Pointwise multiplication operators on weighted Banach spaces of analytic functions

by

J. BONET (Valencia), P. DOMAŃSKI (Poznań) and M. LINDESTRÖM (Åbo)

Abstract. For a wide class of weights we find the approximative point spectrum and the essential spectrum of the pointwise multiplication operator $M_\phi$, $M_\phi(f) = \phi f$, on the weighted Banach spaces of analytic functions on the disc with the sup-norm. Thus we characterise when $M_\phi$ is Fredholm or is an into isomorphism. We also study cyclic phenomena for the adjoint map $M_\phi'$.

1. Introduction. We consider the pointwise multiplication operators $M_\phi$, $M_\phi(f) := \phi f$, where $\phi : D \to \mathbb{C}$ always denotes a bounded non-constant analytic function on the unit disc $D$. These operators are considered on the following Banach spaces of analytic functions:

\begin{align}
H^\infty_v & := \mathcal{H}^\infty_v(D) := \{ f \in \mathcal{H}(D) : \| f \|_v := \sup_{z \in D} v(z) |f(z)| < \infty \}, \\
H^0_v & := \mathcal{H}^0_v(D) := \{ f \in \mathcal{H}(D) : \lim_{|z| \to 1^-} v(z)|f(z)| = 0 \},
\end{align}

endowed with the norm $\| \cdot \|_v$, where $v$ is an arbitrary weight, i.e., a continuous function $v : D \to \mathbb{R}_+$ such that $H^\infty_v$ contains a non-zero function. We will be mostly interested in radial weights (i.e., $v(z) = v(|z|)$) tending to zero at the boundary. In order to include, for instance, typical weights $v(z) = |\text{Im} z|^{\infty}$ on the half plane transferred onto the disc via a conformal isomorphism we have to consider a more general class of weights.

Our purpose is to calculate the spectrum $\sigma(M_\phi)$, the essential spectrum $\sigma_e(M_\phi)$ and the approximative point spectrum $\sigma_{ap}(M_\phi)$ of $M_\phi$, i.e., the sets

1991 Mathematics Subject Classification: Primary 46E15, 47B38.

Key words and phrases: weighted Banach spaces of analytic functions, pointwise multiplication operator, essential norm, closed range, approximative point spectrum, maximal ideal space of $H^\infty$, Shilov boundary, Gleason part, hypercyclic operator, chaotic operator.

The research of the first named author was partially supported by DGESIC, project no. PB 97-0333. The research of the third named author was supported by the Academy of Finland.
of those \( \lambda \in \mathbb{C} \) such that \( M_{\phi} - \lambda \cdot \text{id} = (M_{\phi} - \lambda) \) is not an onto isomorphism, a Fredholm operator, an onto isomorphism, resp. Of course, \( M_{\phi} \) is Fredholm or an onto isomorphism if and only if 0 \( \not\in \sigma_{\phi}(M_{\phi}) \) or 0 \( \not\in \sigma_{\phi}(M_{\phi}) \), resp., so the corresponding calculations are the same as characterizing Fredholm operators and into isomorphisms among all the multipliers \( M_{\phi} \). As we will see the first two tasks are easy (Lemma 2.3, Proposition 2.4) while the third one is much more interesting (see Section 3). We prove that \( \sigma_{\phi}(M_{\phi}) = \varphi(A_{\phi}) \) where \( A_{\phi} \) is a closed subset of the maximal ideal space \( M(H^{\infty}) \) of the algebra \( H^{\infty} \) and \( A_{\phi} \) depends only on the weight \( \nu \) (Theorem 3.5). We then identify \( A_{\phi} \) for weights like \( \nu(z) = (1 - |z|)^{\alpha} \), \( 0 < \alpha < \infty \), on the disc or \( \nu(z) = \text{Im}(z)^{\alpha} \) on the half plane. It turns out that \( A_{\phi} \) equals the set of ideals in \( M(H^{\infty}) \) with a trivial Gleason part (Theorem 3.6). In particular, this implies that for such weights \( M_{\phi} : H^{\infty} \to H^{\infty} \) is an onto isomorphism if and only if \( \varphi = h\delta_{b_1} \cdots \delta_{b_n} \), where \( h \) is invertible in \( H^{\infty} \) and \( b_1, \ldots, b_n \) are interpolating Blaschke products. Let us emphasize that \( \sigma_{\phi}(M_{\phi}) \), unlike \( \sigma_{\phi} \), depends heavily on the weight \( \nu \). In Section 4 we show that for radial weights \( \nu \) tending to zero at the boundary the map adjoint to \( M_{\phi} : H^{\infty} \to H^{\infty} \) has hypercyclic vectors if and only if \( \varphi(\mathbb{D}) \) intersects the unit circle.

Multiplication operators have attracted some attention. See for instance [A], [MS] and [V]. In particular, multiplication operators were used in [S2] to get interpolation results. On the other hand, the spaces \( H^{\infty} \) of analytic functions with controlled growth were studied extensively, for instance, in [BBT], [BS], [RS], [LI], [L2], [S], [S2], [BDLT], [BDL], [DL] or [SW].

The norm topology of \( H^{\infty} \) is stronger than the compact open topology \( \omega \). The latter makes the unit ball compact. It is known that the so-called associated weight

\[
\bar{\nu}(z) = \left( \sup \{|f(z)| : \|f\|_{H^{\infty}} \leq 1 \} \right)^{-1},
\]

is better tied with the space \( H^{\infty} \) than \( \nu \) itself [BBT]. Of course, \( H^{\infty} = H^{\infty} \) isometrically. We say that \( \nu \) is an essential weight if \( \nu \sim \bar{\nu} \), i.e., there is a constant \( C \) such that \( \bar{\nu}(z) \leq C \nu(z) \) for any \( z \in \mathbb{D} \). If a radial weight \( \bar{\nu} \) vanishes on the boundary then \( (H^{\infty})'' = H^{\infty} \) and the unit ball of \( H^{\infty} \) is co-dense in the unit ball of \( H^{\infty} \). See [BS]. On the other hand, if a radial weight \( \nu \) does not vanish on the boundary then \( H^{\infty} = H^{\infty} \) and \( H^{\infty} = \{0\} \).

The following elementary observation is of relevance in the article: if \( H^{\infty} \) (resp. \( H^{\infty} \)) contains a non-zero function, then for any fixed \( z \in \mathbb{D} \) there is \( f \in H^{\infty} \) (resp. \( H^{\infty} \)) vanishing at \( z \). Hence the evaluation functional \( \delta_{\bar{z}} \), \( \delta_{\bar{z}}(f) = f(z) \), is non-zero on \( H^{\infty} \) and \( H^{\infty} \), respectively. In fact, if \( z_1, \ldots, z_n \) are different points in \( \mathbb{D} \), then the evaluation functionals at these points are linearly independent. This can be proved using the following ideas: the product of an element of \( H^{\infty} \) (resp. \( H^{\infty} \)) and an element of \( H^{\infty} \) belongs to \( H^{\infty} \) (resp. \( H^{\infty} \)). On the other hand, if a function \( f \in H^{\infty} \) (resp. \( H^{\infty} \)) has an isolated zero of order \( m \) at \( z_0 \), then the function \( f(z)/(z - z_0)^m \) also belongs to \( H^{\infty} \) (resp. \( H^{\infty} \)) and does not vanish at \( z_0 \).

