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Wold-type extension for N-tuples of commuting contractions

by

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Abstract. Let $(T_1, \ldots, T_N)$ be an N-tuple of commuting contractions on a separable, complex, infinite-dimensional Hilbert space $\mathcal{H}$. We obtain the existence of a commuting N-tuple $(V_1, \ldots, V_N)$ of contractions on a superspace $\mathcal{K}$ of $\mathcal{H}$ such that each V_j extends T_j, $j = 1, \ldots, N$, and the N-tuple $(V_1, \ldots, V_N)$ has a decomposition similar to the Wold–von Neumann decomposition for coisometries (although the V_j need not be coisometries). As an application, we obtain a new proof of a result of Słociński (see [9]).

1. Introduction. Let $\mathcal{H}$ be a complex, separable, infinite-dimensional Hilbert space. We denote by $\mathcal{L}(\mathcal{H})$ the algebra of bounded linear transformations of $\mathcal{H}$. In studying the structure of contractions in $\mathcal{L}(\mathcal{H})$ one of the most potent tools has been the dilation theory and its extension counterpart (see [3], [6], [7]). One of the obstacles to developing a suitable structure theory for N-tuples of commuting contractions is the impossibility of developing a (unitary or isometric) dilation theory for such N-tuples (at least for $N > 2$). Even in the case $N = 2$, the lack of a (joint) Wold-type decomposition for a joint isometric dilation represents a serious obstacle (see [6]).

In this paper we develop a version of an extension (or dilation) theory for N-tuples. The idea is that instead of focusing on the geometric properties of isometries (or coisometries) we focus on its "decomposability", i.e., given an N-tuple of commuting contractions in $\mathcal{L}(\mathcal{H})$, we find an N-tuple of commuting operators on a superspace $\mathcal{K}$ that has a "(joint) Wold-type decomposition".

In order to do this we introduce a new class, denoted by $K_{0,\cdot}$, of N-tuples of operators that, we believe, is the appropriate extension to several variables of the well known class $C_{0,\cdot}$.

This note is organized as follows. Section 2 is dedicated to the class $K_{0,\cdot}$ and its relation to invariant subspaces. In Section 3 we state and prove our main result (the Extension Theorem). Finally, in Section 4 we explore some applications and consequences of our main result.

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2. The class $K_{0,\cdot}$ and invariant subspaces. We say that an N-tuple $T = (T_1, \ldots, T_N)$ of commuting contractions on $\mathcal{H}$ belongs to the class $K_{0,\cdot}$ if

$$\inf_{n\in\mathbb{Z}_+^N} \|T^n x\| = 0, \qquad x \in \mathcal{H}.$$

The T^n is to be understood in the usual multi-index sense, that is, if $n = (n_1, \ldots, n_N)$ is an N-tuple of nonnegative integers, then $T^n := T_1^{n_1} \ldots T_N^{n_N}$. Note that the infimum of the sequence $\{\|T^n x\|\}_{\mathbb{Z}_+^N}$ is equal to its limit. This fact will be of importance in the proof of our main result. Furthermore, we say that $T = (T_1, \ldots, T_N)$ belongs to the class $K_{\cdot,0}$ if $T^* = (T_1^*, \ldots, T_N^*)$ belongs to $K_{0,\cdot}$. The class $K_{0,0}$ is the intersection $K_{0,\cdot} \cap K_{\cdot,0}$. These classes are a natural generalization of the well studied classes $C_{0,\cdot}$, $C_{\cdot,0}$, and $C_{0,0}$. Note that $T = (T_1, \ldots, T_N) \in K_{0,\cdot}$ if and only if $T_1 \cdot \ldots \cdot T_N \in C_{0,\cdot}$.

We now give a generalization to several variables of the following well known result of Sz.-Nagy and C. Foiaş (see [10] or Theorem 2.2 of [1]).

THEOREM 2.1. *Let T be a contraction in $\mathcal{L}(\mathcal{H})$ and assume that for every $x \in \mathcal{H}$, $x \neq 0$, we have*

$$\lim_{n\to\infty} \|T^n x\| \neq 0 \neq \lim_{n\to\infty} \|T^{*n} x\|.$$

Then T is quasisimilar to a unitary operator.

Note that, in what follows, quasisimilarity will be extended to several variables in the strongest sense, i.e., if there is an N-tuple $U = (U_1, \ldots, U_N)$ of unitary operators and operators X, Y, with dense range and one-to-one, such that for any $j = 1, \ldots, N$, $T_j X = X U_j$ and $Y T_j = U_j Y$, we shall say that the N-tuple $T = (T_1, \ldots, T_N)$ is *quasisimilar* to the N-tuple U.

