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The density property for JB^*-triples

by

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Abstract. We obtain conditions on a JB^*-algebra X so that the canonical embedding of X into its associated quasi-invertible manifold has dense range. We prove that if a JB^*-triple has this density property then the quasi-invertible manifold is homogeneous for biholomorphic mappings. Explicit formulae for the biholomorphic mappings are also given.

1. Introduction. There exist, up to biholomorphic equivalence, precisely two one-dimensional simply connected symmetric complex manifolds, and these can be realised as the open unit disc

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}$$

and the Riemann sphere $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. We have the standard inclusions

$$\Delta \to \mathbb{C} \to \overline{\mathbb{C}} \tag{1}$$

and $\mathbb{C}$ is dense in $\overline{\mathbb{C}}$. Moreover, each biholomorphic map of Δ extends to a biholomorphic map of $\overline{\mathbb{C}}$. In the finite-dimensional situation each n-dimensional bounded symmetric domain $\mathcal{D}_b$ determines, and is determined by, its unique compact dual $\mathcal{D}_c$. The domain $\mathcal{D}_b$ can be realised as the open unit ball of a norm on $\mathbb{C}^n$, and $\mathcal{D}_c$, which is an n-dimensional compact symmetric manifold, can be realised as the quasi-invertible manifold associated with $\mathcal{D}_b$ (see [8]) and we have the canonical inclusions

$$\mathcal{D}_b \to \mathbb{C}^n \to \mathcal{D}_c \tag{2}$$

with $\mathbb{C}^n$ dense in $\mathcal{D}_c$. Once again, biholomorphic maps of $\mathcal{D}_b$ extend to biholomorphic maps of $\mathcal{D}_c$.

In the Banach space setting, Kaup [6] has characterised bounded symmetric domains as those complex manifolds which can be realised as the open unit ball of a JB^*-triple. With every JB^*-triple X one can associate a quasi-invertible complex manifold M_X modelled on X and we again have

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the canonical inclusions

$$B_X \to X \to M_X$$

where B_X is the open unit ball of X.

DEFINITION 1.1. The JB^*-triple X has the *density property* if X is dense in M_X.

In this paper we investigate the density property for JB^*-triples. By using J^*-triples, Kaup [5] has shown that each bounded symmetric domain has associated with it a unique simply connected symmetric Banach manifold of compact type. The relationship between this compact type manifold and the above quasi-invertible manifold is still under investigation [1] and the density condition may well feature in the final solution.

We now recall some background information on JB^*-triples.

DEFINITION 1.2. A *JB^*-triple* is a pair $(X, \{\cdot,\cdot,\cdot\})$ consisting of a Banach space X and a continuous real trilinear mapping $\{\cdot,\cdot,\cdot\} : X^3 \to X$ which is complex linear and symmetric in the outer variables, complex antilinear in the middle variable and which satisfies the following axioms:

(i) $\delta(x)\{u,v,w\} = \{\delta(x)u,v,w\} - \{u,\delta(x)v,w\} + \{u,v,\delta(x)w\}$ for $x,u,v,w \in X$,

(ii) $\delta(x)$ is Hermitian for all $x \in X$,

(iii) $\sigma(\delta(x)) \geq 0$ for all $x \in X$,

(iv) $\|\delta(x)\| = \|x\|^2$ for all $x \in X$,

where $\delta(x) \in \mathcal{L}(X)$ is defined by $\delta(x)(y) = \{x,x,y\}$ and σ denotes the operator spectrum. We recall that a linear operator $T \in \mathcal{L}(X)$ is *Hermitian* if $e^{i\lambda T}$ is an isometry for all $\lambda \in \mathbb{R}$. Condition (i) in this definition is known as the *Jordan triple identity*. The above "differential" version of the Jordan triple identity may be linearised to give the equivalent condition

(i′) $\{a,b,\{u,v,w\}\} = \{\{a,b,u\},v,w\} - \{u,\{b,a,v\},w\} + \{u,v,\{a,b,w\}\}$.

Condition (iv) is equivalent to $\|\{x,x,x\}\| = \|x\|^3$. An element $e \in X$ satisfying $\{e,e,e\} = e$ is called a *tripotent*. If $e \,\Box\, e$ is the identity operator on X then e is said to be a *unitary tripotent*.

EXAMPLE 1.3. (i) If H and K are complex Hilbert spaces then $\mathcal{L}(H,K)$ endowed with the triple product

$$\{A,B,C\} := \tfrac{1}{2}(AB^*C + CB^*A) \tag{3}$$

is a JB^*-triple.

(ii) A subtriple of a JB^*-triple X is a Banach subspace which is closed with respect to the triple product, and every closed subtriple is also a JB^*-triple. Using this observation and (3) we see that $\mathcal{K}(H)$, the compact operators on a Hilbert space, is a JB^*-triple and every C^*-algebra is a JB^*-triple. A subtriple of $\mathcal{L}(H,K)$ is called a *J^*-algebra* or a *special JB^*-triple*.

(iii) If Ω is a compact Hausdorff space, then $\mathcal{C}(\Omega)$, the continuous $\mathbb{C}$-valued functions on Ω endowed with the supremum norm, is a JB^*-triple with triple product $\{f,g,h\} = f\bar{g}h$.

Associated with the triple product $\{\cdot,\cdot,\cdot\}$ we can define a number of natural real linear mappings:

$x \,\Box\, y \in \mathcal{L}(X)$ where $x \,\Box\, y(z) = \{x,y,z\}$;

$Q_x \in \mathcal{L}^{\mathbb{R}}(X)$ where $Q_x(y) = \{x,y,x\}$.

Note that $\delta(x) = x \,\Box\, x$ and that $x \,\Box\, y$ is complex linear while Q_x is complex antilinear. The operators $\Box$, δ and Q are continuous (indeed real-analytic) functions of their arguments. The *Bergman operator*, $B(x,y)$, which plays an important role in JB^*-triples, is defined using the above operators.

DEFINITION 1.4. If $(X, \{\cdot,\cdot,\cdot\})$ is a JB^*-triple, we define $B(x,y)$ for $x,y \in X$ as

$$B(x,y) = I_X - 2x \,\Box\, y + Q_x Q_y$$

where I_X is the identity mapping on X.

Clearly, $B(x,y) \in \mathcal{L}(X)$ and $B : X \times X \to \mathcal{L}(X)$ is a real-analytic function of x and y.

EXAMPLE 1.5. (i) If $X = \mathcal{L}(H,K)$ then $B(x,y)(z) = (I_K - xy^*)z(I_H - y^*x)$ for $x,y,z \in \mathcal{L}(H,K)$.

(ii) If $X = \mathcal{C}(K)$ then $B(f,g)(h) = (1 - f\bar{g})^2 h$ for $f,g,h \in \mathcal{C}(K)$.

We say that the pair $(x,y) \in X \times X$ is *quasi-invertible* if $B(x,y)$ is an invertible operator in $\mathcal{L}(X)$. If (x,y) is quasi-invertible, let

$$x^y = B(x,y)^{-1}(x - Q_x(y))$$

and call x^y the *quasi-inverse* of x with respect to y. This concept of quasi-invertibility, as we shall see in Section 4, is related to the classical notion of quasi-inverse in a Jordan algebra.

