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Pointwise multiplication operators on weighted Banach spaces of analytic functions

by

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Abstract. For a wide class of weights we find the approximative point spectrum and the essential spectrum of the pointwise multiplication operator M_φ, $M_\varphi(f) = \varphi f$, on the weighted Banach spaces of analytic functions on the disc with the sup-norm. Thus we characterize when M_φ is Fredholm or is an into isomorphism. We also study cyclic phenomena for the adjoint map M'_φ.

1. Introduction. We consider the pointwise multiplication operators M_φ, $M_\varphi(f) := \varphi f$, where $\varphi : \mathbb{D} \to \mathbb{C}$ always denotes a bounded non-constant analytic function on the unit disc $\mathbb{D}$. These operators are considered on the following Banach spaces of analytic functions:

$$H_v^\infty := H_v^\infty(\mathbb{D}) := \{f \in H(\mathbb{D}) : \|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\}, \tag{1.1}$$

$$H_v^0 := H_v^0(\mathbb{D}) := \{f \in H(\mathbb{D}) : \lim_{|z|\to 1^-} v(z)|f(z)| = 0\}, \tag{1.2}$$

endowed with the norm $\|\cdot\|_v$, where v is an arbitrary *weight*, i.e., a continuous function $v : \mathbb{D} \to \mathbb{R}_+$ such that H_v^∞ contains a non-zero function. We will be mostly interested in *radial* weights (i.e., $v(z) = v(|z|)$) tending to zero at the boundary. In order to include, for instance, typical weights $v(z) = |\mathrm{Im}\, z|^\alpha$ on the half plane transferred onto the disc via a conformal isomorphism we have to consider a more general class of weights.

Our purpose is to calculate the spectrum $\sigma(M_\varphi)$, the essential spectrum $\sigma_{\mathrm{e}}(M_\varphi)$ and the approximative point spectrum $\sigma_{\mathrm{ap}}(M_\varphi)$ of M_φ, i.e., the sets

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of those $\lambda \in \mathbb{C}$ such that $M_\varphi - \lambda \cdot \mathrm{id}$ $(= M_{\varphi-\lambda})$ is not an onto isomorphism, a Fredholm operator, an into isomorphism, resp. Of course, M_φ is Fredholm or an into isomorphism if and only if $0 \notin \sigma_e(M_\varphi)$ or $0 \notin \sigma_{\mathrm{ap}}(M_\varphi)$, resp., so the corresponding calculations are the same as characterizing Fredholm operators and into isomorphisms among all the multipliers M_φ. As we will see the first two tasks are easy (Lemma 2.3, Proposition 2.4) while the third one is much more interesting (see Section 3). We prove that $\sigma_{\mathrm{ap}}(M_\varphi) = \varphi(A_v)$ where A_v is a closed subset of the maximal ideal space $M(H^\infty)$ of the algebra H^∞ and A_v depends only on the weight v (Theorem 3.5). We then identify A_v for weights like $v(z) = (1-|z|)^\alpha$, $0 < \alpha < \infty$, on the disc or $v(z) = |\mathrm{Im}\, z|^\alpha$ on the half plane. It turns out that A_v equals the set of ideals in $M(H^\infty)$ with a trivial Gleason part (Theorem 3.6). In particular, this implies that for such weights $M_\varphi : H_v^\infty \to H_v^\infty$ is an into isomorphism if and only if $\varphi = hb_1 \ldots b_n$, where h is invertible in H^∞ and $b_1, \ldots, b_n$ are interpolating Blaschke products. Let us emphasize that $\sigma_{\mathrm{ap}}(M_\varphi)$, unlike σ and σ_e, depends heavily on the weight v. In Section 4 we show that for radial weights v tending to zero at the boundary the map adjoint to $M_\varphi : H_v^0 \to H_v^0$ has hypercyclic vectors if and only if $\varphi(\mathbb{D})$ intersects the unit circle.

Multiplication operators have attracted some attention. See for instance [A], [MS] and [V]. In particular, multiplication operators were used in [S2] to get interpolation results. On the other hand, the spaces H_v^∞ of analytic functions with controlled growth were studied extensively, for instance, in [BBT], [BS], [RS], [L1], [L2], [S], [S2], [BDLT], [BDL], [DL] or [SW].

The norm topology of H_v^∞ is stronger than the compact open topology co. The latter makes the unit ball compact. It is known that the so-called *associated weight*

$$\widetilde{v}(z) := (\sup\{|f(z)| : \|f\|_v \le 1\})^{-1},$$

is better tied with the space H_v^∞ than v itself [BBT]. Of course, $H_v^\infty = H_{\widetilde{v}}^\infty$ isometrically. We say that v is an *essential weight* if $v \sim \widetilde{v}$, i.e., there is a constant C such that $\widetilde{v}(z) \le Cv(z)$ for any $z \in \mathbb{D}$. If a radial weight v vanishes on the boundary then $(H_v^0)'' = H_v^\infty$ and the unit ball of H_v^0 is co-dense in the unit ball of H_v^∞. See [BS]. On the other hand, if a radial weight v does not vanish on the boundary then $H_v^\infty = H^\infty$ and $H_v^0 = \{0\}$. The following elementary observation is of relevance in the article: *if H_v^∞ (resp. H_v^0) contains a non-zero function, then for any fixed $z \in \mathbb{D}$ there is f in H_v^∞ (resp. H_v^0) not vanishing at z. Hence the evaluation functional δ_z, $\delta_z(f) = f(z)$, is non-zero on H_v^∞ and H_v^0, respectively.* In fact, if $z_1, \ldots, z_n$ are different points in $\mathbb{D}$, then the evaluation functionals at these points are linearly independent. This can be proved using the following ideas: the product of an element of H_v^∞ (resp. H_v^0) and an element of H^∞ belongs to H_v^∞ (resp. H_v^0). On the other hand, if a function $f \in H_v^\infty$ (resp. H_v^0) has an isolated zero of order m at z_0, then the function $f(z)/(z-z_0)^m$ also belongs to H_v^∞ (resp. H_v^0) and does not vanish at z_0.

We also need several facts on H^∞ which are explained in detail in the book of Garnett [G]. We recall some definitions. A sequence (z_n) in $\mathbb{D}$ is called an *interpolating sequence* if for every bounded sequence (β_n) of complex numbers there is an $f \in H^\infty$ for which $f(z_n) = \beta_n$ for all n. A Blaschke product whose zero sequence is an interpolating sequence is called an *interpolating Blaschke product*. We denote by $A(\mathbb{D})$ the *disc algebra* and by $M(H^\infty)$ the *maximal ideal space* of H^∞. The *pseudohyperbolic distance* between two points m and n in $M(H^\infty)$ is defined by

$$\varrho(m,n) = \sup\{|\widehat{f}(n)| : f \in H^\infty,\ \widehat{f}(m) = 0,\ \|f\|_\infty \le 1\},$$

where $\widehat{f}$ denotes the Gelfand transform of f. For the sake of simplicity, we denote the Gelfand transform of f again by f further on. For $\zeta, z \in \mathbb{D}$, $\varrho(\zeta, z) = |\varphi_z(\zeta)|$, where $\varphi_z(\zeta) = (z-\zeta)/(1-\overline{z}\zeta)$. The *Gleason part* of $m \in M(H^\infty)$ is defined by $P(m) = \{n \in M(H^\infty) : \varrho(m,n) < 1\}$. The set $\{m \in M(H^\infty) : P(m) = \{m\}\}$ of trivial Gleason parts is a closed subset of $M(H^\infty)$ that properly contains the Shilov boundary $\Gamma(H^\infty)$ of H^∞ (comp. [G, Ch. X.1]). We denote by $D(z,r)$ and $\Delta(z,r)$ the euclidean disc and the pseudohyperbolic disc of center z and radius r respectively. By $\|\cdot\|_\infty$ we denote the sup-norm on H^∞. Our reference for the Corona Theorem (used e.g. in the proof of 3.7 below) is also [G]. We write $f \sim g$ for two functions f and g if there are positive constants c and C such that $cg \le f \le Cg$. Recall that an operator T is *Fredholm* if it has closed range and both the dimension of its kernel and the codimension of its image are finite. A vector $x \in X$ is called *hypercyclic* for an endomorphism $T : X \to X$ if the orbit $\{T^n x\}$ of x is dense in X.