The proof of the next result is similar to the proof of Theorem 2.1 given in [1]; we include it here for completeness. This result is not new, a more general version of it can be found in [2]. The simplicity of the proofs of the last two results in this section, imitating the proofs of one-variable results, is the main reason we believe that the classes $K_{\cdot,\cdot}$ are the correct generalization of $C_{\cdot,\cdot}$ to several variables.

THEOREM 2.2. *Let $T = (T_1, \ldots, T_N)$ with $T_j \in \mathcal{L}(\mathcal{H})$ and $\|T_j\| \le 1$ for $j = 1, \ldots, N$ be such that*

$$\inf_{n\in\mathbb{Z}_+^N} \|T^n x\| \neq 0, \qquad \inf_{n\in\mathbb{Z}_+^N} \|T^{*n} x\| \neq 0, \qquad x \in \mathcal{H},\ x \neq 0.$$

Then T is quasisimilar to an N-tuple of unitary operators.

Proof. We can define a new norm on $\mathcal{H}$ by the formula

$$\|x\|' = \inf_{n\in\mathbb{Z}_+^N} \|T^n x\|.$$

It is clear that $\|\cdot\|'$ is indeed a norm and that $\|x\|' \le \|x\|$ for every $x \in \mathcal{H}$. Let $\mathcal{K}$ be the completion of $\mathcal{H}$ in this new norm and $X : \mathcal{H} \to \mathcal{K}$ the inclusion operator. Then X is one-to-one and has dense range. For fixed j, we have $\|T_j x\|' = \|x\|'$. Thus, T_j extends to an isometry U_j on $\mathcal{K}$. Since $\operatorname{Ker}(T_j^*) = (0)$ (by the hypothesis on T_j^*), the range of U_j is dense in $\mathcal{K}$. Therefore, $U_j X = X T_j$. With a similar argument applied to T_j^* instead of T_j, we obtain a unitary operator U_j', and an operator Z one-to-one and with dense range, such that $T_j Z = Z U_j'$. Let $Y = ZZ^*X^*$, which is clearly one-to-one and with dense range. Then

$$\begin{aligned} T_j Y &= T_j Z Z^* X^* = Z U_j' Z^* X^* = Z(XZU_j'^{-1})^* = Z(U_j^{-1} U_j X Z U_j'^{-1})^* \\ &= Z(U_j^{-1} X T_j Z U_j'^{-1})^* = Z(U_j^{-1} X Z U_j' U_j'^{-1})^* \\ &= Z(U_j^{-1} X Z)^* = ZZ^*X^*U_j = YU_j. \ \blacksquare \end{aligned}$$

Since any unitary has (nontrivial) hyperinvariant subspaces, and since quasisimilarity preserves the property of having (nontrivial) hyperinvariant subspaces (see [1]), we obtain

THEOREM 2.3. *Let $T = (T_1, \ldots, T_N)$ be an N-tuple of commuting contractions on $\mathcal{H}$. Then either T has a common nontrivial invariant subspace or T belongs to at least one of the classes $K_{0,\cdot}$ and $K_{\cdot,0}$.*

To achieve the necessary reduction one only needs to notice that

$$\mathcal{M} := \{x \in \mathcal{H} : \inf_{n\in\mathbb{Z}_+^N} \|T^n x\| = 0\}$$

is a common invariant subspace for T. If $\mathcal{M} = \mathcal{H}$, then $T \in K_{0\cdot}$. If $\mathcal{M} = (0)$, then a similar argument applied to T^* puts us in the case covered by Theorem 2.2.

3. The Extension Theorem. The following is the main result of this paper.

THEOREM 3.1. *Let $(T_1, \ldots, T_N)$ be an N-tuple of commuting contractions on $\mathcal{H}$. Then there exists a commuting N-tuple $(V_1, \ldots, V_N)$ of contractions on a superspace $\mathcal{K} \supset \mathcal{H}$ with the following properties: Each V_j extends T_j, $j = 1, \ldots, N$, and there is a subspace $\mathcal{M}$ such that with respect to the decomposition $\mathcal{K} = \mathcal{M} \oplus \mathcal{M}^\perp$ we have*

$$V_j = \begin{pmatrix} S_j & 0 \\ 0 & U_j \end{pmatrix},$$

where the (commuting) N-tuple $(S_1, \ldots, S_N)$ is of class $K_{0,\cdot}$, while each U_j is an isometry. Furthermore, the extension is minimal, and therefore unique up to unitary equivalence, i.e., given another extension V' on a superspace

$\mathcal{K}' \supset \mathcal{H}$ with the same properties, we can find a unitary operator W from $\mathcal{K}$ onto its image such that $WV_j = V_j'W$, for $j = 1, \ldots, N$.