On $X \times X$ we define the equivalence relation $\sim$ by $(x,y) \sim (x_1,y_1)$ if, and only if, $(x, y - y_1)$ is quasi-invertible and $x_1 = x^{y-y_1}$. We denote the equivalence class containing (x,y) by $(x : y)$. For each y in X, we let $U_y = \{(x : y) : x \in X\}$ and define $\phi_y : U_y \to X$ by $\phi_y(x : y) = x$. Let $M_X = X \times X/\!\sim$ be the set of all equivalence classes.

PROPOSITION 1.6 [6, 8, 9]. *If X is a JB^*-triple then*

(i) $\phi_y(U_y \cap U_{y_1}) = \{x \in X : (x, y - y_1)$ *is quasi-invertible*$\}$ *is open in* X;

(ii) $\phi_{y_1} \circ \phi_y^{-1} : \phi_y(U_y \cap U_{y_1}) \to \phi_{y_1}(U_y \cap U_{y_1})$ *is the holomorphic mapping*

$$\phi_{y_1} \circ \phi_y^{-1}(x) = x^{y-y_1}$$

and its derivative $d(\phi_{y_1}\circ\phi_y^{-1})(x)=B(x,y-y_1)^{-1}$, *for all* $y,y_1\in X$ *and all* $x\in\phi_y(U_y\cap U_{y_1})$;

(iii) *the collection of charts* $(U_y,\phi_y)_{y\in X}$ *gives the structure of a connected complex Banach manifold to* M_X;

(iv) *the canonical mapping* $\pi: X\times X\to M_X$ *defined by* $\pi(x,y)=(x:y)$ *is holomorphic in* x, *antiholomorphic in* y *and jointly real-analytic in* x *and* y.

We refer to [8] for the algebraic properties of the quasi-inverse, some of which we cite for later convenience. The expression (JPnn) refers to an identity in [8].

(JPA1) If (x,y) is quasi-invertible then $(x,y+z)$ is quasi-invertible if, and only if, (x^y,z) is quasi-invertible and then $x^{y+z}=(x^y)^z$.

(JPA2) If (x,y) is quasi-invertible then $(x+z,y)$ is quasi-invertible if, and only if, (z,y^x) is quasi-invertible and then $(x+z)^y=x^y+B(x,y)^{-1}z^{(y^x)}$.

(JPS) $(B(u,v)x,y)$ is quasi-invertible if, and only if, $(x,B(v,u)y)$ is quasi-invertible and then $(B(u,v)x)^y=B(u,v)x^{B(v,u)y}$.

(JPT) For $t\in\mathbb{C}$, (tx,y) is quasi-invertible if, and only if, $(x,\bar{t}y)$ is quasi-invertible and then $(tx)^y=t(x^{\bar{t}y})$. In particular, $(-x)^y=-(x^{-y})$.

If X is a JB^*-triple and $x\in X$ then X_x denotes the JB^*-subtriple of X generated by x. There is a unique locally compact subset K of $[0,\infty)$ such that $K\cup\{0\}$ is compact, $\mathcal{C}_0(K)$ is isometrically isomorphic to X_x and $x\mapsto \mathrm{id}_K$ (see [6]). The set K is called the *triple spectrum* of the element x. The embedding $\psi_x: X_x\to X$ induces a holomorphic embedding [1]

$$\psi_x: M_{X_x}\to M_X,\qquad (y:z)\mapsto(\psi_x(y):\psi_x(z)).$$

The canonical embedding of X in M_X is given by $\phi_0^{-1}: x\in X\to(x:0)\in M_X$ and we identify X and $\phi_0^{-1}(X)$.

The open unit ball B_X of the JB^*-triple X is the bounded symmetric domain associated with X and we now have a situation similar to (1) and (2) above, namely, the embeddings

$$B_X\to X\to M_X.$$

Recalling Definition 1.1, we say that a JB^*-triple X has the *density property* if $U_0=\phi_0^{-1}(X)$ is dense in M_X.

2. The density property.

We first obtain a simple characterization of the density property. This shows that a JB^*-triple has the density property if, and only if, there exists sufficiently many quasi-invertible pairs, and proves

to be a useful practical tool in identifying spaces with or without the density property.

PROPOSITION 2.1. *If* X *is a* JB^**-triple then the following are equivalent*:

(i) X *has the density property*;

(ii) *for all* x,y *in* X *and* $\varepsilon>0$, *there exists* $z\in X$, $\|z\|<\varepsilon$, *such that* $(x+z,y)$ *is quasi-invertible*;

(iii) *for all* x,y *in* X *and* $\varepsilon>0$, *there exists* $z\in X$, $\|z\|<\varepsilon$, *such that* $(x,y+z)$ *is quasi-invertible*;

(iv) *there exists a dense subset* Z *of* $X\times X$ *consisting of quasi-invertible pairs.*

Proof. Suppose (i) holds. Let $(x:y)\in M_X$ and $\varepsilon>0$. Then (i) implies that there exists $w\in X$ such that $(w,0)\sim(x',y)$ and $\|x-x'\|<\varepsilon$. Let $z=x'-x$. Then $\|z\|<\varepsilon$ and $(w,0)\sim(x+z,y)$. Hence $(x+z,y)$ is quasi-invertible, so (i) implies (ii).

By [8, JP35], (x,y) is quasi-invertible if, and only if, (y,x) is quasi-invertible. Hence (ii)⇔(iii).

Clearly, (iii)⇒(iv). Suppose (iv) holds. Fix $(x:y)\in M_X$. By (iv) there is a sequence of quasi-invertible pairs (x_n,y_n) with $(x_n,y_n)\to(x,y)$ in $X\times X$. By Proposition 1.6(iv) above, $(x_n^{y_n}:0)=(x_n:y_n)\to(x:y)$, so X is dense in M_X and (iv) implies (i). ∎

3. The density property for $\mathcal{C}(K)$ spaces.

We discuss the density property for the commutative JB^*-triple $\mathcal{C}(K)$, where K is a compact Hausdorff space. By Example 1.5(ii), $B(f,g)$ is invertible if, and only if, $f(x)\overline{g(x)}\neq 1$ for all $x\in K$, where $f,g\in\mathcal{C}(K)$. By Proposition 2.1, $\mathcal{C}(K)$ has the density property if, and only if, for all $f,g\in\mathcal{C}(K)$ and all $\varepsilon>0$ we can find $h\in\mathcal{C}(K)$, $\|h\|<\varepsilon$, such that $f(x)\overline{(g(x)+h(x))}\neq 1$ for all x in K.

In the following theorem we identify $f\in\mathcal{C}(K)=\mathcal{C}(K,\mathbb{C})$ with $f\in\mathcal{C}(K,\mathbb{R}^2)$ by means of $f\mapsto(\Re f,\Im f)$ and we use the norm $|x+iy|=\sup(|x|,|y|)$ on $\mathbb{C}$.

THEOREM 3.1. *For* K *a compact Hausdorff space*, $\mathcal{C}(K)$ *has the density property if*, *and only if*, $\dim K\le 1$.