For basic facts on bounded analytic functions and functional analysis we refer to [R2] and [R1] respectively.

2. Boundedness, Fredholm operators and the essential norm of M_φ. We start with the following easy characterization of boundedness.

PROPOSITION 2.1. *Let v be a weight on $\mathbb{D}$. The following statements are equivalent:*

(a) $M_\varphi : H_v^\infty \to H_v^\infty$ *is continuous,*
(b) $\varphi \in H^\infty$.

In this case $\|M_\varphi\| = \|\varphi\|_\infty$. *If* $H_v^0 \ne \{0\}$, *the statements above are equivalent to:*

(c) $M_\varphi : H_v^0 \to H_v^0$ *is continuous (and also here* $\|M_\varphi\| = \|\varphi\|_\infty$*).*

Another observation will be useful to avoid considering operators on H_v^0.

PROPOSITION 2.2. *If $M_\varphi : H_v^0 \to H_w^0$ is bounded and both v and w are radial weights vanishing on the boundary, then $M_\varphi'' = M_\varphi : H_v^\infty \to H_w^\infty$.*

Proof. It is well known that $(H_v^0)'' = H_v^\infty$ and $(H_w^0)'' = H_w^\infty$ (see [BS] and [RS]). Moreover, the evaluation functional δ_z, $\delta_z(f) = f(z)$ on H_v^0, acts on H_v^∞ as the evaluation functional (comp. the analysis in [BDLT]). Since $M_\varphi'(\delta_z) = \varphi(z)\delta_z$, we have, for $f \in H_v^\infty$,

$$\langle M_\varphi'' f, \delta_z\rangle = \langle f, \varphi(z)\delta_z\rangle = f(z)\varphi(z). \blacksquare$$

Note that by Proposition 2.2, for radial weights v vanishing on the boundary, $M_\varphi : H_v^\infty \to H_v^\infty$ has closed range if and only if $M_\varphi : H_v^0 \to H_v^0$ does (see [R1, Thm. 4.14]). A similar fact holds for Fredholm operators M_φ and for isomorphisms. Thus, for radial weights tending to zero at the boundary, one can consider only the case of H_v^∞.

Invertible multiplication operators and the spectrum of M_φ can be characterized very easily.

LEMMA 2.3. *Let v be a weight on $\mathbb{D}$. The following statements are equivalent for $\varphi \in H^\infty$:*

(a) *$M_\varphi : H_v^\infty \to H_v^\infty$ is invertible (or, equivalently, surjective),*

(b) *$1/\varphi \in H^\infty$ (or, equivalently, there exists $\varepsilon > 0$ such that $|\varphi(z)| \ge \varepsilon$ for all $z \in \mathbb{D}$).*

The analogous equivalence holds for H_v^0 if $H_v^0 \ne \{0\}$.

Proof. Observe that the inverse to M_φ must be of the form $M_{1/\varphi}$ and apply Proposition 2.1. ■

Since $\lambda - M_\varphi = M_{\lambda-\varphi}$ the above result shows that the spectrum of M_φ satisfies $\sigma(M_\varphi) = \overline{\varphi(\mathbb{D})} = \varphi(M(H^\infty))$. This implies that M_φ is not compact.

The class of Fredholm operators is another important class of closed range operators. Following Axler [A] we get a characterization of Fredholm multiplication operators.

PROPOSITION 2.4. *Let v be a weight on $\mathbb{D}$ and $\varphi \in H^\infty$. The operator $M_\varphi : H_v^\infty \to H_v^\infty$ is Fredholm if and only if there exists $\varepsilon > 0$ such that $|\varphi(z)| \ge \varepsilon$ for all $1 > |z| \ge 1 - \varepsilon$. Consequently, $\sigma_e(M_\varphi) = \varphi(M(H^\infty) \setminus \mathbb{D})$. The same characterization holds for H_v^0 whenever $H_v^0 \ne \{0\}$.*

Proof. Suppose first there is a sequence $(z_n) \subset \mathbb{D}$ with $|z_n| \to 1$ and $|\varphi(z_n)| \to 0$. Taking a subsequence if necessary, we may assume that (z_n) is an interpolating sequence for H^∞. Thus (see e.g. [G, Ch. VII.1]) for each N there is $\varphi_N \in H^\infty$ such that

$$\varphi_N(z_n) = \begin{cases} 0 & \text{if } n < N, \\ \varphi(z_n) & \text{if } n \ge N, \end{cases}$$

and $\|\varphi_N\|_\infty \le C \sup_{n\ge N} |\varphi(z_n)|$ for some fixed $C > 0$. Let

$$X_N = \{f \in H_v^\infty : f(z_n) = 0 \text{ for all } n \ge N\}.$$

Clearly $\delta_{z_n} \in X_N^\perp \subset (H_v^\infty)'$, $n \ge N$, so $X_N^\perp$ is infinite-dimensional: indeed, the δ_{z_n} are linearly independent in $(H_v^\infty)'$ (and in $(H_v^0)'$ whenever $H_v^0 \ne \{0\}$). Since $(\varphi - \varphi_N)(z_n) = 0$ for $n \ge N$, $\mathrm{range}(M_{\varphi-\varphi_N}) \subset X_N$. Further we conclude from $(H_v^\infty/X_N)' = X_N^\perp$ that H_v^∞/X_N is infinite-dimensional. Therefore $\mathrm{range}(M_{\varphi-\varphi_N})$ has infinite codimension in H_v^∞, so $M_{\varphi-\varphi_N}$ is not Fredholm. Now, $\|\varphi_N\|_\infty \to 0$ as $N \to \infty$. Since the set of non-Fredholm operators is closed, we see from $\|M_{\varphi-\varphi_N} - M_\varphi\| \le \|\varphi_N\|_\infty$ that M_φ is not Fredholm.

Conversely, by the assumption, φ can have only finitely many zeros $z_1, \ldots, z_n$ inside $\mathbb{D}$ with multiplicities $m_1, \ldots, m_n$ respectively. Let $\delta_z^{(k)}(f) := f^{(k)}(z)$ be the evaluation maps for derivatives. Observe that

$$\mathrm{range}(M_\varphi) \subset \bigcap_{i=1}^{n} \bigcap_{k=0}^{m_i-1} \ker \delta_{z_i}^{(k)}.$$

If $g \in \bigcap_{i=1}^n \bigcap_{k=0}^{m_i-1} \ker \delta_{z_i}^{(k)}$, then $g/\varphi \in H(\mathbb{D})$ and therefore the assumption implies that $g/\varphi \in H_v^\infty$. Hence $\mathrm{range}(M_\varphi) = \bigcap_{i=1}^n \bigcap_{k=0}^{m_i-1} \ker \delta_{z_i}^{(k)}$ and M_φ is Fredholm. ■

The *essential norm* of a continuous linear operator T is defined by $\|T\|_e = \inf\{\|T - K\| : K \text{ is compact}\}$. Since $\|T\|_e = 0$ if and only if T is compact, the estimate on $\|M_\varphi\|_e$ proved below shows that $M_\varphi : H_v^\infty \to H_v^\infty$ is non-compact when $\varphi \not\equiv 0$ (comp. the remark after Lemma 2.3). Recall that if $T \in L(E)$, the *essential spectral radius* $r_e(T)$ of T is defined by $r_e(T) = \sup\{|\lambda| : \lambda \in \sigma_e(T)\}$. It can also be calculated by $r_e(T) = \lim_n \|T^n\|_e^{1/n} \le \|T\|_e$ (cf. [C]). Let us see what this means for the continuous multiplication operator M_φ.

