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An application of the Nash–Moser theorem to ordinary differential equations in Fréchet spaces

by

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Abstract. A general existence and uniqueness result of Picard–Lindelöf type is proved for ordinary differential equations in Fréchet spaces as an application of a generalized Nash–Moser implicit function theorem. Many examples show that the assumptions of the main result are natural. Applications are given for the Fréchet spaces $C^\infty(K)$, $\mathcal{S}(\mathbb{R}^N)$, $\mathcal{B}(\mathbb{R}^N)$, $\mathcal{D}_{L_1}(\mathbb{R}^N)$, for Köthe sequence spaces, and for the general class of subbinomic Fréchet algebras.

0. Introduction. The purpose of this paper is to prove a general existence and uniqueness result for ordinary differential equations in Fréchet spaces as an application of the implicit function theorem of Nash–Moser type [21], 4.2.

The well known theorem of Picard–Lindelöf yields the unique solvability of initial value problems for ordinary differential equations in Banach spaces; a proof based on the implicit function theorem is given in [24]. The situation in Fréchet spaces is quite different. A comprehensive survey of the present theory of ordinary differential equations in locally convex spaces is contained in [12]. There are known a lot of negative results showing that a straightforward generalization of the theorem of Picard–Lindelöf (or of Peano's theorem) to Fréchet spaces fails, and positive results are only obtained under rather restrictive assumptions (see e.g. [3]–[6], [11] and the literature cited in [12]).

In the main theorem 4.1 of this paper the following is proved. Let E be a Fréchet space equipped with a fixed fundamental system of norms $|\ |_0 \leq |\ |_1 \leq \ldots$ such that E and $C([-1,1],E)$ have the smoothing property $(\mathrm{S}_\Omega)_t$ (introduced in [19], cf. 1.1) and E satisfies condition (DN) of Vogt ([26], cf. 1.1), and let $f : U \to E$ be a C^2-map defined in an open set $U \subset \mathbb{R} \times E$ such that $|\partial^\alpha f(u)(z)|_n \leq c_n[u;z]_n$ for all $|\alpha| \leq 2$, $n \geq 0$, $u \in U$ and some constants $c_n > 0$; these estimates are formulated in terms of the weighted

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multiseminorms $[\ ]_n$ (introduced in [21], cf. 1.3). Then for $(t_0, y) \in U$ and small enough $a > 0$ the initial value problem

$$x'(t) = f(t, x(t)), \qquad x(t_0) = y,$$

has a unique solution $x \in C^1([t_0 - a, t_0 + a], E)$ which is C^m-depending on y (and also on additional parameters) if f is C^m, $2 \le m \le \infty$.

The assumptions of 4.1 are satisfied for large classes of nonlinear maps on many Fréchet algebras. Section 5 contains examples for the space $C^\infty(K)$ where $K = \overline{\mathring{K}}$ is a compact set in $\mathbb{R}^N$, the space $\mathcal{S}(\mathbb{R}^N)$ of rapidly decreasing functions, the space $\mathcal{B}(\mathbb{R}^N)$ of all C^∞-functions having bounded partial derivatives, and some convolution algebras $(\lambda^1(a), *)$ of Köthe sequence spaces. Moreover, the result applies to polynomial maps in what we call subbinomic Fréchet algebras; a further example is $\mathcal{D}_{L_1}(\mathbb{R}^N)$ (cf. 5.5).

The proof of the main result, Theorem 4.1, is based on a variant of a Nash–Moser type implicit function theorem in Fréchet spaces. Examples 3.1, 3.2 show that—even in the easiest case of linear equations with constant coefficients—any generalization of Picard–Lindelöf's theorem to Fréchet spaces fails if a loss of derivatives is involved. Therefore, the assumptions of Theorem 4.1 do not contain a loss of derivatives. However, very general and precisely described constants are allowed instead in the assumptions of Theorem 4.1 which give many examples not accessible by standard methods.

For instance, Examples 5.4, 5.5 in Köthe sequence spaces and in subbinomic Fréchet algebras including applications to convolution algebras or the nonlocal examples in 5.1, 5.2 where the right hand side f is given by means of an integral operator cannot be obtained by e.g. simply solving ordinary differential equations depending on a parameter or by applying standard Banach space results. Note that if the Fréchet space E is the infinite intersection of Banach spaces E_n and the Banach space result is applied in each E_n yielding a solution in some interval I_n then these intervals I_n may in general shrink to one point and the existence of a solution in the Fréchet space is by no means clear.

The Nash–Moser technique introduced in the proof of Theorem 4.1 is based on a suitable transformation in the time variable and on an application of the implicit function theorem [21], 4.2. This general method is by no means restricted to the case of ordinary differential equations only. For instance, in the forthcoming papers [22], [10] applications containing a loss of derivatives are given for nonlinear parabolic evolution equations [22] and nonlinear Schrödinger equations [10]. Note that such applications require additional properties of the corresponding linear semigroups. The main theorem proved in this paper can be considered as the "optimally general" (in the sense of 3.1, 3.2) result which can be achieved without requiring additional properties of the corresponding linear semigroups.

The proof of Theorem 4.1 uses the implicit function theorem [21], 4.2, which requires properties $(\mathrm{S}_\Omega)_\mathrm{t}$ and (DN) as assumptions on the spaces involved. In Section 2 we give sufficient conditions for both E and $C^n([-1,1], E)$ to have property $(\mathrm{S}_\Omega)_\mathrm{t}$. For instance, it is enough that E is tamely nuclear and has (Ω_{DZ}), or that E is a graded Fréchet–Hilbert space which has (Ω_S), or that E is locally l_1 and has (Ω_{DS}) (see Section 1 for the notation).

In Section 3 the case of linear differential equations is considered. The results of this section are essential for the proof of Theorem 4.1. Most important and of independent interest is the generalization 3.6 of Gronwall's lemma to Fréchet spaces which gives precise estimates for the norms of the solutions of linear differential equations in Fréchet spaces in terms of the right hand side, the initial value and the equation. Lemma 3.6 applies in situations with a loss of derivatives as well (cf. [22], Lemma 5.6).

The assumptions on the mappings in Theorem 4.1 are motivated as follows. The elementary example 3.2 shows that there is a continuous linear diagonal operator A on the space s of rapidly decreasing sequences so that for some $z \in s$ the initial value problem $x'(t) = Ax(t)$, $x(0) = z$, does not admit any C^1-solution near 0 while, on the other hand, A satisfies the strong continuity estimates $|Ax|_t \le C_{t,\varepsilon}|x|_{t+\varepsilon}$ for all $t, \varepsilon > 0$. We thus cannot expect a general existence theorem (under assumptions based on estimates) without supposing normwise tame estimates of the form $|Ax|_n \le c_n|x|_n$ in the case of linear equations $x'(t) = Ax(t)$ with constant coefficients.

The example $x'(t) = f(t, x(t))$, $f(t, x) = x^2$, in the space $C^\infty[0,1]$ shows that in general it does not seem reasonable to suppose local Lipschitz conditions of the form $|f(t,x) - f(t,y)|_n \le L_n|x - y|_n$ since—in contrast to Banach spaces—the constants L_n cannot generally be chosen independently of x, y for all n and all x, y in a neighbourhood of some point. This obstacle is overcome by the use of the weighted multiseminorms $[\ ]_n$ which facilitate a precise description of the occurring constants when estimating the norms of nonlinear maps $f(t, x)$ and their derivatives.

When formulating the assumptions of Theorem 4.1 we proceed similarly to the case of the classical Nash–Moser theorem (cf. [5], [7], [13], [14]): The natural estimates obtained for smooth nonlinear maps of the form $f(u) = F(\cdot, u(\cdot))$, $F \in C^\infty$, and their derivatives on $C^\infty(K)$ for a compact $K = \overline{\mathring{K}} \subset \mathbb{R}^N$ constitute the assumptions of the general result; these estimates are normwise tame in the particular case of linear equations with constant coefficients and can easily be stated using the expressions $[\ ]_n$ (cf. [21], 1.6). Example 5.1 mainly serves to show that the assumptions of Theorem 4.1 are natural.

Acknowledgements. The author thanks L. Frerick for drawing the author's attention to [2], Prop. 4.4 (cf. 2.2), and Prof. D. Vogt for a hint on the proof of Lemma 2.1.

1. Preliminaries. In this section we state some notations and formulate the implicit function theorem which is proved in [21].

