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Generalized fractional linear transformations: convexity and compactness of the image and the pre-image; applications

by

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Abstract. The convexity and compactness in the weak operator topology of the image and pre-image of a generalized fractional linear transformation is established. As an application the exponential dichotomy of solutions to evolution problems of the parabolic type is proved.

Introduction. The present paper consists of three parts. In Section 1 we formulate and prove a number of auxiliary statements describing some basic properties of plus-operators in a Krein space.

In Section 2 we consider generalized fractional linear transformations (g.f.l.t. for brevity) F of the closed unit ball K_+ of the space $L(H_1, H_2)$ of all bounded linear operators acting from H_1 into H_2, where H_1, H_2 are Hilbert spaces. G.f.l.t. of this type are multivalued in general. We show that the image E_A^+ and the so-called pre-image E_A^- of F are convex and compact in the weak operator topology (w.o.t.) (Theorem 2.3). These results extend both the corresponding statements on compactness obtained in [5] under additional restrictions imposed on F, and the theorems on compactness and convexity of the image of F obtained in [6] for the case of single-valued g.f.l.t. (called fractional linear transformations (f.l.t.) in [6]).

In Section 3 we apply the compactness and nonemptieness of E_A^- to the study of the behavior of solutions to evolution problems in a Hilbert space H. Namely we establish (see Theorem 3.1) the exponential dichotomy of solutions for the so-called parabolic case (when the evolution operator is bounded). This result extends Theorem 2.1 of [6], where the corresponding assertion was established for the particular case of a bounded and invertible evolution operator (the so-called hyperbolic case), and Theorems 2.1 of [7] and 3.1 of [8], where only the particular case of a Pontryagin space H was considered. In a way, the present paper completes the series of articles [5]–[8].

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1. Preliminary results. First of all we formulate a statement which is not connected with the indefinite structure of a Krein space and which uses notions and notation of a Hilbert space.

LEMMA 1.1 [6]. *Let* $\mathbf{Y} = \mathbf{Y}(R,P,Q)$ *be the set of all operators* $Y \in L(H_1,H_2)$ *satisfying the inequality*

$$YRY^* + PY^* + YP^* + Q \le 0,$$

where $R \in L(H_1)$, $P \in L(H_1,H_2)$, $Q \in L(H_2)$, $R \ge 0$ *and* $Q^* = Q$. *Then* $\mathbf{Y}$ *is convex and closed in the w.o.t. of the space* $L(H_1,H_2)$.

Now let us consider the case of a Krein space. Let

$$H = H_1 \oplus H_2 \tag{1.1}$$

be a Krein space with an indefinite metric $[x,y] = (Jx,y)$, $x,y \in H$, $J = P_1 - P_2$, where P_1, P_2 ($P_1 + P_2 = I$) are the orthogonal projections onto H_1, H_2 respectively, generated by the decomposition (1.1) and (,) is a Hilbert inner product in H (see, for example, [1]). Set

$$\Re_+ = \{x \in H : [x,x] \ge 0\} \quad \text{and} \quad \Re_- = \{x \in H : [x,x] \le 0\}.$$

A subset $S \subset H$ is called *positive* or *negative* if $x \in \Re_+$ or $x \in \Re_-$ respectively for all $x \in S$. Let M_+ be the set of all maximal (with respect to inclusion) positive subspaces (i.e. closed linear subsets) of H, and M_- the set of all maximal negative subspaces of H. Denote by $S^\perp$ the orthogonal complement of a set S in H: $S^\perp = \{x \in H : [x,y] = 0 \text{ for all } y \in S\}$.

LEMMA 1.2 [4]. $L \in M_+$ *if and only if* $L^\perp \in M_-$.

Now we proceed to plus-operators in a Krein space $H = H_1 \oplus H_2$. A linear bounded operator A is called a *plus-operator* if $A\Re_+ \subset \Re_+$. A is called a *minus-operator* if $A\Re_- \subset \Re_-$. We denote by A^* the adjoint operator to A.

LEMMA 1.3. *The following two conditions are equivalent*:

(a) *there exists* $L_+ \in M_+$ *such that* $A^*L_+ \subset \Re_+$;
(b) *there exists* $L_- \in M_-$ *such that* $AL_- \subset \Re_-$.

Proof. (a)⇒(b). Let $A^*L_+ \subset \Re_+$ for some $L_+ \in M_+$, and let $L_+^1 \in M_+$ be a subspace such that $A^*L_+ \subset L_+^1$. Taking $L_- = L_+^{1\perp}$ we have $(Az,y) = (z,A^*y) = 0$ for all $z \in L_-$ and $y \in L_+$. Hence $AL_- \subset \Re_-$ by Lemma 1.2. The proof of the implication (b)⇒(a) is similar.

