[17] E. Straube, *Harmonic and analytic functions admitting a distribution boundary value*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 11 (1984), 559–591.

Departamento de Matemática Aplicada
Universidad Politécnica de Valencia
C/Vera 14
Valencia 46071, Spain
E-mail: jmotos@mat.upv.es

Instituto de Matemáticas
Universidad Nacional Autónoma de México
Unidad Cuernavaca
A. P. 273-3 Admón. 3
Cuernavaca, Morelos, 62251
México
E-mail: salvador@matcuer.unam.mx

Received June 20, 1998
Revised version April 12, 1999 (4130)

STUDIA MATHEMATICA 137 (1) (1999)

Geometry of oblique projections

by

E. ANDRUCHOW (San Miguel), GUSTAVO CORACH (Buenos Aires) and D. STOJANOFF (Buenos Aires)

Abstract. Let $\mathcal{A}$ be a unital C^*-algebra. Denote by P the space of selfadjoint projections of $\mathcal{A}$. We study the relationship between P and the spaces of projections P_a determined by the different involutions $\#_a$ induced by positive invertible elements $a \in \mathcal{A}$. The maps $\varphi_p : P \to P_a$ sending p to the unique $q \in P_a$ with the same range as p and $\Omega_a : P_a \to P$ sending q to the unitary part of the polar decomposition of the symmetry $2q-1$ are shown to be diffeomorphisms. We characterize the pairs of idempotents $q, r \in \mathcal{A}$ with $\|q-r\| < 1$ such that there exists a positive element $a \in \mathcal{A}$ satisfying $q, r \in P_a$. In this case q and r can be joined by a unique short geodesic along the space of idempotents Q of $\mathcal{A}$.

1. Introduction. Let $\mathcal{H}$ be a Hilbert space with scalar product $\langle\, ,\rangle$. For every bounded positive invertible operator $a : \mathcal{H} \to \mathcal{H}$ consider the scalar product $\langle\, ,\rangle_a$ given by

$$\langle \xi, \eta\rangle_a = \langle \xi, \eta\rangle, \quad \xi, \eta \in \mathcal{H}.$$

It is clear that $\langle\, ,\rangle_a$ induces a norm equivalent to the norm induced by $\langle\, ,\rangle$. With respect to the scalar product $\langle\, ,\rangle_a$, the adjoint of a bounded linear operator $x : \mathcal{H} \to \mathcal{H}$ is

$$x^{\#_a} = a^{-1}x^*a.$$

Thus, x is $\#_a$-selfadjoint if and only if

$$ax = x^*a.$$

Given a closed subspace S of $\mathcal{H}$, denote by $p = P_S$ the orthogonal projection from $\mathcal{H}$ onto S and, for any positive operator a, denote by $\varphi_p(a)$ the unique $\#_a$-selfadjoint projection with range S. In a recent paper, Z. Pasternak-Winiarski [20] proves the analyticity of the map $a \mapsto \varphi_p(a)$ and calculates its Taylor expansion. This study is relevant for understanding reproducing kernels of Hilbert spaces of holomorphic L^2 sections of complex vector bundles and the way they change when the measures and hermitian

1991 *Mathematics Subject Classification*: 46L05, 46C99, 47B15, 53C22, 58B25.
Research partially supported by CONICET, ANPCYT and UBACYT (Argentina).

structures are deformed (see [21], [22]). This type of deformations appears in a natural way when studying quantization of systems where the phase space is a Kähler manifold (Odzijewicz [18], [19]).

In this paper we pose Pasternak-Winiarski's problem in the C^*-algebra setting and use the knowledge of the differential geometry of idempotents, projections and positive invertible elements in order to get more general results in a shorter way.

More precisely, let $\mathcal{A}$ be a unital C^*-algebra, $G = G(\mathcal{A})$ the group of invertible elements of $\mathcal{A}$, $\mathcal{U} = \mathcal{U}_{\mathcal{A}}$ the unitary group of $\mathcal{A}$, $G^+ = \{a \in G : a^* = a,\ a \geq 0\}$ the space of positive invertible elements of $\mathcal{A}$, and

$$Q = Q(\mathcal{A}) = \{q \in \mathcal{A} : q^2 = q\} \quad \text{and} \quad P = P(\mathcal{A}) = \{p \in Q : p = p^*\}$$

the spaces of idempotents and projections of $\mathcal{A}$. The nonselfadjoint elements of Q will be called *oblique projections*. It is well known that Q is a closed analytic submanifold of $\mathcal{A}$, P is a closed real analytic submanifold of Q and G^+ is an open submanifold of

$$\mathcal{S} = \mathcal{S}(\mathcal{A}) = \{b \in \mathcal{A} : b^* = b\},$$

which is a closed real subspace of $\mathcal{A}$ (see [24], [7] or [9] for details).