A Fréchet space E equipped with a fixed sequence $|\ |_0 \le |\ |_1 \le |\ |_2 \le \ldots$ of seminorms defining the topology is called a *graded Fréchet space* (cf. [5]). Graded subspaces and graded quotient spaces are equipped with the induced seminorms, and the product $E \times F$ is graded by $|(x,y)|_k = \max\{|x|_k, |y|_k\}$, $x \in E$, $y \in F$. A linear map $A : E \to F$ between graded Fréchet spaces is called *tame* (cf. [5]) if there exist a fixed integer $b \ge 0$ and constants $c_n > 0$ so that $|Ax|_n \le c_n |x|_{n+b}$ for all n and all $x \in E$; if a choice $b = 0$ is possible then A is called *normwise tame*. E is called a *tame direct summand* of F if there exist tame linear maps $T : E \to F$ and $S : F \to E$ so that $S \circ T = \mathrm{id}_E$; if in addition $T \circ S = \mathrm{id}_F$ then we say that E and F are *tamely isomorphic* (written $E \cong F$ tamely), and T is called a *tame isomorphism*. A short exact sequence $0 \to F \xrightarrow{i} G \xrightarrow{q} E \to 0$ of graded Fréchet spaces with tame linear maps is called *tamely exact* if the induced linear isomorphisms $\bar{i} : F \to iF$ and $\bar{q} : G/iF \to E$ are tame isomorphisms.

In [19] a smoothing property $(\mathrm{S}_\Omega)_\mathrm{t}$ is introduced generalizing the classical concept of smoothing operators (cf. [5], [7], [13]–[15], [28]).

Definition 1.1. Let E be a graded Fréchet space.

(i) E has *property* $(\mathrm{S})_\mathrm{t}$ (smoothing operators) if there exist $b, p \ge 0$ and $c_n > 0$ such that for each $\theta \ge 1$ there is a (not necessarily linear) map $S_\theta : E \to E$ such that

$$|S_\theta x|_n \le c_n \theta^{n+p-k} |x|_k \quad \text{for all } b \le k \le n+p,\ x \in E,$$
$$|x - S_\theta x|_n \le c_k \theta^{n+p-k} |x|_k \quad \text{for all } k \ge n+p,\ x \in E.$$

(ii) E has *property* (Ω_{DZ}) (cf. [17]) if there exist $p \ge 0$ and constants $c_n > 0$ such that for $U_n = \{x \in E : |x|_n \le 1\}$ and all $n \ge p$ and $r > 0$ we have

$$(\Omega_{\mathrm{DZ}}) \qquad U_n \subset c_n \Big(\bigcap_{i=p}^{n} r^{i-p} U_{n-i} \Big) + \Big(\bigcap_{k=-p}^{\infty} c_{n+k} r^{-k-p} U_{n+k} \Big).$$

E has *property* (Ω_{S}) (also called (Ω) in standard form, cf. [30]) resp. property (Ω_{DS}) (cf. [18]) if there are constants $c_n > 0$ such that for all $n \ge 1$ and $r > 0$ we have

$$(\Omega_{\mathrm{S}}) \qquad U_n \subset c_n \Big(r U_{n-1} + \frac{1}{r} U_{n+1} \Big),$$

$$(\Omega_{\mathrm{DS}}) \qquad U_n \subset c_n \Big(((r U_{n-1}) \cap U_n) + \Big(\Big(\frac{1}{r} U_{n+1} \Big) \cap U_n \Big) \Big).$$

(iii) E has the *smoothing property* $(\mathrm{S}_\Omega)_\mathrm{t}$ (cf. [19]) if there exists a tamely exact sequence $0 \to F \to G \to E \times H \to 0$ of graded Fréchet spaces such that F has property (Ω_{DZ}) and G has property $(\mathrm{S})_\mathrm{t}$.

(iv) E has *property* (DN) (cf. [26]) if there exists $b \ge 0$ such that for every n there are a constant $c_n > 0$ and k such that $|x|_n^2 \le c_n |x|_b |x|_{n+k}$ for all $x \in E$.

Remarks 1.2. (i) Properties $(\mathrm{S})_\mathrm{t}$, (Ω_{DZ}) and $(\mathrm{S}_\Omega)_\mathrm{t}$ are preserved by tame isomorphisms, and (DN) by topological isomorphisms. If both E, F have property $(\mathrm{S}_\Omega)_\mathrm{t}$ (or $(\mathrm{S})_\mathrm{t}$, (Ω_{DZ}), respectively) then so does their product $E \times F$.

(ii) Property $(\mathrm{S})_\mathrm{t}$ implies $(\mathrm{S}_\Omega)_\mathrm{t}$, and $(\mathrm{S}_\Omega)_\mathrm{t}$ implies (Ω_{DZ}).

(iii) For example, if $K = \overline{\mathring{K}}$ is a compact set in $\mathbb{R}^N$ which satisfies a generalized cone condition in the sense of Tidten [25] or which is subanalytic in the sense of Bierstone [1] then the space $C^\infty(K)$ has both properties $(\mathrm{S}_\Omega)_\mathrm{t}$ and (DN) (cf. [19], 5.3, 5.4 for $(\mathrm{S}_\Omega)_\mathrm{t}$ and [1], [25] for (DN)). Here $C^\infty(K)$ denotes the space of all C^∞-functions in $\mathring{K}$ such that all partial derivatives $\partial^\alpha f / \partial x^\alpha$ extend continuously to K, equipped with its usual norm system

$$|f|_k = \sup_{|\alpha| \le k} \sup_{x \in K} \left| \frac{\partial^\alpha f}{\partial x^\alpha}(x) \right|, \qquad f \in C^\infty(K).$$

(iv) For any tamely nuclear graded Fréchet space E, properties $(\mathrm{S}_\Omega)_\mathrm{t}$ and (Ω_{DZ}) are equivalent (cf. [19], 4.1). The space E is called *tamely nuclear* (cf. [16], 3.2) if there is a fixed $b \ge 0$ such that the canonical maps $E_{k+b} \to E_k$ are nuclear for all k where E_k denotes the Banach space obtained by completion of $(E/\ker |\ |_k, |\ |_k)$. The spaces $C^\infty(K)$ in (iii) are tamely nuclear (cf. [20], 4.12).

(v) For any graded Fréchet–Hilbert space, property (Ω_{S}) implies $(\mathrm{S}_\Omega)_\mathrm{t}$ (cf. 2.5(ii)). Here Fréchet–Hilbert means that the seminorms are defined by semiscalar products.

(vi) The graded Fréchet space E is called *locally* l_1 if $E_k \cong l_1(J_k)$ for all k and suitable sets J_k. If E is locally l_1 then property (Ω_{DS}) implies $(\mathrm{S}_\Omega)_\mathrm{t}$ (cf. 2.5(iii)).

The assumptions of the implicit function theorem proved in [21], 4.2, are formulated in terms of the following expressions $[\ \ ]_{m,k}$.

Definition 1.3 ([21], 1.1). Let $p, q \ge 0$ be integers with $p + q \ge 1$, and $E_1, \ldots, E_p, F_1, \ldots, F_q$ be graded Fréchet spaces. For $m, k \ge 0$, $x_1 \in E_1, \ldots, x_p \in E_p$, $y_1 \in F_1, \ldots, y_q \in F_q$ put

$$[x_1, \ldots, x_p; y_1, \ldots, y_q]_{m,k} = \sup\{1 \cdot |x_{k_1}|_{m+i_1} \cdot \ldots \cdot |x_{k_r}|_{m+i_r} \cdot |y_1|_{m+j_1} \cdot \ldots \cdot |y_q|_{m+j_q}\}$$

where the "sup" is over all $i_1, \ldots, i_r, j_1, \ldots, j_q \geq 0$ and $1 \leq k_1, \ldots, k_r \leq p$ with $0 \leq r \leq k$ and $i_1 + \ldots + i_r + j_1 + \ldots + j_q \leq k$ (for $r = 0$ the $|x|$-terms are omitted). For $q = 0$ we write $[x_1, \ldots, x_p]_{m,k}$ (where the $|y|$-terms are omitted), and for $p = 0$ we write $[; y_1, \ldots, y_q]_{m,k}$. For $m = 0$ we write $[\ldots]_k = [\ldots]_{0,k}$ in all cases.

In addition, for the vectors $x = (x_1, \ldots, x_p)$ and $y = (y_1, \ldots, y_q)$, in Section 4 we shall use the abbreviation $[x; y]_{m,k} = [x_1, \ldots, x_p; y_1, \ldots, y_q]_{m,k}$ and $[x; y]_k = [x; y]_{0,k}$.

Observe that the expression $[x_1, \ldots, x_p; y_1, \ldots, y_q]_{m,k}$ is a seminorm in each component y_j separately while it is "completely nonlinear" in the x_i-components. The expressions $[\ \ ]_{m,k}$ are increasing in m and in k. For the purely nonlinear expressions we have $[x_1, \ldots, x_p]_{m,0} = 1$ and $[x_1, \ldots, x_p]_{m,k} \geq 1$ for all m, k.

The following theorem on implicit functions is proved in [21]. For the notion of differentiability in Fréchet spaces we refer to [5] or [21].