A plus-operator A with respect to the decomposition (1.1) has the following block-matrix representation:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{1.2}$$

where $A_{ij} \in L(H_j,H_i)$, $i,j = 1,2$, with $A_{ij} = P_iAP_j$.

LEMMA 1.4. *For a plus-operator* A, $A^*H_+ \subset \Re_+$ *if and only if*

$$A_{11}A_{11}^* \ge A_{12}A_{12}^*. \tag{1.3}$$

Proof. Straightforward calculation.

2. Generalized fractional linear transformations. Let A be a plus-operator with block-matrix (1.2). We denote by K_+ the closed unit ball of the space $L(H_1,H_2)$, and by K_- the closed unit ball of $L(H_2,H_1)$. Let $F = F_A$ be a g.f.l.t. of the ball K_+ defined by the block-matrix of the operator A as follows:

$$F(k_+) = \{k_+^1 : k_+^1 \in K_+,\ A_{21} + A_{22}k_+ = k_+^1(A_{11} + A_{12}k_+)\}. \tag{2.1}$$

In general the mapping F is multivalued. Since A is a plus-operator, it follows that if $A_{11} + A_{12}k_+ = 0$ for some $k_+ \in K_+$, then $A_{21} + A_{22}k_+ = 0$. So in this case $F(k_+) = K_+$. If $AL_+ \in M_+$ for all $L_+ \in M_+$, then F becomes single-valued (see, for example, [3]). It is worth recalling that in the case when A is a *bistrict* plus-operator (that is, both A and A^* are *strict* plus-operators: $\inf_{[x,x]=1}[Ax,Ax]$ $(= \mu(A)) > 0$ and $\inf_{[x,x]=1}[A^*x,A^*x]$ $(= \mu(A^*)) > 0$) the formula (2.1) turns into

$$F(k_+) = (A_{21} + A_{22}k_+)(A_{11} + A_{12}k_+)^{-1}$$

(see [11]). Set

$$E_A^+ = \{k_+^1 \in K_+ : k_+^1 \in F(k_+) \text{ for some } k_+ \in K_+\},$$
$$E_A^- = \{k_- \in K_- : A(P_2 + k_-)H_- \subset \Re_-\}.$$

Note that $E_A^+ = \operatorname{Im}F$ $(= F(K_+))$. In the particular case of an invertible bistrict plus-operator A the operator $T = A^{-1}$ is a bistrict minus-operator and it generates the f.l.t. $G = G_T$ of the ball K_-, so in this case $E_A^- = G_T(K_-)$.

THEOREM 2.1. E_A^- *is convex and compact in the w.o.t. of* $L(H_2,H_1)$.

Proof. If $E_A^- = \emptyset$, then the assertion is true.

Suppose $E_A^- \ne \emptyset$. Let $k_- \in E_A^-$. From $A(P_2 + k_-)H_2 \subset \Re_-$ we deduce that there exists k_-^1 such that $A(P_2 + k_-^1)H_2 \subset (P_2 + k_-^1)H_2$. Hence $A_{11}k_- + A_{12} = k_-^1(A_{21}k_- + A_{22})$ and therefore

$$k_-^*(A_{11}^*A_{11} - A_{21}^*A_{21})k_- + k_-^*(A_{11}^*A_{12} - A_{21}^*A_{22}) + (A_{12}^*A_{11} - A_{22}^*A_{21})k_- + (A_{12}^*A_{12} - A_{22}^*A_{22}) \le 0.$$

As $AH_1 \subset \Re_+$ we have $\|A_{11}x_1\| \ge \|A_{21}x_1\|$ for all $x_1 \in H_1$. Hence $A_{11}^*A_{11} \ge A_{21}^*A_{21}$. Now from Lemma 1.1 it follows that the set $(E_A^-)^* = \{k_-^* : k_- \in E_A^-\}$ is convex and compact in the w.o.t. of $L(H_1,H_2)$. As the mapping $* : L(H_1,H_2) \to L(H_2,H_1)$ is an isomorphism with respect to the w.o.t. and $*^2 = \mathrm{Id}$, the set E_A^- is convex and compact in the w.o.t. of $L(H_2,H_1)$.

