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A method of approximate factorization of positive definite matrix functions

by

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Abstract. An algorithm of factorization of positive definite matrix functions of second order is proposed.

1. Formulation of the problem. In [4], [5], [6] Wiener proved that for a positive definite matrix function $S(t) = (f_{ij}(t))_{i,j=\overline{1,r}}$, where $f_{ij}(t)$, $|t| = 1$, are integrable functions on the unit circle of the complex plane with

$$\log\det(S(t)) \in L_1, \tag{1}$$

there exists a factorization

$$S(t) = \chi^+(t) \cdot (\chi^+(t))^*, \tag{2}$$

where χ^+ is an outer matrix function with entries from the Hardy space H_2 and $(\chi^+)^*$ is its adjoint.

Condition (1) is necessary for the existence of such a factorization.

χ^+ is defined up to a constant right unitary multiplier.

In the one-dimensional case the above result is due to Szegő and the factorization can be explicitly given by a formula (see [1]), while in the multidimensional case, $r \geq 2$, Wiener's theorem states the pure existence.

The coefficients of the analytic functions in the factor matrix χ^+ are important for many applications, including the prediction theory of stationary processes constructed by Wiener and Kolmogorov (see [6], [2]). Therefore methods of approximate calculation of these coefficients for a given matrix function $S(t)$ are of great significance. Some of such methods under certain restrictions on $S(t)$ were described in the papers of Wiener and Masani (see [7], [3]). The attempts of other authors to essentially improve these results have not been successful.

In this paper, without imposing any additional restrictions on the matrix function $S(t)$ apart from the necessary and sufficient condition (1) for $S(t)$ to

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be factorizable, we find a new effective factorization algorithm. Namely, we construct a sequence of positive definite matrix functions $S_n(t)$ convergent to $S(t)$ in the L_1 norm and having an explicit factorization

$$S_n(t) = \chi_n^+(t) \cdot (\chi_n^+(t))^*. \tag{3}$$

The convergence of $\chi_n^+(t)$ to $\chi^+(t)$ in the L_2 norm is proved.

In this paper we will only deal with the two-dimensional case which is not only important in itself but also plays a decisive role for higher order matrices.

NOTATION. As usual, L_p is the class of p-integrable complex functions on the unit circle. L_p^+ (resp. L_p^-), $p \geq 1$, is the class of functions from L_p whose negative (resp. positive) Fourier coefficients are all 0. Functions from L_p^+ can be assumed to belong to the Hardy class H_p.

A matrix function is said to be in L_p or L_p^+ if its entries are in this class; a sequence of matrix functions is said to be convergent in the L_p norm if their entries are convergent in this norm.

The "+" or "−" superscript of a function emphasizes that the function belongs to L_p^+ or L_p^-, respectively.

If $f \in L_2$, then $[f]^+$ (resp. $[f]^-$) will denote the function from L_2^+ (resp. L_2^-) which has the same positive (negative) Fourier coefficients as f.

Let E_r be the r-dimensional unit matrix and let

$$D' = \{z \in \mathbb{C} : 0 < |z| < 1\}.$$

2. Construction of $S_n(t)$ and their factorization for the two-dimensional case. A positive definite two-dimensional matrix function has the form

$$S(t) = \begin{pmatrix} a(t) & b(t) \\ \overline{b(t)} & c(t) \end{pmatrix}, \tag{4}$$

where $a, b, c \in L_1$ and $a(t), c(t), a(t)c(t) - |b(t)|^2 \geq 0$ for a.a. t. Condition (1) means that $\log(a(t)c(t) - |b(t)|^2) \in L_1$, which implies that

$$\log a(t), \log\left(\frac{a(t)c(t) - |b(t)|^2}{a(t)}\right) \in L_1.$$

Under these conditions $S(t)$ admits the representation

$$S(t) = \begin{pmatrix} f_1^+(t) & 0 \\ \varphi(t) & f^+(t) \end{pmatrix} \begin{pmatrix} \overline{f_1^+(t)} & \overline{\varphi(t)} \\ 0 & \overline{f^+(t)} \end{pmatrix}, \tag{5}$$

where f_1^+ and f^+ are outer analytic functions from H_2 whose squares of modulus coincide almost everywhere with $a(t)$ and $c(t) - |b(t)|^2/a(t)$, respectively, on the boundary of the unit disk, and

$$\varphi(t) = \overline{b(t)}/\overline{f_1^+(t)}. \tag{6}$$

Observe that $\varphi \in L_2$, since $|\varphi(t)|^2 = |b(t)|^2/a(t) \leq c(t) \in L_1$.