COROLLARY 2.5. *Let v be a weight on $\mathbb{D}$ and let $\varphi \in H^\infty$. Then for $M_\varphi : H_v^\infty \to H_v^\infty$ we have $r_e(M_\varphi) = \|M_\varphi\|_e = \|M_\varphi\| = \|\varphi\|_\infty$. The same holds for H_v^0 if $H_v^0 \ne \{0\}$.*

Proof. It suffices to observe that $r_e(M_\varphi) = \|\varphi\|_\infty$, by Proposition 2.4. ■

3. The closed range property of M_φ. The operator $M_\varphi : H_v^\infty \to H_v^\infty$ is always one-to-one. Therefore the closed range multiplication operators are precisely those operators that are bounded from below (i.e., $\|M_\varphi f\|_v \ge C\|f\|_v$ for some $C > 0$ and every $f \in H_v^\infty$) or, equivalently, are into isomorphisms.

LEMMA 3.1. *Let $\varphi \in H^\infty$. If v and w are two weights on $\mathbb{D}$ and $u := v/w$ is equivalent to an essential weight, then every closed range map M_φ :*

$H_v^\infty \to H_v^\infty$ also has closed range as a map $M_\varphi : H_w^\infty \to H_w^\infty$. An analogous result holds for H_v^0 and $H_w^0 \neq \{0\}$.

Proof. Assume that M_φ does not have closed range on H_w^∞. There are functions $f_n \in H_w^\infty$ with $\|f_n\|_w = 1$ and $\|M_\varphi f_n\|_w \le 1/n$. Clearly there are $z_n \in \mathbb{D}$ such that $|f_n(z_n)|w(z_n) > 1/2$. Let g_n be chosen from H_u^∞, $u = v/w$, such that $\|g_n\|_u \le 1$ and $|g_n(z_n)|u(z_n) \ge m > 0$ for every $n \in \mathbb{N}$. Hence $\|f_n g_n\|_v \ge m/2 > 0$. Now, clearly $\|M_\varphi f_n g_n\|_v \le 1/n$ and the proof is complete. Taking $f_n \in H_w^0$ we get the H_v^0–H_w^0 version of the result. ∎

Now, we study general closed range multiplication operators. The following simple and certainly known fact was pointed out to us by P. Wojtaszczyk. Another proof can be obtained by the machinery of uniform algebras.

PROPOSITION 3.2. *The map $M_\varphi : H^\infty \to H^\infty$ has closed range if and only if φ does not vanish at any point of the Shilov boundary $\Gamma(H^\infty)$ of H^∞.*

REMARKS. 1) Since the Shilov boundary of H^∞ can be identified with the space of maximal ideals of $L^\infty(\partial\mathbb{D})$ [G, V.1.7], a function φ does not vanish on the Shilov boundary if and only if $|\varphi|$ is essentially bounded away from zero on the unit circle.

2) All inner functions φ generate multiplication operators on H^∞ with closed range.

Proof (of Proposition 3.2). Let φ^* be the radial limit of φ. If $A \subseteq \partial\mathbb{D}$ is a subset of positive measure such that $|\varphi^*| < \varepsilon$ on A and $|\varphi^*| \le 1$ elsewhere, then we take the outer function

$$f(z) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\log|f^*(e^{it})|\,dt\right\},$$

where $|f^*| = 1$ on A and $|f^*| = \varepsilon$ on $\partial\mathbb{D}\setminus A$, to get $\|M_\varphi f\|_\infty = \|\varphi^* f^*\|_\infty \le \varepsilon$. Thus if $|\varphi^*|$ is not essentially bounded away from zero then M_φ is not bounded from below. The converse is obvious. ∎

COROLLARY 3.3. *Let v be an (essential) weight on $\mathbb{D}$ and $\varphi \in H^\infty$. If $M_\varphi : H_v^\infty \to H_v^\infty$ has closed range, then φ does not vanish at any point of the Shilov boundary of H^∞.*

REMARK. It is an open problem if the same holds for H_v^0 instead of H_v^∞.

COROLLARY 3.4. *Let φ be an outer function. Then the following assertions are equivalent:*

(a) *$M_\varphi : H_v^\infty \to H_v^\infty$ is invertible for some (or each) weight v;*
(b) *$M_\varphi : H_v^\infty \to H_v^\infty$ is a Fredholm operator for some (or each) weight v;*
(c) *$M_\varphi : H_v^\infty \to H_v^\infty$ is an isomorphism into for some (or each) weight v;*
(d) *φ does not vanish at any point of the Shilov boundary of H^∞;*
(e) *there is $\varepsilon > 0$ such that $|\varphi(z)| > \varepsilon$ for each $z \in \mathbb{D}$.*

REMARK. The equivalences (b)⇔(c)⇔(d) hold for all $\varphi \in A(\mathbb{D})$.

Proof. (e)⇒(a)⇒(b)⇒(c)⇒(d) follow from Lemma 2.3 and Corollary 3.3.

Let φ be an outer function and assume (d). Since φ does not vanish anywhere on the Shilov boundary of H^∞, $|\varphi|$ is essentially bounded away from zero on the unit circle. Since φ is outer, we have

$$\varphi(z) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\log|\varphi(e^{it})|\,dt\right\}.$$

Hence $1/\varphi$ is an outer function generated by $\log|1/\varphi(e^{it})|$ and it belongs to H^∞. Lemma 2.3 applies. ∎

The corollaries above yield that if φ is outer or belongs to the disc algebra $A(\mathbb{D})$, then M_φ has either a closed range on every H_v^∞ or on none of them. By the inner-outer factorization of bounded analytic functions, we have $\varphi = B \cdot m \cdot Q$, where B is a Blaschke product, m is a singular inner function and Q is an outer function. Since φ is assumed to be non-constant, $M_\varphi : H_v^\infty \to H_v^\infty$ has closed range (equivalently, is an into isomorphism) if and only if M_B, M_m and M_Q have closed ranges. Now, we have to look more carefully at inner functions. It turns out that they always generate closed range operators on H^∞ but they need not have closed range on other H_v^∞.

Let X be a commutative unital Banach algebra and $x = (x_1, \ldots, x_n) \in X^n$. The *joint spectrum* $\sigma(x)$ is defined as the set of all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that there do not exist $y_1, \ldots, y_n \in X$ satisfying

$$\sum (x_i - \lambda_i) y_i = 1.$$

Analogously, for a Banach space Y and operators $T_i : Y \to Y$ which mutually commute, we define the *joint approximative point spectrum* $\sigma_{\rm ap}(T)$, $T = (T_1, \ldots, T_n)$, as the set of those $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that

$$(T_1 - \lambda_1, \ldots, T_n - \lambda_n) : Y \to Y^n$$

is not an into isomorphism,

$$(T_1 - \lambda_1, \ldots, T_n - \lambda_n)(y) = ((T_1 - \lambda_1 \cdot \mathrm{id})(y), \ldots, (T_n - \lambda_n \cdot \mathrm{id})(y)).$$

THEOREM 3.5. *For any weight v there is a closed subset A_v, $\Gamma(H^\infty) \subseteq A_v \subseteq M(H^\infty) \setminus \mathbb{D}$, such that $M_\varphi : H_v^\infty \to H_v^\infty$ has closed range if and only if φ does not vanish on A_v (or, equivalently, $\sigma_{\rm ap}(M_\varphi) = \varphi(A_v)$).*

REMARKS. 1) We will see later on (Theorem 3.10) that the upper bound for A_v can be achieved.

