The density property for $JB^*$-triples

by

SEÁN DÍNEN, MICHAEL MACKEY and PAULINE MELLON (Dublin)

Abstract. We obtain conditions on a $JB^*$-algebra $X$ so that the canonical embedding of $X$ into its associated quasi-invertible manifold has dense range. We prove that if a $JB^*$-triple has this density property then the quasi-invertible manifold is homogeneous for biholomorphic mappings. Explicit formulae for the biholomorphic mappings are also given.

1. Introduction. There exist, up to biholomorphic equivalence, precisely two one-dimensional simply connected symmetric complex manifolds, and these can be realised as the open unit disc

$$\Delta = \{ z \in \mathbb{C} : |z| < 1 \}$$

and the Riemann sphere $\overline{\mathbb{C}} := \mathbb{C} \cup \{ \infty \}$. We have the standard inclusions

$$\Delta \hookrightarrow \mathbb{C} \hookrightarrow \overline{\mathbb{C}}$$

and $\mathbb{C}$ is dense in $\overline{\mathbb{C}}$. Moreover, each biholomorphic map of $\Delta$ extends to a biholomorphic map of $\overline{\mathbb{C}}$. In the finite-dimensional situation each $n$-dimensional bounded symmetric domain $\mathcal{D}_b$ determines, and is determined by, its unique compact dual $\mathcal{D}_c$. The domain $\mathcal{D}_b$ can be realised as the open unit ball of a norm on $\mathbb{C}^n$, and $\mathcal{D}_c$, which is an $n$-dimensional compact symmetric manifold, can be realised as the quasi-invertible manifold associated with $\mathcal{D}_b$ (see [8]) and we have the canonical inclusions

$$\mathcal{D}_b \hookrightarrow \mathbb{C}^n \hookrightarrow \mathcal{D}_c$$

with $\mathbb{C}^n$ dense in $\mathcal{D}_c$. Once again, biholomorphic maps of $\mathcal{D}_b$ extend to biholomorphic maps of $\mathcal{D}_c$.

In the Banach space setting, Kaup [6] has characterised bounded symmetric domains as those complex manifolds which can be realised as the open unit ball of a $JB^*$-triple. With every $JB^*$-triple $X$ one can associate a quasi-invertible complex manifold $M_X$ modelled on $X$ and we again have

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the canonical inclusions

\[ B_X \to X \to M_X \]

where \( B_X \) is the open unit ball of \( X \).

**Definition 1.1.** The \( JB^* \)-triple \( X \) has the **density property** if \( X \) is dense in \( M_X \).

In this paper we investigate the density property for \( JB^* \)-triples. By using \( J^* \)-triples, Kaup [5] has shown that each bounded symmetric domain has associated with it a unique simply connected symmetric Banach manifold of compact type. The relationship between this compact type manifold and the above quasi-invertible manifold is still under investigation [1] and the density condition may well feature in the final solution.

We now recall some background information on \( JB^* \)-triples.

**Definition 1.2.** A \( JB^* \)-triple is a pair \( (X, \langle \cdot, \cdot \rangle) \) consisting of a Banach space \( X \) and a continuous real trilinear mapping \( \langle \cdot, \cdot, \cdot \rangle : X^3 \to \mathbb{R} \) which is complex linear and symmetric in the outer variables, complex antilinear in the middle variable and which satisfies the following axioms:

1. \( \delta(x)\{u, v, w\} = \{\delta(x)u, v, w\} - \{u, \delta(x)v, w\} + \{u, v, \delta(x)w\} \) for \( x, u, v, w \in X \),
2. \( \delta(x) \) is Hermitian for all \( x \in X \),
3. \( \sigma(\delta(x)) \geq 0 \) for all \( x \in X \),
4. \( \|\delta(x)\| = \|x\|^3 \) for all \( x \in X \),

where \( \delta(x) \in \mathcal{L}(X) \) is defined by \( \delta(x)(y) = \{x, y, x\} \) and \( \sigma \) denotes the operator spectrum. We recall that a linear operator \( T \in \mathcal{L}(X) \) is **Hermitian** if \( e^{i2\pi T} \) is an isometry for all \( \lambda \in \mathbb{R} \). Condition (i) in this definition is known as the Jordan triple identity. The above “differential” version of the Jordan triple identity may be linearized to give the equivalent condition

\[ \langle e, e, e \rangle = e \]

where \( e \in X \) is the identity operator on \( X \).

**Example 1.3.** (i) If \( H \) and \( K \) are complex Hilbert spaces then \( \mathcal{L}(H, K) \) endowed with the triple product

\[ \{A, B, C\} := \frac{1}{2}(AB^*C + CB^*A) \]

is a \( JB^* \)-triple.

(ii) A subtriple of a \( JB^* \)-triple \( X \) is a Banach subspace which is closed with respect to the triple product, and every closed subtriple is also a \( JB^* \)-triple. Using this observation and (3) we see that \( \mathcal{K}(H) \), the compact operators on a Hilbert space, is a \( JB^* \)-triple and every \( C^* \)-algebra is a \( JB^* \)-triple. A subtriple of \( \mathcal{L}(H, K) \) is called a \( J^* \)-algebra or a special \( JB^* \)-triple.

(iii) If \( \Omega \) is a compact Hausdorff space, then \( C(\Omega) \), the continuous \( \mathbb{C} \)-valued functions on \( \Omega \) endowed with the supremum norm, is a \( JB^* \)-triple with triple product \( \{f, g, h\} = fgh \).

Associated with the triple product \( \langle \cdot, \cdot, \cdot \rangle \) we can define a number of natural real linear mappings:

\[ x \square x \in \mathcal{L}(X) \text{ where } x \square x(x) = \{x, y, x\}; \]
\[ Q_x \in \mathcal{L}(X) \text{ where } Q_x(y) = \{x, y, x\}. \]

Note that \( \delta(x) = x \square x \) and that \( x \square x \) is complex linear while \( Q_x \) is complex antilinear. The operators \( \square, \delta \) and \( Q \) are continuous (indeed real-analytic) functions of their arguments. The **Bergman operator,** \( B(x, y) \), which plays an important role in \( JB^* \)-triples, is defined using the above operators.

