20], Definition 3.1).

THEOREM 11. *If U is a balanced open subset of a Banach space X and V is a family of radial weights on U that satisfies Conditions I and II, then the sequence $(\mathcal{P}V(^mX))_m$ (resp. $(\mathcal{P}V_0(^mX))_m$) is an $\mathcal{S}$-absolute γ-complete decomposition of $\mathcal{H}V(U)$ (resp. of $\mathcal{H}V_0(U)$).*

Proof. As the weights are radial, we apply Proposition 3 to deduce that the projections $P_m : f \in \mathcal{H}V(U) \mapsto P_m f \in \mathcal{P}V(^mX)$ are well defined and continuous. By the very definition, Lemma 10 says that $(\mathcal{P}V(^mX), \tau_V)$ is an absolute Schauder decomposition of $\mathcal{H}V(U)$ (see [13], Definition 3.7). We set $\mathcal{S} := \{(a_m)_m \in \mathbb{C}^{\mathbb{N}} : \limsup_{m\to\infty} |a_m|^{1/m} \le 1\}$. Then $(\mathcal{P}V(^mX), \tau_V)$ is an $\mathcal{S}$-absolute decomposition if and only if

(i) $g = \sum_{m=0}^{\infty} a_m P_m f \in \mathcal{H}V(U)$ for all $f \in \mathcal{H}V(U)$ and all $(a_m) \in \mathcal{S}$,

(ii) $q(f) = \sum_{m=0}^{\infty} |a_m| p_v(P_m f) < \infty$ is well defined and continuous on $\mathcal{H}V(U)$ for all $v \in V$ and all $(a_m) \in \mathcal{S}$.

Let $v \in V$. By Condition II there exist $R > 1$ and $w \in V$ such that $p_v(P_m h) \le (1/R^m) p_w(h)$ for all $h \in \mathcal{H}V(U)$. Given $(a_m) \in \mathcal{S}$ we can find $m_0 \in \mathbb{N}$ such that $|a_m|^{1/m} < (1+R)/2$ for all $m \ge m_0$. For $c > 0$ large enough we have $|a_m| \le c((1+R)/2)^m$ for all $m \in \mathbb{N}_0$, and then

$$\sup_{x\in U} v(x) |a_m P_m f(x)| \le \frac{|a_m|}{R^m} p_w(f) \le c\Big(\frac{1+R}{2R}\Big)^m p_w(f).$$

Since $0 < (1+R)/(2R) < 1$, we have $g \in \mathcal{H}V(U)$ and

$$q(f) = \sum_{m=0}^{\infty} |a_m| p_v(P_m f) \le c \frac{1}{1-(1+R)/(2R)} p_w(f) \quad \text{for all } f \in \mathcal{H}V(U),$$

hence q is well defined and continuous.

Let $Q_m \in \mathcal{P}(^mX)$, $m = 0, 1, \ldots$, be such that the sequence $(\sum_{m=0}^{n} Q_m)_n$ is τ_V-bounded. We verify that it is a τ_V-Cauchy sequence in the Fréchet space $\mathcal{H}V(U)$. By Lemma 10 it is enough to check this for continuous seminorms p on $\mathcal{H}V(U)$ such that $p(f) = \sum_{m=0}^{\infty} p(P_m(f))$. But, for such p, the sequence $(p(\sum_{m=0}^{n} Q_m))_n = (\sum_{m=0}^{n} p(Q_m))_n$ is bounded and hence $\sum_{m=0}^{\infty} p(Q_m) < \infty$.

For $\mathcal{H}V_0(U)$ it is enough to apply Proposition 3 and the fact that this space is a closed subspace of $\mathcal{H}V(U)$. ∎

COROLLARY 12. *If V is a family of rapidly decreasing radial weights on the Banach space X and satisfies Conditions I and II, then $\mathcal{H}V(X) = \mathcal{H}V_0(X)$.*

Proof. Since $\mathcal{H}V_0(X)$ is a closed subspace of $\mathcal{H}V(X)$, the corollary follows from Proposition 6 and Theorem 11. ∎

Corollary 13. *If V is a family of radial weights defined on an open balanced subset U of a Banach space X and satisfies Conditions I and II, then*

(a) *$\mathcal{H}V(U)$ (resp. $\mathcal{H}V_0(U)$) is a reflexive space if and only if each space of the sequence $(\mathcal{P}V(^mX),\tau_V)_m$ (resp. $(\mathcal{P}V_0(^mX),\tau_V)_m$) is reflexive.*

(b) *If $U = X$ and the weights are rapidly decreasing, then $\mathcal{H}V(X)$ is reflexive if and only if each element of the sequence $(\mathcal{P}(^mX),\|\cdot\|)_m$ is a reflexive Banach space.*

Proof. By Theorem 11 the sequence $(\mathcal{P}V(^mX),\tau_V)$ is an $\mathcal{S}$-absolute decomposition of $\mathcal{H}V(U)$ and hence it is a shrinking decomposition ([13], Corollary 3.14). Moreover, it is γ-complete and then [20], Theorem 3.2, gives the conclusion. An analogous argument works for $\mathcal{H}V_0(U)$. (b) follows from Proposition 6 and part (a). ■

This corollary is a generalization to weighted spaces of the one given by Ansemil and Ponte in [3] for the space $(\mathcal{H}_b(X),\tau_b)$. (See also Prieto [27], Corollary 2, which is the approach we follow.)

One example in which the hypothesis of Corollary 13(b) holds was given by Alencar, Aron and Dineen in [2]. They proved in Proposition 4 and the proof of Theorem 6 that the spaces of the sequence $(\mathcal{P}(^mT^*))_m$ are reflexive, where T^* is the original Tsirelson space. Now it is enough to consider the family of weights given in Examples 16(c) and (d). (Conditions to ensure that $(\mathcal{P}(^mX))_m$ are reflexive can be found in [1].)

2. Examples. We present several types of examples of weighted spaces of holomorphic functions defined on Banach spaces. In Examples 14–16 the dimension of the Banach space plays no relevant role. Actually, the weights in 15 and 16 are obtained by composing the norm of the space with a continuous function. On the other hand, the weights in 17 can only be defined in spaces of dimension greater than 1. Finally, the examples in 18 and 19 only make sense in the infinite-dimensional setting.

Example 14. *Let U be an open balanced subset of a Banach space X. The sequence $W_n := \{x \in X : \|x\| < n, \operatorname{dist}(x, X\setminus U) > 1/n\}$, $n = 1,2,\ldots,$ is a fundamental system of U-bounded subsets of U. For each n in $\mathbb{N}$ consider $v_n : U \to [0,1]$ defined as*

$$v_n(x) := \frac{\operatorname{dist}(x, X\setminus W_{n+1})}{\operatorname{dist}(x, X\setminus W_{n+1}) + \operatorname{dist}(x, W_n)}, \qquad x \in U.$$

Then $V := (v_n)_n$ is an increasing sequence of continuous radial weights on U such that $\mathcal{H}V(U) = \mathcal{H}_b(U)$ algebraically and topologically and Conditions I and II are satisfied.