We also need several facts on \( H^{\infty} \) which are explained in detail in the book of Garnett [G]. We recall some definitions. A sequence \( (\nu_n) \) in \( \mathbb{D} \) is called an interpolating sequence if for every bounded sequence \( (\beta_n) \) of complex numbers there is an \( f \in H^{\infty} \) for which \( f(\nu_n) = \beta_n \), for all \( n \). A Blaschke product whose zero sequence is an interpolating sequence is called an interpolating Blaschke product. We denote by \( B(\mathbb{D}) \) the disc algebra and by \( M(H^{\infty}) \) the maximal ideal space of \( H^{\infty} \). The pseudohyperbolic distance between two points \( m \) and \( n \) in \( M(H^{\infty}) \) is defined by

\[
\delta(m, n) = \sup \{|\tilde{f}(n)| : f \in H^{\infty}, \tilde{f}(m) = 0, ||f\|_{\infty} \leq 1\},
\]

where \( \tilde{f} \) denotes the Gelfand transform of \( f \). For the sake of simplicity, we denote the Gelfand transform of \( f \) again by \( f \) further on. For \( \zeta, \xi \in \mathbb{D} \), \( \rho(\zeta, \xi) = |\varphi(\zeta)|, \) where \( \varphi(\zeta) = (z - \zeta)/(1 - \bar{z}\zeta) \). The Gleason part of \( m \in M(H^{\infty}) \) is defined by \( \mathcal{P}(m) = \{n \in M(H^{\infty}) : \overline{\mathbb{D}} \in \mathbb{D} \} \) of trivial Gleason parts is a closed subset of \( M(H^{\infty}) \) that properly contains the Shilov boundary \( \mathcal{B}(H^{\infty}) \) of \( H^{\infty} \) (comp. [G, Ch. X.1]). We denote by \( D(z, r) \) and \( \Delta(z, r) \) the euclidean disc and the pseudohyperbolic disc of center \( z \) and radius \( r \) respectively. By \( ||.||_{H^{\infty}} \) we denote the sup-norm on \( H^{\infty} \). Our reference for the Corona Theorem (used e.g. in the proof of 3.7 below) is also [G]. We write \( f \approx g \) for two functions \( f \) and \( g \) if there are positive constants \( c \) and \( C \) such that \( cg \leq f \leq Cg \).

Recall that an operator \( T \) is Fredholm if it has closed range and both the dimension of its kernel and the codimension of its image are finite. A vector \( x \in X \) is called hypercyclic for an endomorphism \( T : X \to X \) if the orbit \( \{T^n(x)\} \) of \( x \) is dense in \( X \).

For basic facts on bouned analytic functions and functional analysis we refer to [R2] and [R1] respectively.

2. Boundedness, Fredholm operators and the essential norm of \( M_{\phi} \). We start with the following easy characterization of boundedness.

PROPOSITION 2.1. Let \( \nu \) be a weight on \( \mathbb{D} \). The following statements are equivalent:

(a) \( M_{\phi} : H^{\infty} \to H^{\infty} \) is continuous,
(b) \( \varphi \in H^{\infty} \).

In this case \( ||M_{\phi}|| = ||\varphi||_{H^{\infty}} \). If \( H^{\infty} \neq \{0\} \), the statements above are equivalent to:

(c) \( M_{\phi} : H^{\infty} \to H^{\infty} \) is continuous (and also here \( ||M_{\phi}|| = ||\varphi||_{H^{\infty}} \)).

Another observation will be useful to avoid considering operators on \( H^{\infty} \).
PROPOSITION 2.2. If $M_\varphi : H_0^0 \to H_0^0$ is bounded and both $v$ and $w$ are radial weights vanishing on the boundary, then $M_\varphi^w = M_\varphi : H_0^0 \to H_0^0$.

**Proof.** It is well known that $(H_0^0)^\prime = H_0^\infty$ and $(H_0^0)^\prime = H_0^\infty$ (see [BS] and [RS]). Moreover, the evaluation functional $\delta_z, \delta_z(f) = f(z)$ on $H_0^0$, acts on $H_0^\infty$ as the evaluation functional (comp. the analysis in [BDLT]). Since $M_\varphi(\delta_z) = \varphi(z)\delta_z$, we have, for $f \in H_0^\infty$,

$$(M_\varphi^w f, \delta_z) = (f, \varphi(z)\delta_z) = f(z)\varphi(z).$$

Note that by Proposition 2.2, for radial weights $v$ vanishing on the boundary, $M_\varphi : H_0^\infty \to H_0^\infty$ has closed range if and only if $M_\varphi : H_0^0 \to H_0^0$ does (see [R1, Thm. 4.14]). A similar fact holds for Fredholm operators $M_\varphi$ and for isomorphisms. Thus, for radial weights tending to zero at the boundary, one can consider only the case of $H_0^\infty$.

Invertible multiplication operators and the spectrum of $M_\varphi$ can be characterized very easily.

**Lemma 2.3.** Let $v$ be a weight on $\mathbb{D}$. The following statements are equivalent for $\varphi \in H_0^\infty$:

(a) $M_\varphi : H_0^\infty \to H_0^\infty$ is invertible (or, equivalently, surjective);

(b) $1/\varphi \in H_0^\infty$ (or, equivalently, there exists $\varepsilon > 0$ such that $|\varphi(z)| \geq \varepsilon$ for all $z \in \mathbb{D}$).

The analogous equivalence holds for $H_0^0$ if $H_0^0 \neq \{0\}$.

**Proof.** Observe that the inverse to $M_\varphi$ must be of the form $M_{1/\varphi}$ and apply Proposition 2.1. $

Since $\lambda - M_\varphi = \lambda - \varphi$ the above result shows that the spectrum of $M_\varphi$ satisfies $\sigma(M_\varphi) = \varphi(\mathbb{D}) = \varphi(M(H_0^\infty))$. This implies that $M_\varphi$ is not compact.

The class of Fredholm operators is another important class of closed range operators. Following Axler [A] we get a characterization of Fredholm multiplication operators.

**Proposition 2.4.** Let $v$ be a weight on $\mathbb{D}$ and $\varphi \in H_0^\infty$. The operator $M_\varphi : H_0^0 \to H_0^0$ is Fredholm if and only if there exists $\varepsilon > 0$ such that $|\varphi(z)| \geq \varepsilon$ for all $z \in \mathbb{D}$. Consequently, $\sigma_0(M_\varphi) = \varphi(M(H_0^\infty)) \setminus \mathbb{D}$. The same characterization holds for $H_0^\infty$ whenever $H_0^0 \neq \{0\}$.