**Definition 1.4.** If \( (X, \langle \cdot, \cdot \rangle) \) is a \( JB^* \)-triple, we define \( B(x, y) \) for \( x, y \in X \) as

\[ B(x, y) = I_X - 2x \square x + Q_x \]

where \( I_X \) is the identity mapping on \( X \).

Clearly, \( B(x, y) \in \mathcal{L}(X) \) and \( B : X \times X \to \mathcal{L}(X) \) is a real-analytic function of \( x \) and \( y \).

**Example 1.5.** (i) If \( X = \mathcal{L}(H, K) \) then \( B(x, y)(z) = (I_K - xy^*)x(H - y^*x) \) for \( x, y, z \in \mathcal{L}(H, K) \).

(ii) If \( X = C(\Omega) \) then \( B(f, g)(h) = (1 - fg^*)h \) for \( f, g, h \in C(\Omega) \).

We say that the pair \( (x, y) \in X \times X \) is **quasi-invertible** if \( B(x, y) \) is an invertible operator in \( \mathcal{L}(X) \). If \( (x, y) \) is quasi-invertible, let

\[ x^y = B(x, y)^{-1}(x - Q_x(y)) \]

and call \( x^y \) the **quasi-inverse** of \( x \) with respect to \( y \). This concept of quasi-invertibility, as we shall see in Section 4, is related to the classical notion of quasi-inverse in a Jordan algebra.

On \( X \times X \) we define the equivalence relation \( \sim \) by \( (x, y) \sim (x_1, y_1) \) if, and only if, \( (y, x - y_1) \) is quasi-invertible and \( x_1 = x^y - y^t \). We denote the equivalence class containing \( (x, y) \) by \( [x : y] \). For each \( y \), we let \( U_y = \{x : y : x \in X\} \) and define \( \phi_y : U_y \to X \) by \( \phi_y(x : y) = x \). Let \( M_X = X / \sim \) be the set of all equivalence classes.

**Proposition 1.6** [6, 8, 9]. If \( X \) is a \( JB^* \)-triple then

(i) \( \phi_y(U_y \cap U_{y_1}) = \{x \in X : (x, y - y_1) \text{ is quasi-invertible}\} \) is open in \( X \);

(ii) \( \phi_{y_1} \circ \phi_y^{-1} : \phi_y(U_y \cap U_{y_1}) \to \phi_{y_1}(U_{y_1} \cap U_{y_1}) \) is the holomorphic mapping

\[ \phi_{y_1} \circ \phi_y^{-1}(x) = x^{y_1 - y} \]
and its derivative \( d(\phi^{-1} \circ \phi^{-1})(x) = B(x, y - y_1)^{-1} \), for all \( y, y_1 \in X \) and all \( x \in \phi_y(U_y \cap U_{y_1}) \);

(iii) the collection of charts \((U_y, \phi_y)_{y \in X}\) gives the structure of a connected complex Banach manifold to \(M_X\);

(iv) the canonical mapping \( \pi : X \times X \to M_X \) defined by \( \pi(x, y) = (x : y) \) is holomorphic in \( x \), antiholomorphic in \( y \) and jointly real-analytic in \( x \) and \( y \).

We refer to [8] for the algebraic properties of the quasi-inverse, some of which we cite for later convenience. The expression \((\mathfrak{P}m)\) refers to an identity in \([8]\).

(JPA1) If \((x, y)\) is quasi-invertible then \((x, y + x)\) is quasi-invertible if, and only if, \((x, y + x, y)\) is quasi-invertible and then \(x^{y + x} = (x^y)^x\).

(JPA2) If \((x, y)\) is quasi-invertible then \((x + z, y)\) is quasi-invertible if, and only if, \((z, y)^y\) is quasi-invertible and then \((x + z)^y = x^y + B(x, y)^{-1}_x(x^y)\).

(JPS) \((B(u, v)x, y)\) is quasi-invertible if, and only if, \((x, B(u, v)y)\) is quasi-invertible and then \((B(u, v)x, y) = B(u, v)x^B(u, v)y\).

(JPT) For \(t \in \mathbb{C}\), \((tx, y)\) is quasi-invertible if, and only if, \((t, y)\) is quasi-invertible and then \((tx)^y = t(x^y)\). In particular, \((-x)^y = -(x)^{-y}\).

If \(X\) is a \(JB^*\)-triple and \(x \in X\) then \(X_x\) denotes the \(JB^*\)-subtriple of \(X\) generated by \(x\). There is a unique locally compact subset \(K\) of \([0, \infty)\) such that \(K \cup \{0\}\) is compact, \(C_0(K)\) is isometrically isomorphic to \(X_x\) and \(x \mapsto xK(x)\) (see [6]). The set \(K\) is called the \textit{triple spectrum} of the element \(x\).

The embedding \(\psi_x : X_x \to X\) induces a holomorphic embedding \([1]\)

\[\psi_x : X_x \to X, \quad (y : z) \mapsto (\psi_x(y) : \psi_x(x)).\]

The canonical embedding of \(X\) in \(M_X\) is given by \(\phi^{-1}(x) \in X \to (x : 0) \in M_X\) and we identify \(X\) and \(\phi^{-1}(X)\).

The open unit ball \(B_X\) of the \(JB^*\)-triple \(X\) is the bounded symmetric domain associated with \(X\) and we now have a situation similar to (1) and (2) above, namely, the embeddings

\[B_X \to X \to M_X.\]

Recalling Definition 1.1, we say that a \(JB^*\)-triple \(X\) has the density property if \(U_0 = \phi^{-1}(X)\) is dense in \(M_X\).

2. The density property. We first obtain a simple characterization of the density property. This shows that a \(JB^*\)-triple has the density property if, and only if, there exists sufficiently many quasi-invertible pairs, and proves to be a useful practical tool in identifying spaces with or without the density property.