Proof. For each n in $\mathbb{N}$ we have $\operatorname{supp} v_n \subset W_{n+1}$ and $v_n(x) = 1$ for all x in W_n, hence $(v_n)_n$ satisfies Condition I. If we take f to be a holomorphic function on U then

$$\|f\|_{W_n} \le \sup_{x\in U} v_n(x)|f(x)| \le \|f\|_{W_{n+1}}.$$

Hence $\mathcal{H}V(U) = \mathcal{H}_b(U)$ algebraically and topologically. Moreover, for $\varepsilon_n = n^{-2}(n+1)^{-1}$, $(1+\varepsilon_n)W_n \subset W_{n+1}$, and then $v_n(x) \le v_{n+2}((1+\varepsilon_n)x)$ for all x in U and $n = 1,2,\ldots$

Example 15. *Let U be a balanced open subset of a Banach space X and α in $]0,\infty]$ such that $\|x\| < \alpha$ for all x in U. Let $g : [0,\alpha[\to]0,\infty[$ be a continuous function. Define $v(x) := g(\|x\|)$ for x in U. Then v is a radial weight, $V = \{v\}$ satisfies Condition I and $\mathcal{H}v(U)$ is a Banach space contained in $\mathcal{H}_b(U)$.*

Example 16. (a) *If $v : X \to]0,\infty[$ is a radial continuous weight such that $\inf\{v(x) : \|x\| < n\} > 0$ for all $n = 1,2,\ldots$ then the families of continuous radial weights $V := (v_n)_{n=1}^\infty$ and $W := (w_n)_{n=1}^\infty$, defined as $v_n(x) := v(((n+1)/n)x)$ and $w_n(x) := v((1/n)x)$ for x in X, $n = 1,2,\ldots,$ satisfy Conditions I and II and $\mathcal{P}V(^mX) = \mathcal{P}v(^mX) = \mathcal{P}W(^mX)$ algebraically and topologically.*

(b) *If additionally $\mathcal{H}V(X) \ne \bigoplus_{m=0}^k \mathcal{P}V(^mX)$ for all k in $\mathbb{N}$ and $v(\lambda x) \le v(x)$ for all $\lambda \ge 1$ and x in X, then*

$$\mathcal{H}W(X) \subsetneq \mathcal{H}v(X) \subsetneq \mathcal{H}V(X).$$

(c) *A concrete example satisfying all the hypotheses of* (a), (b) *is $v(x) := e^{-\|x\|^p}$, $x \in X$, $p \ge 1$, $V = (e^{-((n+1)/n)\|x\|^p})_{n=1}^\infty$, $W = (e^{-(1/n)\|x\|^p})_{n=1}^\infty$. In that case $(\mathcal{P}(^mX),\|\cdot\|) = (\mathcal{P}v(^mX),\tau_v)$ for all $m = 0,1,\ldots$ by Proposition 6.*

(d) *Another example satisfying only the hypothesis* (a) *is $v(x) := 1/(1+\|x\|^p)$, $p \ge 1$. In that case $(\mathcal{P}v(^mX),\tau_v) = (\mathcal{P}(^mX),\|\cdot\|)$ if $m \le [p]$ and $(\mathcal{P}v(^mX),\tau_v) = \{0\}$ if $m > [p]$.*

Proof. It is immediate that Condition I holds for the families V and W. On the other hand,

$$v_n(x) = v_{n+1}\left(\frac{(n+1)^2}{n(n+2)}x\right), \qquad w_n(x) = w_{n+1}\left(\frac{n+1}{n}x\right),$$

for all $n \in \mathbb{N}$ and all $x \in X$. By Proposition 8, Condition II holds. Consider now an m-homogeneous polynomial $P : X \to \mathbb{C}$. Then $0 \le p_{v_n}(P) = (n/(n+1))^m p_v(P) \le \infty$ and $0 \le p_{w_n}(P) = n^m p_v(P) \le \infty$ for every $n \in \mathbb{N}$. Hence $\mathcal{P}V(^mX) = \mathcal{P}v(^mX) = \mathcal{P}W(^mX)$ algebraically and topologically. If we additionally assume that $v(\lambda x) \le v(x)$ for all $\lambda \ge 1$ and x in X, then $v_n(x) \le v(x) \le w_n(x)$ for all x in X, hence $\mathcal{H}W(X) \subset \mathcal{H}v(X) \subset \mathcal{H}V(X)$.

By Example 15, $\mathcal{H}v(X)$ is a Banach space. But, by Remark 9, under the extra assumptions, $\mathcal{H}W(X)$ and $\mathcal{H}V(X)$ are Fréchet non-Banach spaces. Now an application of the open mapping theorem completes the proof. ∎

The notation of v, V, W of Example 16 will be used again in the next three examples.

EXAMPLE 17. *Let Y be a complemented subspace of a Banach space X, let Z be its complement and $\pi : X \to Y$ the projection of X onto Y. If $t : Y \to]0,\infty[$ and $u : Z \to]0,\infty[$ are two continuous radial weights as in Example* 16(a), *then $v : X \to]0,\infty[$ defined as $v(x) := t(\pi(x))u(x-\pi(x))$ for all x in X is a continuous radial weight satisfying the hypothesis of Example* 16(a).

Concrete examples are:

(a) $v : \mathbb{C}^2 \to]0,\infty[$, $v(x,y) = (1+|x|)^{-p}e^{-|y|^q}$, $p,q \ge 1$. *In that case* $\mathcal{P}V({}^m\mathbb{C}^2) = \mathcal{P}v({}^m\mathbb{C}^2) = \mathcal{P}W({}^m\mathbb{C}^2) = \{\sum_{\alpha+\beta=m} a_{(\alpha,\beta)}x^\alpha y^\beta,\ a_{(\alpha,\beta)} \in \mathbb{C},\ 0 \le \alpha \le p,\ \alpha,\beta \in \mathbb{N}_0\}$.

(b) $v : \mathbb{C}^n \to]0,\infty[$, $v(x_1,\ldots,x_n) := \prod_{j=1}^n (1+|x_j|)^{-p_j}$, $p_j \ge 1$, $j=1,\ldots,n$. *In that case we have* $\mathcal{P}V({}^m\mathbb{C}^n) = \mathcal{P}v({}^m\mathbb{C}^n) = \mathcal{P}W({}^m\mathbb{C}^n) = \{\sum_{\alpha_1+\ldots+\alpha_n=m} a_{(\alpha_1,\ldots,\alpha_n)}x^{\alpha_1}\ldots x^{\alpha_n},\ a_{(\alpha_1,\ldots,\alpha_n)} \in \mathbb{C},\ 0 \le \alpha_j \le p_j,\ \alpha_j \in \mathbb{N}_0,\ j=1,\ldots,n\}$.