Proof. We use the following facts about the covering dimension of a compact topological space K:

(i) if L is a closed subset of K then $\dim L\le\dim K$ ([10, p. 196]);

(ii) $\dim K\ge n$ if, and only if, for any $a,b\in\mathbb{R}$ there exists $f\in\mathcal{C}(K,[a,b]^n)$ such that for all $g\in\mathcal{C}(K,[a,b]^n)$ there exists $x_g\in K$ such that $f(x_g)=g(x_g)$ (see [4]).

Let $\dim K\ge 2$. Consider the pair $(f,1)\in\mathcal{C}(K)\times\mathcal{C}(K)$ where $f\in\mathcal{C}(K,[-2,2]^2)$ has the property described in (ii) above. Suppose we can find

$h \in \mathcal{C}(K)$, $\|h\| \le 1/4$, such that $(f+h,1)$ is quasi-invertible. This would imply

$$(\dagger) \qquad (f+h)(x) \neq 1 \quad \text{for all } x.$$

Since $\|h\| \le 1/4$, we have $3/4 \le \Re(1+h(x)) \le 5/4$ and $-1/4 \le \Im(1+h(x)) \le 1/4$ for all $x \in K$. So $1+h \in \mathcal{C}(K,[-2,2]^2)$. Our choice of f implies that there exists $x_0 \in K$ such that $f(x_0) = 1 + h(x_0)$. This contradicts $(\dagger)$ and hence $\mathcal{C}(K)$ does not have the density property.

Now suppose that $\mathcal{C}(K)$ does not have the density property. Then there exists a pair (f,g) in $\mathcal{C}(K) \times \mathcal{C}(K)$ and $\varepsilon > 0$ such that for all $h \in \mathcal{C}(K)$, $\|h\| \le \varepsilon$, there exists y_h in K satisfying

$$(4) \qquad (f+h)(y_h)\overline{g(y_h)} = 1.$$

We may assume that $\varepsilon < (4(1+\|g\|))^{-1}$ (otherwise replace ε with $\widetilde{\varepsilon} = (5(1+\|g\|))^{-1}$ in the following argument). Then for all h, $\|h\| < \varepsilon$, and y_h as above,

$$|h(y_h)\overline{g(y_h)}| \le \|h\| \cdot \|g\| \le \frac{\|g\|}{4(1+\|g\|)} \le \frac{1}{4}$$

and

$$(5) \qquad |f(y_h)\overline{g(y_h)}| \ge 1 - |h(y_h)\overline{g(y_h)}| \ge \frac{3}{4}.$$

Let $L = \{x \in K : |f(x)\overline{g(x)}| \ge 3/4\}$. Then L is a compact subset of K and we define $k \in \mathcal{C}(L)$ by $k(x) = 1/\overline{g}(x) - f(x)$ for all $x \in L$. Let $k = k_1 + ik_2$ where k_i are continuous real-valued functions. We define $\widetilde{k}_i$, $i = 1,2$, as follows:

$$\widetilde{k}_i(x) = \begin{cases} \varepsilon & \text{if } k_i(x) \ge \varepsilon, \\ k_i(x) & \text{if } -\varepsilon \le k_i(x) \le \varepsilon, \\ -\varepsilon & \text{if } k_i(x) \le -\varepsilon. \end{cases}$$

Clearly, each $\widetilde{k}_i$ is continuous and $\widetilde{k} = \widetilde{k}_1 + i\widetilde{k}_2 \in \mathcal{C}(L,[-\varepsilon,\varepsilon]^2)$.

Let $w \in \mathcal{C}(L,[-\varepsilon,\varepsilon]^2)$. Since compact sets are normal, there exists $\widehat{w} \in \mathcal{C}(K,[-\varepsilon,\varepsilon]^2)$ such that $\widehat{w}|_L = w$. It follows from (4) and (5) that for each $h \in \mathcal{C}(K)$, $\|h\| \le \varepsilon$, there exists $y_h \in L$ such that $h(y_h) = k(y_h)$ and hence there exists $y_w \in L$ such that $w(y_w) = \widehat{w}(z_w) = k(y_w)$. As $w(y_w) \in [-\varepsilon,\varepsilon]^2$, $\widetilde{k}(y_w) = k(y_w)$ and hence $w(y_w) = \widetilde{k}(y_w)$. An application of (ii) implies that $\dim L \ge 2$ and therefore $\dim K \ge 2$. ∎

EXAMPLE 3.2. (i) A scattered compact topological space, e.g. the ordinals Ω with the order topology, is 0-dimensional.

(ii) If X is a totally disconnected topological space, in particular, if X is discrete, then X is 0-dimensional. Since the Stone–Čech compactification preserves dimension, it follows that βX is a 0-dimensional compact space.

Hence $\mathcal{C}(\beta X)$ has the density property. In particular, $\mathcal{C}(\beta\mathbb{N}) \cong \ell^\infty$ has the density property.

(iii) If A is a commutative von Neumann algebra then $A \cong \mathcal{C}(K)$ where K is an extremely disconnected compact Hausdorff space. Since extremely disconnected topological spaces are totally disconnected and hence 0-dimensional, A has the density property. In particular, if K is any compact Hausdorff space then the second dual $\mathcal{C}(K)''$ has the density property. Using Theorem 3.1 we now see that $\mathcal{C}([0,1]^2)$ does not have the density property but its second dual $\mathcal{C}([0,1]^2)''$ does. This shows that the density property is not inherited by subspaces.

(iv) An n-dimensional differentiable manifold is n-dimensional as a topological space [10].

LEMMA 3.3. *Let $X = \mathcal{C}_0(\Omega)$ have the density property. If $I \subset X$ is a closed (algebra) ideal in X then I has the density property.*

Proof. Let $f,g \in I$ and let $\varepsilon > 0$. By the density property of Z there exists $h \in Z$, $\|h\| < \varepsilon$, such that $0 \notin 1 - f(\overline{g+h})(\Omega)$. Upon replacing ε if necessary, we may assume that $\varepsilon < \min\{\|g\|, (2\|g\|)^{-1}\}$. Define a function j as follows: $j(x) = 1$ if $|f(x)| \ge \varepsilon$ and $j(x) = |f(x)|/\varepsilon$ if $|f(x)| < \varepsilon$.