2) Using the same proof we obtain Theorem 3.5 for H_v^0 except the inclusion $\Gamma(H^\infty) \subseteq A_v$.

Proof (of Theorem 3.5). Let $\widetilde{\varphi} = (\varphi_1, \ldots, \varphi_n) \in (H^\infty)^n$ be arbitrary. We define

$$\widetilde{\sigma}(\widetilde{\varphi}) := \sigma_{\mathrm{ap}}(M_{\varphi_1}, \ldots, M_{\varphi_n}),$$

where the spectrum σ_{ap} is calculated in the algebra of endomorphisms of H_v^∞, hence it depends on the weight v. We show that:

1. $\widetilde{\sigma}(\widetilde{\varphi}) \subseteq \sigma(\widetilde{\varphi})$ (the joint spectrum on H^∞);
2. $\widetilde{\sigma}(P(\widetilde{\varphi})) = P(\widetilde{\sigma}(\widetilde{\varphi}))$ for any polynomial map P,

$$P(x_1, \ldots, x_n) = (P_1(x_1, \ldots, x_n), \ldots, P_m(x_1, \ldots, x_n)),$$

where $P_1, \ldots, P_m$ are polynomials of n variables; i.e., $\widetilde{\sigma}$ is a *subspectrum* on H^∞ in the terminology of Żelazko [Z, p. 251].

Assume that $\lambda \notin \sigma(\widetilde{\varphi})$. Then

$$\sum (\varphi_i - \lambda_i) y_i = 1 \quad \text{for some } (y_1, \ldots, y_n) \in (H^\infty)^n.$$

If $\lambda \in \widetilde{\sigma}(\widetilde{\varphi})$, then there is a sequence $(f_k) \subseteq H_v^\infty$, $\|f_k\|_v = 1$, such that

$$\|M_{\varphi_i - \lambda_i} f_k\|_v \to 0 \quad \text{as } k \to \infty \text{ for } i = 1, \ldots, n.$$

This contradicts the observation that

$$\|f_k\| = \Big\| \sum M_{y_i} M_{\varphi_i - \lambda_i} f_k \Big\|_v \le \sum \|M_{y_i}\| \cdot \|M_{\varphi_i - \lambda_i} f_k\|_v \to 0 \quad \text{as } k \to \infty,$$

and proves statement 1.

Condition 2 for a joint approximative point spectrum of operators on any Banach space has been proved in [SZ, Th. 3.4] and, independently, in [ChD, Th. 1]. Thus our condition 2 follows from the definition of $\widetilde{\sigma}$.

Finally, for every commutative Banach algebra X and every subspectrum $\widetilde{\sigma}$ on X there is a compact set $\widetilde{A} \subseteq M(X)$ such that

$$\widetilde{\sigma}(b_1, \ldots, b_n) = \{(b_1(a), \ldots, b_n(a)) : a \in \widetilde{A}\},$$

as shown in [Z, Th. 5.3]. Applying the above fact to $X = H^\infty$ and $\widetilde{\sigma}$ as defined above we get $\widetilde{A}_v \subseteq M(H^\infty)$.

Take $\phi \equiv z$. By Proposition 2.4, $M_{\phi - \lambda}$ has closed range on every H_v^∞ for every $\lambda \in \mathbb{D}$. Thus

$$\sigma_{\mathrm{ap}}(M_\phi) \cap \mathbb{D} = \emptyset$$

and $\phi(\widetilde{A}_v) \cap \mathbb{D} = \emptyset$, which implies that $\mathbb{D} \cap \widetilde{A}_v = \emptyset$.

Let $\lambda \in \varphi(\Gamma(H^\infty))$, $\varphi \in H^\infty$. Then $M_{\varphi - \lambda}$ does not have closed range on H_v^∞ by Corollary 3.3. Thus $\lambda \in \sigma_{\mathrm{ap}}(M_\varphi)$ on H_v^∞. We have shown

$$\varphi(\Gamma(H^\infty)) \subseteq \varphi(\widetilde{A}_v)$$

for any $\varphi \in H^\infty$. It suffices to take $A_v = \widetilde{A}_v \cup \Gamma(H^\infty)$ to complete the proof. ∎

We now try to identify the set A_v from Theorem 3.5 for various weights v.

If $B(z)$ is a Blaschke product with infinitely many zeros, we can apply Proposition 2.4 to deduce that the multiplication operator M_B is not Fredholm on H_v^∞. On the other hand, for many (but not all) weights it has closed range as can be seen from the next result.

It is enough to consider only weights v such that $-\log v$ is subharmonic (since every essential weight satisfies this condition). A special role is played by *moderate* weights v, which are those that satisfy $-\Delta \log v \sim (1 - |z|^2)^{-2}$ (see [BO], [S2]). A radial weight is equivalent to a moderate weight if and only if it is *normal* (see [SW], [L1], [L2] and [DL]), i.e.,

$$\sup_{n\in\mathbb{N}} \frac{v(1-2^{-n})}{v(1-2^{-n-1})} < \infty \quad \text{and} \quad \sup_{k\in\mathbb{N}} \inf_{n\in\mathbb{N}} \frac{v(1-2^{-n})}{v(1-2^{-n-k})} > 1.$$

Indeed, $\Delta \log(1 - |z|^2)^\alpha = -4\alpha(1 - |z|^2)^{-2}$; moreover, for radial weights v, $-\Delta \log v > 0$ if and only if $1/v$ is log-convex (and, then, by [BDL], v is essential). Thus $-\Delta \log v \sim (1 - |z|^2)^{-2}$ means that $v(z)/(1 - |z|^2)^\alpha$ and $(1 - |z|^2)^\beta / v(z)$ are essential weights for suitably chosen α, β, $0 < \alpha, \beta < \infty$. By [DL], if v is normal then there is an equivalent weight w such that

$$\frac{w(z)}{(1-|z|^2)^\alpha} \quad \text{and} \quad \frac{(1-|z|^2)^\beta}{w(z)} \quad \text{for suitable } \alpha, \beta$$

are log-convex, i.e.,

$$-\Delta \log w \sim (1 - |z|^2)^{-2}.$$

A weight v is normal if v tends to zero at the boundary not faster than some weight $(1 - |z|)^\alpha$, $0 < \alpha < \infty$, and not slower than another weight of the same type. All the weights $(1 - |z|)^\alpha$ are moderate, as also are, for instance, the weights $|\mathrm{Im} z|^\alpha$ on the upper half plane transported through the Riemann map onto the unit disc.

THEOREM 3.6. *Let v be a moderate weight. The map $M_\varphi : H_v^\infty \to H_v^\infty$ (or $M_\varphi : H_v^0 \to H_v^0$ if $H_v^0 \neq \{0\}$) has closed range if and only if $\varphi \in H^\infty$ does not vanish on any trivial Gleason part. Consequently, $\sigma_{\mathrm{ap}}(M_\varphi) = \varphi(A)$, where A is the set of all $m \in M(H^\infty)$ with trivial Gleason part.*

REMARKS. (i) A function $\varphi \in H^\infty$ does not vanish on any trivial Gleason part if and only if φ can be factorized as a product $\varphi = uh$, where u is a finite product of interpolating Blaschke products and $h \in H^\infty$ is invertible in H^∞ [Go, Th. 1].