NOTE. After completing this paper the authors have learned that a similar description of $\mathcal{M}^\perp$ for a single operator appears (in a different context) in [4], p. 192.

Proof. For each $x \in \mathcal{H}$ we define
$$\|x\|_{\mathcal{N}} := \inf_{n \in \mathbb{Z}_+^N} \|T^n x\|_{\mathcal{H}}.$$
Note that $\|\cdot\|_{\mathcal{N}}$ is a seminorm (recall that the infimum of the sequence $\{\|T^n x\|\}_{\mathbb{Z}_+^N}$ is equal to its limit). The space $\mathcal{N}$ is defined as the completion of the quotient space of $\mathcal{H}$ by the equivalence relation $\sim$, where $x \sim y$ if $\|x-y\|_{\mathcal{N}} = 0$, for $x, y \in \mathcal{H}$. The space $\mathcal{N}$ is a Hilbert space, with inner product denoted by $\langle \cdot, \cdot \rangle_{\mathcal{N}}$, since $\|\cdot\|_{\mathcal{N}}$ satisfies the parallelogram law. Let $X_{\mathcal{N}}$ denote the inclusion operator from $\mathcal{H}$ into $\mathcal{N}$. The operator $X_{\mathcal{N}}$ has dense range.

We now define a Hermitian, positive semidefinite inner product: for $x, y \in \mathcal{H}$,
$$\langle x, y \rangle_{\mathcal{M}} := \langle x, y \rangle_{\mathcal{H}} - \langle X_{\mathcal{N}} x, X_{\mathcal{N}} y \rangle_{\mathcal{N}}.$$
The space $\mathcal{M}$ is defined as the completion of the quotient Hilbert space of $\mathcal{H}$ by the equivalence relation $\simeq$ defined by $x \simeq y$ if $\langle x-y, x-y \rangle_{\mathcal{M}} = 0$. Let $X_{\mathcal{M}}$ denote the inclusion operator from $\mathcal{H}$ into $\mathcal{M}$. The operator $X_{\mathcal{M}}$ has dense range.

We define $\mathcal{K} := \mathcal{M} \oplus \mathcal{N}$, with inner product
$$\langle x_1 \oplus y_1, x_2 \oplus y_2 \rangle_{\mathcal{K}} := \langle x_1, x_2 \rangle_{\mathcal{M}} + \langle y_1, y_2 \rangle_{\mathcal{N}} \quad \text{for } x_1, x_2 \in \mathcal{M},\ y_1, y_2 \in \mathcal{N}.$$
It is clear that $\mathcal{N} = \mathcal{M}^\perp$ (relative to $\mathcal{K}$). Let $X_{\mathcal{K}} : \mathcal{H} \to \mathcal{K}$ be the isometry $X_{\mathcal{M}} \oplus X_{\mathcal{N}}$, which permits us to identify $\mathcal{H}$ with a subspace of $\mathcal{K}$.

For $x \in \mathcal{H}$, the formula
$$S_j' X_{\mathcal{M}} x = X_{\mathcal{M}} T_j x$$
defines an operator S_j' on a dense subspace of $\mathcal{M}$, and similarly
$$U_j' X_{\mathcal{N}} x = X_{\mathcal{N}} T_j x$$
defines an operator U_j' on a dense subspace of $\mathcal{N} = \mathcal{M}^\perp$. We now define extensions to $\mathcal{K}$ of S_j' and U_j', denoted by S_j and U_j respectively. The operator U_j' defined on a dense manifold $X_{\mathcal{N}}\mathcal{H}$ of $\mathcal{N}$ is an isometry since for all $x \in \mathcal{H}$,
$$\|X_{\mathcal{N}} T_j x\|_{\mathcal{N}} = \inf_{n \in \mathbb{Z}_+^N} \|T^n T_j x\|_{\mathcal{H}} = \inf_{n \in \mathbb{Z}_+^N} \|T^n x\|_{\mathcal{H}} = \|X_{\mathcal{N}} x\|_{\mathcal{N}},$$
and so it can be extended to an isometry U_j on the whole $\mathcal{N}$.