EXAMPLE 18. *Let X be an infinite-dimensional complex Banach space, and let $(x_n^*)_n$ be a sequence of continuous linear forms weak-star convergent to* 0 *in X^* with $\|x_n^*\| = 1$, $n=1,2,\ldots$ The sequence of radial continuous weights $V = (v_n)_{n=1}^\infty$ defined by*

$$v_n(x) := \frac{1}{1+\sum_{k=1}^\infty n^{-k}|x_n^*(x)|^k}, \quad x \in X,\ n=1,2,\ldots,$$

satisfies Conditions I and II, $v_n(x) > 0$ for all x in X and all n in $\mathbb{N}$, but $\inf\{v_n(x) : \|x\| \le r\} = 0$ *for all $r > n$.*

Proof. Fix x in X. Since the sequence $(x_n^*(x))_n$ converges to 0 in $\mathbb{C}$, there exists n_0 in $\mathbb{N}$ such that $|x_n^*(x)| < 1/3$ for all $n \ge n_0$. On the other hand, if $y \in X$, $\|x-y\| < 1/3$, we have $|x_n^*(y)| \le |x_n^*(y-x)| + |x_n^*(x)| \le \|x_n^*\|\|x-y\| + |x_n^*(x)| < 2/3$ for all $n \ge n_0$, hence by the Weierstrass M-criterion, the series $\sum_{k=1}^\infty |x_n^*(y)|^k$ converges uniformly on the open ball of center x and radius $1/3$. Thus $g(x) := 1+\sum_{k=1}^\infty |x_n^*(x)|^k$ is a continuous radial function on X. Since $\|x_n^*\| = 1$, $n=1,2,\ldots$, we have $\sup\{g(x) : \|x\| \le S\} = \infty$ for all $S > 1$, but $\sup\{g(x) : \|x\| \le s\} \le 1/(1-s)$ for all $s < 1$. The claim follows from the fact $v_n(x) = g^{-1}(x/n)$ for all x in X, $n=1,2,\ldots$ ∎

The Josefson–Nissenzweig theorem ([12], Theorem 12.1) ensures the existence of sequences $(x_n^*)_{n=1}^\infty$ as in Example 18. The above example is clearly

derived from the example of an entire function on an infinite-dimensional Banach space which is not of bounded type ([13], Lemma 4.5).

EXAMPLE 19. *Consider in ℓ_1 the continuous weight $v : \ell_1 \to]0,\infty[$ defined as*

$$v((x_n)_n) = \prod_{n=1}^\infty \left(1+\frac{|x_n|}{n}\right)^{-n} \quad \text{for all } x=(x_n) \in \ell_1.$$

Then $\mathcal{H}W(\ell_1) \subsetneq \mathcal{H}v(\ell_1) \subsetneq \mathcal{H}V(\ell_1)$ and, for each m in $\mathbb{N}$,

$$\begin{aligned}(1)\quad \mathcal{P}V({}^m\ell_1) &= \mathcal{P}v({}^m\ell_1) = \mathcal{P}W({}^m\ell_1)\\ &= \Big\{\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})},\,|\alpha|=m} a_\alpha x^\alpha \in \mathcal{P}({}^m\ell_1) :\\ &\qquad a_\alpha = 0 \text{ whenever there exists } j\in\mathbb{N} \text{ with } \alpha_j > j\Big\}\\ &= \Big\{\sum_{\substack{\alpha\in\mathbb{N}_0^{(\mathbb{N})},\,|\alpha|=m\\ 0\le\alpha_j\le j,\,j\in\mathbb{N}}} a_\alpha x^\alpha :\\ &\qquad \sup_{\alpha\in\mathbb{N}_0^{(\mathbb{N})},\,|\alpha|=m} |a_\alpha|R^\alpha \frac{\alpha^{|\alpha|}}{|\alpha|^{|\alpha|}} < \infty \text{ for all } R > 0\Big\}.\end{aligned}$$

Proof. Ryan [31] gives a complete description of the elements of the space $\mathcal{H}_b(\ell_1)$ (see also [23]) in terms of their monomial expansions at 0. In particular, he proves (Theorem 3.3) that for every $m \in \mathbb{N}$, $P \in \mathcal{P}({}^m\ell_1)$ if and only if $P(x) = \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})},\,|\alpha|=m} a_\alpha x^\alpha$ and $\sup_{\alpha\in\mathbb{N}_0^{(\mathbb{N})},\,|\alpha|=m} |a_\alpha|R^\alpha \frac{\alpha^{|\alpha|}}{|\alpha|^{|\alpha|}} < \infty$ for all $R > 0$.

First we prove that v is well defined and continuous on ℓ_1. Since the sequence $((1+\lambda/n)^n)_{n=1}^\infty$ is increasing and convergent to e^λ in $\mathbb{R}$ for all $\lambda \ge 0$, we have

$$e^{\|x\|} = e^{\sum_{n=1}^\infty |x_n|} \ge \prod_{n=1}^\infty \left(1+\frac{|x_n|}{n}\right)^n \ge 1 \quad \text{for all } x=(x_n)_n \in \ell_1.$$

Hence v is well defined and $e^{-\|x\|} \le v(x) \le 1$ for all $x \in \ell_1$. The product $\prod_{n=1}^\infty (1+|x_n|/n)^n$ converges if and only if the series $\sum_{n=1}^\infty n\log(1+|x_n|/n)$ does ([32], 7.27) and in that case $e^{\sum_{n=1}^\infty n\log(1+|x_n|/n)} = \prod_{n=1}^\infty (1+|x_n|/n)^n = v^{-1}((x_n)_n)$ for all $(x_n)_n \in \ell_1$. Now fix $x_0 \in \ell_1$. We have

$$\begin{aligned}(2)\quad v(x)-v(x_0) &= v(x_0)\left(\frac{v(x)}{v(x_0)}-1\right)\\ &= v(x_0)\left(e^{-\sum_{n=1}^\infty n\log\frac{1+|x_n|/n}{1+|x_n^0|/n}}-1\right).\end{aligned}$$

Since $\log(1+\lambda)\le\lambda$ for all $\lambda>-1$, we conclude

$$n\log\left(\frac{1+|x_n|/n}{1+|x_n^0|/n}\right)=n\log\left(\frac{1+|x_n|/n}{1+|x_n^0|/n}-1+1\right)\le n\left(\frac{1+|x_n|/n}{1+|x_n^0|/n}-1\right)$$
$$=n\frac{|x_n|/n-|x_n^0|/n}{1+|x_n^0|/n}\le||x_n|-|x_n^0||.$$

Interchanging x_n and x_n^0, we get

$$\left|n\log\left(\frac{1+|x_n|/n}{1+|x_n^0|/n}\right)\right|\le||x_n|-|x_n^0||\le|x_n-x_n^0|\quad\text{for all } n\in\mathbb{N}.$$

Hence

$$\left|\sum_{n=1}^{\infty}n\log\left(\frac{1+|x_n|/n}{1+|x_n^0|/n}\right)\right|\le\sum_{n=1}^{\infty}\left|n\log\left(\frac{1+|x_n|/n}{1+|x_n^0|/n}\right)\right|\tag{3}$$
$$\le\sum_{n=1}^{\infty}|x_n-x_n^0|=\|x-x_0\|_1.$$

From (2) and (3), v is continuous on X. Finally, since $e^{-\|x\|}\le v(x)\le 1$ for all $x\in\ell_1$, we get $\inf\{v(x):\|x\|\le s\}\ge e^{-s}$ for all $s>0$ and thus v satisfies Condition I.

To complete the proof of Example 19, we need two lemmas.