Since I is a closed ideal in $\mathcal{C}_0(\Omega)$, there exists a (closed) subset Σ of Ω such that $I = \{k \in Z : k|_\Sigma = 0\}$ and as j vanishes with f, j is also in I. Let $h' = jh \in I$. Then $\|h'\| \le \|j\| \cdot \|h\| \le \|h\| \le \varepsilon$. Suppose now that $0 = 1 - f(\overline{g+h'})(x)$ for some $x \in \Omega$. Then either

$$|f(x)| \ge \varepsilon \Rightarrow 1 - f(\overline{g+h'})(x) = 1 - f(\overline{g+h})(x) \neq 0,$$

which gives a contradiction, or

$$|f(x)| < \varepsilon \Rightarrow |f(\overline{g+h'})(x)| \le \varepsilon\|g+h'\| < 1$$

and hence $1 - f(\overline{g+h'})(x) \neq 0$, again giving a contradiction. We conclude that for all $f,g \in I$ and $\varepsilon > 0$ there exists $h' \in I$ with $\|h'\| < \varepsilon$ such that $0 \notin 1 - f(\overline{g+h'})(\Omega)$, that is, I has the density property. ∎

COROLLARY 3.4. *Let X be a JB*-triple, and let $x \in X$. Then X_x has the density property.*

Proof. We know that $X_x \cong \mathcal{C}_0(\Omega)$ for some locally compact subset Ω of $\mathbb{R}$ such that $\Omega \cup \{0\}$ is compact. Since $\Omega \cup \{0\}$ is zero- or one-dimensional, $\mathcal{C}(\Omega \cup \{0\})$ has the density property. But $\mathcal{C}_0(\Omega)$ is a closed ideal of $\mathcal{C}(\Omega \cup \{0\})$ and so has the density property by Corollary 3.3. ∎

We note that if Ω is a locally compact Hausdorff space with $\dim \Omega \le 1$ then, since the Stone–Čech compactification preserves dimension, $\dim \beta\Omega \le 1$ and thus $\mathcal{C}(\beta\Omega)$ has the density property. It then follows from Lemma 3.3 that $\mathcal{C}_0(\Omega)$ has the density property.

4. The JB^*-algebra case. The previous section was devoted to the commutative case (at least from the point of view of operator theory, while one might consider it the associative case from the algebraic perspective). In this section we discuss the density property in the non-commutative setting using JB^*-algebras.

DEFINITION 4.1. A (complex) *Jordan algebra* is a vector space over $\mathbb{C}$ with a bilinear composition law $\circ : A \times A \to A$ satisfying

$$x \circ y = y \circ x, \tag{6}$$

$$(x^2 \circ y) \circ x = x^2 \circ (y \circ x). \tag{7}$$

DEFINITION 4.2. A Banach space $(X, \|\cdot\|)$ which is a complex Jordan algebra equipped with an involution, $*$, is a *JB^*-algebra* if

$$\|x \circ y\| \le \|x\| \cdot \|y\|, \tag{8}$$

$$\|x^*\| = \|x\|, \tag{9}$$

$$\|x \circ x^*\| = \|x\|^2. \tag{10}$$

A JB^*-algebra with identity, e, is called a *unital JB^*-algebra*. By [3, 3.3.9 and 3.8.3] a unit can be added to a JB^*-algebra and the norm extended to give a unital JB^*-algebra. An important example of a unital JB^*-algebra is the space of operators on a complex Hilbert space endowed with the product

$$A \circ B = \tfrac{1}{2}(AB + BA). \tag{11}$$

A *special JB^*-algebra*, or *JC^*-algebra*, is one which can be realised as an algebra of operators on a Hilbert space, with product defined by (11). From the Gelfand–Naimark theorem any C^*-algebra is a JC^*-algebra. A closed subspace of $\mathcal{L}(H)$, where H is a Hilbert space, which is closed under taking adjoints and squares is a JC^*-algebra with product given by (11).

For any JB^*-triple X and any $y \in X$ we may define a binary product on X by

$$x \circ_y z := \{x, y, z\}.$$

This product gives X the structure of a Jordan algebra, which we denote by X^y. Moreover, if y is a unitary tripotent then X^y is a JB^*-algebra. Conversely, given a JB^*-algebra X, we define the triple product

$$\{x, y, z\} = (xy^*)z - (zx)y^* + (y^*z)x \tag{12}$$

and with this product $(X, \{\cdot,\cdot,\cdot\})$ is a JB^*-triple. (If the JB^*-algebra is generated from a JB^*-triple with unitary tripotent, then (12) regains the original triple product.)

Not every JB^*-triple arises in this way and those that do may be characterised geometrically. A JB^*-triple is J^*-isomorphic (i.e. linearly isomorphic by a mapping which preserves the triple product) to a unital JB^*-algebra if, and only if, its unit ball contains a unitary tripotent (see [13, 20.35]). For special JB^*-triples this reduces to the triple containing a unitary operator.

Inverses are defined in a unital JB^*-algebra so that in the special JB^*-algebra case the inverse coincides with the inverse arising from the original product. The standard definition is given in part (i) of the following proposition.

PROPOSITION 4.3. *Let Z be a unital Jordan algebra with involution. If any of the following equivalent conditions are satisfied by the elements x and y in Z then we say that x is invertible with inverse y:*

(i) $xy = 1$ *and* $x^2 y = x$;
(ii) $xy = 1$ *and* $y^2 x = y$;
(iii) Q_x *is invertible and* $y^* = Q_x^{-1}(x)$;
(iv) $Q_x(y^*) = x$ *and* $Q_x(y^{*2}) = 1$ *where* $Q_x(y) = \{x, y, x\}$.

The inverse of x is unique and is written as x^{-1}. Also, $(x^{-1})^{-1} = x$ and $(x^{-1})^* = (x^*)^{-1}$, and the Jordan subalgebra generated by an element x is power associative, i.e. $(x^n)^m = x^{n+m}$ for any integers n and m [14, p. 304, Thm. 5].

EXAMPLE 4.4. Let $X = \mathcal{C}(K)$ where K is a compact Hausdorff space. In Example 1.3 we noted that $\mathcal{C}(K)$ is a JB^*-triple with triple product $\{f, g, h\} = f\overline{g}h$ and involution $g^* = \overline{g}$. If $g \in \mathcal{C}(K)$ then $Q_g(f) = g^2\overline{f}$ and hence Q_g is invertible if and only if $g(x) \neq 0$ for all $x \in K$. In such a case, $1_g := Q_{g^*}^{-1}(g) = (\overline{g})^{-2}\overline{g} = 1/\overline{g}$ is the identity in $\mathcal{C}(K)^g$. An element f in $\mathcal{C}(K)$ is invertible in $\mathcal{C}(K)^g$ with inverse $\widetilde{f}$ if, and only if, $f \circ_g \widetilde{f} = 1_g$, i.e. $f\overline{g}\widetilde{f} = 1/\overline{g}$ and hence $\widetilde{f} = 1/f\overline{g}^2$.

If X is a JB^*-triple, $y \in X$ and Q_y is not invertible then we can adjoin an identity 1_y to the Jordan algebra X^y to obtain a unital Jordan algebra, $X^y_{\#}$. We say that x is *quasi-invertible* in X^y with *quasi-inverse* z if $1_y - x$ is invertible in $X^y_{\#}$ with inverse $1_y + z$. That is,

$$(1_y - x) \circ_y (1_y + z) = 1_y \quad \text{and} \quad (1_y - x)^2 \circ_y (1_y + z) = 1_y - x.$$

In particular, we see that $x - z + x \circ_y z = 0$.

If L is a locally compact Hausdorff space, $X = \mathcal{C}_0(L)$ and $g \in X$, then f is quasi-invertible in $\mathcal{C}_0(L)^g$ if, and only if, $f\overline{g}(x) \neq 1$ for all $x \in L$ and the quasi-inverse f^g of f satisfies $f - f^g + f\overline{g}f^g = 0$. Hence

$$f^g = \frac{f}{1 - f\overline{g}}.$$

Further analysis shows that in an arbitrary JB^*-triple X, an element x is quasi-invertible in X^y if and only if $B(x, y) \in \mathcal{L}(X)$ is invertible and then

$x^y = B(x,y)^{-1}(x - Q_x y)$. Thus, in the terminology of Section 1, the pair $(x,y) \in X \times X$ is quasi-invertible if and only if x is quasi-invertible in X^y.