Now we shall see that $S' = (S_1', \ldots, S_N') \in K_{0,\cdot}$. For $x \in \mathcal{H}$ and an N-tuple $n = (n_1, \ldots, n_N)$ of nonnegative integers, we have
$$\begin{aligned}\|S'^n X_{\mathcal{M}} x\|_{\mathcal{M}}^2 &= \langle S'^n X_{\mathcal{M}} x, S'^n X_{\mathcal{M}} x \rangle_{\mathcal{M}} = \langle X_{\mathcal{M}} T^n x, X_{\mathcal{M}} T^n x \rangle_{\mathcal{M}} \\ &= \langle T^n x, T^n x \rangle_{\mathcal{H}} - \|X_{\mathcal{N}} T^n x\|_{\mathcal{N}}^2 = \|T^n x\|_{\mathcal{H}}^2 - \inf_{k \in \mathbb{Z}_+^N} \|T^k x\|_{\mathcal{H}}^2.\end{aligned}$$
The right hand side is less than or equal to $\|x\|_{\mathcal{H}}^2 - \|X_{\mathcal{N}} x\|_{\mathcal{N}}^2 = \|X_{\mathcal{M}} x\|_{\mathcal{M}}^2$. Hence, taking appropriate n we get the contractivity of S_j' ($j = 1, \ldots, N$). So we can extend them to the whole $\mathcal{M}$. On the other hand, taking the infimum in n we obtain $\inf_{n \in \mathbb{Z}_+^N} \|S^n X_{\mathcal{M}} x\|_{\mathcal{M}} = \inf_{n \in \mathbb{Z}_+^N} \|S'^n X_{\mathcal{M}} x\|_{\mathcal{M}} = 0$. Again, since this holds on the dense subspace $X_{\mathcal{M}}\mathcal{H}$, it must hold on the whole $\mathcal{M}$.

Thus the operator on $\mathcal{K}$ defined by the matrix
$$V_j = \begin{pmatrix} S_j & 0 \\ 0 & U_j \end{pmatrix}$$
has the desired properties.

We now prove the minimality and uniqueness up to unitary equivalence. Indeed, let $V' = (V_1', \ldots, V_N')$ be another extension acting on $\mathcal{K}'$ containing $\mathcal{H}$ as a proper subspace with a decomposition $\mathcal{K}' = \mathcal{M}' \oplus \mathcal{M}'^\perp$ such that with respect to this decomposition
$$V_j' = \begin{pmatrix} \widetilde{S}_j & 0 \\ 0 & \widetilde{U}_j \end{pmatrix},$$
where the (commuting) N-tuple $(\widetilde{S}_1, \ldots, \widetilde{S}_N)$ is of class $K_{0,\cdot}$, while each $\widetilde{U}_j$ is an isometry. Denote by $P_{\mathcal{R}}$ the orthogonal projection onto a subspace $\mathcal{R}$. Let $x \in \mathcal{H}$. Then
$$\begin{aligned}\inf_{n \in \mathbb{Z}_+^N} \|T^n x\|_{\mathcal{H}}^2 &= \inf_{n \in \mathbb{Z}_+^N} \|T^n x\|_{\mathcal{K}'}^2 = \inf_{n \in \mathbb{Z}_+^N} \|V'^n x\|_{\mathcal{K}'}^2 \\ &= \inf_{n \in \mathbb{Z}_+^N} (\|\widetilde{S}^n P_{\mathcal{M}'} x\|_{\mathcal{K}'}^2 + \|\widetilde{U}^n P_{\mathcal{M}'^\perp} x\|_{\mathcal{K}'}^2) \\ &= \inf_{n \in \mathbb{Z}_+^N} \|\widetilde{S}^n P_{\mathcal{M}'} x\|_{\mathcal{K}'}^2 + \|P_{\mathcal{M}'^\perp} x\|_{\mathcal{K}'}^2 = \|P_{\mathcal{M}'^\perp} x\|_{\mathcal{K}'}^2.\end{aligned}$$
Similarly,
$$\inf_{n \in \mathbb{Z}_+^N} \|T^n x\|_{\mathcal{H}}^2 = \inf_{n \in \mathbb{Z}_+^N} (\|S^n P_{\mathcal{M}} x\|_{\mathcal{K}}^2 + \|U^n P_{\mathcal{M}^\perp} x\|_{\mathcal{K}}^2) = \|P_{\mathcal{M}^\perp} x\|_{\mathcal{K}}^2.$$
Thus, $\|P_{\mathcal{M}'^\perp} x\|_{\mathcal{K}'} = \|P_{\mathcal{M}^\perp} x\|_{\mathcal{K}}$ for all $x \in \mathcal{H}$. Since, by construction, $P_{\mathcal{M}^\perp}\mathcal{H}$ is dense in $\mathcal{M}^\perp$ the map $P_{\mathcal{M}^\perp} x \mapsto P_{\mathcal{M}'^\perp} x$ for $x \in \mathcal{H}$ defines a unitary operator $R^\perp$ from $\mathcal{M}^\perp$ onto its image. We observe that
$$\begin{aligned}R^\perp V_j P_{\mathcal{M}^\perp} x &= R^\perp P_{\mathcal{M}^\perp} V_j x = R^\perp P_{\mathcal{M}^\perp} T_j x = P_{\mathcal{M}'^\perp} T_j x \\ &= P_{\mathcal{M}'^\perp} V_j' x = V_j' P_{\mathcal{M}'^\perp} x = V_j' R^\perp P_{\mathcal{M}^\perp} x\end{aligned}$$