LEMMA 20. *Let $p\in\mathbb{N}$ and $\lambda\ge0$.*

(a) *The sequence $((1+\lambda/n)^{n+p})_{n\in\mathbb{N}}$ is decreasing to e^λ if (and only if) $0\le\lambda\le2p$.*

(b) $1/(1+\lambda/n)^n\le(1+\lambda/n)^pe^{-\lambda}$, $0\le\lambda\le2p$.

Proof. For $0\le\lambda\le2p$ we write $a_n=(1+\lambda/n)^{n+p}$, $n=1,2,\ldots$ By using

$$\log\left(\frac{1+u}{1-u}\right)=2\sum_{k=1}^{\infty}\frac{u^{2k-1}}{2k-1},\qquad|u|<1,$$

we have

$$\log a_n=(n+p)\log\left(1+\frac{\lambda}{n}\right)=(n+p)\log\left(\frac{1+\lambda/(2n+\lambda)}{1-\lambda/(2n+\lambda)}\right)$$
$$=2(n+p)\sum_{k=1}^{\infty}\left(\frac{\lambda}{2n+\lambda}\right)^{2k-1}\frac{1}{2k-1}$$
$$=\sum_{k=1}^{\infty}\frac{\lambda^{2k-1}}{(2n+\lambda)^{2k-2}(2k-1)}+(2p-\lambda)\sum_{k=1}^{\infty}\frac{\lambda^{2k-1}}{(2n+\lambda)^{2k-1}(2k-1)}.$$

Since each term is decreasing, $(\log a_n)_{n\in\mathbb{N}}$ is a decreasing sequence, hence (a) holds. (b) is immediate from (a). ∎

The above lemma and its proof are a generalization of Problem 1.172 in [26], where it is proved for $p=1$.

LEMMA 21. *Let $p\in\mathbb{N}$ and let J be a finite subset of $\{p+1,p+2,\ldots\}$. If we consider on $\mathbb{C}^{\mathrm{card}(J)}$ the norm $\|\cdot\|_1$, then the function $f_J:\mathbb{C}^{\mathrm{card}(J)}\to[0,\infty[$ defined as*

$$f_J((x_n)_{n\in J})=\frac{(\sum_{n\in J}|x_n|)^p}{\prod_{n\in J}(1+|x_n|/n)^p}$$

attains a maximum value at points of the closed ball of center 0 and radius $p(p+1)$.

Proof. Let $r:=\max J$, $c:=\mathrm{card}(J)$. For each $x\in\mathbb{C}^c$ there exists $n_0\in J$ such that $\|x\|_1\le c|x_{n_0}|$, hence

$$0\le f_J(x)=\frac{\|x\|_1^p}{\prod_{n\in J}(1+|x_n|/n)^p}\le\frac{\|x\|_1^p}{(1+|x_{n_0}|/n_0)^p}\le\frac{\|x\|_1^p}{(1+\|x\|_1/(cr))^p},$$

thus $f_J(x)$ converges to 0 as $\|x\|_1$ tends to infinity. By a standard argument, f_J attains a maximum value on $\mathbb{C}^c$. Moreover, the function $g((y_n)_{n\in J}):=f_J((y_n^2)_{n\in J})$ attains the same maximum value on $\mathbb{R}^c$. Denote by A the family of all points where this maximum is attained. Since g is differentiable, every point in A is a critical point of g, and so, for every $y\in A$,

$$\frac{\partial g}{\partial y_j}(y)$$
$$=\frac{2y_j\|y\|_2^{2(p-1)}\prod_{n\in J,\,n\ne j}(1+y_n^2/n)^n(1+y_j^2/j)^{j-1}[p(1+y_j^2/j)-\|y\|_2^2]}{(\prod_{n\in J}(1+y_n^2/n)^p)^2}=0$$

where $\|\cdot\|_2$ is the euclidean norm on $\mathbb{R}^n$. As $0\notin A$, if $y=(y_n)_{n\in J}\in A$, then either $y_j=0$ or $p(1+y_j^2/j)=\|y\|_2^2$ and $H:=\{j\in J:y_j\ne0\}$, $H\ne\emptyset$. Hence

$$\|(y_n^2)_n\|_1=\|y\|_2^2=\sum_{j\in J}y_j^2=\sum_{j\in H}y_j^2$$
$$=\sum_{j\in H}j\left(\frac{1}{p}\|y\|_2^2-1\right)=p\frac{\sum_{j\in H}j}{\sum_{j\in H}j-p}\le p(p+1).$$

The last inequality holds since $\emptyset\ne H\subset\{p+1,p+2,\ldots\}$ and $\lambda/(\lambda-p)=1+p/(\lambda-p)\le1+p$ for $\lambda\ge p+1$. ∎

Now we prove (1). Let P be an element of $\mathcal{P}v({}^m\ell_1)$, and $\alpha\in\mathbb{N}_0^{\mathbb{N}}$, $\alpha=(\alpha_1,\ldots,\alpha_r,0,\ldots)$ with $\alpha_r\ne0$ and $|\alpha|=m$. If we define $\widetilde{P}:\mathbb{C}^r\to\mathbb{C}$, $\widetilde{P}(x_1,\ldots,x_r):=P(x_1,\ldots,x_r,0,\ldots)$ and $\widetilde{v}:\mathbb{C}^r\to]0,\infty[$, $\widetilde{v}(x_1,\ldots,x_r):=\prod_{n=1}^r(1+|x_n|/n)^{-n}$ for all $(x_1,\ldots,x_r)\in\mathbb{C}^r$, then clearly $\widetilde{P}\in\mathcal{P}\widetilde{v}(\mathbb{C}^r)$ and, by Example 17(b),

$$\widetilde{P}=\sum_{\beta\in\mathbb{N}^r,\,|\beta|=m} c_\beta x^\beta$$

with $c_\beta=0$ if $\beta_j>j$. By [31], $a_\alpha=c_{(\alpha_1,\ldots,\alpha_r)}$, hence $a_\alpha=0$ if there exists $j\in\mathbb{N}$ such that $\alpha_j>j$.

Conversely, consider a continuous m-homogeneous polynomial $P=\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})},\,|\alpha|=m} a_\alpha x^\alpha$ on ℓ_1 such that $a_\alpha=0$ for all α in $\mathbb{N}_0^{(\mathbb{N})}$ with $\alpha_j>j$ for some j. Since P lies in $\mathcal{P}(^m\ell_1)$ if and only if $\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})},\,|\alpha|=m}|a_\alpha||x|^\alpha<\infty$ for all x in ℓ_1 (see [23], Proposition 3.4, or [31], Theorem 3), given $q>r>m^2+m$ we can write $P(x)=\sum_{\gamma\in\Gamma}x_1^{\gamma_1}\ldots x_{q-1}^{\gamma_{q-1}}P_\gamma(x_q,x_{q+1},\ldots)$ where $\Gamma:=\{\gamma=(\gamma_1,\ldots,\gamma_{q-1})\in\mathbb{N}_0^{q-1}:\gamma_1+\ldots+\gamma_{q-1}\le m,\gamma_1\le 1,\gamma_2\le 2,\ldots,\gamma_{q-1}\le\min\{m,q-1\}\}$, and P_γ is a continuous $(m-|\gamma|)$-homogeneous polynomial on ℓ_1. To simplify the notation we write $\pi:\ell_1\to\ell_1$, $\pi((x_n)_{n\in\mathbb{N}})=(x_q,x_{q+1},\ldots)$ and $Q_\gamma(x):=x_1^{\gamma_1}\ldots x_{q-1}^{\gamma_{q-1}}P_\gamma(\pi(x))$ for all $\gamma\in\ell_1$. Since Γ is a finite set, to show that $P\in\mathcal{P}_v(^m\ell_1)$ it is enough to prove that $Q_\gamma\in\mathcal{P}_v(^m\ell_1)$ for each $\gamma\in\Gamma$. As $\gamma_j\le j$, $M_j:=\sup_{r\ge 0}r^{\gamma_j}/(1+r/j)^j<\infty$ for $j=1,\ldots,q-1$. If we write $M_\gamma:=M_1\ldots M_{q-1}$, then