PROPOSITION 4.5. *Let Z be a JB^*-algebra with identity* 1. *If Z has the density property then the invertible elements are dense in Z.*

Proof. By (12) we have $z^* = \{1, z, 1\}$,

$$B(x,1)(z) = z - 2xz + \{x, z^*, x\} = z - 2xz + 2(xz)x - x^2 z$$

and

$$\begin{aligned} Q_{1-x}(z^*) &= \{1 - x, z^*, 1 - x\} \\ &= 2(z - xz)(1 - x) - (1 - x)^2 z = z - 2xz + 2(xz)x - x^2 z \end{aligned}$$

for all $x, z \in Z$. Hence $(x, 1)$ is a quasi-invertible pair if and only if Q_{1-x} is invertible. From the equivalence (i)⇔(ii) in Proposition 2.1 this shows that if Z has the density property then the invertibles are dense in Z. ∎

Next, we prove the converse.

PROPOSITION 4.6. *Let Z be a JB^*-algebra with identity* 1 *and suppose the invertible elements of Z are dense in Z. Then Z has the density property.*

Proof. Let x, y be arbitrary elements of Z. Let $\varepsilon > 0$ and choose $x' \in Z$ such that $\|x - x'\| < \varepsilon$ and x' is invertible. Then $Q_{x'}$ is invertible. Again by density of the invertibles we can choose an invertible element z in Z with $\|z - (x' - Q_{x'}y)\| < \varepsilon/\|Q_{x'}^{-1}\|$. Let $y' = Q_{x'}^{-1}(x' - z)$. Then $x' - Q_{x'}y' = x' - (x' - z)$ is invertible and

$$\begin{aligned} \|y - y'\| = \|y - Q_{x'}^{-1}(x' - z)\| &= \|Q_{x'}^{-1}(Q_{x'}y - x' + z)\| \\ &\le \|Q_{x'}^{-1}\| \cdot \|z - (x' - Q_{x'}y)\| < \varepsilon. \end{aligned}$$

Now consider $B(x', y')$. By (JP23), $B(x', y')Q_{x'} = Q_{x' - Q_{x'}y'}$ and since x' and $x' - Q_{x'}y'$ are invertible, so too are the operators $Q_{x'}$ and $Q_{x' - Q_{x'}y'}$. Hence $B(x', y')$ is invertible and since ε is arbitrary we have shown that there is a dense subset of quasi-invertible pairs in $Z \times Z$. This finishes the proof by Proposition 2.1. ∎

Combining Propositions 4.5 and 4.6 we obtain the following.

THEOREM 4.7. *A JB^*-algebra Z with identity has the density property if, and only if, the invertibles are dense in Z.*

Rieffel [12] in studying K-theory for C^*-algebras introduced *topological stable rank*, tsr, and proved the following: *A Banach algebra A with identity and continuous involution has* $\operatorname{tsr}(A) = 1$ *if, and only if, the invertibles are dense in A.* Hence, for the case of C^*-algebras with identity we can restate Proposition 4.6 as follows.

THEOREM 4.8. *A C^*-algebra A with identity has the density property if, and only if,* $\operatorname{tsr}(A) = 1$.

This means of course that, by the results in [12], Theorem 4.8 contains Theorem 3.1 as a special case. We included the proof of 3.1 because of its directness. The results in [12] lead to a number of examples, which we list further on, and also motivates the study of topological stable rank in JB^*-triples.

We now seek to remove the hypothesis of an identity from some of the previous results.

LEMMA 4.9. *If Z is a JB^*-algebra then the following are equivalent:*

(a) *the quasi-invertible elements are dense in Z;*
(b) *the quasi-invertible elements are dense in $Z_{\#}$;*
(c) *the invertible elements are dense in $Z_{\#}$.*

Proof. Since translation by the identity is a continuous operation, (b) and (c) are equivalent. Suppose (a) holds. Given $x + \alpha 1 \in Z_{\#}$ and $\varepsilon > 0$, choose a complex number $\gamma \ne -1$ such that $|\alpha - \gamma/(1+\gamma)| < \varepsilon$ and choose $x' \in Z$ quasi-invertible such that $\|x' - (1+\gamma)x\| \le \varepsilon|1+\gamma|$. Since x' is quasi-invertible in Z, it follows that $1 - x'$, $1 + \gamma - (x' + \gamma)$ and $1 - (x'/(1+\gamma) + \gamma/(1+\gamma))$ are all invertible in $Z_{\#}$. Hence $x'/(1+\gamma) + \gamma/(1+\gamma)$ is quasi-invertible and

$$\left\| x + \alpha - \frac{x'}{1+\gamma} - \frac{\gamma}{1+\gamma} \right\| \le \left\| x - \frac{x'}{1+\gamma} \right\| + \left| \alpha - \frac{\gamma}{1+\gamma} \right| \le 2\varepsilon,$$

which implies (b).

Now suppose (b) holds. Given $x \in Z$ and $\varepsilon > 0$ we can find $x' \in Z$ and $\alpha \in \mathbb{C}$ such that $x' + \alpha$ is quasi-invertible, $\|x - x'\| < \varepsilon$ and

$$|\alpha| < \min\left(\frac{\varepsilon}{\|x\| + \varepsilon}, \frac{1}{2} \right).$$

Hence $|\alpha(\|x\| + \varepsilon)| < \varepsilon$ and $(1+\alpha)^{-1} < 2$.

Since $x' + \alpha$ is quasi-invertible, we see that

$$1 - (x' + \alpha) = 1 + \alpha - x' = (1+\alpha)\left(1 - \frac{x'}{1+\alpha}\right)$$

is invertible and hence $x'/(1+\alpha) \in Z$ is quasi-invertible. The estimate

$$\left\| x - \frac{x'}{1+\alpha} \right\| = \left\| x - x' + x'\left(1 - \frac{1}{1+\alpha}\right) \right\| \le \varepsilon + (\|x\| + \varepsilon)\left| \frac{\alpha}{1+\alpha} \right| < \varepsilon + 2\varepsilon$$

completes the proof. ∎

THEOREM 4.10. *If Z is a JB^*-algebra such that the quasi-invertibles are dense in Z then Z has the density property.*

Proof. By Lemma 4.9 and Theorem 4.7, $Z_\#$ has the density property, so letting $x,y\in Z$ and $\varepsilon>0$, there exists $x'\in Z$ and $\alpha\in\mathbb{C}$, $\|x-x'\|<\varepsilon$, and $|\alpha|<\varepsilon$ such that $x'+\alpha 1$ is quasi-invertible in $Z^y_\#$. Extend $Z^y_\#$, if necessary, to obtain a JB^*-algebra $(Z^y_\#)_\#$ with identity 1_y. Thus $1_y-(x'+\alpha 1)$ is invertible in $(Z^y_\#)_\#$.