for every $x \in \mathcal{H}$ and $j = 1, \ldots, N$. Hence, $R^{\perp} V_j|_{\mathcal{M}^{\perp}} = V_j' R^{\perp}|_{\mathcal{M}^{\perp}}$ for $j = 1, \ldots, N$.

Since for $x \in \mathcal{H}$, we have

$$\|P_{\mathcal{M}}x\|_{\mathcal{K}}^2 + \|P_{\mathcal{M}^{\perp}}x\|_{\mathcal{K}}^2 = \|x\|_{\mathcal{H}}^2 = \|P_{\mathcal{M}'}x\|_{\mathcal{K}'}^2 + \|P_{\mathcal{M}'^{\perp}}x\|_{\mathcal{K}'}^2,$$

we get $\|P_{\mathcal{M}}x\|_{\mathcal{K}} = \|P_{\mathcal{M}'^{\perp}}\|_{\mathcal{K}'}$, and so the map $P_{\mathcal{M}}x \mapsto P_{\mathcal{M}'}x$ for $x \in \mathcal{H}$ defines a unitary operator R from $\mathcal{M}^{\perp}$ onto its image. As before, we obtain $RV_j|_{\mathcal{M}} = V_j' R|_{\mathcal{M}}$.

Defining the unitary operator W on $\mathcal{K}$ as $y \mapsto RP_{\mathcal{M}}y \oplus R^{\perp}P_{\mathcal{M}'^{\perp}}y$, we obtain

$$WV_j = V_j' W \quad \text{for } j = 1, \ldots, N.$$

Thus, each V_j is unitarily equivalent to an operator acting on $R\mathcal{M} \oplus R^{\perp}\mathcal{M}^{\perp} \subset \mathcal{K}$, and the N-tuple $WVW^* = (WV_1W^*, \ldots, WV_NW^*)$ is an extension of T of the desired type acting on a, perhaps proper, subspace of $\mathcal{K}'$. ■

A similar proof can also be obtained in a different way (we give a sketch): Let $B = T_1 \cdot \ldots \cdot T_N$ and define $\varrho(x) := \lim \|B^n x\|$ for $x \in \mathcal{H}$. Then ϱ is a seminorm on $\mathcal{H}$ that comes from a semidefinite scalar product; take $\mathcal{N}$ to be the completion of $\mathcal{H}$ with respect to it. The map $x \in \mathcal{H} \mapsto x \in \mathcal{N}$ is a contraction denoted by $A : \mathcal{H} \to \mathcal{N}$; define $D_A := (I - A^*A)^{1/2}$ and $\mathcal{M} = \mathcal{D}_A := \overline{D_A\mathcal{H}}$ (the defect operator and defect space of A respectively). Then $x \mapsto D_Ax \oplus Ax$ is an isometric embedding of $\mathcal{H}$ into $\mathcal{K} = \mathcal{M} \oplus \mathcal{N}$. Finally, we define $S_j \in \mathcal{L}(\mathcal{M})$ by $S_jD_Ax = D_AT_jx$, and $U_jAx = AT_jx$ (both on a dense set, of course). Then, defining V_j as in the previous proof, one can check all the desired properties.