REMARK 2.2. The set E_A^- can be empty. A simple example is the following: $H = H_1 \oplus H_2$ with both H_1 and H_2 one-dimensional and

$$A = \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix},$$

where $|\alpha| < |\beta|$. Evidently, the operator A is noninvertible and nonstrict. (Note that if $|\alpha| = |\beta| \neq 0$, then $E_A^- = \{-\alpha/\beta\} \neq \emptyset$). The next example presents a more complicated situation when A is an invertible strict plus-operator, and as above $E_A^- = \emptyset$:

$$H_1 = \mathrm{CLin}\{e_j\}_{j=1}^{\infty}, \quad H_2 = \mathrm{Clin}\{e_j\}_{j=-\infty}^{0}, \quad H = H_1 \oplus H_2,$$

A is a bounded linear operator on H defined by the formula

$$Ae_j = e_{j+1}, \quad j \in \mathbb{Z}.$$

In view of Lemmas 1.3 and 1.4 to obtain the nonemptieness of E_A^- it is sufficient to impose on A the restriction (1.3). It is interesting that the same condition (1.3) enables us to establish the convexity and compactness of E_A^+ in the w.o.t.

THEOREM 2.3. *Suppose a plus-operator A satisfies the condition* (1.3). *Then both E_A^+ and E_A^- are nonempty, convex and compact in the w.o.t. of $L(H_1, H_2)$ and $L(H_2, H_1)$ respectively.*

Proof. Lemmas 1.3 and 1.4 imply $E_A^- \neq \emptyset$. The convexity and compactness of E_A^- in the w.o.t. were established in Theorem 2.1. Let us pass to E_A^+. We have $Y \in E_A^+$ if and only if $A_{21} + A_{22}k_+ = Y(A_{11} + A_{12}k_+)$ for some $k_+ \in K_+$. Hence $YA_{11} - A_{21} = (A_{22} - YA_{12})k_+$. Since $\|k_+\| \leq 1$, we obtain

$$(A_{22} - YA_{12})(A_{22} - YA_{12})^* \geq (YA_{11} - A_{21})(YA_{11} - A_{21})^*$$

or

$$Y(A_{11}A_{11}^* - A_{12}A_{12}^*)Y^* + Y(A_{12}A_{22}^* - A_{11}A_{21}^*) + (A_{22}A_{12}^* - A_{21}A_{11}^*)Y^* + (A_{21}A_{21}^* - A_{22}A_{22}^*) \leq 0.$$

Now the assertion on E_A^+ follows from Lemma 1.1.

COROLLARY 2.4. *Let A be a bistrict plus-operator. Then both E_A^+ and E_A^- are nonempty, convex and compact in the w.o.t. of $L(H_1, H_2)$ and $L(H_2, H_1)$ respectively.*

Proof. Since A is a bistrict plus-operator we have $A^*H_1 \subset \Re_+$ (see [11]). From Lemma 1.4 it follows that the condition (1.3) holds. Now the assertion follows from Theorem 2.3.

3. Applications to evolution problems. Consider a differential equation

$$\frac{dx}{dt} = A(t)x \tag{3.1}$$

in a Hilbert space H with an inner product $(\ ,\ )$. Let the operators $A(t)$ be selfadjoint and have a common dense domain $D \subset H$ for $t \in \mathbb{R}^+ = [0, \infty)$. The Cauchy problem (3.1) is assumed to be *uniformly well posed*: there exists a bounded linear operator $U(t)$ (an *evolution operator*) such that for every solution $x(t)$ to (3.1) with $x(0) = x_0 \in D$ we have $x(t) = U(t)x_0$. If y_0 does not belong to D, then $y(t) = U(t)y_0$ is called a *generalized solution*.

The results of Section 2 enable us to generalize Theorem 2.1 of [6], where the evolution operator $U(t)$ was assumed to be invertible. In this section we will establish an analogous statement without this assumption.

Let $L_{2,w}(\mathbb{R}^+, H)$ be the set of functions $x : \mathbb{R}^+ \to H$ which are Bochner square integrable with respect to a positive locally integrable weight $w = w(t)$. Denote by N the set of generalized solutions belonging to $L_{2,w}(\mathbb{R}^+, H)$. Set $N_0 = \{h \in H : h = y(0),\ y \in N\}$. Let $[x, y]_t$ be the indefinite metric on H (depending on t) given by

$$[x, y]_t = (J(t)x, y), \quad x, y \in H,$$

where $J(t) = P_1(t) - P_2(t)$, $P_1(t) = \int_{+0}^{+\infty} dE_\lambda(t)$, $P_2(t) = \int_{-\infty}^{0} dE_\lambda(t)$, and $E_\lambda(t)$ is the spectral function of $\{A(t)\}$. For every $t \in \mathbb{R}^+$ we denote by C_t^- (so-called *bicone*) the set

$$C_t^- = \{y_0 \in H : [U(t)y_0, U(t)y_0] \leq 0\}.$$

A bicone C_t^- is said to be of *rank* $d \leq \infty$ if it contains a subspace $L \subset H$ with $\dim L = d$, and does not contain subspaces of greater dimensions (see [9], [10]).