Proposition 2.1. If \(X\) is a \(JB^*\)-triple then the following are equivalent:

(i) \(X\) has the density property;
(ii) for all \(x, y \in X\) and \(\varepsilon > 0\), there exists \(z \in X\), \(\|z\| < \varepsilon\), such that \((x + z, y)\) is quasi-invertible;
(iii) for all \(x, y \in X\) and \(\varepsilon > 0\), there exists \(z \in X\), \(\|z\| < \varepsilon\), such that \((x + z, y)\) is quasi-invertible;
(iv) there exists a dense subset \(Z\) of \(X \times X\) consisting of quasi-invertible pairs.

Proof. Suppose (i) holds. Let \((x : y) \in M_X\) and \(\varepsilon > 0\). Then (i) implies that there exists \(w \in X\) such that \((w, 0) \sim (x', y')\) and \(\|w - x'\| < \varepsilon\). Hence \(x' \sim x\) has \(\|x' - x\| < \varepsilon\) and \((w, 0) \sim (x, y)\).

3. The density property for \(C(K)\) spaces. We discuss the density property for the commutative \(JB^*\)-triple \(C(K)\), where \(K\) is a compact Hausdorff space. By Example 1.5(ii), \(B(f, g)\) is invertible if, and only if, \(f(x)g(x) \neq 1\) for all \(x \in K\), where \(f, g \in C(K)\). By Propostion 2.1, \(C(K)\) has the density property if, and only if, for all \(f, g \in C(K)\) and all \(c > 0\) we can find \(h \in C(K)\), \(\|h\| < c\), such that \(f(x)g(x + h(x)) \neq 1\) for all \(x \in K\).

In the following theorem we identify \(f \in C(K)\) with \(f \in C(K, \mathbb{R})\) by means of \(f \mapsto (\mathfrak{A}f, \mathfrak{B}f)\) and we use the norm \(\|x + iy\| = \sup\{\|x\|, |y|\}\) on \(\mathbb{C}\).

Theorem 3.1. For \(K\) a compact Hausdorff space, \(C(K)\) has the density property if, and only if, \(\dim K \leq 1\).

Proof. We use the following facts about the covering dimension of a compact topological space \(K\):

(i) if \(L\) is a closed subset of \(K\) then \(\dim L \leq \dim K \leq \{10, p. 196\}\);
(ii) \(\dim K \geq n\) if, and only if, for any \(a, b \in \mathbb{R}\) there exists \(f \in C(K, [a, b]^n)\) such that for all \(g \in C(K, [a, b]^n)\) there exists \(x_g \in K\) such that \(f(x_g) = g(x_g)\) (see [4]).

Let \(\dim K \geq 2\). Consider the pair \((f, 1) \in C(K) \times C(K)\) where \(f \in C(K, [-2, 2]^2)\) has the property described in (ii) above. Suppose we can find
$h \in C(K)$, $\|h\| \leq 1/4$, such that $(f+h,1)$ is quasi-invertible. This would imply

$$(f+h)(x) \neq 1 \quad \text{for all } x.$$  

Since $\|h\| \leq 1/4$, we have $3/4 \leq R(1+h(x)) \leq 5/4$ and $-1/4 \leq 3(1+h(x)) \leq 1/4$ for all $x \in K$. So $1 + h \in C(K,[-2,2])$. Our choice of $f$ implies that there exists $x_0$ in $K$ such that $f(x_0) = 1 + h(x_0)$. This contradicts (i) and hence $C(K)$ does not have the density property.

Now suppose that $C(K)$ does not have the density property. Then there exists a pair $(f,g)$ in $C(K) \times C(K)$ and $\varepsilon > 0$ such that for all $h \in C(K)$, 

$$\|h\| \leq \varepsilon,$$

there exists $y_h$ in $K$ satisfying

$$4 \frac{|g|}{4(1+\|g\|)} = 1.$$  

We may assume that $\varepsilon < (4(1+\|g\|))^{-1}$ (otherwise replace $\varepsilon$ with $\varepsilon = 0 < \frac{1}{4(1+\|g\|)}$). Then for all $h$, $\|h\| < \varepsilon$, and $y_h$ as above,

$$\|h(y_h)g(y_h)\| \leq \|h\| \cdot \|g\| \leq \frac{\|g\|}{4(1+\|g\|)} \leq \frac{1}{4}$$

and

$$\|f(y_h)g(y_h)\| \geq 1 - \|h(y_h)g(y_h)\| \geq 3 \frac{4}{4}.$$  

Let $L = \{ x \in K : |f(x)\| \geq 3/4 \}$. Then $L$ is a compact subset of $K$ and we define $k \in C(L)$ by $k(x) = 1/\|g\| - f(x)$ for all $x \in L$. Let $k = k_1 + ik_2$ where $k_i$ are continuous real-valued functions. We define $k_i$, $i = 1,2$, as follows:

$$k_i(x) = \begin{cases} 
\varepsilon & \text{if } k_i(x) \geq \varepsilon, \\
0 & \text{if } -\varepsilon \leq k_i(x) \leq \varepsilon, \\
-\varepsilon & \text{if } k_i(x) \leq -\varepsilon.
\end{cases}$$

Clearly, each $k_i$ is continuous and $k = k_1 + ik_2 \in C(L,[-\varepsilon,\varepsilon]_2)$.

Let $w \in C(L,[-\varepsilon,\varepsilon]_2)$. Since compact sets are normal, there exists $\tilde{w} \in C(K,[-\varepsilon,\varepsilon]_2)$ such that $\tilde{w}|L = w$. It follows from (4) and (5) that for each $h \in C(K)$, $\|h\| \leq \varepsilon$, there exists $y_h$ in $L$ such that $h(y_h) = k(y_h)$ and hence there exists $y_w$ in $L$ such that $w(y_w) \geq \varepsilon$. As $w(y_w) \in [-\varepsilon,\varepsilon]_2$, $k(w(y_w)) = k(y_w)$ and hence $w(y_w) = k(y_w)$. An application of (ii) implies that dim $L \geq 2$ and therefore dim $K \geq 2$. 