Using Proposition I.6.2 of [10] one can extend the N-tuple $(U_1, \ldots, U_N)$ of isometries to an N-tuple of unitaries acting on a larger space. Thus we get the following result, which we shall call the Extension Theorem:

COROLLARY 3.2. *Let $(T_1, \ldots, T_N)$ be an N-tuple of commuting contractions on $\mathcal{H}$. Then there exists a commuting N-tuple $(B_1, \ldots, B_N)$ of contractions on a superspace $\mathcal{K} \supset \mathcal{H}$ with the following properties: Each B_j extends T_j, $j = 1, \ldots, N$, and there is a subspace $\mathcal{M}$ such that with respect to the decomposition $\mathcal{K} = \mathcal{M} \oplus \mathcal{M}^{\perp}$ we have*

$$B_j = \begin{pmatrix} S_j & 0 \\ 0 & U_j \end{pmatrix},$$

where the (commuting) N-tuple $(S_1, \ldots, S_N)$ is of class $K_{0,\cdot}$, while each U_j is a unitary operator. Furthermore, the space $\mathcal{K}$ is the smallest one with this property, and therefore unique up to unitary equivalence.

We point out that in the case of a single contraction ($N = 1$), the *Wold-type extension* given in Corollary 3.2 is smaller, in general, than the minimal

coisometric extension (with its Wold–von Neumann decomposition). For example, one can consider the case of a $C_{0,\cdot}$ contraction, which is its own Wold-type extension, but not its own minimal coisometric extension.

REMARK 3.3. One can obtain from Corollary 3.2 a direct construction of the minimal coisometric extension of a single contraction T, by extending the $C_{0,\cdot}$ part of the Wold-type extension of T^* to a backward shift (see, for example, Section 2 of [1]). (The adjoint of such an extension is the minimal isometric dilation of T.) Moreover, extending the unilateral shift to the bilateral one, we get a simple construction of the minimal unitary dilation of T.

Vice versa, given a contraction T and the minimal isometric dilation B^* of T^* we can obtain the Wold-type extension of T by restricting the backward shift part of B to the proper minimal subspace given in the proof of Theorem 3.1. It can be seen that if T is a coisometry, then it is its own Wold-type extension. The situation for pairs of commuting contractions ($N = 2$) is similar but much harder to discuss thoroughly because of the lack of a joint Wold–von Neumann decomposition for pairs of commuting coisometries. We leave the details for future work.

4. Diagonalization of isometries. The extension $V = (V_1, \ldots, V_N)$ of $T = (T_1, \ldots, T_N)$ is somewhat analogous to the coisometric extension for a single contraction (or for pairs). We can thus obtain a dilation theory for N-tuples of commuting contractions by applying the result above to $T^* := (T_1^*, \ldots, T_N^*)$. The dilation (or extension) of the N-tuple T may, obviously, lack the geometric or norm properties of the single contraction (or the pair) case, but will have a decomposition similar to that provided by the Wold–von Neumann decomposition (which the pair case lacked). We remark that the dilations so constructed would, of course, be power or strong dilations (i.e., dilations for all powers of T). This may be enough to extend several results known in the single contraction case to the case of N-tuples of contractions. We have succeeded in using the Extension Theorem to show that an N-tuple having the polydisk as a spectral set and rich (Harte) spectrum has a (common) invariant subspace. This extends previously known results even in the case of pairs (where the spectral set hypothesis is always satisfied) and it closes the first chapter of the Theory of Dual Algebras generated by N-tuples of commuting contractions. The proof of this result is somewhat technical and long. For these reasons, it is postponed to another article ([5]).

In this section we show (Propositions 4.1 and 4.2) that the Wold-type extension always diagonalizes isometries. We will also use the Extension Theorem (Corollary 3.2) to study the following problem: Given two commuting isometries V_1, V_2 acting on $\mathcal{H}$, let $\mathcal{M}$ be the subspace reducing V_1 to the shift part in its Wold decomposition; give conditions under which $\mathcal{M}$ is also reducing for V_2. In [8] the second named author showed that it is not always reducing. We have already pointed out that this is the main difficulty in using the joint coisometric extension of pairs of commuting contractions in the Theory of Dual Algebras and that this difficulty was removed by the authors (see [5]) by using the Extension Theorem (Corollary 3.2). Now we show that we can get a well known result of Słociński [9], which implies that the subspace is reducing if the isometries *doubly commute* (i.e., V_1 commutes with V_2 and V_2^*).