$$\begin{aligned}|Q_\gamma(x)v(x)|&=\frac{|x_1|^{\gamma_1}}{(1+|x_1|)}\cdots\frac{|x_{q-1}|^{\gamma_{q-1}}}{(1+|x_{q-1}|/(q-1))^{q-1}}\left|\frac{P\gamma(\pi(x))}{\prod_{n=q}^\infty(1+|x_n|/n)^n}\right|\\&\le M_\gamma|P_\gamma(\pi(x))|v(\pi(x))\le M_\gamma\|P_\gamma\|\|\pi(x)\|_1^{m-|\gamma|}v(\pi(x))\end{aligned}$$

for all $x\in\ell_1$.

We put $t:=m-|\gamma|$. If $t=0$, then $|Q_\gamma(x)|v(x)\le M_\gamma\|P_\gamma\|v(\pi(x))\le M_\gamma\|P_\gamma\|$ for all $x\in\ell_1$. If $t>0$ then

$$\sup_{x\in\ell_1}\|\pi(x)\|_1^t v(\pi(x))=\sup_{k\in\mathbb{N}}\sup_{\|x\|_1\le k}\|\pi(x)\|_1^t v(\pi(x))\in[0,\infty],$$

but

$$\begin{aligned}\sup_{\|x\|_1\le k}\|\pi(x)\|_1^t v(\pi(x))&=\sup_{s\ge q,\ s\in\mathbb{N}}\ \sup_{|x_q|+\ldots+|x_s|\le k}\frac{(\sum_{n=q}^s|x_n|)^t}{\prod_{n=q}^s(1+|x_n|/n)^n}\\&\le\sup_{s\ge q}\ \sup_{(x_q,\ldots,x_s)\in\mathbb{C}^{s-q+1}}f_{\{q,\ldots,s\}}(x_q,\ldots,x_s)\\&=\sup_{s\ge q}\ \sup_{\|(x_q,\ldots,x_s)\|_1\le t^2+t}f_{\{q,\ldots,s\}}(x_q,\ldots,x_s).\end{aligned}$$

The last equality holds by Lemma 21. Now we have $q>r>m^2+m\ge t^2+t$, hence we can apply Lemma 20 to $|x_n|\le\|(x_q,\ldots,x_s)\|_1\le t^2+t<r$, $n=q,\ldots,s$, to get

$$\begin{aligned}\sup_{\|x\|_1\le k}\|\pi(x)\|_1^t v(\pi(x))&\le\sup_{s\ge q}\ \sup_{\|(x_q,\ldots,x_s)\|_1\le r}\Big(\sum_{n=q}^s|x_n|\Big)^t\prod_{n=q}^s\Big(1+\frac{|x_n|}{n}\Big)^r e^{-|x_n|}\\&\le\sup_{\|\pi(x)\|_1\le r}\|\pi(x)\|_1^t\prod_{n=q}^\infty\Big(1+\frac{|x_n|}{n}\Big)^r e^{-\|\pi(x)\|_1};\end{aligned}$$

thus

$$\sup_{x\in\ell_1}\|\pi(x)\|_1^t v(\pi(x))\le\sup_{\|x\|_1\le r}\|\pi(x)\|_1^t\prod_{n=q}^\infty\Big(1+\frac{|x_n|}{n}\Big)^r e^{-\|\pi(x)\|_1}.$$

Since $\log(1+\lambda)\le\lambda$ for all $\lambda>-1$,

$$\begin{aligned}\prod_{n=q}^\infty\Big(1+\frac{|x_n|}{n}\Big)^r&=\prod_{n=q}^\infty e^{r\log(1+|x_n|/n)}\le e^{\sum_{n=q}^\infty(r/n)|x_n|}\\&\le e^{(r/q)\sum_{n=q}^\infty|x_n|}=e^{(r/q)\|\pi(x)\|_1}\end{aligned}$$

for all $x\in\ell_1$. Finally, as $r/q-1<0$,

$$\begin{aligned}\sup_{x\in\ell_1}|Q_\gamma(x)|v(x)&\le M_\gamma\|P_\gamma\|\sup_{x\in\ell_1}\|\pi(x)\|_1^t v(\pi(x))\\&\le M_\gamma\|P_\gamma\|\sup_{\|\pi(x)\|_1\le r}\|\pi(x)\|_1^t e^{(r/q-1)\|\pi(x)\|_1}\\&\le M_\gamma\|P_\gamma\|\sup_{x\in\ell_1}\|\pi(x)\|_1^t e^{(r/q-1)\|\pi(x)\|_1}<\infty\end{aligned}$$

for all $\gamma\in\Gamma$. ∎

Examples 18 and 19 show some situations that are meaningless in the finite-dimensional setting. Moreover, the fact that $\mathcal{H}W(X)\subsetneq\mathcal{H}v(X)\subsetneq\mathcal{H}V(X)$ in Examples 16 and 19 emphasizes that asking for the polynomials to be contained in the weighted space of holomorphic functions is much more subtle for infinite-dimensional spaces than for finite-dimensional ones as we noted after Corollary 5. We refer to [8] for other interesting (even non-radial) examples of weights on an open subset of $\mathbb{C}^n$ $(n\in\mathbb{N})$.

3. A representation of the (LB)-predual of $\mathcal{H}V(X)$. In this section we deal with an increasing sequence $V=(v_n)_{n\in\mathbb{N}}$ of continuous weights satisfying Condition I. We consider the following topologies on $\mathcal{H}V(X)$: let τ_c be the topology of uniform convergence on compact subsets of $GV(X)$, let τ_0 be the compact open topology and let τ_{bcl} the finest locally convex topology that agrees with τ_0 on τ_V-bounded sets.