Assume $\varphi = \varphi^+ + \varphi^-$, where $\varphi^+ \in L_2^+$ and $\varphi^- \in L_2^-$, and rewrite (5) as

$$S(t) = \begin{pmatrix} f_1^+(t) & 0 \\ \varphi^+(t) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varphi^-(t) & f^+(t) \end{pmatrix} \begin{pmatrix} 1 & \overline{\varphi^-(t)} \\ 0 & \overline{f^+(t)} \end{pmatrix} \begin{pmatrix} \overline{f_1(t)} & \overline{\varphi^+(t)} \\ 0 & 1 \end{pmatrix}.$$

Let

$$\varphi_n^-(t) = \sum_{k=0}^{n} \gamma_k t^{-k}, \qquad n = 1, 2, \ldots, \tag{7}$$

where $\varphi^- \sim \sum_{k=0}^{\infty} \gamma_k t^{-k}$, and let $S_n(t)$, $n = 1, 2, \ldots$, be the following sequence of positive definite matrix functions:

$$S_n(t) = \begin{pmatrix} f_1^+(t) & 0 \\ \varphi^+(t) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varphi_n^-(t) & f^+(t) \end{pmatrix} \begin{pmatrix} 1 & \overline{\varphi_n^-(t)} \\ 0 & \overline{f^+(t)} \end{pmatrix} \begin{pmatrix} \overline{f_1(t)} & \overline{\varphi^+(t)} \\ 0 & 1 \end{pmatrix}. \tag{8}$$

Obviously, $\|S_n - S\|_{L_1} \to 0$. In the remaining part of this section we will construct the factorization of the matrix function $S_n(t)$ (n is assumed to be fixed).

We search for a unitary matrix function $U_n(t)$,

$$U_n(t) \cdot (U_n(t))^* = E_2, \tag{9}$$

with $\det(U_n(t)) = 1$ almost everywhere such that

$$\begin{pmatrix} 1 & 0 \\ \varphi_n^-(t) & f^+(t) \end{pmatrix} \cdot U_n(t) \in L_2^+. \tag{10}$$

A unitary matrix with determinant 1 is of the form

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1.$$

Thus condition (10) takes the form

$$\begin{pmatrix} \alpha_n^+(t) & \beta_n^+(t) \\ \varphi_n^-(t)\alpha_n^+(t) - f^+(t)\overline{\beta_n^+(t)} & \varphi_n^-(t)\beta_n^+(t) + f^+(t)\overline{\alpha_n^+(t)} \end{pmatrix} \in L_2^+. \tag{11}$$

Since $\varphi_n^-(t)$ has only n nonzero negative coefficients the desired functions α_n^+ and β_n^+ must be polynomials of the same order n. Thus

$$U_n(t) = \begin{pmatrix} \alpha_n^+(t) & \beta_n^+(t) \\ -\overline{\beta_n^+(t)} & \overline{\alpha_n^+(t)} \end{pmatrix}, \tag{12}$$

where

$$\alpha_n^+(t) = \sum_{k=0}^{n} a_k t^k, \qquad \beta_n^+(t) = \sum_{k=0}^{n} b_k t^k \tag{13}$$

and

(14) $$|\alpha_n^+(t)|^2+|\beta_n^+(t)|^2=1,\qquad |t|=1.$$

Rewrite condition (11) as a system

(15) $$\begin{cases}\varphi_n^-(t)\alpha_n^+(t)-f^+(t)\overline{\beta_n^+(t)}=\Psi_{1n}^+(t),\\ \varphi_n^-(t)\beta_n^+(t)+f^+(t)\overline{\alpha_n^+(t)}=\Psi_{2n}^+(t),\end{cases}$$

where Ψ_{1n}^+ and Ψ_{2n}^+ are some functions from L_2^+. Equating the negative Fourier coefficients of the functions in (15) to 0, we construct a system of linear equations and show that it has a nontrivial solution.