PROPOSITION 4.1. *Let $(T_1,\ldots,T_N)$ be an N-tuple of commuting contractions on $\mathcal{H}$ and let W be an isometry on $\mathcal{H}$ commuting with each T_j, $j=1,\ldots,N$. Let $(V_1,\ldots,V_N)$ be the extension of $(T_1,\ldots,T_N)$ to $\mathcal{K}=\mathcal{M}\oplus\mathcal{M}^\perp$ given by Theorem 3.1. Then there is a unique (up to unitary equivalence) isometry $\widetilde{U}$ on $\mathcal{K}$ commuting with V_j, $j=1,\ldots,N$, and extending W such that the subspace $\mathcal{M}$ is reducing for $\widetilde{U}$.*

Proof. Using the notation of the proof of Theorem 3.1 we define operators $W'_{\mathcal{M}}$ and $W'_{\mathcal{N}}$ on dense subspaces of $\mathcal{M}$ and $\mathcal{N}$ respectively as follows:

$$W'_{\mathcal{M}}X_{\mathcal{M}}x = X_{\mathcal{M}}Wx, \quad x\in\mathcal{K},$$
$$W'_{\mathcal{N}}X_{\mathcal{N}}x = X_{\mathcal{N}}Wx, \quad x\in\mathcal{K},$$

It is easy to see that $W'_{\mathcal{M}}$ commutes with all S'_j and $W'_{\mathcal{N}}$ commutes with all U'_j, $j=1,\ldots,N$. Moreover, the operator $W'_{\mathcal{N}}$ defined on a dense manifold $X_{\mathcal{N}}\mathcal{H}$ of $\mathcal{N}$ is an isometry since for all $x\in\mathcal{H}$,

$$\|X_{\mathcal{N}}Wx\|_{\mathcal{N}} = \inf_{n\in\mathbb{Z}_+^N}\|T^nWx\|_{\mathcal{H}} = \inf_{n\in\mathbb{Z}_+^N}\|T^nx\|_{\mathcal{H}} = \|X_{\mathcal{N}}x\|_{\mathcal{N}},$$

and so it can be extended to an isometry $W_{\mathcal{N}}$ on the whole $\mathcal{N}$ commuting with all U'_j, $j=1,\ldots,N$.

By the definition of the norm in $\mathcal{K}$ we can easily see that $W'_{\mathcal{M}}$ is an isometry on a dense manifold $X_{\mathcal{M}}\mathcal{H}$ of $\mathcal{M}$ and so it extends to an isometry $W_{\mathcal{M}}$ on $\mathcal{M}$ commuting with all S'_j, $j=1,\ldots,N$.

Thus the operator on $\mathcal{K}$ defined by the matrix

$$\widetilde{U} = \begin{pmatrix} W_{\mathcal{M}} & 0 \\ 0 & W_{\mathcal{N}} \end{pmatrix}$$

has the desired properties. ■

By an easy modification of the proof of Proposition 4.1, using Proposition I.6.2 of [10], and the observation that isometries commuting on a dense manifold must commute on the whole space, we also get the following

PROPOSITION 4.2. *Let $(T_1,\ldots,T_N)$ be an N-tuple of commuting contractions on $\mathcal{H}$ and let $\{S_\alpha\}_{\alpha\in I}$ be a collection of commuting isometries on $\mathcal{H}$ which also commute with each T_j. Let $(V_1,\ldots,V_N)$ be the extension of $(T_1,\ldots,T_N)$ to $\mathcal{K}=\mathcal{M}\oplus\mathcal{M}^\perp$ given by Corollary 3.2. Then there is a collection $\{U_\alpha\}_{\alpha\in I}$ of commuting isometries on $\mathcal{K}$ which also commute with all V_j and extend $\{S_\alpha\}$ such that the subspace $\mathcal{M}$ is reducing for all U_α.*

We obtain the following result:

THEOREM 4.3. *Let V be an isometry on $\mathcal{H}$ and W a coisometry commuting with V. Then we have the following decomposition reducing V and W:*

$$\mathcal{H} = \mathcal{H}_{\mathrm{SB}}\oplus\mathcal{H}_{\mathrm{SU}}\oplus\mathcal{H}_{\mathrm{UB}}\oplus\mathcal{H}_{\mathrm{UU}},$$

where

(1) *$V|_{\mathcal{H}_{\mathrm{SB}}}$ is a shift, $W|_{\mathcal{H}_{\mathrm{SB}}}$ is a backward shift,*
(2) *$V|_{\mathcal{H}_{\mathrm{SU}}}$ is a shift, $W|_{\mathcal{H}_{\mathrm{SU}}}$ is unitary,*
(3) *$V|_{\mathcal{H}_{\mathrm{UB}}}$ is unitary, $W|_{\mathcal{H}_{\mathrm{UB}}}$ is a backward shift,*
(4) *$V|_{\mathcal{H}_{\mathrm{UU}}}$ and $W|_{\mathcal{H}_{\mathrm{UU}}}$ are unitary.*

Proof. By the Wold–von Neumann decomposition for V we get

$$\mathcal{H} = \mathcal{H}_{\mathrm{S}}\oplus\mathcal{H}_{\mathrm{U}},$$

where $\mathcal{H}_{\mathrm{S}}$ reduces V to a shift and $\mathcal{H}_{\mathrm{U}}$ reduces V to a unitary operator. Consequently, $V^*|_{\mathcal{H}_{\mathrm{S}}}$ is a backward shift and $V^*|_{\mathcal{H}_{\mathrm{U}}}$ is unitary.