THEOREM 1.4 ([21], 4.2). *Let E, F, G be graded Fréchet spaces such that $E, F \in (\mathrm{S}_\Omega)_\mathrm{t}$ and $E, F, G \in (\mathrm{DN})$. Let $U \subset E$ and $V \subset F$ be open sets and $x_0 \in U$, $y_0 \in V$. Let $f : U \times V \to G$ be a C^2-map, $f = f(x,y)$, and $f(x_0, y_0) = 0$. Assume that for any $w \in U \times V$ the partial derivative $f_y(w) : F \to G$ is bijective and for some fixed $d \geq 0$ and suitable $c_n > 0$ the following estimates hold for all $n \geq 0$:*

$$(1) \qquad |f'(w)x|_n \leq c_n[w; x]_{d,n}, \qquad w \in U \times V,\ x \in E \times F,$$

$$(2) \qquad |f_y(w)^{-1}z|_n \leq c_n[w; z]_{d,n}, \qquad w \in U \times V,\ z \in G,$$

$$(3) \qquad |f''(w)\{x, x\}|_n \leq c_n[w; x, x]_{d,n}, \qquad w \in U \times V,\ x \in E \times F.$$

Then there exist open neighbourhoods $U_0 \subset U$ of x_0, $V_0 \subset V$ of y_0 and a C^2-map $h : U_0 \to V_0$ such that $\{(x,y) \in U_0 \times V_0 : f(x,y) = 0\} = \{(x, h(x)) : x \in U_0\}$. In addition, if f is C^n then so is h, for $n \in \mathbb{N} \cup \{\infty\}$, $n \geq 2$.

2. The smoothing property for the spaces $C^n([0,1], E)$. Let E be a graded Fréchet space, let $J = [0,1]$, and let $C^n(J, E)$ denote the space of all n times continuously differentiable functions $f : J \to E$, $n = 0, 1, \ldots$; for $n = 0$ we write $C(J, E) = C^0(J, E)$. Then $C^n(J, E)$ is a graded Fréchet space with

$$|f|_n = \sup_{i=0}^{n} \sup_{x \in J} |f^{(i)}(x)|_n, \qquad f \in C^n(J, E).$$

The goal of this section is to give sufficient conditions on E for both E and $C^n(J, E)$ to have the smoothing property $(\mathrm{S}_\Omega)_\mathrm{t}$. Since the map

$$\Phi : C^n(J, E) \to C(J, E) \times E^n, \qquad \Phi(f) = (f^{(n)}, f(0), \ldots, f^{(n-1)}(0)),$$

is a normwise tame isomorphism it is enough to consider the case $n = 0$ (cf. 1.2(i)).

LEMMA 2.1. *If the graded Fréchet space E has property (Ω_S) (or (Ω_DS) or (Ω_DZ)) then so does $C(J, E)$.*

Proof. We show the assertion for (Ω_S); the same proof gives the result for (Ω_DS) or (Ω_DZ). We assume that $U_n \subset c_n(rU_{n-1} + r^{-1}U_{n+1})$ for all $n \geq 1$ and all $0 < r < 1$ where $U_n = \{x \in E : |x|_n \leq 1\}$. Let $f \in C(J, E)$ with $|f|_n \leq 1$ and $0 < r < 1$ be fixed. We choose an integer p such that $|f(x) - f(y)|_n \leq r$ for all $x, y \in J$ with $|x - y| \leq 1/(p+1)$, and define a continuous partition of unity $(\phi_j)_{j=0}^{p+1}$ in $C(J)$ as follows. For $j = 0, \ldots, p+1$ and $x \in J$ we put

$$\phi_j(x) = \begin{cases} 1 & \text{if } \dfrac{j}{p+1} \leq x \leq \dfrac{2j+1}{2(p+1)}, \\[2mm] 0 & \text{if } x \leq \dfrac{2j-1}{2(p+1)} \text{ or } x \geq \dfrac{j+1}{p+1}, \end{cases}$$

and extend ϕ_j continuously by straight line segments. Then $\phi_j \in C(J)$, $0 \leq \phi_j \leq 1$, $\sum_{j=0}^{p+1} \phi_j(x) = 1$ for $x \in J$, and for any $x \in J$ there is a unique maximal $j = j(x)$ such that $\phi_j(x) + \phi_{j+1}(x) = 1$. We put $x_j = 4j + 1/(4(p+1))$ for $j = 0, \ldots, p$ and $x_{p+1} = 1$. Notice that $|x_{j+1} - x_j| \leq 1/(p+1)$ and $|x - x_{j(x)}| \leq 1/(p+1)$. We choose a decomposition $f(x_j) = a_j + b_j$ in E such that $|a_j|_{n-1} \leq c_n r$ and $|b_j|_{n+1} \leq c_n r^{-1}$, $j = 0, \ldots, p+1$. We put $h = \sum_{j=0}^{p+1} \phi_j b_j \in C(J, E)$. We now obtain the assertion since for $x \in J$ and $j = j(x)$ we have

$$\begin{aligned} |h(x)|_{n+1} &\leq |\phi_j(x) b_j|_{n+1} + |(1 - \phi_j)(x) b_{j+1}|_{n+1} \leq c_n r^{-1}, \\ |f(x) - h(x)|_{n-1} &\leq |f(x) - f(x_j)|_{n-1} + |\phi_{j+1}(x)(f(x_{j+1}) - f(x_j))|_{n-1} \\ &\quad + |\phi_j(x) a_j|_{n-1} + |(1 - \phi_j(x)) a_{j+1}|_{n-1} \\ &\leq (c_n + 2)r. \end{aligned}$$

For graded Fréchet spaces E, F the *ε-product* $F\varepsilon E = L(E'_\gamma, F)$ is a graded Fréchet space with $|u|_k = \sup_{e' \in U_k^\circ} |u(e')|_k$, $u \in F\varepsilon E$, where $U_k = \{x \in E : |x|_k \leq 1\}$ and E'_γ denotes the dual space of E equipped with the topology of uniform convergence on compact subsets of E (cf. [8]). We have $C(J, E) \cong C(J)\varepsilon E$ normwise tamely by means of the mapping $C(J, E) \to C(J)\varepsilon E$, $f \mapsto u_f$, $u_f(e') = e' \circ f$ (cf. [8], 16.6). Notice that $C(J) \mathbin{\widetilde{\otimes}_\varepsilon} E = C(J)\varepsilon E$ where $C(J) \mathbin{\widetilde{\otimes}_\varepsilon} E$ denotes the closure in $C(J)\varepsilon E$ of the finite tensors $\sum_{i=1}^n f_i \otimes e_i$, $e_i \in E$, $f_i \in C(J)$.

LEMMA 2.2. *Let $0 \to F \xrightarrow{i} G \xrightarrow{q} E \to 0$ be a tamely exact sequence of graded Fréchet spaces. Then the maps $I(f) = i \circ f$ and $Q(g) = q \circ g$ define* *a tamely exact sequence*

$$0 \to C(J,F) \xrightarrow{I} C(J,G) \xrightarrow{Q} C(J,E) \to 0.$$

Proof. We only have to prove the tame surjectivity of Q because the other assertions are obvious. For that it suffices to show that the map $\mathrm{id}\,\widetilde{\otimes}_\varepsilon q : C(J)\,\widetilde{\otimes}_\varepsilon\, G \to C(J)\,\widetilde{\otimes}_\varepsilon\, E$ is tamely surjective. This immediately follows from [2], Proposition 4.4.

For $\delta > 0$ and a Banach space B let $s_\delta(B)$ denote the graded Fréchet space

$$s_\delta(B) = \Big\{x = (x_j)_{j=1}^\infty \subset B : |x|_k = \sum_{j=1}^\infty \|x_j\| j^{\delta k} < \infty \text{ for all } k\Big\}.$$

In particular, we write $s(B) = s_1(B)$ and $s_\delta = s_\delta(\mathbb{K})$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

PROPOSITION 2.3. *Let E be a graded Fréchet space. If there is a tamely exact sequence*

$$0 \to s_\delta(B) \to s_\delta(B) \to E \times H \to 0$$

for some graded Fréchet space H and $\delta > 0$ then both E and $C(J,E)$ have property $(\mathrm{S}_\Omega)_{\mathrm{t}}$.

Proof. This holds for E by definition since $s_\delta(B)$ has both properties (Ω_{DZ}) and $(\mathrm{S})_{\mathrm{t}}$. Lemma 2.2 gives a tamely exact sequence

$$0 \to C(J, s_\delta(B)) \to C(J, s_\delta(B)) \to C(J,E) \times C(J,H) \to 0.$$

The result follows since $C(J, s_\delta(B)) \cong C(J)\,\widetilde{\otimes}_\varepsilon\, s_\delta(B) \cong s_\delta(C(J)\,\widetilde{\otimes}_\varepsilon\, B)$ tamely.