(ii) If $m \in M(H^\infty)$ has a non-trivial Gleason part then there is a closed range operator $M_\varphi : H_v^\infty \to H_v^\infty$ with φ vanishing on m. Indeed, by [G, Th. 2.4, p. 413], there is an interpolating sequence (α_n) such that m belongs to its closure. We take as φ the corresponding Blaschke product.

LEMMA 3.7. *Let $\varphi \in H^\infty$. Then φ does not vanish on any trivial Gleason part if and only if for every $r < 1$ there is $\varepsilon > 0$ such that for any $z \in \mathbb{D}$ there is $p \in \mathbb{D}$ such that $\varrho(z, p) \le r$ but $|\varphi(p)| \ge \varepsilon$.*

REMARK. In fact, formally we obtain more: if φ vanishes on some trivial Gleason part then for any $r<1$ and any $\varepsilon>0$ there is a pseudohyperbolic disc of radius r on which $|\varphi|$ is not greater than ε.

Let us note that it suffices to consider inner functions φ which are finite products of interpolating Blaschke products. The necessity part then follows from Lemma 1 of [KL] (see also [H, Lemma 4.2]) or more explicitly from [N, Lemma 1].

The following straightforward sufficiency argument was kindly provided by R. Mortini.

Proof of sufficiency in 3.7. Let φ vanish at a trivial point $x\in M(H^\infty)$. By the Corona Theorem, there is a net $(z_\alpha)\subseteq\mathbb{D}$ tending to x. Recall that $\varphi_z(\zeta)=(z-\zeta)/(1-\bar{z}\zeta)$. By [H, Th. 4.3], φ_{z_α} tends pointwise to a constant map $L:\mathbb{D}\to M(H^\infty)$, $L(z)\equiv x$. This implies that $\varphi\circ\varphi_{z_\alpha}$ is a bounded net of analytic functions tending pointwise (so uniformly on compact sets) to zero. Thus for any $r<1$ we have $\sup_{|z|<r}|\varphi(\varphi_{z_\alpha}(z))|\to 0$ as α increases. This completes the proof since $\varphi_{z_\alpha}(r\mathbb{D})$ is a pseudohyperbolic disc of radius r. ∎

LEMMA 3.8. *For every $\alpha>0$ and every $n\in\mathbb{N}$ the function $(1-r)^\alpha r^n$ is increasing for r from 0 to $n/(\alpha+n)$ and then it decreases.*

Proof. It suffices to analyze the function $\log((1-e^t)^\alpha e^{tn})$ for $t\in(-\infty,0)$. ∎

LEMMA 3.9. *Let $v(z)=(1-|z|)^\alpha$, $\alpha>0$. For every $\varepsilon>0$ we can find $r<1$ such that for each $z_0\in\mathbb{D}$ there is $f\in H_v^0$, $\|f\|_v=1$, such that for every $z\in\mathbb{D}$ with $\varrho(z,z_0)>r$ we have*

$$|f(z)|v(z)<\varepsilon.$$

Proof. Assume that $2^{n_0}-1>1/\alpha$. Without loss of generality we may consider only points z_0 with $|z_0|>1-2^{-n_0}$. Let

$$1-2^{-n}<|z_0|\le 1-2^{-n-1}\quad\text{for some } n\ge n_0.$$

We choose $k\ge 1$, $k\in\mathbb{N}$, such that $\alpha(2^n-1)<k\le\alpha(2^{n+1}-1)$, which implies

$$1-2^{-n}<\frac{k}{\alpha+k}\le 1-2^{-n-1}.$$

By Lemma 3.8, there is a positive a_k such that the function $g_k(z):=a_kz^k$ satisfies

1. $\|g_k\|_v=1$;
2. $\left|g_k\left(\frac{k}{\alpha+k}\right)\right|v\left(\frac{k}{\alpha+k}\right)=1.$

We define the function f as follows:

$$f(z):=\frac{1}{2}\left(\frac{z_0}{|z_0|}-\varphi_{z_0}(z)\right)g_k(z).$$

Since the first factor is of modulus ≤ 1, we have $\|f\|_v\le 1$. Moreover,

$$\begin{aligned}|f(z_0)|v(z_0)&=\frac{1}{2}|g_k(z_0)|v(z_0)=\frac{1}{2}\frac{|g_k(z_0)|v(z_0)}{|g_k(\frac{k}{\alpha+k})|v(\frac{k}{\alpha+k})}\\&\ge\frac{1}{2}\frac{g_k(1-2^{-n})v(1-2^{-n-1})}{g_k(1-2^{-n-1})v(1-2^{-n})}=\left(\frac{1}{2}\right)^{\alpha+1}\frac{(1-2^{-n})^k}{(1-2^{-n-1})^k}\\&\ge\left(\frac{1}{2}\right)^{\alpha+1}\left(1-\frac{1}{2^{n+1}-1}\right)^{\alpha(2^{n+1}-1)}\\&\ge\left(\frac{1}{2}\right)^{2\alpha+1}\frac{1}{(1+\frac{1}{2^{n+1}-2})^{\alpha(2^{n+1}-2)}}\ge\frac{1}{2}(4e)^{-\alpha}.\end{aligned}$$

Choose a natural number l such that

$$l>1+\frac{1-\alpha^{-1}\log\varepsilon}{\log 2}$$

and consider a euclidean disc C contained in $\mathbb{D}$, inner tangential to $\partial\mathbb{D}$ at 1 and such that the euclidean disc $D(0,1-2^{-l})$ is inner tangential to C at $-(1-2^{-l})$. There is $r\in(0,1)$ such that

$$D(0,r)\supseteq C\setminus D(1,2\varepsilon),$$

where $D(x,R)$ denotes the euclidean disc of radius R and center x.

We will show that outside $\varphi_{z_0}(D(0,r))$ which is the pseudohyperbolic disc of center z_0 and radius r the function f multiplied by the weight has modulus $\le\varepsilon$.

Observe that the map $t\mapsto|(t-s)/(1-ts)|$, $0\le t<s$, $0<s<1$, is decreasing, and we have

$$\begin{aligned}\varrho\left(z_0,\frac{z_0}{|z_0|}(1-2^{-n-l})\right)&=\left|\frac{z_0-\frac{z_0}{|z_0|}(1-2^{-n-l})}{1-|z_0|(1-2^{-n-l})}\right|\\&\le\left|\frac{1-2^{-n}-(1-2^{-n-l})}{1-(1-2^{-n})(1-2^{-n-l})}\right|\le 1-2^{-l}.\end{aligned}$$

This implies that $\varphi_{z_0}(\frac{z_0}{|z_0|}C)\supseteq D(0,1-2^{-n-l})$. Thus if $z\notin\varphi_{z_0}(D(0,r))$, then either

$$|z|\ge 1-2^{-n-l}\quad\text{or}\quad z\in\varphi_{z_0}(D(z_0/|z_0|,2\varepsilon)).$$

Fix such a z. By Lemma 3.8, if $|z| > 1-2^{-n-l}$ we obtain

$$|f(z)|v(z) \le |g_k(z)|v(z) \le \frac{g_k(1-2^{-n-l})v(1-2^{-n-l})}{g_k(1-2^{-n-1})v(1-2^{-n-1})}$$
$$= (2^{-l+1})^\alpha \frac{(1-2^{-n-l})^k}{(1-2^{-n-1})^k} \le (2^{-l+1})^\alpha \left(1+\frac{1}{2^{n+1}-1}\right)^{\alpha(2^{n+1}-1)}$$
$$\le (2^{-l+1})^\alpha e^\alpha \le \varepsilon.$$