For each $\alpha=(\alpha_n)_{n\in\mathbb{N}}$, $\alpha_n>0$, $n\in\mathbb{N}$, let

$$D_\alpha:=\{f\in\mathcal{H}V(U):p_n(f):=p_{v_n}(f)\le\alpha_n\text{ for all }n\in\mathbb{N}\}.$$

Since $\tau_0 \le \tau_V$ and D_α is τ_V-bounded, D_α is τ_0-bounded. Since τ_0-convergence implies pointwise convergence, D_α is also τ_0-closed. Hence, by Montel's theorem, it is τ_0-compact. So, $\{D_\alpha\}_\alpha$ is a fundamental system of absolutely convex bounded and τ_0-compact sets (property (BBC) in [6]). It is not difficult to see that

$$(V_{n,\varepsilon} := \{f \in \mathcal{H}V(U) : p_n(f) \le \varepsilon\})_{n\in\mathbb{N},\, \varepsilon>0}$$

is a 0-neighbourhood base of absolutely convex τ_0-closed sets in $(\mathcal{H}V(U), \tau_V)$ (property (CNC) in [6]). So, by Corollary 5 of [6],

$$GV(U) := \{\phi \in \mathcal{H}V(U)' : \phi|_{D_\alpha} \text{ is } \tau_0\text{-continuous for all } \alpha = (\alpha_n), \alpha_n > 0\},$$

endowed with the topology of uniform convergence on the sets D_α, is a complete barrelled (DF)-space such that its strong dual is topologically isomorphic to $(\mathcal{H}V(U), \tau_V)$. This duality allows us to consider the weak-star topology on $\mathcal{H}V(U)$. In particular, when $V = \{v\}$, $Gv(U) := GV(U)$ is a Banach space whose strong dual is topologically isomorphic to the Banach space $\mathcal{H}v(U)$.

Proposition 22. (i) $\tau_0 \le \tau_{\mathrm{bcl}} = \tau_{\mathrm{c}} \le \tau_V$ *on* $\mathcal{H}V(X)$.
(ii) $(\mathcal{H}V(X), \tau_V)$ *and* $(\mathcal{H}V(X), \tau_{\mathrm{bcl}})$ *have the same bounded sets.*
(iii) $GV(X) = (\mathcal{H}V(X), \tau_{\mathrm{bcl}})'_{\mathrm{b}}$.

Proof. We only prove the equality in (i). Since $GV(X)$ is barrelled, τ_{bcl} is the finest locally convex topology which agrees with the weak-star topology on equicontinuous sets. So, by ([21], 21.9(7)), $\tau_{\mathrm{bcl}} = \tau_{\mathrm{c}}$. ∎

Since each v_n is continuous, the set $U_n := \{x \in X : v_n(x) > 0\}$ $(n \in \mathbb{N})$ is open. We define $r_n : \mathcal{H}V(X) \to \mathcal{H}v_n(U_n)$, and for $m > n$, $r_{n,m} : \mathcal{H}v_m(U_m) \to \mathcal{H}v_n(U_n)$, by $f \mapsto f|_{U_n}$. In a natural way we can show that $\mathcal{H}V(X) = \mathrm{proj}_n \mathcal{H}v_n(U_n)$ algebraically and topologically. This projective limit is not reduced in general. For example, $\mathcal{H}(\mathbb{C}) = \mathcal{H}_{\mathrm{b}}(\mathbb{C}) = \mathrm{proj}_n \mathcal{H}^\infty(n\Delta)$, where Δ is the open unit disc in $\mathbb{C}$, is not reduced.

Our main goal in this section is to prove that $GV(X) = \mathrm{ind}_n Gv_n(U_n)$. To do this we consider the transposed mapping $r_n^{\mathrm{t}} : \mathcal{H}v_n(U_n)' \to \mathcal{H}V(X)'$.

Proposition 23. (i) $r_n^{\mathrm{t}}(Gv_n(U_n)) \subset GV(X)$.
(ii) $(r_n^{\mathrm{t}}|_{Gv_n(U_n)})^{\mathrm{t}} = r_n$.
(iii) *If, in addition, each* v_n $(n \in \mathbb{N})$ *is radial,* $v_n(tx) > 0$ *for all* $t \in \mathbb{C}$ *with* $|t| \le 1$ *whenever* $v_n(x) > 0$ *and if* $\mathcal{H}V(X)$ *contains the polynomials, then* $r_n^{\mathrm{t}}|_{Gv_n(U_n)}$ *is injective.*

Proof. (i) For $\phi \in Gv_n(U_n)$ and $C \subset \mathcal{H}V(X)$ τ_V-bounded, we see that $r_n(C)$ is bounded, so $r_n^{\mathrm{t}}(\phi)|_C = \phi \circ r_n|_C = \phi|_{r_n(C)}$ is τ_0-continuous, hence $r_n^{\mathrm{t}}(\phi) \in GV(X)$.

(ii) For $f \in \mathcal{H}V(X)$ and $\phi_n \in Gv_n(U_n)$ we have

$$\begin{aligned}(r_n^{\mathrm{t}}|_{Gv_n(U_n)})^{\mathrm{t}}(f)(\phi_n) &= f \circ r_n^{\mathrm{t}}|_{Gv_n(U_n)}(\phi_n)\\ &= f(\phi_n \circ r_n) = (\phi_n \circ r_n)(f) = r_n(f)(\phi_n).\end{aligned}$$

(iii) We show that r_n has weak-star dense range. By assumption U_n is balanced. Let f be a function in $\mathcal{H}v_n(U_n)$. Since the Cesàro means $C_m(f)$ of the partial sums of the Taylor series of f belong to $\mathcal{H}V(X)$ for all $m = 0, 1, \ldots$ and the sequence $(r_n(C_m(f)))_m$ τ_{bcl}-converges to f and $(\mathcal{H}v_n(U_n), \tau_{\mathrm{bcl}})' = Gv_n(U_n)$, it follows that $(r_n(C_m(f)))_m$ weak-star converges to f. ∎

In an analogous way, if we take the transposed mappings $r_{n,m}^{\mathrm{t}} : \mathcal{H}v_n(U_n)' \to \mathcal{H}v_m(U_m)'$ $(n < m)$ then $r_{n,m}^{\mathrm{t}}$ satisfies 23(i)–(iii) with the obvious changes.

So, by taking transposed mappings the commutative diagram

$$\begin{tikzcd} & \mathcal{H}V(X) \arrow[dl, "r_m"'] \arrow[dr, "r_n"] & \\ \mathcal{H}v_m(U_m) \arrow[rr, "r_{n,m}"'] & & \mathcal{H}v_n(U_n) \end{tikzcd}$$

is transformed into

$$\begin{tikzcd} & GV(X) & \\ Gv_m(U_m) \arrow[ur, "r_m^{\mathrm{t}}"] & & Gv_n(U_n) \arrow[ul, "r_n^{\mathrm{t}}"'] \arrow[ll, "r_{n,m}^{\mathrm{t}}"] \end{tikzcd}$$

Our next lemma contains all the conditions we need to get an (LB)-structure on $GV(X)$. All of them are very natural and are satisfied in the usual cases (see Example 16(c)). Set $B_n := \{f \in \mathcal{H}v_n(U_n) : \|f\|_{v_n} \le 1\}$.

Lemma 24. *Let X be a Banach space and let $V = (v_n)_n$ be an increasing sequence of radial rapidly decreasing continuous weights which satisfies Conditions I, II and is such that for each $n \in \mathbb{N}$, if $v_n(x) \ne 0$ then $v_n(tx) \ne 0$ for all $t \in \mathbb{C}$ with $|t| \le 1$. Then for each $\varepsilon > 0$ and each $n \in \mathbb{N}$ there exist $m > n$ and $\alpha = (\alpha_j)_j$, $\alpha_j > 0$, such that every $f \in B_m$ has the decomposition*

$$f = r_m\Big(\sum_{j=0}^{N-1} P_j f\Big) + \sum_{j=N}^{\infty} P_j f,$$

where N is given by the conditions $r_{n,m}(\sum_{j=N}^{\infty} P_j f) \in (\varepsilon/2)B_n$ and $\sum_{j=0}^{N-1} P_j f \in (\varepsilon/2)D_\alpha$.