Since the invertible elements form an open set, we may suppose ε is sufficiently small for $1_y-\alpha 1$ to be invertible with inverse $\sum_{n=0}^\infty(\alpha 1)^n$ in $(Z^y_\#)_\#$.

It now follows from $1_y-\alpha 1-x'=(1_y-\alpha 1)(1_y-(1_y-\alpha 1)^{-1}x')$ that $(1_y-\alpha 1)^{-1}x'$ is quasi-invertible in $(Z^y_\#)_\#$ and $(1_y-\alpha 1)^{-1}x'=\sum_{n=0}^\infty(\alpha 1)^n x$. It is easily verified that $(\alpha 1)^n=\alpha^n(y^*)^{n-1}$ in $(Z^y_\#)_\#$ for $n\ge 1$, and $(\alpha 1)^0=1_y$. Hence

$$(1_y-\alpha 1)^{-1}(x')=\sum_{n=0}^{\infty}\alpha^n(y^*)^{n-1}x' \quad \text{in } Z.$$

Also, we see that

$$\|1_y-(1_y-\alpha.1)^{-1}\|=\Big\|\sum_{n=1}^{\infty}(\alpha 1)^n\Big\|\le\|\alpha 1\|\Big\|\sum_{n=0}^{\infty}(\alpha 1)^n\Big\|$$
$$\le\varepsilon\sum_{n=0}^{\infty}|\alpha|^n<\varepsilon(1-\alpha)^{-1}<\frac{\varepsilon}{1-\varepsilon}<2\varepsilon$$

(assuming $\varepsilon<1/2$) and so $\|(1_y-\alpha 1)^{-1}x'-x\|\le 2\varepsilon\|x'\|+\varepsilon\le\varepsilon(2\|x\|+2\varepsilon+1)$. Letting $x''=(1_y-\alpha 1)^{-1}x'$, we find that x'' is quasi-invertible in $(Z^y_\#)_\#$ and its quasi-inverse is $w+\beta 1$ ($w\in Z$, $\beta\in\mathbb{C}$). Then

$$(1_y-\alpha 1)^{-1}x'+w+\beta 1-(x'')\circ_y(w+\beta 1)=0$$

gives

$$x''+w-\{x'',y,w\}-\{x'',y,\beta 1\}+\beta 1=0$$

or

$$\underbrace{x''+w-\{x'',y,w\}-\beta x''y^*}_{\in Z}+\beta 1=0.$$

Hence $\beta=0$ and the quasi-inverse of x'' is in Z. We have thus shown that x'' is quasi-invertible in Z^y. Hence (x'',y) is a quasi-invertible pair and $\|x-x''\|<\varepsilon(2\|x\|+2\varepsilon+1)$. Since ε is arbitrarily small, an application of 2.1 completes the proof. ∎

EXAMPLE 4.11. (a) The following spaces have the density property:

(i) $\mathcal{K}(H)$, the compact operators on a Hilbert space,
(ii) any $\mathcal{AF}$ (approximately finite) C^*-algebra,
(iii) $c_0(\{A_n\})$, where each A_n is a C^*-algebra with the density property for each n,
(iv) $A\otimes\mathcal{K}(H)$ where A is a C^*-algebra with the density property.

(b) The following spaces do not have the density property:

(i) $\mathcal{L}(H)$, the bounded linear operators on an infinite-dimensional Hilbert space,
(ii) $C^*(S)$, the C^*-algebra with identity generated by the unilateral shift $S:H\to H$, $Se_i=e_{i+1}$, where $(e_i)_{i=1}^\infty$ is a basis for the Hilbert space H,
(iii) any unital C^*-subalgebra of $\mathcal{L}(H)$ which contains a Fredholm operator of non-zero index,
(iv) any unital C^*-algebra containing two isometries with orthogonal ranges.

All of these examples are given in [12] and follow from Theorems 4.7 and 4.8. Some of them also follow from earlier results. For example, Rickart [11, p. 279] shows that a non-invertible linear operator in $\mathcal{L}(H)$ with closed range lies in the interior of the singular elements of $\mathcal{L}(H)$ (this gives example (b)(i) and (ii)), while it is well known that the Fredholm operators of a given index form an open set of operators and this implies (b)(iii). Moreover, by [12, Proposition 3.1] the invertible, left invertible, and right invertible elements in a C^*-algebra A coincide if $\mathrm{tsr}(A)=1$, and hence any C^*-algebra in which these do not coincide (e.g. (b)(i), (ii) and (iii)) fails to have the density property.

To see directly that $\mathcal{K}(H)$ has the density property we return to Example 1.5(i) and use spectral theory. If $x,y\in\mathcal{K}(H)$ then $xy^*\in\mathcal{K}(H)$ and hence $\sigma(xy^*)$ is countable, so there exists λ close to 1 such that $\lambda\notin\sigma(xy^*)$. This implies that $\lambda-xy^*=\lambda(1-(x/\lambda)y^*)$ and hence $(1-(x/\lambda)y^*)$ are invertible operators. By 1.5(i) and Proposition 2.1 this implies that $\mathcal{K}(H)$ has the density property. More generally, one can show, using the same proof, the following result.

PROPOSITION 4.12. *If Z is a C^*-algebra and $\sigma(x)$ has empty interior for all $x\in Z$ then Z has the density property.*

5. Complete holomorphic vector fields on M_X. We now study the holomorphic structure of the quasi-invertible manifold, M_X, of a JB^*-triple X. In particular, we are interested in the consequences of the density property for the geometry of the manifold. A fundamental result will be that M_X is a homogeneous manifold whenever X has the density property. The proof of this fact is based on the ability to extend translations on X to biholomorphic maps of M_X. This leads to an examination of a conjecture by Dorfmeister [2] asserting that translations on X extend to biholomorphic maps on the manifold M_X if, and only if, a certain algebraic condition on X is satisfied. We prove this conjecture in one direction.

We begin by following the arguments of Loos [8] for the finite-dimensional case, where three different types of biholomorphic map are defined. For each $a \in X$, define a map $\widetilde{t}_a : M_X \to M_X$ by

$$\widetilde{t}_a(x : y) = (x : y + a).$$

It is easy to see that $\widetilde{t}_a$ is well defined on M_X and is biholomorphic with inverse $\widetilde{t}_{-a}$ (see [8, 8.4(a)]).

The second type of biholomorphic map is an extension of a JB^*-triple automorphism. The set of automorphisms of the JB^*-triple X is denoted by $\mathrm{Aut}(X)$. The *structure group* of X (see [7]) is the set of all $f \in \mathrm{GL}(X)$ for which there exists $\bar{f} \in \mathrm{GL}(X)$ satisfying $f\{x, y, z\} = \{fx, \bar{f}y, fz\}$ and $\bar{f}\{x, y, z\} = \{\bar{f}x, fy, \bar{f}z\}$. It is easy to check that $\bar{f}$ is uniquely determined by f and that the map $f \mapsto \bar{f}$ is antiholomorphic, of period two, and has $\mathrm{Aut}(X)$ as its fixed point set. Moreover, the structure group forms a complex Lie group (being an algebraic subgroup of the complex Lie group $\mathrm{GL}(X) \times \mathrm{GL}(\overline{X})$ where $\overline{X}$ is the complex conjugate Banach space of X), as opposed to $\mathrm{Aut}(X)$, which is a real Lie group. We denote the structure group of X by $\mathrm{Aut}(X, \overline{X})$. Compare [8, Section 3]. Any element $f \in \mathrm{Aut}(X, \overline{X})$ extends to a biholomorphic map on the manifold M_X via $(x : y) \overset{f}{\mapsto} (fx : \bar{f}y)$.