For simplicity, the matrix notation will be used:

$$\Gamma_n=\begin{pmatrix}\gamma_0&\gamma_1&\cdots&\gamma_{n-1}&\gamma_n\\ \gamma_1&\gamma_2&\cdots&\gamma_n&0\\ \cdot&\cdot&\cdots&\cdot&\cdot\\ \gamma_n&0&\cdots&0&0\end{pmatrix},\qquad F_n=\begin{pmatrix}l_0&l_1&\cdots&l_{n-1}&l_n\\ 0&l_0&\cdots&l_{n-2}&l_{n-1}\\ \cdot&\cdot&\cdots&\cdot&\cdot\\ 0&0&\cdots&0&l_0\end{pmatrix}$$

$$A_n=\begin{pmatrix}a_0\\a_1\\ \vdots\\ a_n\end{pmatrix},\quad B_n=\begin{pmatrix}b_0\\b_1\\ \vdots\\ b_n\end{pmatrix},\quad \mathbf{0}=\begin{pmatrix}0\\0\\ \vdots\\ 0\end{pmatrix},\quad \mathbf{1}=\begin{pmatrix}1\\0\\ \vdots\\ 0\end{pmatrix},$$

where γ_k, $k=1,\ldots,n$, are defined by (7) and

$$f^+(z)=\sum_{k=0}^{\infty}l_kz^k.$$

The corresponding system is

(16) $$\begin{cases}\Gamma_n\cdot A_n-F_n\cdot\overline{B}_n=\mathbf{0},\\ \Gamma_n\cdot B_n+F_n\cdot\overline{A}_n=\mathbf{1}\end{cases}$$

(to avoid a trivial solution we take $\Psi_{2n}^+(0)=1$).

Since f^+ is an outer analytic function, $1/f^+$ is analytic in D. Hence

$$\frac{1}{f^+(z)}=\sum_{k=0}^{\infty}d_kz^k,\qquad |z|<1,$$

where $d_0=(f^+(0))^{-1}\neq 0$, and

$$F_n^{-1}=\begin{pmatrix}d_0&d_1&d_2&\cdots&d_{n-1}&d_n\\ 0&d_0&d_1&\cdots&d_{n-2}&d_{n-1}\\ 0&0&d_0&\cdots&d_{n-3}&d_{n-2}\\ \cdot&\cdot&\cdot&\cdots&\cdot&\cdot\\ 0&0&0&\cdots&0&d_0\end{pmatrix}.$$

By defining B_n from the first equation of (16) and inserting it into the second equation, we get

$$B_n=\overline{F}_n^{-1}\cdot\overline{\Gamma}_n\cdot\overline{A}_n,\qquad \Gamma_n\cdot\overline{F}_n^{-1}\cdot\overline{\Gamma}_n\cdot\overline{A}_n+F_n\cdot\overline{A}_n=\mathbf{1}.$$

Thus

(17) $$(F_n^{-1}\Gamma_n\cdot\overline{F}_n^{-1}\overline{\Gamma}_n+E_n)\overline{A}_n=F_n^{-1}\cdot\mathbf{1}.$$

But the matrix $\Theta=F_n^{-1}\Gamma_n$ is symmetric, since

$$\Theta_{ij}=\Theta_{ji}=\begin{cases}0 & \text{for } i+j>n,\\ \displaystyle\sum_{k=0}^{n-(i+j)}d_k\gamma_{i+j+k} & \text{for } i+j\le n.\end{cases}$$

Thus $\Theta\cdot\overline{\Theta}$ is positive definite and the determinant of the left matrix of (17) is not 0 (moreover, all eigenvalues of this matrix are greater than 1). Thus by defining $\overline{A}_n$ from (17) we will find the coefficients a_k, b_k, $k=0,1,\ldots,n$.