THEOREM 2.4. *Let E be a graded Fréchet space.*

(i) *If E is tamely nuclear with property (Ω_{DZ}) then there exist $\delta > 0$ and a tamely exact sequence $0 \to s_\delta \to s_\delta \to E \to 0$.*

(ii) *If E is a graded Fréchet-Hilbert space with property (Ω_{S}) then there exist a Hilbert space H and a tamely exact sequence $0 \to s(H) \to s(H) \to E \times s(H) \to 0$.*

(iii) *If E is locally l_1 with property (Ω_{DS}) then there exist a Banach space B and a tamely exact sequence $0 \to s(B) \to s(B) \to E \times s(B) \to 0$.*

Proof. (i) is proved in [17], 7.1, and (ii) is shown in [19], 4.7, as an application of the tame splitting theorem [23]. For the proof of (iii) we assume $E_k \cong l_1(J_k)$ and put

$$B = \Big\{x = (x_k)_{k=1}^\infty \in \prod_{k=1}^\infty E_k : \|x\| = \sum_{k=1}^\infty |x_k|_{E_k} < \infty\Big\}.$$

Notice that the Banach space B is isomorphic to $l_1(J)$ where $J = \bigcup_k (J_k \times \{k\})$. The proof of [19], 4.6, gives a tamely exact sequence $0 \to E \to B^{\mathbb{N}} \to B^{\mathbb{N}} \to 0$ (cf. [29], 3.1). By means of [18], 3.3, the pairs $(s(B), s(B))$ and $(s(B), E)$ are tame splitting pairs since $s(B) = \Lambda^1(M, a)$ for $M = \mathbb{N} \times J$ and $a(n,j) = n$ in the notation of [18] (a pair (F,G) of graded Fréchet spaces is called a *tame splitting pair* if all tamely exact sequences of the form $0 \to G \to H \xrightarrow{q} F \to 0$ tamely split, i.e. q admits a tame linear right inverse). Therefore, the assertion follows from [19], 4.4 (cf. [27], 3.3).

COROLLARY 2.5. *Let E be a graded Fréchet space such that either* (i) *or* (ii) *or* (iii) *holds.*

(i) *E is tamely nuclear and has property (Ω_{DZ}).*

(ii) *E is a graded Fréchet-Hilbert space and has property (Ω_{S}).*

(iii) *E is locally l_1 and has property (Ω_{DS}).*

Then the spaces E and $C^n(J,E), n \in \mathbb{N}_0$, have property $(\mathrm{S}_\Omega)_{\mathrm{t}}$.

3. Linear equations. Some results on linear ordinary differential equations in Fréchet spaces are now stated which are important for the application of the implicit function theorem in the next section. For $0 \le \alpha_0 \le \alpha_1 \le \ldots \nearrow \infty$ we consider the power series space

$$\Lambda^1_\infty(\alpha) = \Big\{x = (x_j)_{j=1}^\infty \subset \mathbb{K} : |x|_k = \sum_{j=1}^\infty |x_j| e^{k\alpha_j} < \infty \text{ for all } k\Big\}$$

of infinite type. Recall that $\Lambda^1_\infty(\alpha)$ is nuclear if and only if $\sup_j (\log j)/\alpha_j < \infty$.

As a motivation for the assumptions of our main theorem we prove the following

THEOREM 3.1. *Let $E = \Lambda^1_\infty(\alpha)$ be nuclear and $(d_j)_{j=1}^\infty \subset \mathbb{R}$. Assume that the diagonal operator $A : E \to E$, $A(x_j)_j = (d_j x_j)_j$, is continuous and linear. For $z \in E$ consider*

$$(*) \qquad y'(t) = Ay(t), \qquad y(0) = z.$$

Then the following are equivalent:

(i) *Problem* $(*)$ *has for each $z \in E$ a unique global solution $y \in C^1(\mathbb{R}, E)$.*

(ii) *For any $z \in E$ there are $\varepsilon > 0$ and $y \in C^1((-\varepsilon,\varepsilon), E)$ which solves* $(*)$ *in $(-\varepsilon,\varepsilon)$.*

(iii) *There is $C > 0$ so that $|d_j| \le C\alpha_j$ for all large j.*

Proof. We note that A is continuous if and only if there is k so that $\sup_j |d_j| e^{-k\alpha_j} < \infty$.

We consider the initial value problems $y_j'(t) = d_j y_j(t)$, $y_j(0) = z_j$, with the unique solutions $y_j(t) = e^{d_j t} z_j$. For an open interval $I = (-\varepsilon, \varepsilon)$, $(*)$ has a (unique) solution in I if and only if $(e^{d_j t} z_j)_j \in E$ for all $t \in I$. Thus (iii)$\Rightarrow$(i) and (i)$\Rightarrow$(ii) are clear.

For the proof of (ii)$\Rightarrow$(iii) we may suppose that for any $z \in E$ there is $\varepsilon > 0$ such that $(e^{\varepsilon |d_j|} z_j)_j \in E$. We assume that (iii) is not true and choose an increasing sequence $\phi(j)$ such that $|d_{\phi(j)}|/\alpha_{\phi(j)} \nearrow \infty$ and a decreasing sequence $\varepsilon_j > 0$ such that $\varepsilon_j \to 0$ and $\varepsilon_j |d_{\phi(j)}|/\alpha_{\phi(j)} \nearrow \infty$. We define $(x_k)_k$ by $x_{\phi(j)} = e^{-\varepsilon_j |d_{\phi(j)}|}$ and $x_k = 0$ otherwise. Then $(x_k)_k \in E$ since $E = \Lambda_\infty^1(\alpha)$ is nuclear. However, for any $\varepsilon > 0$ we have $e^{\varepsilon |d_{\phi(j)}|} |x_{\phi(j)}| \to \infty$, which is a contradiction.

It is easy to see that the equivalence (i)$\Leftrightarrow$(iii) holds without supposing nuclearity.

Example 3.2. Let $E = s := \Lambda_\infty^1(\log j)$ and $A(x_j) = (d_j x_j)_j$ where $d_j = (\log j)^3$. Then A is even "very tame", i.e. for any $t \geq 0$ and $\varepsilon > 0$ there is $C = C_{t,\varepsilon} > 0$ so that $|Ax|_t \leq C|x|_{t+\varepsilon}$ for all $x \in E$. On the other hand, for $z_j = j^{-\log j}$ we have $z = (z_j)_j \in E$, and the initial value problem $(*)$ does not admit any solution $y \in C^1((-\varepsilon, \varepsilon), E), \varepsilon > 0$.

The previous results indicate that a general existence theorem is—even in the case of linear equations—hardly to be expected without supposing normwise tame estimates. Also uniqueness may fail when only tame estimates are assumed.

Example 3.3. Let $E = C^\infty[0,1]$ and $A : E \to E$, $A(f) = f'$. Then the initial value problem $y'(t) = Ay(t)$, $y(0) = f_0$, $f_0 \in E$, has a solution $y(t)(x) = f_0(t + x)$ for any extension $f_0 \in C^\infty(\mathbb{R})$. However, the extension and thus the solution are by no means unique. Observe that in general the formal expansion $e^{tA} f = \sum t^k f^{(k)}/k!$ does not converge.

Next we consider positive results. For a graded Fréchet space E, the space

$$\mathrm{LNT}(E) = \{A : E \to E \text{ linear} : |A|_n := \sup\{|Ax|_n : |x|_n \leq 1\} < \infty \text{ for all } n\}$$

of all normwise tame linear endomorphisms of E ($= L(E)$ if E is a Banach space) is a graded Fréchet space for the grading $\|A\|_k = \sup_{n=0}^k |A|_n$. For the algebra $(\mathrm{LNT}(E), \circ)$ with unit element I we have $|BA|_n \leq |B|_n |A|_n$ and $\|BA\|_n \leq \|B\|_n \|A\|_n$.

The theorem of Picard–Lindelöf is well known for (linear) differential equations in Banach spaces. Due to the existence of global solutions, the generalization to linear equations in Fréchet spaces with coefficients in $\mathrm{LNT}(E)$ is obvious:

Proposition 3.4. *Let E be a graded Fréchet space, $I \subset \mathbb{R}$ an open interval, $t_0 \in I$, and let $A : I \to \mathrm{LNT}(E)$ and $b : I \to E$ be continuous maps. Then the initial value problem*

$$y'(t) = A(t)\,y(t) + b(t), \quad y(t_0) = y_0,$$

has for any $y_0 \in E$ a unique solution $y \in C^1(I, E)$.

The next step is to derive a priori estimates for the norms of the solutions of linear differential equations in Fréchet spaces. For that a suitable generalization of the classical Gronwall lemma will be proved. The following lemma is well known.