**Proof.** Suppose first there is a sequence $(z_n) \subset \mathbb{D}$ with $|z_n| \to 1$ and $|\varphi(z_n)| \to 0$. Taking a subsequence if necessary, we may assume that $(z_n)$ is an interpolating sequence for $H_0^\infty$. Thus (see e.g. [G, Ch. VII.1]) for each $N$ there is $\varphi_N \in H_0^\infty$ such that

$$\varphi_N(z_n) = \begin{cases} 0 & \text{if } n < N, \\ \varphi(z_n) & \text{if } n \geq N, \end{cases}$$

and $||\varphi_N||_\infty \leq C \sup_{n \geq N} |\varphi(z_n)|$ for some fixed $C > 0$. Let

$$X_N = \{ f \in H_0^\infty : f(z_n) = 0 \text{ for all } n \geq N \}.$$

Clearly $z_n \in X_N \subset (H_0^\infty)^\prime$, $n \geq N$, so $X_N$ is infinite-dimensional: indeed, the $\delta_{z_n}$ are linearly independent in $(H_0^\infty)^\prime$ (and in $(H_0^0)^\prime$ whenever $H_0^0 \neq \{0\}$). Since $(\varphi - \varphi_N)(z_n) = 0$ for $n \geq N$, $\text{range}(M_\varphi - \varphi_N) \subset X_N$. Further we conclude from $(H_0^\infty/X_N)^\prime = X_N^\prime$ that $H_0^\infty/X_N$ is infinite-dimensional. Therefore range($M_\varphi - \varphi_N$) has infinite codimension in $H_0^\infty$, so $M_\varphi - \varphi_N$ is not Fredholm. Now, $||\varphi_N||_\infty \to 0$ as $N \to \infty$. Since the set of non-Fredholm operators is closed, we see from $||M_\varphi - \varphi_N - M_\varphi|| \leq ||\varphi_N||_\infty$ that $M_\varphi$ is not Fredholm.

Conversely, by the assumption, $\varphi$ can have only finitely many zeros $z_1, \ldots, z_n$ inside $\mathbb{D}$ with multiplicities $m_1, \ldots, m_n$ respectively. Let $\delta_{z_i}^{(k)}(f) := f^{(k)}(z_i)$ be the evaluation maps for derivatives. Observe that

$$\text{range}(M_\varphi) = \bigcap_{i=1}^n \bigcap_{k=0}^{m_i-1} \ker \delta_{z_i}^{(k)}.$$

If $g \in \bigcap_{i=1}^n \bigcap_{k=0}^{m_i-1} \ker \delta_{z_i}^{(k)}$, then $g/\varphi \in H(\mathbb{D})$ and therefore the assumption implies that $g/\varphi \in H_0^\infty$. Hence range($M_\varphi$) $= \bigcap_{i=1}^n \bigcap_{k=0}^{m_i-1} \ker \delta_{z_i}^{(k)}$ and $M_\varphi$ is Fredholm.

The essential norm of a continuous linear operator $T$ is defined by $||T||_e = \inf(||T - K|| : K \text{ is compact})$. Since $||T||_e = 0$ if and only if $T$ is compact, the estimate on $||M_\varphi||_e$ proved below shows that $M_\varphi : H_0^\infty \to H_0^\infty$ is non-compact when $\varphi \neq 0$ (comp. the remark after Lemma 2.3). Recall that if $T \in L(E)$, the essential spectral radius $r_e(T)$ of $T$ is defined by $r_e(T) = \sup(|\lambda| : \lambda \in \sigma_0(T))$. It can also be calculated by $r_e(T) = \lim_{n \to \infty} ||T^n||^{1/n} \leq ||T||_e$ (cf. [C]). Let us see what this means for the continuous multiplication operator $M_\varphi$.

**Corollary 2.5.** Let $v$ be a weight on $\mathbb{D}$ and $\varphi \in H_0^\infty$. Then for $M_\varphi : H_0^\infty \to H_0^\infty$ we have $r_e(M_\varphi) = ||M_\varphi||_e = ||M_\varphi|| = ||\varphi||_\infty$. The same holds for $H_0^0$ if $H_0^0 \neq \{0\}$.

**Proof.** It suffices to observe that $r_e(M_\varphi) = ||\varphi||_\infty$, by Proposition 2.4.

3. The closed range property of $M_\varphi$. The operator $M_\varphi : H_0^\infty \to H_0^\infty$ is always one-to-one. Therefore the closed range multiplication operators are precisely those operators that are bounded from below (i.e., $||M_\varphi f||_v \geq C||f||_v$, for some $C > 0$ and every $f \in H_0^\infty$) or, equivalently, are into isomorphisms.

**Lemma 3.1.** Let $\varphi \in H_0^\infty$. If $v$ and $w$ are two weights on $\mathbb{D}$ and $u := v/w$ is equivalent to an essential weight, then every closed range map $M_\varphi$:
$H_v^\infty \to H_v^\infty$ also has closed range as a map $M_\varphi : H_v^\infty \to H_v^\infty$. An analogous result holds for $H_0^\infty$ and $H_0^\infty \neq \{0\}$.

Proof. Assume that $M_\varphi$ does not have closed range on $H_v^\infty$. There are functions $f_n \in H_v^\infty$ with $\|f_n\|_w = 1$ and $\|M_\varphi f_n\|_w \leq 1/n$. Clearly there are $z_n \in \mathbb{D}$ such that $|f_n(z_n)|/w(z_n) > 1/2$. Let $g_n$ be chosen from $H_0^\infty$, $u = w/w$, such that $\|g_n\|_w \leq 1$ and $|g_n(z_n)|/u(z_n) \geq m > 0$ for every $n \in \mathbb{N}$. Hence $\|g_n f_n\|_w \geq m/2 > 0$. Now, clearly $\|M_\varphi g_n f_n\|_w \leq 1/n$ and the proof is complete. Taking $f_n \in H_0^\infty$ we get the $H_0^\infty-H_v^\infty$ version of the result. □

Now, we study general closed range multiplication operators. The following simple and certainly known fact was pointed out to us by P. Wojtaszczyk. Another proof can be obtained by the machinery of uniform algebras.

**Proposition 3.2.** The map $M_\varphi : H_v^\infty \to H_v^\infty$ has closed range if and only if $\varphi$ does not vanish at any point of the Shilov boundary $\Gamma(H_v^\infty)$ of $H_v^\infty$.

**Remarks.**

1) Since the Shilov boundary of $H_v^\infty$ can be identified with the space of maximal ideals of $L_v^\infty(\mathbb{D})$ (G, V.1.7), a function $\varphi$ does not vanish on the Shilov boundary if and only if $|\varphi|$ is essentially bounded away from zero on the unit circle.

2) All inner functions $\varphi$ generate multiplication operators on $H_v^\infty$ with closed range.