Let us now show that the equality

(18) $$|\alpha_n^+(t)|^2+|\beta_n^+(t)|^2=\text{const},\qquad |t|=1,$$

holds for polynomials of the form (13) which satisfy (15). It follows from (15) that

$$f^+(t)(|\alpha_n^+(t)|^2+|\beta_n^+(t)|^2)=\Psi_{2n}^+(t)\alpha_n^+(t)-\Psi_{1n}^+(t)\beta_n^+(t).$$

Therefore

(19) $$|\alpha_n^+(t)|^2+|\beta_n^+(t)|^2=\frac{1}{f^+(t)}(\Psi_{2n}^+(t)\alpha_n^+(t)-\Psi_{1n}^+(t)\beta_n^+(t)).$$

Although equation (19) holds for almost all t from ∂D, one can consider the right side of this equality as the boundary values of an analytic function Φ:

$$\Phi^+(z)=\frac{1}{f^+(z)}(\Psi_{2n}^+(z)\alpha_n^+(z)-\Psi_{1n}^+(z)\beta_n^+(z)),\qquad z\in D.$$

Since $f^+(z)$ is an outer analytic function, $\Phi^+(z)$ remains in the subclass N^+ of Nevanlinna's class and since we know that $\Phi^+(z)|_{|z|=1}\in L_\infty$ because of (19), we can conclude that $\Phi^+\in H_\infty$ (see [1, Theorem 2.11]). But the left side of equality (19) is positive. So the boundary values of a function from H_∞ are positive almost everywhere. This implies that Φ^+ is constant and (18) holds.

Having solutions a_k and b_k, $k=0,1,\ldots,n$, of (16), we can obtain the value of the constant after substituting $t=1$ into (18):

$$\text{const}=\Big|\sum_{k=0}^{n}a_k\Big|^2+\Big|\sum_{k=0}^{n}b_k\Big|^2.$$

Then we normalize the coefficients so that (14) hold and the matrix (12) be unitary.

Now we are ready to show that

(20) $$\chi_n^+(t)=\begin{pmatrix} f_1^+(t) & 0\\ \varphi^+(t) & 1\end{pmatrix}\begin{pmatrix} 1 & 0\\ \varphi_n^-(t) & f^+(t)\end{pmatrix}U_n(t).$$

Equality (3) holds because of (8), (9) and (20). To show that $\chi_n^+(t)$ is an outer analytic matrix function, observe that

(21) $$\det(\chi_n^+(z))=f_1^+(z)f^+(z),\qquad |z|<1.$$

Indeed, equality (20) can be continued naturally in D', using the definitions of the functions (7) and (13) for $|t|<1$ and assuming

$$\overline{\alpha_n^+}(t)=\sum_{k=0}^{n}\overline{a}_k t^{-k},\quad \overline{\beta_n^+}(t)=\sum_{k=0}^{n}\overline{b}_k t^{-k},\qquad 0<|t|<1.$$

Then

$$\det(U_n(t))=1,\qquad t\in D',$$

since (14) holds, and it follows from (20) that (21) is valid for $z\in D'$. But since we know a priori that both sides of (21) are analytic, they are equal in the entire disk.

REMARK. It follows from the above arguments that if the matrix function (4) is such that the function φ defined by (6) has only a finite number of nonzero Fourier negative coefficients, then the factorization of $S(t)$ can be constructed in explicit form. In our opinion, this is the only case when explicit factorization of a positive definite matrix function of second order is possible.

3. Convergence of χ_n^+.

As mentioned above, the factorization (2) is defined up to a constant unitary multiplier. Namely, if we require the factor matrix at the origin $\chi^+(0)$ to be positive definite, then the factorization is unique.