Proof. By Proposition 6(b), $\mathcal{H}V(X)$ contains all polynomials. First note that for each $p \in \mathbb{N}$ the open set U_p is balanced. Thus the Taylor series expansion of each $f \in \mathcal{H}v_p(U_p)$ converges pointwise to f and $r_p(\sum_{j=0}^{N} P_j f) \in \mathcal{H}v_p(U_p)$, hence

$$\sum_{j=N}^{\infty} P_j f = f - r_p\Big(\sum_{j=0}^{N} P_j f\Big) \in \mathcal{H}v_p(U_p) \quad \text{for all } N \in \mathbb{N}.$$

On the other hand, for all $f \in \mathcal{H}v_p(U_p)$ and all $p \in \mathbb{N}$ the following properties hold:

(a) We have

$$\begin{aligned}\|r_p(P_j f)\|_{v_p} &= \sup_{x\in U_p} v_p(x)|r_p(P_j f)(x)| \le \sup_{x\in U_p} v_p(x) \sup_{|t|\le 1} |P_j f(tx)| \\ &\le \sup_{x\in U_p} v_p(x) \sup_{|t|\le 1} |f(tx)| = \sup_{x\in U_p} v_p(x) \sup_{|t|=1} |f(tx)| \\ &= \sup_{x\in U_p} \sup_{|t|=1} v_p(tx)|f(tx)| = \|f\|_{v_p}.\end{aligned}$$

(b) Given $n \in \mathbb{N}$, we select $m > n$ and $t > 1$ according to Condition II. Then

$$\begin{aligned}t^j \|r_n(P_j f)\|_{v_n} &= t^j \sup_{x\in U_n} v_n(x)|r_n(P_j f)(x)| \\ &= t^j \sup_{x\in X} v_n(x)|P_j f(x)| \le \sup_{x\in X} v_m(x)|P_j f(x)| \\ &= \sup_{x\in U_m} v_m(x)|P_j f(x)| = \|r_m(P_j f)\|_{v_m} \le \|f\|_{v_m}\end{aligned}$$

for all $f \in \mathcal{H}v_m(U_m)$.

Let $f \in B_m$. By (b), the series $\sum_{j=0}^{\infty} r_n(P_j f)$ converges to $r_{n,m}(f)$ in $(\mathcal{H}v_n(U_n), \|\cdot\|_{v_n})$ and

$$\Big\|\sum_{j=N}^{\infty} r_n(P_j f)\Big\|_{v_n} \le \sum_{j=N}^{\infty} \|r_n(P_j f)\|_{v_n} \le \sum_{j=N}^{\infty} \frac{1}{t^j}\|f\|_{v_m} \le \sum_{j=N}^{\infty} \frac{1}{t^j},$$

which is less than $\varepsilon/2$ for N large enough.

Without loss of generality we may assume that $\inf_{x\in B} v_1(x) > 0$, where B is the closed unit ball of X.

By Proposition 6(b), for each $k \in \mathbb{N}$, $p_k|_{\mathcal{P}(^jX)}$ is a continuous seminorm on $\mathcal{P}(^jX)$, hence there exists $a_{k,j} > 0$ such that $p_k(P) \le a_{k,j}\|P\|$ for every $P \in \mathcal{P}(^jX)$. Moreover, since $\|P_j f\| \le c p_1(P_j f)$, where $c := 1/\inf_{x\in B} v_1(x)$, it follows that

$$\begin{aligned}\sum_{j=0}^{N-1} p_k(P_j f) &\le \sum_{j=0}^{N-1} a_{k,j}\|P_j f\| \le \sum_{j=0}^{N-1} a_{k,j} c p_1(P_j f) \\ &\le \sum_{j=0}^{N-1} a_{k,j} c p_{v_m}(P_j f) = \sum_{j=0}^{N-1} a_{k,j} c \|r_m(P_j f)\|_{v_m} \\ &\le \sum_{j=0}^{N-1} a_{k,j} c \|f\|_{v_m} \le \sum_{j=0}^{N-1} a_{k,j} c.\end{aligned}$$

If we put $\alpha_k := (2/\varepsilon)c\sum_{j=0}^{N-1} a_{k,j} > 0$ then $p_k(\sum_{j=0}^{N-1} P_j f) \le (\varepsilon/2)\alpha_k$, thus $\sum_{j=0}^{N-1} P_j f \in (\varepsilon/2)D_\alpha$, $\alpha := (\alpha_k)_k$. ■

PROPOSITION 25. *Under the hypotheses of Lemma* 24,

$$D_\alpha^\circ \cap r_n^{\mathrm t}(A_n) \subset \varepsilon r_m^{\mathrm t}(A_m),$$

where A_k is the closed unit ball of $Gv_k(U_k)$ for all $k \in \mathbb{N}$ and D_α° is the polar of D_α in $GV(X)$.

Proof. Let $\phi \in D_\alpha^\circ \cap r_n^{\mathrm t}(A_n)$ and $\phi_n \in A_n$ be such that $\phi = r_n^{\mathrm t}(\phi_n)$. We put $\phi_m := r_{n,m}^{\mathrm t}(\phi_n) \in Gv_m(U_m)$. Then $\phi = r_n^{\mathrm t}(\phi_n) = r_m^{\mathrm t}(r_{n,m}^{\mathrm t}(\phi_n)) = r_m^{\mathrm t}(\phi_m)$. To conclude it suffices to show that $\phi_m \in \varepsilon A_m$. By Lemma 24,

$$f = r_m\Big(\sum_{j=0}^{N-1} P_j f\Big) + \sum_{j=N}^{\infty} P_j f$$

with $\sum_{j=0}^{N-1} P_j f \in (\varepsilon/2)D_\alpha$ and $r_{n,m}(\sum_{j=N}^{\infty} P_j f) \in (\varepsilon/2)B_n$. Hence

$$\begin{aligned}|\phi_m(f)| = |r_{n,m}^{\mathrm t}(\phi_n)(f)| &= \Big|\phi_n \circ r_{n,m}\Big(r_m\Big(\sum_{j=0}^{N-1} P_j f\Big) + \sum_{j=N}^{\infty} P_j f\Big)\Big| \\ &= \Big|\phi_n\Big(r_n\Big(\sum_{j=0}^{N-1} P_j f\Big)\Big) + \phi_n\Big(r_{n,m}\Big(\sum_{j=N}^{\infty} P_j f\Big)\Big)\Big| \\ &\le \Big|r_n^{\mathrm t}(\phi_n)\Big(\sum_{j=0}^{N-1} P_j f\Big)\Big| + \frac{\varepsilon}{2} \le \varepsilon. \ ■\end{aligned}$$

COROLLARY 26. *Under the hypotheses of Lemma* 24 *for each $n \in \mathbb{N}$ there exists $m > n$ such that $GV(X)$ and $r_m^{\mathrm t} Gv_m(U_m)$, endowed with the induced topology via $r_m^{\mathrm t}$, induce the same topology on $r_n^{\mathrm t}(A_n)$.*

THEOREM 27. *Under the hypotheses of Lemma* 24,

$$GV(X) = \mathrm{ind}_n Gv_n(U_n)$$

holds algebraically and topologically. This inductive limit is boundedly retractive.