**Example 3.2.** (i) A scattered compact topological space, e.g. the ordinals with the order topology, is 0-dimensional.

(ii) If $X$ is a totally disconnected topological space, in particular, if $X$ is discrete, then $X$ is 0-dimensional. Since the Stone–Čech compactification preserves dimension, it follows that $\beta X$ is a 0-dimensional compact space.

Hence $C(\beta X)$ has the density property. In particular, $C(\beta N) \cong \mathbb{R}$ has the density property.

(iii) If $A$ is a commutative von Neumann algebra then $A \cong C(K)$ where $K$ is an extremely disconnected compact Hausdorff space. Since extremely disconnected topological spaces are totally disconnected and hence 0-dimensional, $A$ has the density property. In particular, if $K$ is any compact Hausdorff space then the second dual $C(K)^{''}$ has the density property. Using Theorem 3.1 we now see that $C([0,1]^2)$ does not have the density property but its second dual $C([0,1]^2)^{''}$ does. This shows that the density property is not inherited by subspaces.

(iv) An $n$-dimensional differentiable manifold is $n$-dimensional as a topological space.

**Lemma 3.3.** Let $X = C_0(\Omega)$ have the density property. If $I \subset X$ is a closed (algebra) ideal in $X$ then $I$ has the density property.

**Proof.** Let $f,g \in I$ and let $\varepsilon > 0$. By the density property of $Z$ there exists $h \in Z$, $\|h\| < \varepsilon$, such that $0 \neq 1 - f(g + h)$. Upon replacing $f$ if necessary, we may assume that $\varepsilon < \min\{\|g\|, \|g\|^{-1}\}$. Define a function $j$ as follows: $j(x) = 1$ if $|f(x)| \geq \varepsilon$ and $j(x) = |f(x)|/\varepsilon$ if $|f(x)| < \varepsilon$.

Since $I$ is a closed ideal in $C_0(\Omega)$, there exists a (closed) subset $\Sigma$ of $\Omega$ such that $I = \{ k \in Z : k|\Sigma = 0 \}$ and as $j$ vanishes with $f$, $j$ is also in $I$. Let $h' = jh \in I$. Then $|h'| \leq |j| \cdot |h| \leq |h| \leq \varepsilon$. Suppose now that $0 = 1 - f(g + h')(x)$ for some $x \in \Omega$. Then either

$$|f(x)| \geq \varepsilon \Rightarrow 1 - f(g + h')(x) = 1 - f(g + h)(x) \neq 0,$$

which gives a contradiction, or

$$|f(x)| < \varepsilon \Rightarrow |f(g + h')(x)| \leq \varepsilon \|g + h'\| < 1$$

and hence $1 - f(g + h')(x) \neq 0$, again giving a contradiction. We conclude that for all $f,g \in I$ and $\varepsilon > 0$ there exists $h' \in I$ with $|h'| \leq \varepsilon$ such that $0 \neq 1 - f(g + h')(x)$, that is, $I$ has the density property.

**Corollary 3.4.** Let $X$ be a $J_b^*$-triple, and let $x \in X$. Then $X_x$ has the density property.

**Proof.** We know that $X_x \cong C_0(\Omega)$ for some locally compact subset $\Omega$ of $\mathbb{R}$ such that $\Omega \cup \{0\}$ is compact. Since $\Omega \cup \{0\}$ is zero- or one-dimensional, $C(\Omega \cup \{0\})$ has the density property. But $C_0(\Omega)$ is a closed ideal of $C(\Omega \cup \{0\})$ and so has the density property by Corollary 3.3.

We note that if $\Omega$ is a locally compact Hausdorff space with dim $\Omega \leq 1$ then, since the Stone–Čech compactification preserves dimension, dim $\beta \Omega \leq 1$ and thus $C(\beta \Omega)$ has the density property. It then follows from Lemma 3.3 that $C_0(\Omega)$ has the density property.
4. The $JB^*$-algebra case. The previous section was devoted to the commutative case (at least from the point of view of operator theory, while one might consider it the associative case from the algebraic perspective). In this section we discuss the density property in the non-commutative setting using $JB^*$-algebras.

**Definition 4.1.** A (complex) Jordan algebra is a vector space over $\mathbb{C}$ with a bilinear composition law $\circ : A \times A \to A$ satisfying

\begin{align*}
(6) & \quad x \circ y = y \circ x, \\
(7) & \quad (x^2 \circ y) \circ z = x^2 \circ (y \circ z).
\end{align*}

**Definition 4.2.** A Banach space $(X, \| \cdot \|)$ which is a complex Jordan algebra equipped with an involution, $\ast$, is a $JB^*$-algebra if

\begin{align*}
(8) & \quad \|x \circ y\| \leq \|x\| \cdot \|y\|, \\
(9) & \quad \|x^\ast\| = \|x\|, \\
(10) & \quad \|x \circ x^\ast\| = \|x\|^2.
\end{align*}

A $JB^*$-algebra with identity, $e$, is called a unital $JB^*$-algebra. By [3, 3.3.9 and 3.8.3] a unit can be added to a $JB^*$-algebra and the norm extended to give a unital $JB^*$-algebra. An important example of a unital $JB^*$-algebra is the space of operators on a complex Hilbert space endowed with the product

$$ A \circ B = \frac{1}{2}(AB + BA). $$

An special $JB^*$-algebra, or $JC^*$-algebra, is one which can be realised as an algebra of operators on a Hilbert space, with product defined by (11). From the Gelfand–Naimark theorem any $C^*$-algebra is a $JC^*$-algebra. A closed subspace of $A(B, H)$, where $H$ is a Hilbert space, which is closed under taking adjoints and squares is a $JC^*$-algebra with product given by (11).