**Proof** (of Proposition 3.2). Let $\varphi^*$ be the radial limit of $\varphi$. If $A \subseteq \partial \mathbb{D}$ is a subset of positive measure such that $|\varphi^*| < \varepsilon$ on $A$ and $|\varphi^*| \leq 1$ elsewhere, then we take the outer function

$$f(z) = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log|\varphi(e^{it})| \, dt \right\},$$

where $|f^*| = 1$ on $A$ and $|f^*| = \varepsilon$ on $\partial \mathbb{D}\setminus A$, to get $\|M_\varphi f\|_\infty = \|\varphi^* f^*\|_\infty \leq \varepsilon$. Thus if $|\varphi^*|$ is not essentially bounded away from zero then $M_\varphi$ is not bounded from below. The converse is obvious. □

**Corollary 3.3.** Let $\varphi$ be an (essential) weight on $\mathbb{D}$ and $\varphi \in H_v^\infty$. If $M_\varphi : H_v^\infty \to H_v^\infty$ has closed range, then $\varphi$ does not vanish at any point of the Shilov boundary of $H_v^\infty$.

**Remark.** It is an open problem if the same holds for $H_0^\infty$ instead of $H_v^\infty$.

**Corollary 3.4.** Let $\varphi$ be an outer function. Then the following assertions are equivalent:

(a) $M_\varphi : H_v^\infty \to H_v^\infty$ is invertible for some (or each) weight $v$;
(b) $M_\varphi : H_v^\infty \to H_v^\infty$ is a Fredholm operator for some (or each) weight $v$;
(c) $M_\varphi : H_v^\infty \to H_v^\infty$ is an isomorphism into for some (or each) weight $v$;
(d) $\varphi$ does not vanish at any point of the Shilov boundary of $H_v^\infty$;
(e) there is $\varepsilon > 0$ such that $|\varphi(x)| > \varepsilon$ for each $x \in \mathbb{D}$.

**Remark.** The equivalences (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) hold for all $\varphi \in A(\mathbb{D})$.

**Proof.** (e) $\Leftrightarrow$ (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) follow from Lemma 2.3 and Corollary 3.3.

Let $\varphi$ be an outer function and assume (d). Since $\varphi$ does not vanish anywhere on the Shilov boundary of $H_v^\infty$, $|\varphi|$ is essentially bounded away from zero on the unit circle. Since $\varphi$ is outer, we have

$$\varphi(x) = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + x}{e^{it} - x} \log|\varphi(e^{it})| \, dt \right\}.$$

Hence $1/\varphi$ is an outer function generated by $\log(1/|\varphi(e^{it})|)$ and it belongs to $H_v^\infty$. Lemma 2.3 applies. □

The corollaries above yield that if $\varphi$ is outer or belongs to the disc algebra $A(\mathbb{D})$, then $M_\varphi$ has either a closed range on every $H_v^\infty$ or on none of them. By the inner-outer factorization of bounded analytic functions, we have $\varphi = B \cdot m \cdot Q$, where $B$ is a Blaschke product, $m$ is a singular inner function and $Q$ is an outer function. Since $\varphi$ is assumed to be non-constant, $M_\varphi : H_v^\infty \to H_v^\infty$ has closed range (equivalently, is an isomorphism) if and only if $M_B$, $M_m$ and $M_Q$ have closed ranges. Now, we have to look more carefully at inner functions. It turns out that they always generate closed range operators on $H_v^\infty$ but they need not have closed range on other $H_v^\infty$.

Let $X$ be a commutative unital Banach algebra and $x = (x_1, \ldots, x_n) \in X^n$. The joint spectrum $\sigma(x)$ is defined as the set of all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that there do not exist $y_1, \ldots, y_n \in X$ satisfying

$$\sum (x_i - \lambda_i y_i) = 1.$$

Analogously, for a Banach space $Y$ and operators $T_i : Y \to Y$ which mutually commute, we define the joint approximative point spectrum $\sigma_{ap}(T)$, $T = (T_1, \ldots, T_n)$, as the set of those $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that

$$(T_1 - \lambda_1, \ldots, T_n - \lambda_n) : Y \to Y^n$$

is not an isomorphism,

$$(T_1 - \lambda_1, \ldots, T_n - \lambda_n)(y) = ((T_1 - \lambda_1 \cdot id)(y), \ldots, (T_n - \lambda_n \cdot id)(y)).$$

**Theorem 3.5.** For any weight $v$ there is a closed subset $A_v \cap \Gamma(H_v^\infty) \subseteq A_v \subseteq M(H_v^\infty) \setminus \mathbb{D}$, such that $M_\varphi : H_v^\infty \to H_v^\infty$ has closed range if and only if $\varphi$ does not vanish on $A_v$ (or, equivalently, $\sigma_{ap}(M_\rho) = \varphi(A_v)$).

**Remarks.**

1) We will see later on (Theorem 3.10) that the upper bound for $A_v$ can be achieved.

2) Using the same proof we obtain Theorem 3.5 for $H_0^\infty$ except the inclusion $\Gamma(H_v^\infty) \subseteq A_v$. 
Proof (of Theorem 3.5). Let \( \tilde{\phi} = (\varphi_1, \ldots, \varphi_n) \in (H^\infty)^n \) be arbitrary. We define
\[
\tilde{\sigma} = \sigma_{ap}(M_{\varphi_1}, \ldots, M_{\varphi_n}),
\]
where the spectrum \( \sigma_{ap} \) is calculated in the algebra of endomorphisms of \( H^\infty \), hence it depends on the weight \( \nu \). We show that:
1. \( \tilde{\sigma} = \sigma(\tilde{\phi}) \) (the joint spectrum on \( H^\infty \));
2. \( \tilde{\sigma}(P(\tilde{\phi})) = P(\tilde{\sigma}(\tilde{\phi})) \) for any polynomial map \( P \),
\[
P((x_1, \ldots, x_n)) = (P_1(x_1, \ldots, x_n), \ldots, P_m(x_1, \ldots, x_n)),
\]
where \( P_1, \ldots, P_m \) are polynomials of \( n \) variables; i.e., \( \tilde{\sigma} \) is a subspectrum on \( H^\infty \) in the terminology of Želazko [Z, p. 251].
Assume that \( \lambda \notin \tilde{\sigma} \). Then
\[
\sum_i (\varphi_i - \lambda_i)y_i = 1 \quad \text{for some } (y_1, \ldots, y_n) \in (H^\infty)^n.
\]
If \( \lambda \notin \tilde{\sigma} \), then there is a sequence \( (f_k) \subseteq H^\infty \), \( \|f_k\|_\nu = 1 \), such that
\[
\|M_{\varphi_i - \lambda}f_k\|_\nu \rightarrow 0 \quad \text{as } k \rightarrow \infty \quad \text{for } i = 1, \ldots, n.
\]
This contradicts the observation that
\[
\|f_k\| = \left\| \sum_i M_{\varphi_i}M_{\varphi_i - \lambda}f_k \right\|_\nu \leq \sum_i \|M_{\varphi_i}\| \cdot \|M_{\varphi_i - \lambda}f_k\|_\nu \rightarrow 0 \quad \text{as } k \rightarrow \infty,
\]
and proves statement 1.
Condition 2 for a joint approximative point spectrum of operators on any Banach space has been proved in [SZ, Th. 3.4] and, independently, in [ChD, Th. 1]. Thus our condition 2 follows from the definition of \( \tilde{\sigma} \).
Finally, for every commutative Banach algebra \( X \) and every subspectrum \( \tilde{\sigma} \) on \( X \) there is a compact set \( \tilde{A} \subseteq M(X) \) such that
\[
\tilde{\sigma}(b_1, \ldots, b_n) = \{b_1(a), \ldots, b_n(a) : a \in \tilde{A}\},
\]
as shown in [Z, Th. 5.3]. Applying the above fact to \( X = H^\infty \) and \( \tilde{\sigma} \) as defined above we get \( \tilde{A} \subseteq M(H^\infty) \).
Take \( \phi \equiv x \). By Proposition 2.4, \( M_{\phi - \lambda} \) has closed range on every \( H^\infty \) for every \( \lambda \in \mathbb{D} \). Thus
\[
\sigma_{ap}(M_{\phi}) \cap \mathbb{D} = \emptyset
\]
and \( \phi(\tilde{A}_0) \cap \mathbb{D} = \emptyset \), which implies that \( \mathbb{D} \cap \tilde{A}_0 = \emptyset \).
Let \( \lambda \in \phi(\Gamma(H^\infty)) = \phi \in H^\infty \). Then \( M_{\phi - \lambda} \) does not have closed range on \( H^\infty \) by Corollary 3.3. Thus \( \lambda \in \sigma_{ap}(M_{\phi}) \) on \( H^\infty \). We have shown
\[
\phi(\Gamma(H^\infty)) \subseteq \phi(\tilde{A}_0)
\]
for any \( \phi \in H^\infty \). It suffices to take \( A_0 = \tilde{A}_0 \cup \Gamma(H^\infty) \) to complete the proof.