We define a fibration

$$\varphi : P \times G^+ \to Q$$

which coincides, when $\mathcal{A} = L(\mathcal{H})$, with the map $(p, a) \mapsto \varphi_p(a)$, the unique $\#_a$-selfadjoint projection with the same range as p. This allows us to study the analyticity of Pasternak-Winiarski's map in both variables p, a. The rich geometry of Q, P and G^+ gives an amount of information which may be useful in the problems that motivated [20].

Along our paper we use the fact that every $p \in Q$ induces a representation α_p of elements of $\mathcal{A}$ by 2×2 matrices given by

$$\alpha_p(a) = \begin{pmatrix} pap & pa(1-p) \\ (1-p)ap & (1-p)a(1-p) \end{pmatrix}.$$

Under this homomorphism p can be identified with

$$\begin{pmatrix} 1_{p\mathcal{A}p} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and all idempotents q with the same range of p have the form

$$q = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$$

for some $x \in p\mathcal{A}(1-p)$. This trivial remark shortens many proofs in a drastic way and the analyticity of some maps (for example $\varphi : P \times G^+ \to Q$) follows immediately.

The content of the paper is the following. Section 2 presents some preliminary material including the matrix representations mentioned above and the description of the adjoint operation induced by each positive invertible (element or operator) a.

In Section 3 we study the map $\varphi_p = \varphi(p, \cdot) : G^+ \to Q$, which is Pasternak-Winiarski's map when $\mathcal{A}$ is $L(\mathcal{H})$ and p is the orthogonal projection P_S onto a closed subspace $S \subseteq \mathcal{H}$. For $a \in G^+$, let $P_a = P_a(\mathcal{A})$ denote the set of all $\#_a$-selfadjoint projections. This is a subset of Q and Section 4 starts the study of the relationship between $P = P_1$ and P_a and the way they are located in Q. In particular, we show that $\varphi_\alpha = \varphi(\cdot, a) : P \to P_a$ is a diffeomorphism and compute its tangent map. Another interesting map is the following: for $q \in P_a$, $\varepsilon = 2q - 1$ is a reflection, i.e. $\varepsilon^2 = 1$, which admits in $\mathcal{A}$ a polar decomposition $\varepsilon = \lambda\varrho$, with $\lambda \in G^+$ and ϱ a unitary element of $\mathcal{A}$. It is easy to see that $\varrho = \varrho^* = \varrho^{-1}$ so that $p = \frac{1}{2}(\varrho + 1) \in P$. In Section 5 we prove that the map $\Omega_a : P_a \to P$ given by $\Omega_a(q) = p$ is a diffeomorphism and study the movement of P given by the composition $\Omega_a \circ \varphi_\alpha : P \to P$. We also characterize the orbit of p under these movements, i.e.

$$\mathcal{O}_p := \{r \in P : \Omega_a \circ \varphi_a(p) = r \text{ for some } a \in G^+\}.$$

In recent years several papers have appeared which study the length of curves in P and Q (see [25], [3], [23], [7], [2], for example). It is known that P and the fibres of $\Omega : Q \to P$ are geodesically complete and their geodesics are short curves (for convenient Finsler metrics, see [7]). For a fixed $p \in P$, let us call those directions around p which produce geodesics along P (resp. along the fibre $\Omega^{-1}(p)$) *horizontal* (resp. *vertical*). In Section 6 we show that there exist short geodesics in many other directions (not only the horizontal and the vertical ones).

This paper, which originated from a close examination of Pasternak-Winiarski's work, is part of the program of understanding the structure of the space of idempotent operators. For a sample of the vast bibliography on the subject the reader is referred to the papers by Afriat [1], Kovarik [15], Zemánek [29], Porta and Recht [24], Gerisch [11], Corach [6] and the references therein. Applications of oblique projections to complex, harmonic and functional analysis and statistics can be found in the papers by Kerzman and Stein [13], [14], Pták [27], Coifman and Murray [5] and Mizel and Rao [17], among others.