If $z \in \varphi_{z_0}(D(z_0/|z_0|, 2\varepsilon))$ we get

$$|f(z)|v(z) \le \frac{1}{2}\left|\frac{z_0}{|z_0|} - \varphi_{z_0}(z)\right| \le \varepsilon,$$

since $\varphi_{z_0}(z) \in D(z_0/|z_0|, 2\varepsilon)$. ■

Proof of 3.6. By [S1, Th. 2] (comp. [S2, Th. 6]) every moderate weight is essential. For every moderate weight v we find α and β such that both $v(z)/(1-|z|)^\alpha$ and $(1-|z|)^\beta/v(z)$ are moderate. Thus, by Lemma 3.1, it suffices to prove our result for power weights $v(z) = (1-|z|)^\alpha$ for all α, $0 < \alpha < \infty$. By Corollary 3.4 (since the points on the Shilov boundary have trivial Gleason parts), it suffices to prove the result for φ an inner function and, by Proposition 2.2, only for the space H_v^∞.

Sufficiency. Since $v(z) = (1-|z|)^\alpha$, Lemmas 1 and 4 of [DL] imply the existence of $0 < R < 1$ and $1 < C < \infty$ such that for any $f \in H_v^\infty$,

$$|f(z) - f(p)| \le \frac{4C\|f\|_v}{Rv(z)}\varrho(z,p) \quad \text{and} \quad \left|\frac{v(z)}{v(p)}\right| \le C$$

for all z, p with $\varrho(z,p) \le R/2$. Assume that φ does not vanish on any trivial Gleason part in $M(H^\infty)$. Take $f \in H_v^\infty$ with $\|f\|_v = 1$. There is $z \in \mathbb{D}$ such that $|f(z)|v(z) > 1/2$. Apply Lemma 3.7 for $\delta < R/(16C)$ to find $\varepsilon > 0$ and $p \in \mathbb{D}$ with $\varrho(z,p) < \delta < R/2$ such that $|\varphi(p)| \ge \varepsilon$. Now,

$$|f(p)v(p)\varphi(p)| \ge \varepsilon|f(p)v(z)|\left|\frac{v(p)}{v(z)}\right|$$
$$\ge \frac{\varepsilon}{C}(||f(z)v(z)| - |f(z)v(z) - f(p)v(z)||) \ge \frac{\varepsilon}{4C}.$$

Necessity. Let φ, $\|\varphi\|_\infty \le 1$, vanish on some trivial part. By Lemma 3.7 and the remark below it, there is a pseudohyperbolic disc of radius $r < 1$ such that $|\varphi| < \varepsilon$ on that disc. By Lemma 3.9, we find a function $f \in H_v^0$ with $\|f\|_v = 1$ such that $|fv| < \varepsilon$ outside the corresponding disc. Thus $\|M_\varphi f\|_v < \varepsilon$ and M_φ cannot have closed range. ■

Now, we identify A_v for a weight tending to zero very rapidly. To do this we define the *lower norm* of the operator T as

$$L(T) := \inf\{\|Tf\| : \|f\| = 1\}.$$

THEOREM 3.10. *Let $\varphi_w(z) = (w-z)/(1-\overline{w}z)$ and $v(z) := e^{-1/(1-|z|)}$. The lower norm of $M_{\varphi_w} : H_v^\infty \to H_v^\infty$ tends to zero as $|w| \to 1$.*

It follows that $A_v = M(H^\infty) \setminus \mathbb{D}$ for the weight considered.

COROLLARY 3.11. *The map $M_\varphi : H_v^\infty \to H_v^\infty$ (or $M_\varphi : H_v^0 \to H_v^0$) for the weight v defined above is an into isomorphism if and only if it is Fredholm, i.e., there is $\varepsilon > 0$ such that $|\varphi(z)| \ge \varepsilon$ for $|z| > 1-\varepsilon$ or, equivalently, φ does not vanish on $M(H^\infty) \setminus \mathbb{D}$. Consequently, $\sigma_{\rm ap}(M_\varphi) = \varphi(M(H^\infty) \setminus \mathbb{D})$.*

REMARK. It follows that $M_\varphi : H_v^\infty \to H_v^\infty$ has closed range if and only if $\varphi = hb$ where h is invertible in H^∞ and b is a finite Blaschke product.

Proof (of Corollary 3.11). It is easily seen that $e^{-1/(1-|z|)}/(1-|z|)$ is equivalent to an essential weight because $(1-t)e^{1/(1-t)}$ is log-convex (comp. [BDL]). Assume that M_φ has closed range. Then by Lemma 3.1 and Theorem 3.6, φ does not vanish on trivial Gleason parts, i.e., $\varphi = hb$, where h is invertible in H^∞ and b is a Blaschke product. If b were an infinite Blaschke product then b would have a factor φ_w with w arbitrarily close to the boundary. This leads to a contradiction because the lower norm of M_b is clearly less than or equal to the lower norm of M_{φ_w}. ■

COROLLARY 3.12. *Let $\varphi \in H^\infty$. Then $M_\varphi : H_v^\infty \to H_v^\infty$ (resp. $M_\varphi : H_v^0 \to H_v^0$) has closed range for every weight v on $\mathbb{D}$ if and only if $M_\varphi : H_w^\infty \to H_w^\infty$ (resp. $M_\varphi : H_w^0 \to H_w^0$) is Fredholm for some (equivalently, every) weight w on $\mathbb{D}$.*

Proof of Theorem 3.10. Since the weight v is radial, all the rotations are isometries and without loss of generality we may assume that w is positive. Let $w \in [1-1/n, 1-1/(n+1)]$. We define the function $f_n : \mathbb{D} \to \mathbb{C}$ by

$$f_n(z) := \frac{e^n}{n^n(1-z)^n}.$$

Observe that $\|f_n\|_v = 1$. Indeed, $f_n(1-1/n)v(1-1/n) = 1$ and $f_n(t)v(t)$ increases for $t \in (0, 1-1/n)$ and decreases for $t \in (1-1/n, 1)$ (calculate the derivative of $\log f_n(t)v(t)$).

Choose $0 < \varepsilon < 1/2$ and calculate

$$f_n\left(1-\frac{1}{(1\pm\varepsilon)n}\right)v\left(1-\frac{1}{(1\pm\varepsilon)n}\right) = \left(\frac{1\pm\varepsilon}{e^{\pm\varepsilon}}\right)^n.$$

We define

$$C(\varepsilon,n) = \max\left(\left(\frac{1+\varepsilon}{e^\varepsilon}\right)^n, \left(\frac{1-\varepsilon}{e^{-\varepsilon}}\right)^n\right).$$

Clearly $C(\varepsilon,n) \to 0$ as $n \to \infty$ for any fixed ε. Since v decreases as the modulus of the argument increases while $|f_n|$ increases as the distance of the argument from 1 decreases,

$$|f_n(z)|v(z) \le C(\varepsilon, n)$$

for z on the boundary of

$$M := D\left(0, 1 - \frac{1}{(1+\varepsilon)n}\right) \cap D\left(1, \frac{1}{(1-\varepsilon)n}\right).$$

Then using essentially the same argument we obtain the same inequality for all $z \notin M$.