Lemma 3.5 (Gronwall). *Let $d > 0$, $a, b \geq 0$, and let $u \in C(J, \mathbb{R})$ where $J = [0, d]$. If $u(t) \leq a + b\int_0^t u(s)\,ds$ for all $t \in J$ then $u(t) \leq ae^{bt}$ for all $t \in J$.*

Lemma 3.6. *Let $J = [0, d]$, $d > 0$, and assume that $u_k \in C(J, \mathbb{R})$, $k = 0, 1, \ldots$, satisfy, for some real numbers $0 \leq A_k \leq A_{k+1}$, $0 \leq B_k$ and constants $1 \leq D_k \leq D_{k+1}$, the estimates*

$$u_k(t) \leq A_k + D_k \sum_{i=0}^{k} \int_0^t B_{k-i} u_i(s)\,ds \quad \text{for all } t \in J,\ k = 0, 1, \ldots$$

where $B_i B_j \leq D_{i+j} B_{i+j}$ for all $i, j \geq 0$. Then there exist constants $C_k > 0$ where C_k only depends on k, d, B_0, D_k such that

$$u_k(t) \leq C_k \sum_{i=0}^{k} (1 + B_{k-i}) A_i \quad \text{for all } t \in J,\ k = 0, 1, \ldots$$

Proof. For $k = 0$ we have $u_0(t) \leq A_0 + B_0 D_0 \int_0^t u_0(s)\,ds$; hence 3.5 gives the estimate $u_0(t) \leq C_0(1 + B_0)A_0$ for $C_0 = e^{B_0 D_0 d}$. If the assertion is proved for $k - 1$ then we get

$$\begin{aligned} u_k(t) &\leq A_k + dD_k \sum_{i=0}^{k-1} B_{k-i} C_i \Big\{ \sum_{j=0}^{i} (1 + B_{i-j}) A_j \Big\} + B_0 D_k \int_0^t u_k(s)\,ds \\ &\leq A_k + C_k' \sum_{i=0}^{k-1} B_{k-i} A_i + B_0 D_k \int_0^t u_k(s)\,ds. \end{aligned}$$

Now we apply Gronwall's lemma once again to obtain the result.

PROPOSITION 3.7. *Let E be a graded Fréchet space, and let $A : J \to \mathrm{LNT}(E)$ and $b : J \to E$ be continuous where $J = [0, d]$, $d > 0$. Assume that*

$$|A(t)x|_k \le D_k \sum_{i=0}^{k} B_{k-i}|x|_i,$$

$$|b(t)|_k \le b_k \quad \text{for all } t \in J,\ x \in E,\ k = 0, 1, \ldots,$$

where $0 \le b_k \le b_{k+1}, 1 \le D_k \le D_{k+1}$ and $B_i B_j \le D_{i+j} B_{i+j}$ for all i, j. Then for $y_0 \in E$ the unique solution $y \in C^1(J, E)$ of the problem $y'(t) = A(t)y(t) + b(t)$, $y(0) = y_0$, satisfies the inequalities

$$|y(t)|_k \le C_k \sum_{i=0}^{k} (1 + B_{k-i})(b_i + |y_0|_i)$$

for all k with constants $C_k = C(k, d, B_0, D_k)$.

Proof. Considering the corresponding integral equation, for $t \in J$ we obtain

$$|y(t)|_k \le |y_0|_k + db_k + D_k \sum_{i=0}^{k} \int_0^t B_{k-i}|y(s)|_i\, ds.$$

We put $A_k = |y_0|_k + db_k$ and $u_k(t) = |y(t)|_k$; then 3.6 gives the result.

4. The main result. If E, F are graded Fréchet spaces and $U \subset E$ is open then a continuous $f : (U \subset E) \to F$ is called a *C^m-map* for an integer $m \ge 0$ if all directional derivatives $f^{(n)} : U \times E^n \to F$, $(u; e_1, \ldots, e_n) \mapsto f^{(n)}(u)\{e_1, \ldots, e_n\}$, $0 \le n \le m$, exist and are continuous (cf. [5], I.3); in this case $f^{(n)}(u)\{e_1, \ldots, e_n\}$ is completely symmetric and separately linear in $e_1, \ldots, e_n$ (cf. [5], I.3.6.2). We say f is C^∞ if f is C^m for all m.

Let E, P be graded Fréchet spaces, let $U \subset \mathbb{R} \times E \times P$ be an open set, and let $f : U \to E$ be a C^m-map, $f = f(t, x, p)$. We can then consider the partial derivative $\frac{\partial^i}{\partial t^i}\frac{\partial^j}{\partial x^j}\frac{\partial^k}{\partial p^k} f$ for $0 \le i + j + k \le m$ as a map $\partial_t^i \partial_x^j \partial_p^k f : U \times \mathbb{R}^i \times E^j \times P^k \to E$ (or as a map $U \times E^j \times P^k \to E$, cf. [5], I.3.4). Using multiindices we can write $\partial^\alpha f = \partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_p^{\alpha_3} f$ for $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \le m$; then $\partial^\alpha\partial^\beta f = \partial^\beta\partial^\alpha f$ if $|\alpha| + |\beta| \le m$ (cf. [5], I.3.5). For $a > 0$ and $y \in E$, $p \in P$ we consider the initial value problem

$$\text{(P)} \qquad \begin{cases} x'(t) = f(t, x(t), p), & t \in [t_0 - a, t_0 + a], \\ x(t_0) = y. \end{cases}$$

We now formulate and prove the main result of this paper; for the corresponding theorem in Banach spaces see [24].

THEOREM 4.1. *Let E, P be graded Fréchet spaces satisfying* (DN) *and* $(\mathrm{S}_\Omega)_t$ *such that $C([-1, 1], E)$ has property* $(\mathrm{S}_\Omega)_t$. *Let $U \subset \mathbb{R} \times E \times P$ be open and $(t_0, x_0, p_0) \in U$. Let $f : U \to E$ be a C^2-map, $f = f(t, x, p)$, such that for any $n \ge 0$ there is $c_n > 0$ such that for all $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$ with $|\alpha| \le 2$ and all $u \in U$, $z \in \mathbb{R}^{\alpha_1} \times E^{\alpha_2} \times P^{\alpha_3}$ we have*

$$(*) \qquad |\partial^\alpha f(u)(z)|_n \le c_n [u; z]_n.$$

(i) *Then there are $a > 0$ and open neighbourhoods $U(x_0)$ of x_0 and $U(p_0)$ of p_0 such that for any $(y, p) \in U(x_0) \times U(p_0)$ problem* (P) *has a unique solution $x \in C^1([t_0 - a, t_0 + a], E)$.*

(ii) *If f is in addition C^m, $2 \le m \le \infty$, then $x \in C^{m+1}([t_0 - a, t_0 + a], E)$, and the solution map can be chosen C^m as a map $U(x_0) \times U(p_0) \to C^1([t_0 - a, t_0 + a], E)$.*

(iii) *If x_1, x_2 are C^1-solutions of* (P) *for t in any interval J then $x_1 = x_2$ in J.*

(iv) *The solution map $(t, y, p) \mapsto x(t; y, p)$ is C^1 on $(t_0 - a, t_0 + a) \times U(x_0) \times U(p_0)$. Moreover, there is a maximal open neighbourhood U_1 of (t_0, x_0, p_0) in U where this solution map exists and is C^1.*

REMARK. For $u = (t, x, p)$ and $z = (t_1, \ldots, t_{\alpha_1}, x_1, \ldots, x_{\alpha_2}, p_1, \ldots, p_{\alpha_3})$ the term $[u; z]_n$ in $(*)$ is defined (cf. 1.3) by $[u; z]_n = [t, x, p; t_1, \ldots, t_{\alpha_1}, x_1, \ldots \ldots, x_{\alpha_2}, p_1, \ldots, p_{\alpha_3}]_n$.

Proof (of 4.1). By means of the transformation $t = t_0 + as$, $z(s) = x(t_0 + as) - y$, $s \in J = [-1, 1]$, the initial value problem (P) is equivalent to the problem

$$\text{(P1)} \qquad \begin{cases} z'(s) = af(t_0 + as, z(s) + y, p), & s \in J, \\ z(0) = 0. \end{cases}$$

The map $\Phi : (W \subset C^1(J, E) \times \mathbb{R} \times E \times P) \to C(J, E) \times E$ defined by

$$\Phi(z, a, y, p) = (z'(s) - af(t_0 + as, z(s) + y, p), z(0)), \qquad s \in J,$$

is a C^2-map defined in the open subset $W \subset C^1(J, E) \times \mathbb{R} \times E \times P$ where

$$W = \{(z, a, y, p) : (t_0 + as, z(s) + y, p) \in U \text{ for all } s \in J\}.$$

We have $(0, 0, x_0, p_0) \in W$, $\Phi(0, 0, x_0, p_0) = (0, 0)$, and (P1) holds if and only if $\Phi(z, a, y, p) = (0, 0)$. For the partial derivative Φ_z we get

$$\Phi_z(z, a, y, p)(w) = (w'(s) - af_x(t_0 + as, z(s) + y, p)w(s), w(0)), \qquad s \in J,$$

for any $(z, a, y, p) \in W$ and $w \in C^1(J, E)$. For $u \in C(J, E)$, $u_0 \in E$ we want to solve the equation $\Phi_z(z, a, y, p)(w) = (u, u_0)$ in order to apply the implicit function theorem 1.4. The assumptions in 1.4 on the spaces are satisfied since E, P and $C^1(J, E)$ all have (DN) and $(\mathrm{S}_\Omega)_t$ (cf. 1.2(i)). We have to consider the linear initial value problem

$$\text{(LP)} \qquad \begin{cases} w'(s) = A(s)w(s) + u(s), & s \in J, \\ w(0) = u_0, \end{cases}$$

where $A(s) := a\, f_x(t_0 + as, z(s) + y, p) \in \mathrm{LNT}(E)$ by $(*)$. Problem (LP) has a unique solution $w \in C^1(J, E)$ by 3.4 since $A \in C(J, \mathrm{LNT}(E))$. Hence $\Phi_z(z, a, y, p) : C^1(J, E) \to C(J, E) \times E$ is bijective for all $(z, a, y, p) \in W$.