We now try to identify the set \( A_0 \) from Theorem 3.5 for various weights \( \nu \).

If \( B(z) \) is a Blaschke product with infinitely many zeros, we can apply Proposition 2.4 to deduce that the multiplication operator \( M_B \) is not Fredholm on \( H^\infty \). On the other hand, for many (but not all) weights it has closed range as can be seen from the next result.

It is enough to consider only weights \( \nu \) such that \(- \log \nu \) is subharmonic (since every essential weight satisfies this condition). A special role is played by moderate weights \( \nu \), which are those that satisfy \( -\Delta \log \nu \sim (1 - |z|^2)^{2\alpha} \) (see [BO], [S2]). A radial weight is equivalent to a moderate weight if and only if it is normal (see [SW], [L1], [L2] and [DL]), i.e.,
\[
\sup_{n \in \mathbb{N}} \frac{\nu(1 - 2^{-n})}{\nu(1 - 2^{-(n-1)})} < \infty \quad \text{and} \quad \sup_{n \in \mathbb{N}} \frac{\nu(1 - 2^{-n})}{\nu(1 - 2^{-(n-2)})} < \infty.
\]
Indeed, \( \Delta \log(1 - |z|^2)^{2\alpha} = -4\alpha(1 - |z|^2)^{-2}; \) moreover, for radial weights \( \nu \), \( -\Delta \log \nu > 0 \) if and only if \( 1/\nu \) is log-convex (and, then, by [BDL], \( \nu \) is essential). Thus \(-\Delta \log \nu \sim (1 - |z|^2)^{-2(\alpha)} \) means that \( \nu(z)/(1 - |z|^2)^{2\alpha} \) and \((1 - |z|^2)^{-\beta}/\nu(z)\) are essential weights for suitably chosen \( \alpha, \beta, 0 < \alpha, \beta < \infty \). By [DL], if \( \nu \) is normal then there is an equivalent weight \( \nu \) such that
\[
\frac{\nu(z)}{(1 - |z|^2)^\alpha} \quad \text{and} \quad \frac{(1 - |z|^2)^\beta}{\nu(z)}
\]
are log-convex, i.e.,
\[
-\Delta \log \nu \sim (1 - |z|^2)^{-2}.
\]
A weight \( \nu \) is normal if \( \nu \) tends to zero at the boundary not faster than some weight \( (1 - |z|^2)^\alpha \), \( 0 < \alpha < \infty \), and not slower than another weight of the same type. All the weights \( (1 - |z|^2)^\alpha \) are moderate, as also are, for instance, the weights \( |\mu z|^\alpha \) on the upper half plane transported through the Riemann map onto the unit disc.

Theorem 3.6. Let \( \nu \) be a moderate weight. The map \( M_{\phi} : H^\infty \rightarrow H^\infty \) (or \( M_{\varphi} : H^\infty \rightarrow H^\infty \) if \( H^\infty \neq \{0\} \)) has closed range if and only if \( \varphi \in H^\infty \) does not vanish on any trivial Gleason part. Consequently, \( \sigma_{ap}(M_{\phi}) = \varphi(A) \), where \( A \) is the set of all \( \nu \in M(H^\infty) \) with trivial Gleason part.

Remarks. (i) A function \( \varphi \in H^\infty \) does not vanish on any trivial Gleason part if and only if \( \varphi \) can be factorized as a product \( \varphi = u\chi \), where \( u \) is a finite product of interpolating Blaschke products and \( \chi \) is in \( H^\infty \) and \( h \) is invertible in \( H^\infty \) [Go, Th. 1].
(ii) If \( m \in M(H^\infty) \) has a non-trivial Gleason part then there is a closed range operator \( M_{\phi} : H^\infty \rightarrow H^\infty \) with \( \varphi \) vanishing on \( m \). Indeed, by [G, Th. 2.4, p. 413], there is an interpolating sequence \( (\alpha_n) \) such that \( \alpha_n \) belongs to its closure. We take as \( \varphi \) the corresponding Blaschke product.

Lemma 3.7. Let \( \varphi \in H^\infty \). Then \( \varphi \) does not vanish on any trivial Gleason part if and only if for every \( \varepsilon > 0 \) such that for any \( z \in \mathbb{D} \) there is \( \rho \in \mathbb{D} \) such that \( \varphi(z) \leq \rho \) but \( |\varphi(\rho)| \geq \varepsilon \).
We define the function $f$ as follows:

$$f(z) := \frac{1}{2} \left( \frac{z_0}{|z_0|} - \varphi_0(z) \right) g_k(z).$$

Since the first factor is of modulus $\leq 1$, we have $\|f\|_v \leq 1$. Moreover,

$$|f(z_0)|v(z_0) = \frac{1}{2} \left| g_k(z_0) \right| v(z_0) = \frac{1}{2} \left| \frac{g_k(z_0)}{1+\alpha k} \right| v\left( \frac{k}{1+\alpha k} \right)$$

$$\geq \frac{1}{2} \frac{g_k(1-2^{-n})}{2g_k(1-2^{-n-1})} \left( \frac{1}{1+\alpha k} \right) \left( \frac{1}{1+\alpha (2n+1)-1} \right) \frac{1}{\alpha (2n+1)} \left( \frac{1}{1+\alpha (2n+1)-2} \right) \geq \frac{1}{2} (4\varepsilon)^{-n}.$$

Choose a natural number $\lambda$ such that

$$t > 1 + \frac{1 - \alpha^{-1} \log \varepsilon}{\log 2}$$

and consider a euclidean disc $C$ contained in $D$, inner tangential to $\partial D$ at 1 such that the euclidean disc $D(0, 1 - 2^{-t})$ is inner tangential to $C$ at $-(1 - 2^{-t})$. There is $r \in (0, 1)$ such that

$$D(0, r) \supset C \setminus D(1, 2e),$$

where $D(x, R)$ denotes the euclidean disc of radius $R$ and center $x$.