Proof. Let $A \subset GV(X)$ be a bounded set. So, its polar A° in $\mathcal{H}V(X)$ is a 0-neighbourhood in $GV(X)'_b = \mathcal{H}V(X)$. Since $\mathcal{H}V(X) = \mathrm{proj}_n \mathcal{H}v_n(U_n)$ there exists a finite set $F \subset \mathbb{N}$ such that

$$\bigcap_{j\in F} \{f \in \mathcal{H}V(X) : p_j(f) \le \varepsilon\} \subset A^\circ$$

for some $\varepsilon > 0$. If $n := \max_{j\in F} j$, then

$$\bigcap_{j\in F} \{f \in \mathcal{H}V(X) : p_j(f) \le \varepsilon\} = \{f \in \mathcal{H}V(X) : p_n(f) \le \varepsilon\} = r_n^{-1}(\varepsilon B_n),$$

and

$$\varepsilon(r_n^{\mathrm t}|_{Gv_n(U_n)}(A_n))^\circ = ((r_n^{\mathrm t}|_{Gv_n(U_n)})^{\mathrm t})^{-1}(\varepsilon A_n^\circ) = r_n^{-1}(\varepsilon B_n) \subset A^\circ,$$

where the first polar is in $\mathcal{H}V(X)$ and the second in $\mathcal{H}v_n(U_n)$, and A_n is defined as in Proposition 25. Taking polars in $GV(X)$ and using the bipolar theorem, we get

$$A \subset A^{\circ\circ} \subset \frac{1}{\varepsilon}(r_n^{\mathrm t}|_{Gv_n(U_n)}(A_n))^{\circ\circ} = \frac{1}{\varepsilon}\overline{r_n^{\mathrm t}(A_n)}^{\tau_{GV(X)}}.$$

By Corollary 26, there exists $m > n$ such that $GV(X)$ and $r_m^{\mathrm t}Gv_m(U_m)$ induce the same topology on $r_n^{\mathrm t}(A_n)$. Since $r_m^{\mathrm t}Gv_m(U_m)$ is complete with the topology induced via $r_m^{\mathrm t}$, we have

$$\overline{r_n^{\mathrm t}(A_n)}^{\tau_{GV(X)}} = \overline{r_n^{\mathrm t}(A_n)}^{\tau_{r_m^{\mathrm t}(Gv_m(U_m))}}$$

and so

$$A \subset \frac{1}{\varepsilon}\overline{r_n^{\mathrm t}(A_n)}^{\tau_{r_m^{\mathrm t}(Gv_m(U_m))}} \subset r_m^{\mathrm t}(Gv_m(U_m)).$$

Thus each bounded subset of $GV(X)$ is contained in some $r_m^{\mathrm t}(Gv_m(U_m))$ and bounded there. Hence

$$GV(X) = \bigcup_{n=1}^{\infty} r_n^{\mathrm t}Gv_n(U_n)$$

and the spaces $GV(X)$ and $\mathrm{ind}_n r_n^{\mathrm t}Gv_n(U_n)$ induce the same topology on each bounded subset of $GV(X)$. In fact, if $A \subset GV(X)$ is bounded, there exists $n \in \mathbb{N}$ such that A is contained and bounded in $r_n^{\mathrm t}(Gv_n(U_n))$, and there exists $m > n$ such that $GV(X)$ and $r_m^{\mathrm t}(Gv_m(U_m))$ induce the same topology on A. On the other hand, if τ_{ind} denotes the inductive limit topology then $\tau_{\mathrm{ind}}|_A \le \tau_{r_m^{\mathrm t}(Gv_m(U_m))}|_A$ and since

$$\tau_{r_m^{\mathrm t}(Gv_m(U_m))}|_A = \tau_{GV(X)}|_A \le \tau_{\mathrm{ind}}|_A \le \tau_{r_m^{\mathrm t}(Gv_m(U_m))}|_A,$$

it follows that $\mathrm{ind}_j r_j^{\mathrm t}(Gv_j(U_j))$ and $r_m^{\mathrm t}(Gv_m(U_m))$ induce the same topology on A.

Finally, since $GV(X)$ is a (DF)-space, a theorem of Grothendieck ([19], Thm. 3) guarantees that the identity mapping $GV(X) \to \mathrm{ind}_n r_n^{\mathrm t}(Gv_n(U_n))$ is continuous. ∎

The next corollary can be obtained as a consequence of the above theorem and Corollary 5(d) of [6]. (It also follows from Theorem 11, Proposition 6(b) and [15], Theorem 3.1).

COROLLARY 28. *Under the hypotheses of Lemma* 24, $\mathcal{H}V(X)$ *is a quasinormable Fréchet space.*

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On operator ideals related to (p,σ)-absolutely continuous operators

by

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Abstract. We study tensor norms and operator ideals related to the ideal $\mathcal{P}_{p,\sigma}$, $1 < p < \infty$, $0 < \sigma < 1$, of (p,σ)-absolutely continuous operators of Matter. If α is the tensor norm associated with $\mathcal{P}_{p,\sigma}$ (in the sense of Defant and Floret), we characterize the $(\alpha')^t$-nuclear and $(\alpha')^t$- integral operators by factorizations by means of the composition of the inclusion map $L^r(\mu) \to L^1(\mu)+L^p(\mu)$ with a diagonal operator $B_w : L^\infty(\mu) \to L^r(\mu)$, where r is the conjugate exponent of $p'/(1-\sigma)$. As an application we study the reflexivity of the components of the ideal $\mathcal{P}_{p,\sigma}$.

1. Introduction. The ideal $\mathcal{P}_{p,\sigma}$ of (p,σ)-absolutely continuous operators was introduced by Matter [8] in order to get a classification of the absolutely continuous operators previously defined by Niculescu [10]. Since $\mathcal{P}_{p,\sigma}$ is a maximal ideal, it is interesting to study the tensor norm α (or the transposed α^t) associated with $\mathcal{P}_{p,\sigma}$ and the properties of the operator ideals naturally related to α. The results obtained could be applied to study the metric properties of α as well as some topological properties of the components of $\mathcal{P}_{p,\sigma}$. As far as we know, this work has not been done yet. Concretely, the main questions can be reduced to the following:

1. Find the tensor norm α such that $(E\hat{\otimes}_\alpha F)' = \mathcal{P}_{p,\sigma}(F,E')$ for every pair of Banach spaces E and F.
2. Characterize the α-nuclear and α-integral operators.

In this spirit, we have characterized in [6] the tensor norm $g_{p',\sigma}$ which solves question 1. *In the present paper we give a full answer to problem* 2. Although this can be done without any reference to tensor products (Definitions 2 and 4 below have a meaning in the context of purely operator ideals), we have chosen the tensorial approach for two reasons. The first one

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