We have to verify the estimates (1), (2), (3) in 1.4; here (2) can be considered as a priori estimates for the solutions of (LP). When proving these estimates we may assume—shrinking U if necessary—that $|(z, a, y, p)|_0 \le C$ for some fixed $C > 0$ and all $(z, a, y, p) \in W$.

To show (1) we observe that $\Phi'(u)\{w, b, x, q\} = \Phi_z(u)w + \Phi_a(u)b + \Phi_y(u)x + \Phi_p(u)q$ for all $u \in W$ and $w \in C^1(J, E)$, $b \in \mathbb{R}$, $x \in E$, $q \in P$. From $(*)$ we obtain the estimates $|\Phi_z(u)w|_n \le c_n[u; w]_n$ for all n. The other partial derivatives are given by

$$\begin{aligned} \Phi_a(z, a, y, p)b &= (-bf(t_0 + as, z(s) + y, p) - absf_t(t_0 + as, z(s) + y, p), 0), \\ \Phi_y(z, a, y, p)x &= (-af_x(t_0 + as, z(s) + y, p)x, 0), \\ \Phi_p(z, a, y, p)q &= (-af_p(t_0 + as, z(s) + y, p)q, 0). \end{aligned}$$

Hence $(*)$ implies $|\Phi_a(u)b|_n \le c_n[u; b]_n$, $|\Phi_y(u)x|_n \le c_n[u; x]_n$ and $|\Phi_p(u)q|_n \le c_n[u; q]_n$. This gives $|\Phi'(u)v|_n \le c_n[u; v]_n$ for $u \in W$ and $v \in C^1(J, E) \times \mathbb{R} \times E \times P$ and hence (1).

To show (3) we have to estimate $|\Phi''(u)\{v, v\}|_n$ for $u \in W$ and $v \in C^1(J, E) \times \mathbb{R} \times E \times P$. Now $\Phi''(u)\{v, v\}$ can be calculated as a sum (of 16 summands resp. 10 different summands) involving the second partial derivatives $\Phi_{zz}, \Phi_{aa}, \Phi_{yy}, \Phi_{pp}$ and $\Phi_{za}, \Phi_{zy}, \Phi_{zp}, \Phi_{ay}, \Phi_{ap}, \Phi_{yp}$. It is hence enough to prove the desired estimates for these second partial derivatives. For instance,

$$\Phi_{zz}(z, a, y, p)\{w, w\} = (-af_{xx}(t_0 + as, z(s) + y, p)\{w(s), w(s)\}, 0), \qquad s \in J.$$

Now $(*)$ gives $|\Phi_{zz}(u)\{w, w\}|_n \le c_n[u; w, w]_n$ for $u \in W$, $w \in C^1(J, E)$. Analogous estimates for the other second partial derivatives yield $|\Phi''(u)\{v, v\}|_n \le c_n[u; v, v]_n$ for $u \in W$, $v \in C^1(J, E) \times \mathbb{R} \times E \times P$, proving (3).

It remains to prove (2). Fix $u \in C(J, E)$ and $u_0 \in E$ and assume that $w \in C^1(J, E)$ is a solution of (LP). We have to estimate $|w|_n$ in terms of u, u_0 and (z, a, y, p) where $|(z, a, y, p)|_0 \le C$. From $(*)$ we obtain

$$\sup_{s \in J} |A(s)x|_n \le c_n[(z, a, y, p); x]_n \le c_n \sup_{i=0}^{n} [z, a, y, p]_{n-i} |x|_i, \qquad x \in E.$$

We can apply 3.7 with $B_i = [z, a, y, p]_i$ noting that $[u]_i[u]_j \le [u]_{i+j}$ to get

$$\begin{aligned} \sup_{s \in J} |w(s)|_n &\le c_n \sup_{i=0}^{n} [z, a, y, p]_{n-i} (\sup_{s \in J} |u(s)|_i + |u_0|_i) \\ &\le c_n[(z, a, y, p); (u, u_0)]_n. \end{aligned}$$

For the first derivative of w, from the differential equation we obtain

$$\begin{aligned} \sup_{s \in J} |w'(s)|_n &\le \sup_{s \in J} (|A(s)w(s)|_n + |u(s)|_n) \\ &\le c_n \sup_{s \in J} (\sup_{i=0}^{n} [z, a, y, p]_{n-i} |w(s)|_i + |u(s)|_n) \\ &\le c_n[(z, a, y, p); (u, u_0)]_n. \end{aligned}$$

This shows $|w|_n = |\Phi_z(z, a, y, p)^{-1}(u, u_0)|_n \le c_n[(z, a, y, p); (u, u_0)]_n$ and this yields (2).

We are therefore allowed to apply the implicit function theorem 1.4 to the function Φ and obtain open neighbourhoods U_0 of $(0, x_0, p_0)$ in $\mathbb{R} \times E \times P$ and V_0 of 0 in $C^1(J, E)$ with $V_0 \times U_0 \subset W$ and a C^2-map $\Psi_1 : U_0 \to V_0$ such that the equation $\Phi(z, a, y, p) = (0, 0)$ is uniquely solved in V_0 by $z = \Psi_1(a, y, p)$ for $(a, y, p) \in U_0$. We hence find $\delta > 0$ and open neighbourhoods $U(x_0)$ of x_0 and $U(p_0)$ of p_0 such that $\Psi_1 : (0, \delta) \times U(x_0) \times U(p_0) \to V_0$ is a C^2-map and $z = \Psi_1(a, y, p)$ is a solution of (P1) which is unique in V_0. Applying the above transformation $z(s) = x(t_0 + as) - y$ we can choose a fixed $a > 0$ and a C^2-map $\Psi : U(x_0) \times U(p_0) \to C^1([t_0 - a, t_0 + a], E)$ such that $x = \Psi(y, p)$ is a solution of (P). This proves the existence part of (i). If f is C^m then the differential equation implies that x is C^{m+1}; hence 1.4 implies (ii).

To show uniqueness we choose V_0 as above and assume first that x_1, x_2 are any C^1-solutions of (P) for $y = x_0$, $p = p_0$ and some $a > 0$. The transformed functions $z_j(s) = x_j(t_0 + as) - x_0$ are solutions of (P1). Since $a \mapsto x_j(t_0 + as) - x_0 \in C^1(J, E)$ is a continuous map which is 0 for $a = 0$ we can find a small $a > 0$ so that $z_1, z_2 \in V_0$, hence $z_1 = z_2$. This shows that $x_1 = x_2$ in a small neighbourhood of t_0. This local uniqueness also implies global uniqueness since for $t_1 > t_0$ with $x_1(t_1) \ne x_2(t_1)$ and $t_2 := \inf\{t_1 > t_0 : x_1(t_1) \ne x_2(t_1)\}$ we have $x_1(t_2) = x_2(t_2)$ by continuity and hence $x_1 = x_2$ near t_2 by local uniqueness, which is a contradiction.

Since the evaluation map $C^1(J, E) \times J \to E$ is C^1, a straightforward continuation argument gives (iv), and the theorem is proved.

REMARK. For C^∞-maps f satisfying $(*)$ for all α, the above theorem can also be proved using the space $C^\infty([-1, 1], E)$ in place of $C^1([-1, 1], E)$. Then $C^\infty([-1, 1], E)$ must be supposed to have the smoothing property $(\mathrm{S}_\Omega)_t$, and the proof of the estimates (1), (2), (3) in 1.4 is technically more difficult. In this case, one obtains a solution map which is C^∞ considered as a map $U(x_0) \times U(p_0) \to C^\infty([t_0 - a, t_0 + a], E)$.

5. Examples. The following example shows that the assumptions of 4.1 are natural.

EXAMPLE 5.1. Let $E = P = C^\infty(K)$ where the compact set $K = \overline{\mathring{K}} \subset \mathbb{R}^N$ either has a Lipschitz boundary or is subanalytic. Then $C^\infty(K)$ has properties (DN) and $(\mathrm{S}_\Omega)_t$ and is tamely nuclear (cf. 1.2); hence $C(J,E)$ has $(\mathrm{S}_\Omega)_t$ by 2.5(i).