We will show that outside $\varphi_0(D(0, r))$ which is the pseudohyperbolic disc of center $z_0$ and radius $r$ the function $f$ multiplied by the weight has modulus $\leq \varepsilon$. Observe that the map $t \mapsto |(t-s)/(1-ts)|$, $0 \leq t < s$, $0 < s < 1$, is decreasing, and we have

$$\varphi_0 \left( \frac{z_0}{|z_0|}(1-2^{-n-i}) \right) \leq \frac{z_0 - \frac{z_0}{|z_0|}(1-2^{-n-1})}{1-|z_0|(1-2^{-n-1})} \leq 1-2^{-n+1}.$$

This implies that $\varphi_0(D(0, 1 - 2^{-n-1})) \supset D(0, 1 - 2^{-n-1})$. Thus if $z \notin \varphi_0(D(0, r))$, then either

$$|z| \geq 1 - 2^{-n-1} \quad \text{or} \quad z \in \varphi_0(D(z_0/|z_0|, 2\varepsilon)).$$
Fix some $a$. By Lemma 3.8, if $|z| > 1 - 2^{-n-l}$ we obtain
\[
|f(z)v(z)| \leq |g(z)v(z)| \leq \frac{g(z)(1 - 2^{-n-l})v(1 - 2^{-n-l})}{g(1 - 2^{-n-l})v(1 - 2^{-n-l})}
\]
\[
= \left(2^{-l+1}\alpha \frac{1 - 2^{-n-l}}{1 - 2^{-n-l}}\right)^{k} \leq \left(2^{-l+1}\alpha \frac{1}{1 + 2^{-n-l} - 1}\right)^{(2m+1-l)} \leq (2^{-l+1}\alpha \epsilon) \leq \epsilon.
\]

If $z \in \varphi_{x_0}(D(x_0/|x_0|, 2|z|))$ we get
\[
|f(z)v(z)| \leq \frac{1}{2} |x_0| - \varphi_{x_0}(z) \leq \epsilon,
\]

since $\varphi_{x_0}(z) \in D(x_0/|x_0|, 2|z|)$. \hfill $\blacksquare$

**Proof of 3.6.** By [S1, Th. 2] (comp. [S2, Th. 6]) every moderate weight is essential. For every moderate weight $v$ we find $\alpha$ and $\beta$ such that both $v(z)/(1 - |z|)^\alpha$ and $(1 - |z|)^\beta/v(z)$ are moderate. Thus, by Lemma 3.1, it suffices to prove our result for power weights $v(z) = (1 - |z|)^\alpha$ for all $\alpha > 0$. By Corollary 3.4 (since the points on the Shilov boundary have trivial Gleason parts), it suffices to prove the result for $\varphi$ an inner function and, by Proposition 2.2, only for the space $H_v^\infty$.

**Sufficiency.** Since $v(z) = (1 - |z|)^\alpha$, Lemmas 1 and 4 of [DL] imply the existence of $0 < R < 1$ and $1 < C < \infty$ such that for any $f \in H_v^\infty$,
\[
|f(z) - f(p)| \leq 4C \left(\frac{|f(z)|}{Ro(z)}\right)^\alpha \varphi(z, p) \quad \text{and} \quad \left|\frac{v(z)}{v(p)}\right| \leq C
\]
for all $z, p$ with $\varphi(z, p) \leq R/2$. Assume that $\varphi$ does not vanish on any trivial Gleason part in $M(H_v^\infty)$. Take $f \in H_v^\infty$ with $\|f\|_v = 1$. There is $z \in \mathbb{D}$ such that $|f(z)v(z)| > 1/2$. Apply Lemma 3.7 for $\delta < R/(16C)$ to find $\epsilon > 0$ and $p \in \mathbb{D}$ with $\varphi(z, p) < \delta < R/2$ such that $|\varphi(p)| > \epsilon$. Now,
\[
|f(p)v(p)\varphi(p)| \geq \epsilon |f(p)v(z)| \left|\frac{v(p)}{v(z)}\right|
\]
\[
\geq \frac{\epsilon}{C} |f(z)v(z) - f(z)v(z) - f(p)v(z)| \geq \frac{\epsilon}{4C}.
\]

**Necessity.** Let $\varphi$, $\|\varphi\|_{\infty} \leq 1$, vanish on some trivial part. By Lemma 3.7 and the remark below it, there is a pseudohyperbolic disc of radius $\tau < 1$ such that $|\varphi| < \epsilon$ on that disc. By Lemma 3.9, we find a function $f \in H_v^\infty$ with $\|f\|_v = 1$ such that $|f| < \epsilon$ outside the corresponding disc. Thus $\|\varphi f\|_v < \epsilon$ and $\varphi$ cannot have closed range. \hfill $\blacksquare$

Now, we define $A_\epsilon$ for a weight tending to zero very rapidly. To do this we define the **lower norm** of the operator $T$ as
\[
L(T) := \inf\{||Tf|| : ||f|| = 1\}.
\]

**Theorem 3.10.** Let $\varphi_w(z) = (w - z)/(1 - \overline{w}z)$ and $v(z) := e^{-1/(1 - |z|)}$. The lower norm of $M_{\varphi_w}$ : $H_v^\infty \to H_v^\infty$ tends to zero as $|w| \to 1$.

It follows that $A_v = M(H_v^\infty) \setminus \mathbb{D}$ for the weight considered.

**Corollary 3.11.** The map $M_{\varphi} : H_v^\infty \to H_v^\infty$ (or $M_{\varphi} : H_v^0 \to H_v^0$) for the weight $v$ defined above is an into isomorphism if and only if it is Fredholm, i.e., there is $\epsilon > 0$ such that $|\varphi(z)| \geq \epsilon$ for $|z| > 1 - \epsilon$ or, equivalently, $\varphi$ does not vanish on $M(H_v^\infty) \setminus \mathbb{D}$. Consequently, $c_{ap}(M_{\varphi}) = \varphi(M(H_v^\infty) \setminus \mathbb{D})$.

**Remark.** It follows that $M_{\varphi} : H_v^\infty \to H_v^\infty$ has closed range if and only if $v = bh$ where $h$ is invertible in $H_v^\infty$ and $b$ is a finite Blaschke product.

**Proof** (of Corollary 3.11). It is easily seen that $e^{-1/(1 - |z|)}$ is equivalent to an essential weight because $(1 - t)e^{1/(1 - t)}$ is log-convex (comp. [BDL]). Assume that $M_{\varphi}$ has closed range. Then by Lemma 3.1 and Theorem 3.6, $\varphi$ does not vanish on trivial Gleason parts, i.e., $\varphi = bh$, where $h$ is invertible in $H_v^\infty$ and $b$ is a Blaschke product. If $b$ were an infinite Blaschke product then $b$ would have a factor $\varphi_w$ with $w$ arbitrarily close to the boundary. This leads to a contradiction because the lower norm of $M_{\varphi}$ is clearly less than or equal to the lower norm of $M_{\varphi_w}$.