For any $JB^*$-triple $X$ and any $y \in X$ we may define a binary product on $X$ by

$$ x \circ y := \{x, y, z\}. $$

This product gives $X$ the structure of a Jordan algebra, which we denote by $X^y$. Moreover, if $y$ is a unitary tripotent then $X^y$ is a $JB^*$-algebra. Conversely, given a $JB^*$-algebra $X$, we define the triple product

$$ (x, y, z) = (xy^*)z - (zx)yx + (yx^*)x $$

and with this product $(X, \langle \cdot, \cdot, \cdot \rangle)$ is a $JB^*$-triple. (If the $JB^*$-algebra is generated from a $JB^*$-triple with unitary tripotent, then (12) regains the original triple product.)

Not every $JB^*$-triple arises in this way and those that do may be characterised geometrically. A $JB^*$-triple is $J^*$-isomorphic (i.e. linearly isomorphic by a mapping which preserves the triple product) to a unital $JB^*$-algebra if, and only if, its unit ball contains a unitary tripotent (see [13, 20.35]). For special $JB^*$-triples this reduces to the triple containing a unitary operator.

Inverses are defined in a unital $JB^*$-algebra so that in the special $JB^*$-algebra case the inverse coincides with the inverse arising from the original product. The standard definition is given in part (i) of the following proposition.

**Proposition 4.3.** Let $Z$ be a unital Jordan algebra with involution. If any of the following equivalent conditions are satisfied by the elements $x$ and $y$ in $Z$ then we say that $x$ is invertible with inverse $y$:

(i) $xy = 1$ and $x^2y = x$;
(ii) $xy = 1$ and $y^2x = y$;
(iii) $Q_x$ is invertible and $y^* = Q_x^{-1}(x)$;
(iv) $Q_x(y^*) = x$ and $Q_x(yy^*) = 1$ where $Q_x(y) = \{x, y, x\}$.

The inverse of $x$ is unique and is written as $x^{-1}$. Also, $(x^{-1})^{-1} = x$ and $(x^{-1})^* = (x^*)^{-1}$, and the Jordan subalgebra generated by an element $x$ is power associative, i.e. $(x^n)^m = x^{n+m}$ for any integers $n$ and $m$ [14, p. 304, Thm. 5].

**Example 4.4.** Let $X = C(K)$ where $K$ is a compact Hausdorff space. In Example 1.3 we noted that $C(K)$ is a $JB^*$-triple with triple product $\{f, g, h\} = fg\bar{h}$ and involution $g^* = \bar{g}$. If $g \in C(K)$ then $Q_g(f) = \bar{g}f$ and hence $Q_g$ is invertible if and only if $g(x) \neq 0$ for all $x \in K$. In such a case, $1_g := Q_g^{-1}(g) = (\bar{g})^{-1}\bar{g} = 1/\bar{g}$ is the identity in $C(K)^0$. An element $f$ in $C(K)$ is invertible in $C(K)^0$ with inverse $\tilde{f}$ if, and only if, $f \circ_\ast \tilde{f} = 1_g$, i.e. $f\bar{g} = 1/\bar{g}$ and hence $\tilde{f} = 1/fg^2$.

If $X$ is a $JB^*$-triple, $y \in X$ and $Q_y$ is not invertible then we can adjoin an identity $1_y$ to the Jordan algebra $X^y$ to obtain a unital Jordan algebra, $X^y_{1_y}$. We say that $x$ is quasi-invertible in $X^y$ with quasi-inverse $z$ if $1_y - x$ is invertible in $X^y_{1_y}$ with inverse $1_y + z$. That is,

$$ (1_y - x) \circ_\ast (1_y + z) = 1_y \quad \text{and} \quad (1_y - x)^2 \circ_\ast (1_y + z) = 1_y - x. $$

In particular, we see that $x - z + x \circ_\ast z = 0$.

If $L$ is a locally compact Hausdorff space, $X = C_0(L)$ and $g \in X$, then $f$ is quasi-invertible in $C_0(L)^0$ if, and only if, $f\bar{g}(x) \neq 1$ for all $x \in L$ and the quasi-inverse $f^0$ of $f$ satisfies $f - f^0 = f\bar{g}^2$. Hence

$$ f^0 = \frac{f}{1 - f\bar{g}}. $$

Further analysis shows that in an arbitrary $JB^*$-triple $X$, an element $x$ is quasi-invertible in $X^y$ if and only if $B(x, y) \in C(X)$ is invertible and then
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Theorem 4.8. A C$^*$-algebra $A$ with identity has the density property if, and only if, $\text{tsr}(A) = 1$.

This means of course that, by the results in [12], Theorem 4.8 contains Theorem 3.1 as a special case. We included the proof of 3.1 because of its directness. The results in [12] lead to a number of examples, which we list further on, and also motivates the study of topological stable rank in JB$^*$-triples.

We now seek to remove the hypothesis of an identity from some of the previous results.

Lemma 4.9. If $Z$ is a JB$^*$-algebra then the following are equivalent:

(a) the quasi-invertible elements are dense in $Z$;
(b) the quasi-invertible elements are dense in $Z_{\#}$;
(c) the invertible elements are dense in $Z_{\#}$.

Proof. Since translation by the identity is a continuous operation, (b) and (c) are equivalent. Suppose (a) holds. Given $x + \alpha \in Z_{\#}$ and $\varepsilon > 0$, choose a complex number $\gamma \neq -1$ such that $|\alpha - \gamma/(1 + \gamma)| < \varepsilon$ and choose $x' \in Z$ quasi-invertible such that $||x' + (1 + \gamma)|| \leq 1 + \gamma$. Since $x'$ is quasi-invertible in $Z$, it follows that $1 - x'/(1 + \gamma) = (x' + \gamma) - 1/(1 + \gamma)$ and $1 - x'/(1 + \gamma)$ are all invertible in $Z_{\#}$. Hence $x'/(1 + \gamma) + \gamma/(1 + \gamma)$ is quasi-invertible and

$$||x + \alpha - x'/(1 + \gamma) - \gamma/(1 + \gamma)|| \leq ||x - x'|| + ||\alpha - \gamma/(1 + \gamma)|| \leq 2\varepsilon,$$

which implies (b).