Suppose that $J(t)$ is strongly differentiable. Consider the derivative of the solution $x(t)$ to (3.1) along the trajectory:

$$[x(t), x(t)]_t' = 2\mathrm{Re}[A(t)x(t), x(t)]_t + (J'(t)x(t), x(t)).$$

Hereafter we will assume that $[x(t), x(t)]_t'$ is qualified positive (see the condition (3.2) below).

THEOREM 3.1. *Suppose that the Cauchy problem* (3.1) *is uniformly well posed, and the metric $[\ ,\ ]_t$ satisfies the following conditions*:

(a) *$J(t)$ is strongly differentiable, the limit $\lim_{t\to\infty} \dim P_2(t)H = d_-$ exists and*

$$\inf_{\|z\|=1} \{\mathrm{Re}[A(t)z, z]_t + \tfrac{1}{2}(J'(t)z, z)\} \geq w(t), \quad t \in \mathbb{R}^+; \tag{3.2}$$

(b) *for every* $t \in \mathbb{R}^+$,

$$U_{11}(t)U_{11}^*(t) \geq U_{12}(t)U_{12}^*(t), \tag{3.3}$$

where $U_{ij}(t) = P_i(t)U(t)P_j(t)$, $i, j = 1, 2$.

Then the generalized solution $y(t) = U(t)y_0$, $y_0 \in H$, *has the following properties*:

1) $N_0 \supset C_\infty^- = \bigcap_{t\in\mathbb{R}^+} C_t^-$, *where* C_∞^- *is a bicone of rank* d_-;

2) *for any* $y(t) \in N$,

$$\int_t^\infty w(s)\|y(s)\|^2\,ds \leq I(y)\exp\Big(-2\int_0^t w(s)\,ds\Big),$$

where $I(y) = \int_0^\infty w(s)\|y(s)\|^2\,ds$;

3) *for any* $y_0 \in N \setminus C_\infty^-$,

$$\|y(t)\| \geq [y_0, y]_0 \exp\Big(2\int_0^t w(s)\,ds\Big), \quad t \in \mathbb{R}^+. \tag{3.4}$$

COROLLARY 3.2. *Let the conditions of Theorem* 3.1 *be satisfied, and*

$$\int_0^\infty w(t)\,dt = \infty.$$

Then all the statements 1)–3) *are true, and moreover*, N_0 *is a closed subspace of* H *with* $\dim N_0 = d_-$.

Proof. Denote by $U(t,s)$ the operator assigning to each $y \in D$ the value $y(t,s)$ of the solution to the equation (3.1) which satisfies the initial condition $y(s,s) = y_0$. For brevity we denote $U(t,0)$ by $U(t)$. From (3.2) we get

$$[U(t,\tau)y_0, U(t,\tau)y_0]_t - [y_0,y_0]_\tau \geq 2\int_\tau^t w(s)\|U(s,\tau)y_0\|^2\,ds \tag{3.5}$$

for any $\tau < t$ ($\in \mathbb{R}^+$) and $y_0 \in D$. By continuity of $U(t,\tau)$ the inequality (3.5) holds for any $y \in H$. Hence we obtain (keeping in mind $\|U(t)y_0\|^2 \geq [U(t)y_0, U(t)y_0]_t$ and setting $y(t) = U(t)y_0$)

$$\|y(t)\|^2 \geq 2\int_0^t w(s)\|y(s)\|^2\,ds + [y_0,y_0]_0.$$

Taking $y_0 \in H \setminus C_0^-$ and arguing as in the Bellman–Gronwall lemma (see [2]) we get (3.4) (see [6]).

Now let us turn to the bicones C_t^-. In view of (3.3) and Lemma 1.3 the bicone C_t^- is of rank $d = \dim H_2(t) = \dim P_2(t)H$ and is closely related to the set $E_{U(t)}^-$. Namely, $k_-^t \in E_{U(t)}^-$ if and only if the maximal negative subspace $L_-^t = H_2(t) + k_-^t H_2(t) \subset C_t^-$. By Theorem 2.1 we see that $E_{U(t)}^-$ is nonempty, convex and compact in the w.o.t. Now using the property of $\dim P_2(t)H$ (see condition (a)) it is easy to check (by letting $t \to \infty$) that C_∞^- is a bicone of rank d_-. The remaining part of the proof is the same as the corresponding part of the proof of Theorem 2.1 of [6].

The proof of Corollary 3.2 is the same as that of Corollary 2.1 of [6].

Our last remark is that Theorem 3.1 also generalizes Theorem 2.1 of [7] and Theorem 3.1 of [8].

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