**Corollary 3.12.** Let $\varphi \in H_v^\infty$. Then $M_{\varphi} : H_v^\infty \to H_v^\infty$ (resp. $M_{\varphi} : H_v^0 \to H_v^0$) has closed range for every weight $v \in \mathbb{D}$ if and only if $M_{\varphi} : H_v^\infty \to H_v^\infty$ (resp. $M_{\varphi} : H_v^0 \to H_v^0$) is Fredholm for some (equivalently, every) weight $w \in \mathbb{D}$.

**Proof of Theorem 3.10.** Since the weight $v$ is radial, all the rotations are isometries and without loss of generality we may assume that $w$ is positive. Let $w \in [1 - 1/n, 1 - 1/(n + 1)]$. We define the function $f_n : \mathbb{D} \to \mathbb{D}$ by
\[
f_n(z) := \frac{e^n}{n^\alpha(1 - z)^n}.
\]

Observe that $\|f_n\|_v = 1$. Indeed, $f_n(1 - 1/n)v(1 - 1/n) = 1$ and $f_n(t)v(t)$ increases for $t \in (0, 1 - 1/n)$ and decreases for $t \in (1 - 1/n, 1)$ (calculate the derivative of $e^n(1 - e^z)^n$). Choose $0 < \epsilon < 1/2$ and calculate
\[
f_n \left(1 - \frac{1}{(1 \pm \epsilon)n}\right) v \left(1 - \frac{1}{(1 \pm \epsilon)n}\right) = \left(1 + \frac{\epsilon}{e^n}\right)^n.
\]

We define
\[
C(\epsilon, n) := \max \left(\frac{1 + \epsilon}{e^n}, \frac{1 - \epsilon}{e^n}\right)^n.
\]

Clearly $C(\epsilon, n) \to 0$ as $n \to \infty$ for any fixed $\epsilon$. Since $v$ decreases as the modulus of the argument increases while $|f_n|$ increases as the distance of
the argument from 1 decreases,
\[ |f_n(x)|v(x) \leq C(\varepsilon, n) \]
for \( x \) on the boundary of
\[ M := D \left( 0, 1 - \frac{1}{(1 + \varepsilon)n} \right) \cap D \left( 1, \frac{1}{(1 - \varepsilon)n} \right). \]
Then using essentially the same argument we obtain the same inequality for all \( x \notin M \).

We determine the intersection of the boundaries of the two discs in the definition of \( M \). We have to solve the system
\[
\begin{align*}
|z| &= 1 - \frac{1}{(1 + \varepsilon)n}, \\
|1 - z| &= \frac{1}{(1 - \varepsilon)n}.
\end{align*}
\]
Taking \( z = x + iy \) we obtain
\[
\begin{align*}
x^2 + y^2 &= 1 - \frac{2}{(1 + \varepsilon)n} + \frac{1}{(1 + \varepsilon)^2 n^2}, \\
(1 - x)^2 + y^2 &= \frac{1}{(1 - \varepsilon)^2 n^2}.
\end{align*}
\]
Subtracting the first equation from the second one we easily obtain
\[ x = 1 - \frac{2\varepsilon}{(1 - \varepsilon)^2 n^2} - \frac{1}{(1 + \varepsilon)n}, \]
\[ y^2 = \frac{1}{(1 - \varepsilon)^2 n^2} - \frac{1}{(1 + \varepsilon)^2 n^2} - \frac{4\varepsilon^2}{(1 - \varepsilon)^2 n^4} - \frac{4\varepsilon}{(1 + \varepsilon)^2 n^2}. \]
In particular,
\[ y^2 \leq \frac{1}{(1 - \varepsilon)^2 n^2} - \frac{1}{(1 + \varepsilon)^2 n^2} = \frac{4\varepsilon}{(1 - \varepsilon)^2 n^2}. \]
This implies that \( M \) is contained in the rectangle with vertices
\[ S := \left( 1 - \frac{1}{(1 + \varepsilon)n}, \frac{-2\varepsilon}{(1 - \varepsilon)n} \right). \]

We estimate
\[
\begin{align*}
\varphi(S, 1 - \frac{1}{n}) &= \frac{(1 - \frac{1}{(1 + \varepsilon)n})^2 + \frac{4\varepsilon}{(1 + \varepsilon)^2 n^2} + \frac{2\varepsilon}{(1 + \varepsilon)^2 n^2}}{(1 + \varepsilon)^2 n^2} \leq \frac{\varepsilon^2(1 + \varepsilon)^2 + 4\varepsilon}{(1 - \varepsilon)^2 n^2} \\
&\leq \frac{5\varepsilon}{(1 - \varepsilon)^2} \leq 20\varepsilon.
\end{align*}
\]
Since the pseudohyperbolic disc is convex, \( \Delta(1 - 1/n, 5\sqrt{\varepsilon}) \) contains the above rectangle, hence
\[ \Delta \left( 1 - \frac{1}{n}, 5\sqrt{\varepsilon} \right) \supseteq M. \]
Since
\[ \varepsilon \left( 1 - \frac{1}{n}, 1 - \frac{1}{n + 1} \right) = \frac{1}{2n}, \]
we get
\[ \Delta \left( \omega, 5\sqrt{\varepsilon} + \frac{1}{2n} \right) \supseteq M. \]
Finally, if we take \( \varepsilon \) and \( n_0 \) such that \( 5\sqrt{\varepsilon} + 1/(2n_0) < \delta \), then for \( n > n_0 \) such that \( C(\varepsilon, n) < \delta \) we have
\[ |\varphi_w(x) : f_n(x)|v(x) < \delta \quad \text{for} \quad x \in \mathbb{D}. \]
Indeed, \( |\varphi_w(x)| = g(w, z) < \delta \) for \( z \in M \) while \( |f_n(x)|v(x) < \delta \) for \( z \notin M \).

**Open Problems.**
(a) Show that if \( \Delta \log v(x)(1 - |x|^2)^2 \to \infty \) as \( |x| \to 1 \) then \( A_o = M(H^\infty) \setminus \mathbb{D} \).
(b) Show that if \( \Delta \log v(x)(1 - |x|^2)^2 \to 0 \) as \( |x| \to 1 \) then \( A_o = I(H^\infty) \).

4. Hypercyclicity of the transpose multiplication operator \( M'_w \).
In [GS] Godefroy–Shapiro studied the hypercyclicity of the transpose multiplication operators on Hilbert spaces of analytic functions and obtained a sufficient and necessary condition. In this section we show that this condition is also necessary and sufficient for transpose multipliers on \( \mathcal{H}_0^0 \)'s. Our main result in this section is

**Theorem 4.1.** Let \( v \) be a radial weight on \( \mathbb{D} \) such that \( \lim_{|x| \to 1} v(x) = 0 \) and let \( \varphi \in H^\infty \). Then \( M'_w : (\mathcal{H}_0^0)' \to (\mathcal{H}_0^0)' \) has a dense, invariant hypercyclic vector manifold if and only if
\[ \varphi(\mathbb{D}) \cap \partial \mathbb{D} \neq \emptyset. \]

To prove this result we shall use a result due to Bourdon [B] saying that if \( T \in L(E) \) is hypercyclic, then there is a dense, invariant vector manifold in \( E \) consisting entirely, except for zero, of vectors which are hypercyclic for \( T \). Accordingly, it is enough to prove the following result.