The third type of biholomorphic maps on M_X that we consider are possible extensions of translations on X, namely, extensions of the map $(x : 0) \mapsto (x + a : 0)$ on X. The naive attempt at an extension, $(x : y) \mapsto (x + a : y)$, is not well defined on M_X and the finite-dimensional approach of Loos [8, 8.4(c)], which depends on the fact that holomorphic vector fields on a compact manifold are complete, no longer applies. The following conjecture, made by Dorfmeister [2], introduces a property on the JB^*-triple X which, by openness of the set of invertible operators, is easily seen to be implied by the density property.

DEFINITION 5.1. A JB^*-triple X satisfies *condition* (D) if for all $x, y, z \in X$ there exists $c \in X$ such that (z, c) and $(x, y - c)$ are quasi-invertible.

CONJECTURE 5.2. *The map* $(x : 0) \mapsto (x + u : 0)$ *can be extended to a biholomorphic map of* M_X *for all* $u \in X$ *if, and only if, the* JB^**-triple* X *satisfies condition* (D).

We prove this conjecture in one direction.

THEOREM 5.3. *If* X *is a* JB^**-triple which satisfies condition* (D) *then for all* x, y *and* a *in* X, *there exists* $c \in X$ *such that*

$$t_a(x : y) := (B(a, c)(a^c + x^{y-c}) : c^a) \tag{13}$$

is well defined on M_X. *Moreover,* $t_a(x : y)$ *does not depend on the choice of* c, t_a *is a well defined biholomorphic mapping on* M_X *with inverse* t_{-a} *and* $t_a(x : 0) = (x + a : 0)$ *for all* $x \in X$.

REMARK. The formula in (13) was obtained by the following procedure. The constant vector field ξ_a on X given by $\xi_a(x : 0) = a\frac{\partial}{\partial z}$ generates the one-parameter group of translations $(x : 0) \mapsto (x + ta : 0)$, $t \in \mathbb{R}$, and moreover, each ξ_a extends to a holomorphic vector field on M_X by the formula $\xi_a(x : y) = B(x, y)a\frac{\partial}{\partial z}$ on the chart U_y (compare [8, 8.4(c)]). To obtain (13) it suffices to show that this vector field is complete on M_X and to take the exponential.

If x^y exists, it is easily verified that $\alpha(t) = ((x^y + ta)^{-y} : y)$ is an integral curve of ξ_a subject to the initial condition $\alpha(0) = (x : y)$. To show that α can be defined on all of $\mathbb{R}$ it is necessary to find a new expression for $\alpha(t)$ and we consider

$$\begin{aligned} ((x^y + ta)^{-y} : y) &= ((x^y + ta)^{-z} : z) \quad (\text{for any } z \in X) \\ &= ((x^y + ta)^{-(c^{ta})} : c^{ta}) \\ &= (B(ta, c)[(ta)^c + x^{y-c}] : c^{ta}). \end{aligned} \tag{14}$$

The final expression on the right hand side gives (13) when $t = 1$ and, under the hypothesis of condition (D), such an element c can be found.

Proof of 5.3. Given (x, y) and $a \in X$, we can find, using condition (D), $c \in X$ such that (a, c) and $(x, y - c)$ are quasi-invertible pairs. If $(x : y) = (x_1 : y_1)$ then x^{y-y_1} exists and equals x_1. By (JPA1), $x_1^{y_1-c} = (x^{y-y_1})^{y_1-c} = x^{y-c}$ exists and therefore $t_a(x : y) = t_a(x_1 : y_1)$. Hence t_a will be well defined if the formula in (13) is independent of the choice of c. In other words, if $d \in X$ is another element such that a^d and x^{y-d} exist, we must check that $(B(a, c)[a^c + x^{y-c}])^{c^a - d^a}$ exists and equals $B(a, d)(a^d + x^{y-d})$.

Notice first that, by (JPA1) and (JPT), $(-a^c)^c$ exists and equals $-a$, $c^{(-a^c)}$ exists and equals $c - Q_c a$ and, using the fact that $(-d)^{(-a^c)^c} = -d^a$ exists, we find by (JPA2) that $(c - d)^{(-a^c)}$ exists and

$$\begin{aligned} (c - d)^{(-a^c)} &= c^{(-a^c)} + B(c, -a^c)^{-1}((-d)^{(-a^c)^c}) \\ &= c - Q_c a + B(c, a)(-d^a) = B(c, a)(c^a - d^a). \end{aligned} \tag{15}$$

Since $a^d = (a^c)^{-(c-d)} = -((-a^c)^{c-d})$ by (JPT), and $x^{y-d} = (x^{y-c})^{c-d}$, it follows, again by (JPA2), that $(a^c + x^{y-c})^{(c-d)^{-a^c}}$ exists and

$$a^d + x^{y-d} = (x^{y-c})^{c-d} - (-a^c)^{c-d} = B(-a^c, c - d)^{-1}((a^c + x^{y-c})^{(c-d)^{-a^c}}.$$

Using the fact that

$$\begin{aligned} B(-a^c, c - d) &= B(-a^c, c)B((-a^c)^c, -d) \quad \text{by (JP33)} \\ &= B(a, c)^{-1}B(-(a^c)^{-c}, -d) = B(a, c)^{-1}B(-a, -d) \\ &= B(a, c)^{-1}B(a, d) \end{aligned}$$

we get

$$(16)\quad B(a,d)(a^d + x^{y-d}) = B(a,d)B(a,d)^{-1}B(a,c)((a^c + x^{y-c})^{(c-d)^{-a^c}}) = B(a,c)((a^c + x^{y-c})^{(c-d)^{-a^c}}).$$

Now (15) and (16) give

$$(17)\quad B(a,d)(a^d + x^{y-d}) = B(a,c)((a^c + x^{y-c})^{B(c,a)(c^a - d^a)}).$$

Finally, by (JPS), we see that $(B(a,c)(a^c + x^{y-c}))^{c^a - d^a}$ exists and (17) becomes

$$B(a,d)(a^d + x^{y-d}) = (B(a,c)(a^c + x^{y-c}))^{c^a - d^a},$$

which is precisely what we required.