Let $J=[t_0-a,t_0+a]$, let $A\subset J\times K\times\mathbb{K}\times\mathbb{K}$ be relatively open, and let $F\in C^\infty(\bar A,\mathbb{K})$, $F=F(t,u,x,p)$. Then the set

$$U=\{(x,p)\in E\times P:(t,u,x(u),p(u))\in A\text{ for all } t\in J,\ u\in K\}$$

is open. We assume that $(x_0,p_0)\in U$ and consider the map

$$f:J\times U\to E,\qquad f(t,x,p)(u)=F(t,u,x(u),p(u)).$$

Then f is a C^∞-map; if, in addition, A is relatively compact then f satisfies the estimates $(*)$ in 4.1 (even for all α). This is proved in [21], 1.5, 1.6, considering x,p as mappings $x(t,u)=x(u)$, $p(t,u)=p(u)$, $(t,u)\in J\times K$ (cf. [7]). Therefore, for suitable small $a>0$ and for (y,p) near (x_0,p_0) the initial value problem (P) has a unique solution $x\in C^\infty(J,E)$ which is C^∞-depending on (y,p). Moreover, 4.1(i)–(iv) apply to this problem. Writing $x(t,u)=x(t)(u)$ and omitting p this also implies for $y\in E$ the unique solvability of the Cauchy problem

$$(\mathrm{CP})\qquad\begin{cases}\dfrac{\partial x}{\partial t}(t,u)=F(t,u,x(t,u)), & t\in[t_0-a,t_0+a],\\ x(t_0,u)=y(u).\end{cases}$$

More generally, for $A\subset J\times K^2\times\mathbb{K}^3$ and $F\in C^\infty(\bar A,\mathbb{K})$ we can consider the map

$$f(t,x,p)(u)=\int_K F(t,u,w,x(u),x(w),p(u))\,dw.$$

Then $f:J\times E\times P\to E$ is C^∞ and satisfies $(*)$ for all α. Hence 4.1 implies the unique solvability of all initial value problems (P) for f.

Example 5.1 can also be considered for C^∞-maps on the space $E=C^\infty(X)$ for a compact C^∞-manifold X. In fact, in [20], 4.14, it is shown that $C^\infty(X)\cong s_{1/N}$ tamely, $N=\dim(X)$; the estimates $(*)$ follow from [7] for C^∞-maps as in 5.1.

The assumptions of 4.1 are satisfied for smooth maps in a lot of other Fréchet algebras.

EXAMPLE 5.2. We consider the graded Fréchet space of rapidly decreasing functions

$$\begin{aligned}\mathcal{S}&=\mathcal{S}(\mathbb{R}^N)\\ &=\{x\in C^\infty(\mathbb{R}^N):|x|_k=\sup_{|\alpha|\le k}\sup_{u\in\mathbb{R}^N}(1+|u|)^k|\partial^\alpha x(u)|<\infty\text{ for all }k\},\end{aligned}$$

which is tamely isomorphic to $s_{1/N}$ ([20], 5.5). We have $|x\cdot y|_k\le c_k[;x,y]_k$, $x,y\in\mathcal{S}$. Let $\mathbb{K}=\mathbb{R}$, $J=[t_0-a,t_0+a]$, and let $F\in C^\infty(J\times\mathbb{R}^N\times\mathbb{R})$, $F=F(t,u,y)$, be a function such that for any compact $I\subset\mathbb{R}$ all partial derivatives $\partial^\alpha F$ $(\alpha\in\mathbb{N}_0^{N+2})$ are bounded on $J\times\mathbb{R}^N\times I$. We assume that $(\partial/\partial t)^i(\partial/\partial u)^\beta F(t,u,0)=0$ for all $t\in J$, $u\in\mathbb{R}^N$, $\beta\in\mathbb{N}_0^N$ and $i=0,1,2$ (e.g. we can put $F(t,u,y)=\phi(t,u)\psi(y)$ where $\phi\in\mathcal{B}(J\times\mathbb{R}^N)$ and $\psi\in C^\infty(\mathbb{R})$ with $\psi(0)=0$). We define

$$f:J\times\mathcal{S}\to\mathcal{S},\quad f(t,x)(u)=F(t,u,x(u)),\quad t\in J,\ x\in\mathcal{S},\ u\in\mathbb{R}^N.$$

We put $V=\{x\in\mathcal{S}:|x|_0\le C_0\}$ where $C_0>0$ is a constant. For $t\in J$ and $x\in\mathcal{S}$ we have $f(t,x)\in\mathcal{S}$ since $|F(t,u,y)|\le c|y|$ for $|y|\le C_0$ and hence

$$\sup_{u\in\mathbb{R}^N}|F(t,u,x(u))|(1+|u|)^k\le c\sup_{u\in\mathbb{R}^N}|x(u)|(1+|u|)^k\le c|x|_k,\quad x\in V,$$

and the chain rule for higher derivatives (cf. [21], 1.4) implies the estimates $|f(t,x)|_k\le c_k[x]_k$ for $t\in J$, $x\in V$. The same argument gives $\partial_t^i f(t,x)=\partial_t^i F(t,\cdot,x(\cdot))\in\mathcal{S}$ for $i=0,1,2$. The proof of [21], 1.5, shows that $f:J\times\mathcal{S}\to\mathcal{S}$ is a C^2-map and $\partial_t^i\partial_x^j f(t,x)\{y_1,\ldots,y_j\}(u)=\partial_t^i\partial_y^j F(t,u,x(u))y_1(u)\cdot\ldots\cdot y_j(u)$ for all $0\le i+j\le 2$, $t\in J$, $x\in\mathcal{S}$, $y_1,\ldots,y_j\in\mathcal{S}$. This formula implies for $0\le i+j\le 2$ the inequalities

$$|\partial_t^i\partial_x^j f(t,x)\{y_1,\ldots,y_j\}|_k\le c_k[x;y_1,\ldots,y_j]_k,\qquad x\in V,\ y_1,\ldots,y_j\in\mathcal{S}.$$

Thus 4.1 yields for any $y\in\mathcal{S}$ a unique solution $x\in C^1([t_0-a,t_0+a],\mathcal{S})$ of

$$(\mathrm{IP})\qquad\begin{cases}x'(t)=f(t,x(t)), & t\in[t_0-a,t_0+a],\\ x(t_0)=y,\end{cases}$$

for some suitable small $a>0$, and 4.1(i)–(iv) apply. Writing $x(t,u)=x(t)(u)$ we also obtain a unique solution $x\in C^1(J,\mathcal{S})$ of the Cauchy problem (CP).

It is not difficult to see that also other conditions on F are sufficient. For instance, it is enough that for $F\in C^\infty(J\times\mathbb{R}^N\times\mathbb{R})$, $F=F(t,u,y)$, the functions $(1+|u|)^k|\partial^\alpha F(t,u,y)|$ are bounded on $J\times\mathbb{R}^N\times I$ for any compact $I\subset\mathbb{R}$ and all $\alpha\in\mathbb{N}_0^{N+2}$, $k\ge0$. More generally, we can consider nonlocal examples as well. We take $F\in C^\infty(J\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R})$, $F=F(t,u,w,y,z)$, and suppose that for any compact $I\subset\mathbb{R}$ and $\alpha\in\mathbb{N}_0^{N+3}$, $k\ge0$ we have

$$\sup_{t\in J}\sup_{y,z\in I}\sup_{u\in\mathbb{R}^N}(1+|u|)^k\int_{\mathbb{R}^N}|\partial^\alpha_{t,u,y,z}F(t,u,w,y,z)|\,dw<\infty.$$

Then the nonlocal map $f:J\times\mathcal{S}\to\mathcal{S}$ defined by

$$f(t,x)(u)=\int_{\mathbb{R}^N}F(t,u,w,x(u),x(w))\,dw$$

is a C^∞-map which satisfies $(*)$ for all α; hence Theorem 4.1 applies to f.

The preceding example can easily be generalized to the spaces $K\{M_p\}$ in the sense of Gel'fand and Shilov (cf. [20], 5); more general maps F could be considered as well.