Now suppose (b) holds. Given $x \in Z$ and $\varepsilon > 0$ we can find $x' \in Z$ and $\alpha \in \mathbb{C}$ such that $x' + \alpha$ is quasi-invertible, $||x - x'|| \leq \varepsilon$ and

$$|\alpha| < \min \left( \frac{\varepsilon}{||x'|| + \varepsilon}, \frac{1}{2} \right).$$

Hence $|\alpha( ||x|| + \varepsilon) | < \varepsilon$ and $(1 + \alpha)^{-1} < 2$.

Since $x' + \alpha$ is quasi-invertible, we see that

$$1 - (x' + \alpha) = 1 + \alpha - x' = (1 + \alpha) \left( 1 - \frac{x'}{1 + \alpha} \right)$$

is invertible and hence $x'/(1 + \alpha) \in Z$ is quasi-invertible. The estimate

$$||x - x'/(1 + \alpha)|| = ||x - x' + x' \left( 1 - \frac{1}{1 + \alpha} \right)|| \leq \varepsilon + (||x'|| + \varepsilon) \frac{\alpha}{1 + \alpha} < \varepsilon + 2\varepsilon$$

completes the proof.

Theorem 4.10. If $Z$ is a JB$^*$-algebra such that the quasi-invertibles are dense in $Z$ then $Z$ has the density property.
The density property for JB*-triples

(b) The following spaces do not have the density property:
(i) \( \mathcal{L}(H) \), the bounded linear operators on an infinite-dimensional Hilbert space,
(ii) \( C^*(S) \), the C*-algebra with identity generated by the unilateral shift \( S : H \to H \), \( Se_i = e_{i+1} \), where \( (e_i)_{i=1}^{\infty} \) is a basis for the Hilbert space \( H \),
(iii) any unital C*-subalgebra of \( \mathcal{L}(H) \) which contains a Fredholm operator of non-zero index,
(iv) any unital C*-algebra containing two isometries with orthogonal ranges.

All of these examples are given in [12] and follow from Theorems 4.7 and 4.8. Some of them also follow from earlier results. For example, Rickart [11, p. 279] shows that a non-invertible linear operator in \( \mathcal{L}(H) \) with closed range lies in the interior of the singular elements of \( \mathcal{L}(H) \) (this gives example (b)(i) and (ii)), while it is well known that the Fredholm operators of a given index form an open set of operators and this implies (b)(iii). Moreover, by [12, Proposition 3.1] the invertible, left invertible, and right invertible elements in a C*-algebra \( A \) coincide if \( \text{tr}(A) = 1 \), and hence any C*-algebra in which these do not coincide (e.g. (b)(i), (ii) and (iii)) fails to have the density property.

To see directly that \( \mathcal{K}(H) \) has the density property we return to Example 1.5(i) and use spectral theory. If \( x, y \in \mathcal{K}(H) \) then \( x^*y \in \mathcal{K}(H) \) and hence \( \sigma(xy^*) \) is countable, so there exists \( \lambda \) close to 1 such that \( \lambda \notin \sigma(xy^*) \). This implies that \( \lambda - xy^* = \lambda(1 - (x/\lambda)y)^* \) and hence \( (1 - (x/\lambda)y)^* \) are invertible operators. By 1.5(i) and Proposition 2.1 this implies that \( \mathcal{K}(H) \) has the density property. More generally, one can show, using the same proof, the following result.

**Proposition 4.12.** If \( Z \) is a C*-algebra and \( \sigma(x) \) has empty interior for all \( x \in Z \) then \( Z \) has the density property.

5. Complete holomorphic vector fields on \( M_X \). We now study the holomorphic structure of the quasi-invertible manifold, \( M_X \), of a JB*-triple \( X \). In particular, we are interested in the consequences of the density property for the geometry of the manifold. A fundamental result will be that \( M_X \) is a homogeneous manifold whenever \( X \) has the density property. The proof of this fact is based on the ability to extend translations on \( X \) to biholomorphic maps of \( M_X \). This leads to an examination of a conjecture by Dorfmeister [2] asserting that translations on \( X \) extend to biholomorphic maps on the manifold \( M_X \) if and only if a certain algebraic condition on \( X \) is satisfied. We prove this conjecture in one direction.
We begin by following the arguments of Loos [8] for the finite-dimensional case, where three different types of biholomorphic map are defined. For each \( a \in X \), define a map \( t_a : M_X \to M_X \) by

\[
t_a(x : y) = (x : y + a).
\]

It is easy to see that \( t_a \) is well defined on \( M_X \) and is biholomorphic with inverse \( t_{-a} \) (see [8, 8.4(a)]).

The second type of biholomorphic map is an extension of a \( JB^* \)-triple automorphism. The set of automorphisms of the \( JB^* \)-triple \( X \) is denoted by \( \text{Aut}(X) \). The structure group of \( X \) (see [7]) is the set of all \( f \in GL(X) \) for which there exists \( \tilde{f} \in GL(X) \) satisfying \( f(x, y, z) = \{fx, fy, fz\} \) and \( \tilde{f}(x, y, z) = \{fx, fy, \tilde{f}z\} \). It is easy to check that \( f \) is uniquely determined by \( \tilde{f} \) and that the map \( f \mapsto \tilde{f} \) is antiholomorphic, of period two, and has \( \text{Aut}(X) \) as its fixed point set. Moreover, the structure group forms a complex Lie group (being an algebraic subgroup of the complex Lie group \( GL(X) \times GL(X) \times GL(X) \) by \( X \) is the complex conjugate Banach space of \( X \), as opposed to \( \text{Aut}(X) \), which is a real Lie group. We denote the structure group of \( X \) by \( \text{Aut}(X, X) \). Compare [8, Section 3]. Any element \( f \in \text{Aut}(X, X) \) extends to a biholomorphic map on the manifold \( M_X \) via \( (x : y) \mapsto (fx : \bar{f}y) \).