We can assume that the functions f_1^+ and f^+ in (5) are positive at 0, which implies that

$$\det(\chi_n^+(0))=f_1^+(0)f^+(0)>0$$

(see (21)). We also have $\Psi_{1n}^+(0)=0$ and $\Psi_{2n}^+(0)>0$ (see (15), (16)), which means that in the second row of the matrix $\chi_n^+(0)$ the first entry is zero and the second one is positive:

$$(\chi_n^+(0))_{21}=0,\quad (\chi_n^+(0))_{22}>0,\qquad n=1,2,\ldots$$

The factorization (2) of the matrix function (4) for which

(22) $$\det(\chi^+(0))>0,\quad (\chi^+(0))_{21}=0,\quad (\chi^+(0))_{22}>0$$

is also unique. We will show that χ_n^+ converges to this χ^+ in the L_2 norm,

(23) $$\|\chi_n^+-\chi^+\|_{L^2}\to 0.$$

We will prove that the entries $\alpha_n^+(t)$ and $\beta_n^+(t)$ of the constructed unitary matrix functions $U_n(t)$ (see (12)) converge in measure, and thus obtain (23).

Observe first that if some subsequences

(24) $$(\alpha_n^+)_{n\in N_0},\qquad (\beta_n^+)_{n\in N_0},$$

$N_0\subset\mathbb{N}$, are convergent in measure to α and β, respectively, then

$$L_\infty^+\ni\alpha\equiv\alpha^+,\qquad L_\infty^+\ni\beta\equiv\beta^+$$

and, moreover,

$$\varphi^-\alpha^+-f^+\overline{\beta^+},\varphi^-\beta^++f^+\overline{\alpha^+}\in L_2^+.$$

Hence, under these conditions, we have

$$\chi^+(t)=\begin{pmatrix} f_1^+(t)\alpha^+(t) & f_1^+(t)\beta^+(t)\\ \varphi(t)\alpha^+(t)-f^+(t)\overline{\beta^+(t)}, & \varphi(t)\beta^+(t)+f^+(t)\overline{\alpha^+(t)}\end{pmatrix},$$

since the passage to the limit in (3) implies the validity of (2), while the equality

$$\lim_{N_0\ni n\to\infty}\chi_n^+(z)=\chi^+(z),\qquad |z|<1,$$

yields

$$\det\chi^+(z)=f_1(z)f(z)$$

(see (21)), so that the conditions in (22) are also satisfied.

Let us now show that for each $N_1\subset\mathbb{N}$ there exists $N_0\subset N_1$ such that (24) converges in measure. Since $\chi^+(t)$ is unique, this will complete the proof of convergence.

Hankel's operator $H_{\varphi^-}:H_\infty^+\to L_2^-$ defined by

$$H_{\varphi^-}(\alpha^+)=[\varphi^-\alpha^+]^-$$

is compact, since H_{φ^-} is the limit of the finite-dimensional operators $H_{\varphi_n^-}$ in the operator norm. Thus a convergent subsequence $[\varphi^-\alpha_n^+]^-_{n\in N_2}$, $N_2\subset N_1$, can be extracted from $[\varphi^-\alpha_n^+]^-_{n\in N_1}$. Then $[\varphi_n^-\alpha_n^+]^-_{n\in N_2}$ is also convergent and this implies the convergence of $[f^+\overline{\beta_n^+}]^-_{n\in N_2}$ by the first equation of (15). Considering now the operator $H_{f^+}^*:H_\infty^-\to L_2^+$,

$$H_{f^+}^*(\overline{\beta^+})=[f^+\overline{\beta^+}]^+$$

it becomes clear that a convergent subsequence $[f^+\overline{\beta_n^+}]^+_{n\in N_0}$, $N_0\subset N_2$, can be extracted from $[f^+\overline{\beta_n^+}]^+_{n\in N_2}$. So $[f^+\overline{\beta_n^+}]_{n\in N_0}$ is convergent in L_2, which implies the convergence of β_n^+ in measure.

The convergence of α_n^+ is proved in a similar manner.

Thus the validity of (23) is shown. The authors have also obtained some results on the rate of this convergence.

Cases of dimension greater than two will be discussed in a forthcoming paper.

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