Applying now Corollary 3.2 to V^*, we get the same decomposition since the extension is trivial in this case, which follows directly from the proof of Corollary 3.2 or from the fact that the extension in Corollary 3.2 is minimal.

By Proposition 4.1, the subspaces $\mathcal{H}_{\mathrm{S}}$ and $\mathcal{H}_{\mathrm{U}}$ reduce the isometry W^*. Now we use the Wold–von Neumann decomposition for each part of W^*. So

$$\mathcal{H}_{\mathrm{S}} = \mathcal{H}_{\mathrm{SB}}\oplus\mathcal{H}_{\mathrm{SU}}, \qquad \mathcal{H}_{\mathrm{U}} = \mathcal{H}_{\mathrm{UB}}\oplus\mathcal{H}_{\mathrm{UU}},$$

where $\mathcal{H}_{\mathrm{SB}}$, $\mathcal{H}_{\mathrm{UB}}$ reduce W^* to a shift, and $\mathcal{H}_{\mathrm{SU}}$, $\mathcal{H}_{\mathrm{UU}}$ reduce W^* to a unitary operator. Consequently, the first two subspaces must reduce W to a backward shift, and the last two to a unitary operator.

By the same argument as before, all parts of the decomposition obtained must reduce V, which implies the desired decomposition. ■

Now we state, as an easy corollary, the result of Słociński:

THEOREM 4.4. *Let (V,W) be a pair of doubly commuting isometries on $\mathcal{H}$. Then we have the following decomposition reducing V and W:*

$$\mathcal{H} = \mathcal{H}_{\mathrm{SS}}\oplus\mathcal{H}_{\mathrm{SU}}\oplus\mathcal{H}_{\mathrm{UU}}\oplus\mathcal{H}_{\mathrm{UU}},$$

where

(1) *$V|_{\mathcal{H}_{\mathrm{SS}}}$ and $W|_{\mathcal{H}_{\mathrm{SS}}}$ are (forward) shifts,*
(2) *$V|_{\mathcal{H}_{\mathrm{SU}}}$ is a shift, $W|_{\mathcal{H}_{\mathrm{SU}}}$ is unitary,*
(3) *$V|_{\mathcal{H}_{\mathrm{US}}}$ is unitary, $W|_{\mathcal{H}_{\mathrm{US}}}$ is a backward shift,*
(4) *$V|_{\mathcal{H}_{\mathrm{UU}}}$ and $W|_{\mathcal{H}_{\mathrm{UU}}}$ are unitary.*

Theorem 4.4 also implies Theorem 4.3, since it can be shown that if an isometry commutes with a coisometry, then they must doubly commute. This is an analog of the Fuglede theorem for another class of operators. It is probably known but the authors have been unable to find a reference for it. We shall present it here as a consequence of Theorem 4.3. Independent proofs are possible and simpler, nevertheless, we feel Theorem 4.3 presents a complete picture of the problem.

PROPOSITION 4.5. *Let $V \in \mathcal{L}(\mathcal{H})$ be an isometry and let $W \in \mathcal{L}(\mathcal{H})$ be a coisometry. If V commutes with W, then it commutes with W^*.*

Proof. It is easy to see that if an operator commutes with a unitary then they doubly commute. Therefore, by Theorem 4.3, we only have to show that if a forward shift S commutes with a backward shift B then they doubly commute. With respect to some decomposition of the space on which it acts, S has the form

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ I & 0 & 0 & 0 & \dots \\ 0 & I & 0 & 0 & \dots \\ 0 & 0 & I & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where I is the identity operator in a space of dimension equal to the multiplicity of S. Since B commutes with S, it must have the form

$$B = \begin{pmatrix} B_1 & 0 & 0 & 0 & \dots \\ B_2 & B_1 & 0 & 0 & \dots \\ B_3 & B_2 & B_1 & 0 & \dots \\ B_4 & B_3 & B_2 & B_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with respect to the same decomposition. But B can be a coisometry only if $B_j = 0$ for $j = 2, 3, \dots$ Thus B, being diagonal, must doubly commute with S. ∎

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