EXAMPLE 5.3. We consider the graded Fréchet space

$$\mathcal{B}=\mathcal{B}(\mathbb{R}^N)=\{x\in C^\infty(\mathbb{R}^N):|x|_k=\sup_{|\alpha|\le k}\sup_{u\in\mathbb{R}^N}|\partial^\alpha x(u)|<\infty \text{ for all } k\},$$

which is tamely isomorphic to $s_{1/N}(l_\infty)$ (cf. [20], 5.4(i)). Let $\mathbb{K}=\mathbb{R}$, $J=[t_0-a,t_0+a]$, and let $F\in C^\infty(J\times\mathbb{R}^N\times\mathbb{R})$, $F=F(t,u,y)$, be such that all partial derivatives $\partial^\alpha F$ are bounded on $J\times\mathbb{R}^N\times I$ for any compact interval $I\subset\mathbb{R}$. Then $f:J\times\mathcal{B}\to\mathcal{B}$ defined by $f(t,x)(u)=F(t,u,x(u))$, $t\in J$, $x\in\mathcal{B}$, $u\in\mathbb{R}^N$, is a C^∞-map. If $C_0>0$ is a constant and $V=\{x\in\mathcal{B}:|x|_0\le C_0\}$ then $\sup_{u\in\mathbb{R}^N}|f(t,x)(u)|\le C$ for all $x\in V$. The chain rule gives the estimates $|f(t,x)|_k\le c_k[x]_k$ for all $x\in V$ and k with suitable $c_k>0$. The Leibniz rule implies $|xy|_k\le c_k[;x,y]_k$ for $x,y\in\mathcal{B}$, and as in 5.2 we obtain

$$|\partial_t^i\partial_x^j f(t,x)\{y_1,\dots,y_j\}|_k\le c_k[x;y_1,\dots,y_j]_k,\qquad x\in V,\ y_1,\dots,y_j\in\mathcal{B}.$$

Hence Theorem 4.1 can be applied and for any $y\in\mathcal{B}$ the initial value problem (IP) has a unique solution $x\in C^\infty([t_0-a,t_0+a],\mathcal{B})$ for some $a>0$. The same holds for the Cauchy problem (CP). Nonlocal examples as in 5.2 can be considered as well.

EXAMPLE 5.4. Let $0\le a_{j,k}\le a_{j,k+1}$, $j\in\mathbb{Z}$, be a Köthe matrix with $\sup_k a_{j,k}>0$ for all j, and consider the Köthe sequence space

$$\lambda=\lambda^1(a)=\Big\{x=(x_j)_{j\in\mathbb{Z}}:|x|_k=\sum_{j\in\mathbb{Z}}|x_j|a_{j,k}<\infty \text{ for all } k\Big\}.$$

We assume that $\lambda\in(\mathrm{DN})$ (cf. [26], 2.3) and $a_{j,k}^2\ge c_k^{-1}a_{j,k-1}a_{j,k+1}$ for some constants $c_k>0$. Then $\lambda\in(\Omega_{\mathrm{DS}})$ (cf. [17], 3.7) and hence $\lambda\in(\mathrm{S}_\Omega)_{\mathrm{t}}$ and $C([-1,1],\lambda)\in(\mathrm{S}_\Omega)_{\mathrm{t}}$ by 2.5(iii). We assume that for some constants $c_k>0$ we have

$$a_{i+j,k}\le c_k\sum_{l=0}^k a_{i,l}a_{j,k-l}\quad \text{for all } i,j\in\mathbb{Z},\ k=0,1,\dots$$

Then $(\lambda^1(a),*)$ is a convolution algebra for $x*y=(\sum_{j\in\mathbb{Z}}x_{n-j}y_j)_{n\in\mathbb{Z}}=y*x$ since

$$|x*y|_k=\sum_{n\in\mathbb{Z}}\Big|\sum_{i+j=n}x_{n-j}y_j\Big|a_{n,k}\le c_k\sum_{l=0}^k\Big(\sum_i|x_i|a_{i,l}\Big)\Big(\sum_j|y_j|a_{j,k-l}\Big)$$
$$\le c_k'[;x,y]_k.$$

For instance, a power series space with $a_{j,k}=e^{k\alpha_{|j|}}$, $0\le\alpha_0\le\alpha_1\le\dots\nearrow\infty$, satisfies all these conditions if $e^{\alpha_{i+j}}\le e^{\alpha_i}+e^{\alpha_j}$ for all $i,j\ge0$ (cf. [9], 2.4.5) since $e^{k\alpha_{|i+j|}}\le(e^{\alpha_{|i|}}+e^{\alpha_{|j|}})^k\le\sum_{l=0}^k\binom{k}{l}e^{\alpha_{|i|}l}e^{\alpha_{|j|}(k-l)}$; this holds e.g. for $\alpha_j=\log(1+j)$ and $\lambda=s$.

We put $J=[t_0-a,t_0+a]$ and choose an integer $m\ge1$ and $\phi_1,\dots,\phi_m\in C^\infty(J)$. We define $f:J\times\lambda\to\lambda$ by $f(t,x)=\sum_{n=1}^m\phi_n(t)x^n$, $t\in J$, $x\in\lambda$, where $x^1:=x$ and $x^{n+1}:=x*x^n$ (clearly, more general mappings f could also be considered). Then f is a C^∞-map and

$$\partial_t^i\partial_x^j f(t,x)\{y_1,\dots,y_j\}=\sum_{n=j}^m\phi_n^{(i)}(t)c_{n,j}x^{n-j}*y_1*\dots*y_j$$

for some constants $c_{n,j}$. This implies

$$|\partial_t^i\partial_x^j f(t,x)\{y_1,\dots,y_j\}|_k\le c_k[x;y_1,\dots,y_j]_k$$

for suitable $c_k>0$ and $|x|_0\le C_0$. Hence, for any $y\in\lambda$, (IP) has a unique local solution $x\in C^\infty(J,\lambda)$ for $a>0$ small enough, and 4.1(i)–(iv) apply.

The preceding example can immediately be generalized to more abstract situations. Let E be a graded Fréchet space which is an algebra for a product $\cdot:E\times E\to E$. We call $(E,(|\ \ |_k)_k,\cdot)$ a *subbinomic Fréchet algebra* if there are constants $c_k>0$ such that $|x\cdot y|_k\le c_k[;x,y]_k$ for all $x,y\in E$ and all k; and we call $(E,(|\ \ |_k)_k,\cdot)$ a *submultiplicative Fréchet algebra* if $|x\cdot y|_k\le c_k|x|_k|y|_k$ for all $x,y\in E$ and all k. Notice that a power series space $\Lambda_\infty^1(\alpha)$ of infinite type equipped with the convolution product is subbinomic if $e^{\alpha_{i+j}}\le e^{\alpha_i}+e^{\alpha_j}$ and is submultiplicative if $\alpha_{i+j}\le\alpha_i+\alpha_j$.

In view of the preceding examples we obtain the following general rule of thumb: Theorem 4.1 applies to subbinomic Fréchet algebras but not to submultiplicative ones.

EXAMPLE 5.5. Let E be a commutative subbinomic Fréchet algebra such that E and $C([-1,1],E)$ satisfy $(\mathrm{S}_\Omega)_{\mathrm{t}}$ and E has (DN). Let $J=[t_0-a,t_0+a]$, $\phi_1,\dots,\phi_m\in C^\infty(J)$. Then $f:J\times E\to E$ defined by

$$f(t,x)=\sum_{n=1}^m\phi_n(t)x^n,\qquad t\in J,\ x\in E,$$

is a C^∞-map which satisfies (*) in 4.1 for $|x|_0\le C_0$. Hence 4.1 gives a unique local solution $x\in C^\infty(J,E)$ of (IP) for any $y\in E$ and $a>0$ small enough, and 4.1(i)–(iv) apply.

As a further concrete example, we can consider the space

$$\mathcal{D}_{L_1}(\mathbb{R}^N)=\Big\{f\in C^\infty(\mathbb{R}^N):|f|_k=\sup_{|\alpha|\le k}\int_{\mathbb{R}^N}|\partial^\alpha f(x)|\,dx<\infty \text{ for all } k\Big\},$$

which is a commutative subbinomic Fréchet algebra for the convolution product defined by $(g*h)(x)=\int_{\mathbb{R}^N}g(x-y)h(y)\,dy$ and which satisfies the assumptions of 4.1 by 2.3 since $\mathcal{D}_{L_1}(\mathbb{R}^N)\cong s_{1/N}(l_1)$ tamely (cf. [20], 5.4(ii)). Hence Theorem 4.1 can be applied e.g. to mappings $f:J\times\mathcal{D}_{L_1}(\mathbb{R}^N)\to\mathcal{D}_{L_1}(\mathbb{R}^N)$ defined as above.

On the other hand, the Fréchet algebra $H(\mathbb{C})$ of entire functions is submultiplicative for $|x|_k = \sup\{|x(u)| : |u| \le k\}$; however, the initial value problem $x'(t) = x(t)^2$, $x(0) = y$, in $H(\mathbb{C})$ does not admit any C^1-solution near 0 e.g. for $y(u) = u$ (consider $x(t)(u) = 1/(-t+1/u)$ for $0 < t < 1/u$ and $u \to \infty$). Notice that the map $f : H(\mathbb{C}) \to H(\mathbb{C})$, $f(x) = x^2$, only satisfies $|f(x)|_k \le |x|_k^2 \le [x]_{2k}$, which is not good enough to apply Theorem 4.1.

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