The third type of biholomorphic maps on \( M_X \) that we consider are possible extensions of translations on \( X \), namely, extensions of the map \( (x : 0) \mapsto (x + a : 0) \) on \( X \). The naive attempt at an extension, \( (x : y) \mapsto (x + a : y) \), is not well defined on \( M_X \) and the finite-dimensional approach of Loos [8, 8.4(c)], which depends on the fact that holomorphic vector fields on a compact manifold are complete, no longer applies. The following conjecture, made by Dorfmeister [2], introduces a property on the \( JB^* \)-triple \( X \) which, by openness of the set of invertible operators, is easily seen to be implied by the density property.

**Definition 5.1.** A \( JB^* \)-triple \( X \) satisfies condition (D) if for all \( x, y, z \in X \) there exists \( c \in X \) such that \( (x, c) \) and \( (x, y - c) \) are quasi-invertible.

**Conjecture 5.2.** The map \( (x : 0) \mapsto (x + a : 0) \) can be extended to a biholomorphic map of \( M_X \) for all \( u \in X \) if, and only if, the \( JB^* \)-triple \( X \) satisfies condition (D).

We prove this conjecture in one direction.

**Theorem 5.3.** If \( X \) is a \( JB^* \)-triple which satisfies condition (D) then for all \( x, y, a \in X \), there exists \( c \in X \) such that

\[
t_a(x : y) := (B(a, c)(a^c + x^{c-a}) : c^a)
\]

is well defined on \( M_X \). Moreover, \( t_a(x : y) \) does not depend on the choice of \( c \), \( t_a \) is a well defined biholomorphic mapping on \( M_X \) with inverse \( t_{-a} \) and \( t_a(x : 0) = (x + a : 0) \) for all \( x \in X \).

**Remark.** The formula in (13) was obtained by the following procedure. The constant vector field \( \xi_0 \) on \( X \) given by \( \xi_0(x : 0) = a^c \) generates the one-parameter group of translations \( (x : 0) \mapsto (x + a : 0) \), \( t \in \mathbb{R} \), and moreover, each \( \xi_0 \) extends to a holomorphic vector field on \( M_X \) by the formula \( \xi_0(x : y) = B(x, y)a^c \) on the chart \( U_0 \) (compare [8, 8.4(c)]). To obtain (13) it suffices to show that this vector field is complete on \( M_X \) and to take the exponential.

If \( x^y \) exists, it is easily verified that \( \alpha(t) = (x^y + ta)^{-y} : y \) is an integral curve of \( \xi_0 \) subject to the initial condition \( \alpha(0) = (x : y) \). To show that \( \alpha \) can be defined on all of \( \mathbb{R} \) it is necessary to find a new expression for \( \alpha(t) \) and we consider

\[
\begin{align*}
((x^y + ta)^{-y} : y) &= ((x^y + ta)^{-y} : z) \quad \text{for any } z \in X \\
&= ((x^y + ta)^{-y} : c^a) \\
&= B(ta, c)(t^a)^{-y} : (c^a).
\end{align*}
\]

The final expression on the right hand side gives (13) when \( t = 1 \) and, under the hypothesis of condition (D), such an element \( c \) can be found.

**Proof of 5.3.** Given \( (x, y) \) and \( a \in X \), we can find, using condition (D), \( c \in X \) such that \( (a, c) \) and \( (x, y - c) \) are quasi-invertible pairs. If \( (x : y) = (x_1 : y_1) \), then \( x^{y_1} - x \) exists and equals \( x_1 \). By (JP1), \( x^{y_1} - x = (x^{y_1} - x_1)^{-c} = x^{y_1} - x \) exists and therefore \( t_a(x : y) = t_a(x_1 : y_1) \). Hence \( t_a \) will be well defined if the formula in (13) is independent of the choice of \( c \). In other words, if \( d \in X \) is another element such that \( a^d \) and \( x^{d} \) exist, we must check that \( B(a, c)(a^c + x^{c-a}) \) exists and equals \( B(a, d)(a^d + x^{d}) \).

Notice first that, by (JP1) and (JPT), \( (a^{-c})^c \) exists and equals \( -a \), \( (a^{-c})^c \) exists and equals \( c - Q_c a \), and, using the fact that \( (d^c)^c = d^c \) exists, we find by (JP2) that \( (c - d)^{-c} \) exists and

\[
(c - d)^{-c} = (a^{-c}) + B(c_1 - a^{-1}(d^{-c})^c)
\]

\[
= c - Q_c a + B(c, a)(d^c - a^c).
\]

Since \( a^d = (a^c)^{-c} \) by (JPT), and \( x^{d} = (x^{c-a})^{-c} \), it follows, again by (JP2), that \( (a^c + x^{c-a})^{-c} \) exists and

\[
a^d + x^{d} = (x^{c-a})^{-c} + (a^{-c})^{-c} = B(a, c)(a^{-c} + x^{c-a})^{-c}.
\]

Using the fact that

\[
B(-a^c, c - d) = B(-a^c, c)B((-a^c)^{-c}, -d)
\]

\[
= B(a, c)^{-1}B((-a^c)^{-c}, -d) = B(a, c)^{-1}B(-a, -d)
\]

\[
= B(a, c)^{-1}B(a, d)
\]
we get
\[ B(a, d)(a^d + x^d) = B(a, d)B(a, d)^{-1}B(a, c)((a^c + x^c)(c^{-d})^{-s}) = B(a, c)((a^c + x^c)(c^{-d})^{-s}). \]

Now (15) and (16) give
\[ B(a, d)(a^d + x^d) = B(a, c)((a^c + x^c)^B(a, c)c^{(c^{-d})}). \]

Finally, by (JPS), we see that \( B(a, c)(a^c + x^c) \) exists and (17) becomes
\[ B(a, d)(a^d + x^d) = B(a, c)(a^c + x^c)^{c^{-d}}, \]

which is precisely what we required.