If $(x, y-c)$ is quasi-invertible then $(x', y-c)$ is quasi-invertible for all x' near x. Hence, Proposition 1.6(ii) implies that t_a is holomorphic. If $y = 0$ then we may take $c = 0$ in (13) and we see that $t_a(x:0) = (a+x:0)$. This also shows that $t_{-a} \circ t_a(x:0) = (x:0)$ and since $t_{-a} \circ t_a$ is holomorphic, the identity principle for holomorphic mappings shows that t_a is biholomorphic with inverse t_{-a}. ∎

COROLLARY 5.4. *Let X be a JB*-triple satisfying condition* (D). *Then, for all $a \in X$, the vector field ξ_a is complete on the manifold M_X.*

Proof. Clearly, the mapping $s \in \mathbb{R} \mapsto t_{sa}$ defines a one-parameter subgroup of biholomorphic mappings on M_X which generates the vector field ξ_a, so ξ_a is complete on M. ∎

COROLLARY 5.5. *Let X be a JB*-triple with the density property. Then, for all $a \in X$, the vector field ξ_a is complete on the manifold M_X.*

COROLLARY 5.6. *Let X be a JB*-triple satisfying condition* (D). *Then M_X is a complex homogeneous manifold.*

Proof. For $(x:a) \in M_X$, we have $(x:a) = \widetilde{t}_a t_x(0:0)$. ∎

Another consequence of condition (D) (and hence of density) is that, similar to the finite-dimensional case mentioned in the introduction, biholomorphic maps of the open unit ball, B_X, extend to give biholomorphic maps on M_X.

COROLLARY 5.7. *Let X be a JB*-triple satisfying condition* (D). *Every biholomorphic automorphism of the open unit ball B_X of X has an extension to a biholomorphic map on M_X.*

Proof. Every automorphism $g \in \mathrm{Aut}(B_X)$ has a decomposition $g = kp$ where k is a JB*-triple automorphism of X (i.e. a surjective linear isometry of X) restricted to the unit ball, and p takes the form $p = p_c$ ($c = p(0)$) where

$$p_c(z) = c + B(c,c)^{1/2}(I + z \,\square\, c)^{-1}z.$$

See [7] for details. Clearly, k extends to M_X as we saw earlier. Note that $(I + z \,\square\, c)^{-1}z = z^{-c}$ so $p_c(z) = t_c \circ B(c,c)^{1/2} \circ \widetilde{t}_{-c}(z)$ and knowing that t_c and $\widetilde{t}_{-c}$ both extend to biholomorphic maps on M_X, we must simply check that the linear map $B(c,c)^{1/2}$ extends biholomorphically to M_X. However, as a corollary to [7, Theorem 3.5], $B(c,c)^{1/2}$ is in the structure group of X and hence extends to M_X. ∎

We conclude by showing that the converse to Corollary 5.5 is false, and it follows that either the density property is strictly stronger than condition (D), or Conjecture 5.2 is false. Our counterexample occurs in the JB*-triple $\mathcal{C}(\overline{\Delta})$, which by 2.1 does not have the density property although the constant vector fields on $\mathcal{C}(\overline{\Delta})$ do extend to the quasi-invertible manifold. To see this, we make use of a simple lemma.

LEMMA 5.8. *Given $f, g \in \mathcal{C}(\overline{\Delta})$ such that, for any $x \in \Delta$, $f(x)$ and $g(x)$ are not both zero, there exists $h \in \mathcal{C}(\overline{\Delta})$ such that $0 \notin f + gh(\overline{\Delta})$.*

COROLLARY 5.9. *Let $X = \mathcal{C}(\overline{\Delta})$. For all $a \in X$ the map $(x:0) \mapsto (x+a:0)$ extends to a biholomorphic map on M_X and hence the vector field ξ_a is complete on M_X.*

Proof. We know that if the vector field ξ_a is complete on M_X then the automorphism t_a takes the symbolic form (see (14))

$$t_a(x:y) = ((x^y + a)^{-y} : y) = ((x^y + a)^{-b} : b).$$

The above lemma shows that for any x, y and a in X, we can choose $b \in X$ such that $(x^y + a)^{-b}$ exists in the following sense. In X, $x^y = x/(1 - x\overline{y})$ exists if $0 \notin 1 - x\overline{y}(\overline{\Delta})$. Symbolically,

$$x^y + a = \frac{x + a(1 - x\overline{y})}{1 - x\overline{y}}$$

and

$$(x^y + a)^{-b} = \frac{x + a(1 - x\overline{y})}{1 - x\overline{y}}\left(1 + \frac{x + a(1 - x\overline{y})}{1 - x\overline{y}}\overline{b}\right)^{-1} = \frac{x + a(1 - x\overline{y})}{\underbrace{1 - x\overline{y}}_{f} + \overline{b}\underbrace{(x + a(1 - x\overline{y}))}_{g}}.$$

Letting $f = 1 - x\overline{y}$ and $g = x + a(1 - x\overline{y})$ we see that f and g cannot take the value 0 simultaneously, so by the previous lemma, there exists a continuous function h with $0 \notin f + gh(\overline{\Delta})$. Letting $b = \overline{h}$, we have $0 \notin 1 - x\overline{y} + \overline{b}(x + a(1 - x\overline{y}))$ and thus

$$\frac{x + a(1 - x\overline{y})}{1 - x\overline{y} + \overline{b}(x + a(1 - x\overline{y}))}$$

exists. Next define t_a using this fact:

$$t_a(x:y) := \left(\frac{x+a(1-x\overline{y})}{1-x\overline{y}+\overline{b}(x+a(1-x\overline{y}))} : b\right).$$

It can be checked as in Theorem 5.3 that t_a is independent of the b chosen and of the representation of the equivalence class $(x:y)$. Finally, t_a is a biholomorphic map on the manifold M_X and for $\gamma \in \mathbb{R}$ sufficiently small, $t_{\gamma a}$ coincides with the flow of ξ_a at time γ through the point $(x:y)$. By uniqueness of this flow we see that ξ_a is a complete holomorphic vector field and $\exp(\xi_a) = t_a$. ∎

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H^∞ functional calculus in real interpolation spaces

by

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Abstract. Let A be a linear closed densely defined operator in a complex Banach space X. If A is of type ω (i.e. the spectrum of A is contained in a sector of angle 2ω, symmetric around the real positive axis, and $\|\lambda(\lambda I - A)^{-1}\|$ is bounded outside every larger sector) and has a bounded inverse, then A has a bounded H^∞ functional calculus in the real interpolation spaces between X and the domain of the operator itself.

1. Introduction. The concept of H^∞ functional calculus has been developed by various authors; we recall the papers [1]–[3], [5]–[8]. In the present paper we follow the definitions of [3] and [7].

An H^∞ functional calculus can be defined for closed, linear, one-to-one operators with dense domain and dense range, having resolvent set that contains $\mathbb{R}^-$ and with resolvent that decreases in a maximal way on $\mathbb{R}^-$ (i.e. $\|\lambda(\lambda I - A)^{-1}\|$ is bounded). If for every $f \in H^\infty$ the operator $f(A)$ is bounded, with norm not exceeding a constant times the supremum of f, then we say that A has a bounded H^∞ functional calculus. In general it is not easy to prove that an operator has a bounded H^∞ functional calculus and there are several examples of operators that do not enjoy this property (see [8]).

In this paper we prove that every linear, closed, densely defined operator A in a complex Banach space X, with resolvent satisfying the above-mentioned hypotheses and such that $0 \in \varrho(A)$, has a bounded H^∞ functional calculus in the real interpolation spaces between X and the domain of the operator itself.

2. Preliminary definitions and results. We now recall some definitions and results from [7] and [3], to which we refer for the details.

For $\theta \in [0, \pi[$ we put

$$S_\theta = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \le \theta\} \cup \{0\}$$

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