We determine the intersection of the boundaries of the two discs in the definition of M. We have to solve the system

$$\begin{cases} |z| = 1 - \dfrac{1}{(1+\varepsilon)n}, \\ |1-z| = \dfrac{1}{(1-\varepsilon)n}. \end{cases}$$

Taking $z = x + iy$ we obtain

$$\begin{cases} x^2 + y^2 = 1 - \dfrac{2}{(1+\varepsilon)n} + \dfrac{1}{(1+\varepsilon)^2 n^2}, \\ (1-x)^2 + y^2 = \dfrac{1}{(1-\varepsilon)^2 n^2}. \end{cases}$$

Subtracting the first equation from the second one we easily obtain

$$x = 1 - \frac{2\varepsilon}{(1-\varepsilon^2)^2 n^2} - \frac{1}{(1+\varepsilon)n},$$

hence

$$y^2 = \frac{1}{(1-\varepsilon)^2 n^2} - \frac{1}{(1+\varepsilon)^2 n^2} - \frac{4\varepsilon^2}{(1-\varepsilon^2)^4 n^4} - \frac{4\varepsilon}{(1+\varepsilon)^3 (1-\varepsilon)^2 n^3}.$$

In particular,

$$y^2 \le \frac{1}{(1-\varepsilon)^2 n^2} - \frac{1}{(1+\varepsilon)^2 n^2} = \frac{4\varepsilon}{(1-\varepsilon^2)^2 n^2}.$$

This implies that M is contained in the rectangle with vertices

$$S := \left(1 - \frac{1}{(1\pm\varepsilon)n}, \frac{\pm 2\sqrt{\varepsilon}}{(1-\varepsilon^2)n}\right).$$

We estimate

$$\varrho^2\left(S, 1 - \frac{1}{n}\right) = \frac{\left(\frac{1}{n} - \frac{1}{(1\pm\varepsilon)n}\right)^2 + \frac{4\varepsilon}{(1-\varepsilon^2)^2 n^2}}{\left(\frac{(1\pm\varepsilon)+1-\frac{1}{n}}{(1\pm\varepsilon)n}\right)^2 + \left(\frac{2\sqrt{\varepsilon}(1-\frac{1}{n})}{(1-\varepsilon^2)n}\right)^2} \le \frac{\varepsilon^2(1\pm\varepsilon)^2 + 4\varepsilon}{(1-\varepsilon^2)^2}$$
$$\le \frac{5\varepsilon}{(1-\varepsilon)^2} \le 20\varepsilon.$$

Since the pseudohyperbolic disc is convex, $\Delta(1 - 1/n, 5\sqrt{\varepsilon})$ contains the above rectangle, hence

$$\Delta\left(1 - \frac{1}{n}, 5\sqrt{\varepsilon}\right) \supseteq M.$$

Since

$$\varrho\left(1 - \frac{1}{n}, 1 - \frac{1}{n+1}\right) = \frac{1}{2n}$$

we get

$$\Delta\left(w, 5\sqrt{\varepsilon} + \frac{1}{2n}\right) \supseteq M.$$

Finally, if we take ε and n_0 such that $5\sqrt{\varepsilon} + 1/(2n_0) < \delta$, then for $n > n_0$ such that $C(\varepsilon, n) < \delta$ we have

$$|\varphi_w(z)| \cdot |f_n(z)|v(z) < \delta \quad \text{for } z \in \mathbb{D}.$$

Indeed, $|\varphi_w(z)| = \varrho(w, z) < \delta$ for $z \in M$ while $|f_n(z)|v(z) < \delta$ for $z \notin M$. ∎

Open problems. (a) Show that if $\Delta \log v(z)(1-|z|^2)^2 \to \infty$ as $|z| \to 1$ then $A_v = M(H^\infty) \setminus \mathbb{D}$.

(b) Show that if $\Delta \log v(z)(1-|z|^2)^2 \to 0$ as $|z| \to 1$ then $A_v = \Gamma(H^\infty)$.

4. Hypercyclicity of the transpose multiplication operator M'_φ. In [GS] Godefroy–Shapiro studied the hypercyclicity of the transpose multiplication operators on Hilbert spaces of analytic functions and obtained a sufficient and necessary condition. In this section we show that this condition is also necessary and sufficient for transpose multipliers on $(H_v^0)'$. Our main result in this section is

Theorem 4.1. *Let v be a radial weight on $\mathbb{D}$ such that $\lim_{|z|\to 1} v(z) = 0$ and let $\varphi \in H^\infty$. Then $M'_\varphi : (H_v^0)' \to (H_v^0)'$ has a dense, invariant hypercyclic vector manifold if and only if*

$$\varphi(\mathbb{D}) \cap \partial\mathbb{D} \neq \emptyset.$$

To prove this result we shall use a result due to Bourdon [B] saying that *if $T \in L(E)$ is hypercyclic, then there is a dense, invariant vector manifold in E consisting entirely, except for zero, of vectors which are hypercyclic for T*. Accordingly, it is enough to prove the following result.

Proposition 4.2. *Let v be a radial weight on $\mathbb{D}$ such that $\lim_{|z|\to 1} v(z) = 0$ and let $\varphi \in H^\infty$. Then $M'_\varphi : (H_v^0)' \to (H_v^0)'$ is hypercyclic if and only if*

$$\varphi(\mathbb{D}) \cap \partial\mathbb{D} \neq \emptyset.$$

Proof. Assume first that $\varphi(\mathbb{D})$ does not meet $\partial\mathbb{D}$. Since $\varphi(\mathbb{D})$ is an open, connected subset of $\mathbb{C}$, either $\varphi(\mathbb{D}) \subset \mathbb{D}$ or $\varphi(\mathbb{D}) \cap \overline{\mathbb{D}} = \emptyset$. In the first case, we have $\|M'_\varphi\| = \|\varphi\|_\infty \le 1$. Then M'_φ cannot be hypercyclic. The latter case follows from the former, since the inverse $M'_{1/\varphi}$ of M'_φ is hypercyclic if and only if M'_φ is hypercyclic ([GS, p. 234]).

Conversely, assume that $\varphi(\mathbb{D})\cap\partial\mathbb{D}\neq\emptyset$. Since $\varphi(\mathbb{D})$ is a connected subset of $\mathbb{C}$, the open sets

$$A:=\{z\in\mathbb{D}:|\varphi(z)|<1\}\quad\text{and}\quad B:=\{z\in\mathbb{D}:|\varphi(z)|>1\}$$

are both non-empty. Consider

$$U:=\operatorname{span}\{\delta_z:z\in A\}\subset(H_v^0)'\quad\text{and}\quad V:=\operatorname{span}\{\delta_z:z\in B\}\subset(H_v^0)'.$$

We show that both are dense in $(H_v^0)'$. To see this, let $f\in(H_v^0)''=H_v^\infty$ satisfy $\langle f,u\rangle=0$ for all $u\in U$. Then $f(z)=0$ for all $z\in A$, so f is identically zero. The proof for V is the same.

We now write $T:=M'_\varphi:(H_v^0)'\to(H_v^0)'$, and observe that for each n, $T^n=M'_{\varphi^n}$ and $M_{\varphi^n}f(z)=\varphi(z)^nf(z)$ for $f\in H_v^0$.

We are now going to apply the hypercyclicity criterion from [GS, Cor. 1.5]:

Let E be a Banach space and let $T\in L(E)$. Suppose that (T^n) tends to zero pointwise on a dense subset Z of E. If there is a dense subset Y of E and a map $S:Y\to Y$ such that $TS=\mathrm{id}_Y$ and (S^n) tends to zero pointwise on Y, then T is hypercyclic.