**Proposition 4.2.** Let \( v \) be a radial weight on \( \mathbb{D} \) such that \( \lim_{|x| \to 1} v(x) = 0 \) and let \( \varphi \in H^\infty \). Then \( M'_w : (\mathcal{H}_0^0)' \to (\mathcal{H}_0^0)' \) is hypercyclic if and only if \( \varphi(\mathbb{D}) \cap \partial \mathbb{D} \neq \emptyset \).

**Proof.** Assume first that \( \varphi(\mathbb{D}) \) does not meet \( \partial \mathbb{D} \). Since \( \varphi(\mathbb{D}) \) is an open, connected subset of \( \mathbb{C} \), either \( \varphi(\mathbb{D}) \subset \mathbb{D} \) or \( \varphi(\mathbb{D}) \cap \partial \mathbb{D} \neq \emptyset \). In the first case, we have \( ||M'_w|| = ||\varphi||_\infty \leq 1 \). Then \( M'_w \) cannot be hypercyclic. The latter
case follows from the former, since the inverse $M_{\varphi}^{-1}$ of $M_{\varphi}$ is hypercyclic if and only if $M_{\varphi}$ is hypercyclic ([GS, p. 234]).

Conversely, assume that $\varphi(\mathbb{D}) \cap \partial \mathbb{D} \neq \emptyset$. Since $\varphi(\mathbb{D})$ is a connected subset of $\mathbb{C}$, the open sets

$$A := \{ z \in \mathbb{D} : |\varphi(z)| < 1 \} \quad \text{and} \quad B := \{ z \in \mathbb{D} : |\varphi(z)| > 1 \}$$

are both non-empty. Consider

$$U := \text{span}\{ \delta_z : z \in A \} \subset (H^0_\varphi)' \quad \text{and} \quad V := \text{span}\{ \delta_z : z \in B \} \subset (H^0_\varphi)'$$

We show that both are dense in $(H^0_\varphi)'$. To see this, let $f \in (H^0_\varphi)'' = H^\infty$ satisfy $(f, \psi) = 0$ for all $\psi \in U$. Then $f(z) = 0$ for all $z \in A$, so $f$ is identically zero. The proof for $V$ is the same.

We now write $T := M_{\varphi}' : (H^0_\varphi)' \to (H^0_\varphi)'$, and observe that for each $n$, $T^n = M_{\varphi^n}$ and $M_{\varphi^n}f(z) = \varphi(z)^n f(z)$ for $f \in H^0_\varphi$.

We are now going to apply the hypercyclicity criterion from [GS, Cor. 1.5]:

Let $E$ be a Banach space and let $T \in L(E)$. Suppose that $(T^n)$ tends to zero pointwise on a dense subset $Z$ of $E$. If there is a dense subset $Y$ of $E$ and a map $S : Y \to Y$ such that $TS = S^2$ and $(S^n)$ tends to zero pointwise on $Y$, then $T$ is hypercyclic.

(a) Since $T^n(\delta_z) = \varphi(z)^n \delta_z$, we get $||T^n(\delta_z)|| = ||\varphi(z)^n|| \delta_z||$. Consequently, for $z \in A$, $\lim_n ||T^n(\delta_z)|| = 0$. Thus $(T^n)$ converges to zero on $U$.

(b) Observe that $\{ \delta_z : z \in \mathbb{D} \}$ is linearly independent, since $H^\infty \subset H^0_\varphi$. We can define $S : V \to V$ by setting $S\delta_z = \varphi(z)^{-1} \delta_z$, $z \in \mathbb{D}$. This map is well defined as $\{ \delta_z : z \in B \}$ is linearly independent and $|\varphi(z)| > 1$ for all $z \in B$. Clearly, $S^n\delta_z = \varphi(z)^{-n} \delta_z$ for all $z \in B$, so $(S^n)$ tends to zero on $V$.

(c) Since $TS\delta_z = \delta_z$ for all $z \in B$, we conclude that $TS = S^2$. This completes the proof.

**Remark.** Under the conditions of Theorem 4.1, if $M_{\varphi}' : (H^0_\varphi)' \to (H^0_\varphi)'$ is hypercyclic, then it is chaotic, i.e., it also has a dense set of periodic points. Indeed (comp. [GS, 6.2]), by assumption $\varphi(\mathbb{D}) \cap \partial \mathbb{D} \neq \emptyset$. There is a relatively compact set $G$ with $\overline{G} \subseteq \mathbb{D}$ such that $\varphi(G)$ intersects $\partial \mathbb{D}$. This intersection contains a non-trivial arc of the circle, which contains infinitely many roots of unity. Their preimages form an infinite subset $E$ of $G$ which has a limit point in $\mathbb{D}$. The subspace $H := \text{span}\{ \delta_z : z \in E \}$ is dense in $(H^0_\varphi)'$ for the norm topology, since every element $f \in H^\infty = (H^0_\varphi)'$ which vanishes on $E$ is identically zero. Since every element in $H$ is periodic for $M_{\varphi}'$, the conclusion follows.

**Acknowledgements.** The authors are greatly indebted to R. Mortini, A. Seip and A. Soltysiak for discussions on the topic of the paper. We especially thank R. Mortini for his consent to use the argument in the proof of Lemma 3.7.

**References**


An exponential estimate for convolution powers

by

ROGER L. JONES (Chicago, IL)

Abstract. We establish an exponential estimate for the relationship between the ergodic maximal function and the maximal operator associated with convolution powers of a probability measure.

1. Introduction. Let $\tau : X \to X$ denote a measurable, invertible, ergodic point transformation from a probability space $(X, \Sigma, \mu)$ to itself. For $f \in L^1(X)$, define

$$f^*(x) = \sup_{m,n\geq 0} \frac{1}{m+n+1} \sum_{k=-m}^{n} |f(\tau^k x)|.$$

Let $\mu$ denote a probability measure on $\mathbb{Z}$ and define

$$\mu f(x) = \sum_{j=-\infty}^{\infty} \mu(j)f(\tau^j x).$$

For $n > 1$ define

$$\mu^n f(x) = \mu(\mu^{n-1} f)(x).$$

(See [2] for a discussion of these averaging operators, and conditions associated with a.e. convergence for $f \in L^p$, $p > 1$. Also see [1] where for a large class of measures, $\mu$, Bellow and Calderón establish a.e. convergence for all $f \in L^1$.)

In [2] the following condition was introduced.

**Definition 1.1.** A probability measure $\mu$ on $\mathbb{Z}$ has bounded angular ratio if $|\bar{\mu}(\gamma)| = 1$ only for $\gamma = 1$, and

$$\sup_{|\gamma|=1} \frac{|\bar{\mu}(\gamma) - 1|}{1 - |\bar{\mu}(\gamma)|} < \infty.$$