If \((x, y - c)\) is quasi-invertible then \((x', y - c)\) is quasi-invertible for all \(x'\) near \(x\). Hence, Proposition 1.6(ii) implies that \(t_a\) is holomorphic. If \(y = 0\) then we may take \(c = 0\) in (13) and we see that \(t_a(x : 0) = (a + x : 0)\). This also shows that \(t_a \circ t_a(x : 0) = (x : 0)\) and since \(t_a \circ t_a\) is holomorphic, the identity principle for holomorphic mappings shows that \(t_a\) is biholomorphic with inverse \(t_{-a}\).

**Corollary 5.4.** Let \(X\) be a \(JB^*\)-triple satisfying condition (D). Then, for all \(a \in X\), the vector field \(\xi_a\) is complete on the manifold \(M_X\).

**Proof.** Clearly, the mapping \(s \in \mathbb{R} \mapsto t_a\) defines a one-parameter subgroup of biholomorphic mappings on \(M_X\) which generates the vector field \(\xi_a\), so \(\xi_a\) is complete on \(M\).

**Corollary 5.5.** Let \(X\) be a \(JB^*\)-triple with the density property. Then, for all \(a \in X\), the vector field \(\xi_a\) is complete on the manifold \(M_X\).

**Corollary 5.6.** Let \(X\) be a \(JB^*\)-triple satisfying condition (D). Then \(M_X\) is a complex homogeneous manifold.

**Proof.** For \((x : a) \in M_X\), we have \((x : a) = t_a(0 : 0)\).

Another consequence of condition (D) (and hence of density) is that, similar to the finite-dimensional case mentioned in the introduction, biholomorphic maps of the open unit ball, \(B_X\), extend to give biholomorphic maps on \(M_X\).

**Corollary 5.7.** Let \(X\) be a \(JB^*\)-triple satisfying condition (D). Every biholomorphic automorphism of the open unit ball \(B_X\) of \(X\) has an extension to a biholomorphic map on \(M_X\).

**Proof.** Every automorphism \(g \in \text{Aut}(B_X)\) has a decomposition \(g = kp\) where \(k\) is a \(JB^*\)-triple automorphism of \(X\) (i.e. a surjective linear isometry of \(X\)) restricted to the unit ball, and \(p\) takes the form \(p = p_e (c = p(0))\) where
\[ p_e(z) = c + B(c, c)^{1/2}(I + z \square c)^{-1}z. \]

See [7] for details. Clearly, \(k\) extends to \(M_X\) as we saw earlier. Note that \((I + z \square c)^{-1}z = c^{-1}z\) so \(p_e(z) = t_e \circ B(c, c)^{1/2} \circ t_e^{-1}(z)\) and knowing that \(t_e\) and \(t_{-c}\) both extend to biholomorphic maps on \(M_X\), we must simply check that the linear map \(B(c, c)^{1/2}\) extends biholomorphically to \(M_X\). However, as a corollary to [7, Theorem 3.3], \(B(c, c)^{1/2}\) is in the structure group of \(X\) and hence extends to \(M_X\).

We conclude by showing that the converse to Corollary 5.5 is false, and it follows that either the density property is strictly stronger than condition (D), or Conjecture 5.2 is false. Our counterexample occurs in the \(JB^*\)-triple \(C(\Delta)\), which by 2.1 does not have the density property although the constant vector fields on \(C(\Delta)\) do extend to the quasi-invertible manifold. To see this, we make use of a simple lemma.

**Lemma 5.8.** Given \(f, g \in C(\Delta)\) such that, for any \(x \in \Delta\), \(f(x)\) and \(g(x)\) are not both zero, there exists \(h \in C(\Delta)\) such that \(0 \not\in f + gh(\Delta)\).

**Corollary 5.9.** Let \(X = C(\Delta)\). For all \(a \in X\) the map \((x : 0) \mapsto (x + a : 0)\) extends to a biholomorphic map on \(M_X\) and hence the vector field \(\xi_a\) is complete on \(M_X\).

**Proof.** We know that if the vector field \(\xi_a\) is complete on \(M_X\) then the automorphism \(t_a\) takes the symbolic form (see (14))
\[ t_a(x : y) = ((x^y + a)^{-y} : y) = ((x^y + a)^{-b} : b). \]

The above lemma shows that for any \(x, y\) and \(a\) in \(X\), we can choose \(b \in X\) such that \((x^y + a)^{-b}\) exists in the following sense. In \(X^y = x/(1 - xy)\) exists if \(0 \not\in 1 - xy(\Delta)\). Symbolically,
\[ x^y + a = \frac{x + a(1 - xy)}{1 - xy} \]
and
\[ (x^y + a)^{-b} = \frac{x + a(1 - xy)}{1 - xy} = \left(1 + \frac{x + a(1 - xy)}{1 - xy}\right)^{-1} = \frac{x + a(1 - xy)}{1 - y + \hat{b}(x + a(1 - xy))}. \]

Letting \(f = 1 - xy\) and \(g = x + a(1 - xy)\) we see that \(f\) and \(g\) cannot take the value 0 simultaneously, so by the previous lemma, there exists a continuous function \(h\) with \(0 \not\in f + gh(\Delta)\). Letting \(b = h\), we have \(0 \not\in 1 - xy + \hat{b}(x + a(1 - xy))\) and thus
\[ \frac{x + a(1 - xy)}{1 - xy + \hat{b}(x + a(1 - xy))} \]
exists. Next define \(t_a\) using this fact:
It can be checked as in Theorem 5.3 that \( t_a \) is independent of the \( b \) chosen and of the representation of the equivalence class \((x : y)\). Finally, \( t_a \) is a biholomorphic map on the manifold \( M_X \) and for \( \gamma \in \mathbb{R} \) sufficiently small, \( t_{a,\gamma} \) coincides with the flow of \( \xi_a \) at time \( \gamma \) through the point \((x : y)\). By uniqueness of this flow we see that \( \xi_a \) is a complete holomorphic vector field and \( \exp(\xi_a) = t_a \).

References


Department of Mathematics
University College Dublin
Belfield, Dublin 4, Ireland
E-mail: sean.dineen@ucd.ie
Michael.Mackey@ucd.ie
Pauline.Mellon@ucd.ie

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