(a) Since $T^n(\delta_z)=\varphi(z)^n\delta_z$, we get $\|T^n(\delta_z)\|=|\varphi(z)|^n\|\delta_z\|$. Consequently, for $z\in A$, $\lim_n\|T^n(\delta_z)\|=0$. Thus (T^n) converges pointwise to zero on U.

(b) Observe that $\{\delta_z:z\in\mathbb{D}\}$ is linearly independent, since $H^\infty\subset H_v^0$. We can define $S:V\to V$ by setting $S\delta_z=\varphi(z)^{-1}\delta_z$, $z\in\mathbb{D}$. This map is well defined as $\{\delta_z:z\in B\}$ is linearly independent and $|\varphi(z)|>1$ for all $z\in B$. Clearly, $S^n\delta_z=\varphi(z)^{-n}\delta_z$ for all $z\in B$, so (S^n) tends pointwise to zero on V.

(c) Since $TS\delta_z=\delta_z$ for all $z\in B$, we conclude that $TS=\mathrm{id}_V$. This completes the proof. ■

REMARK. *Under the conditions of Theorem 4.1, if $M'_\varphi:(H_v^0)'\to(H_v^0)'$ is hypercyclic, then it is chaotic*, i.e., it also has a dense set of periodic points. Indeed (comp. [GS, 6.2]), by assumption $\varphi(\mathbb{D})\cap\partial\mathbb{D}\neq\emptyset$. There is a relatively compact set G with $\overline{G}\subseteq\mathbb{D}$ such that $\varphi(G)$ intersects $\partial\mathbb{D}$. This intersection contains a non-trivial arc of the circle, which contains infinitely many roots of unity. Their preimages form an infinite subset E of G which has a limit point in $\mathbb{D}$. The subspace $H:=\operatorname{span}\{\delta_z:z\in E\}$ is dense in $(H_v^0)'$ for the norm topology, since every element $f\in H_v^\infty=(H_v^0)''$ which vanishes on E is identically zero. Since every element in H is periodic for M'_φ, the conclusion follows.

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References

[A] S. Axler, *Multiplication operators on Bergman spaces*, J. Reine Angew. Math. 336 (1982), 26–44.

[BO] B. Berndtsson and J. Ortega-Cerdà, *On interpolation and sampling in Hilbert spaces of analytic functions*, ibid. 464 (1995), 109–128.

[BBT] K. D. Bierstedt, J. Bonet and J. Taskinen, *Associated weights and spaces of holomorphic functions*, Studia Math. 127 (1998), 70–79.

[BS] K. D. Bierstedt and W. H. Summers, *Biduals of weighted Banach spaces of analytic functions*, J. Austral. Math. Soc. Ser. A 54 (1993), 70–79.

[BDL] J. Bonet, P. Domański and M. Lindström, *Essential norm and weak compactness of composition operators on weighted Banach spaces of analytic functions*, Canad. Math. Bull. 42 (1999), 139–148.

[BDLT] J. Bonet, P. Domański, M. Lindström and J. Taskinen, *Composition operators between weighted Banach spaces of analytic functions*, J. Austral. Math. Soc. 64 (1998), 101–118.

[B] P. S. Bourdon, *Invariant manifolds of hypercyclic vectors*, Proc. Amer. Math. Soc. 118 (1993), 845–847.

[ChD] M. D. Choi and C. Davis, *The spectral mapping theorem for joint approximative point spectrum*, Bull. Amer. Math. Soc. 80 (1974), 317–321.

[C] J. B. Conway, *A Course in Functional Analysis*, Springer, Berlin, 1990.

[DL] P. Domański and M. Lindström, *Sets of interpolation and sampling for weighted Banach spaces of holomorphic functions*, preprint, 1998.

[G] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, 1981.

[GS] G. Godefroy and J. H. Shapiro, *Operators with dense, invariant, cyclic vector manifold*, J. Funct. Anal. 98 (1991), 229–269.

[Go] P. Gorkin, *Functions not vanishing on trivial Gleason parts of Douglas algebras*, Proc. Amer. Math. Soc. 104 (1988), 1086–1090.

[H] K. Hoffman, *Bounded analytic functions and Gleason parts*, Ann. of Math. 86 (1967), 74–111.

[KL] A. Kerr-Lawson, *Some lemmas on interpolating Blaschke products and a correction*, Canad. J. Math. 21 (1969), 531–534.

[L1] W. Lusky, *On the structure of $Hv_0(D)$ and $hv_0(D)$*, Math. Nachr. 159 (1992), 279–289.

[L2] —, *On weighted spaces of harmonic and holomorphic functions*, J. London Math. Soc. (2) 51 (1995), 309–320.

[MS] G. McDonald and C. Sundberg, *Toeplitz operators on the disc*, Indiana Univ. Math. J. 28 (1979), 595–611.

[N] A. Nicolau, *Finite products of interpolating Blaschke products*, J. London Math. Soc. 50 (1994), 520–531.

[RS] L. A. Rubel and A. L. Shields, *The second duals of certain spaces of analytic functions*, J. Austral. Math. Soc. 11 (1970), 276–280.

[R1] W. Rudin, *Functional Analysis*, McGraw-Hill, 1974.

[R2] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, 1974.

[S] K. Seip, *Beurling type density theorems in the unit disk*, Invent. Math. 113 (1993), 21–39.

[S1] —, *On Korenblum's density condition for the zero sequences of $A^{-\alpha}$*, J. Anal. Math. 67 (1995), 307–322.

[S2] —, *Developments from nonharmonic Fourier series*, in: Proc. ICM 1998, Vol. II, 713–722.

[Sh] J. O. Shapiro, *Composition Operators and Classical Function Theory*, Springer, 1993.

[SW] A. L. Shields and D. L. Williams, *Bounded projections and the growth of harmonic conjugates in the disk*, Michigan Math. J. 29 (1982), 3–25.

[SZ] Z. Słodkowski and W. Żelazko, *On joint spectra of commuting families of operators*, Studia Math. 50 (1974), 127–148.

[V] D. Vukotić, *Pointwise multiplication operators between Bergman spaces on simply connected domains*, Indiana Univ. Math. J., to appear.

[Z] W. Żelazko, *An axiomatic approach to joint spectra I*, Studia Math. 64 (1979), 249–261.

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An exponential estimate for convolution powers

by

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Abstract. We establish an exponential estimate for the relationship between the ergodic maximal function and the maximal operator associated with convolution powers of a probability measure.

1. Introduction. Let $\tau : X \to X$ denote a measurable, invertible, ergodic point transformation from a probability space (X, Σ, m) to itself. For $f \in L^1(X)$, define

$$f^\star(x) = \sup_{m,n\geq 0} \frac{1}{m+n+1} \sum_{k=-m}^{n} |f(\tau^k x)|.$$

Let μ denote a probability measure on $\mathbb{Z}$ and define

$$\mu f(x) = \sum_{j=-\infty}^{\infty} \mu(j) f(\tau^j x).$$

For $n > 1$ define

$$\mu^n f(x) = \mu(\mu^{n-1} f)(x).$$

(See [2] for a discussion of these averaging operators, and conditions associated with a.e. convergence for $f \in L^p$, $p > 1$. Also see [1] where for a large class of measures, μ, Bellow and Calderón establish a.e. convergence for all $f \in L^1$.)

In [2] the following condition was introduced.

Definition 1.1. A probability measure μ on $\mathbb{Z}$ has *bounded angular ratio* if $|\widehat{\mu}(\gamma)| = 1$ only for $\gamma = 1$, and

$$\sup_{|\gamma|=1} \frac{|\widehat{\mu}(\gamma) - 1|}{1 - |\widehat{\mu}(\gamma)|} < \infty.$$

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