2. Preliminary results. Let $\mathcal{H}$ be a Hilbert space, $\mathcal{A} \subset L(\mathcal{H})$ a unital C^*-algebra, $G = G(\mathcal{A})$ the group of invertible elements and $\mathcal{U}_{\mathcal{A}}$ the unitary group of $\mathcal{A}$,

If S is a closed subspace of $\mathcal{H}$ and q is a bounded linear projection onto S, then

(1) $$p = qq^*(1-(q-q^*)^2)^{-1}$$

is the unique sefadjoint projection onto S. Note that, by this formula, $p \in \mathcal{A}$ when $q \in \mathcal{A}$. Several different formulas are known for p (see [11], p. 294); perhaps the simplest one is the so-called *Kerzman–Stein formula*

(2) $$p = q(1+q-q^*)^{-1}$$

(see [13], [14] or [5]). However, for the present purposes, (1) is more convenient. We denote by

(3) $$Q = Q(\mathcal{A}) = \{q \in \mathcal{A} : q^2 = q\},$$
$$P = P(\mathcal{A}) = \{p \in \mathcal{A} : p = p^* = p^2\}$$

the spaces of idempotents and projections of $\mathcal{A}$. Given a fixed closed subspace S of $\mathcal{H}$, we denote by

(4) $$Q_S = Q_S(\mathcal{A}) = \{q \in Q(\mathcal{A}) : q(\mathcal{H}) = S\}$$

the space of idempotents of $\mathcal{A}$ with range S. Note that, by (1), Q_S is not empty if and only if the projection $p = p_S$ onto S belongs to $\mathcal{A}$. We shall make this assumption.

It is easy to see that two idempotents $q, r \in Q$ have the same range if and only if $qr = r$ and $rq = q$. Therefore the space Q_S of (4) can be characterized as

$$Q_S = Q_p = \{q \in Q : qp = p,\ pq = q\}.$$

In what follows, we adopt this notation Q_p, emphasizing the role of p rather than S. This enables us to simplify many computations. Moreover, this operator algebraic viewpoint allows one to get the results below independently of the representation of $\mathcal{A}$.

Recall some facts about matrix representations. Every $p \in Q$ induces a representation α_p of elements of $\mathcal{A}$ by 2×2 matrices given by

(5) $$\alpha_p(a) = \begin{pmatrix} pap & pa(1-p) \\ (1-p)ap & (1-p)a(1-p) \end{pmatrix}.$$

If $p \in P$ the representation preserves the involution $*$. For simplicity we identify a with $\alpha_p(a)$ and $\mathcal{A}$ with its image under α_p. Observe that, with this convention,

(6) $$p = \begin{pmatrix} 1_{p\mathcal{A}p} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Moreover, $q \in Q_S = Q_p$ if and only if there exists $x \in p\mathcal{A}(1-p)$ such that

(7) $$q = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}.$$

Indeed, let $q = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Q_p$. Then

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = p = qp = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix},$$

hence $a = 1$ and $c = 0$. On the other hand,

$$q = pq = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix},$$

so $d = 0$ and b can be anything. We summarize this information in the following:

2.1. Proposition. *The space Q_p can be identified with $p\mathcal{A}(1-p)$ by means of the affine map*

(8) $$Q_p \to p\mathcal{A}(1-p), \quad q \mapsto q - p.$$

Proof. Clearly, the affine map defined in (8) is injective. By (7) it is well defined and onto. ∎

In the Hilbert space $\mathcal{H}$, every scalar product which is equivalent to the original $\langle , \rangle$ is determined by a unique positive invertible operator $a \in L(\mathcal{H})$ by means of

(9) $$\langle \xi, \eta \rangle_a = \langle a\xi, \eta \rangle, \quad \xi, \eta \in \mathcal{H}.$$

For this scalar product the adjoint $x^{\#_a}$ of $x \in L(\mathcal{H})$ is easily seen to be

(10) $$x^{\#_a} = a^{-1}x^*a$$

where $*$ denotes the adjoint operation for the original scalar product. Operators which are selfadjoint for some $\#_a$ have been considered by Lax [16] and Dieudonné [10]. A geometrical study of families of C^*-involutions has been done by Porta and Recht [26].

Denote by $G^+ = G^+(\mathcal{A})$ the set of all positive invertible elements of $\mathcal{A}$. Every $a \in G^+$ induces as in (10) a continuous involution $\#_a$ on $\mathcal{A}$ by means of $x^{\#_a} = a^{-1}x^*a$, for $x \in \mathcal{A}$. $\mathcal{A}$ is a C^*-algebra with the involution $\#_a$ and the corresponding norm $\|x\|_a = \|a^{1/2}xa^{-1/2}\|$ for $x \in \mathcal{A}$. The mapping $x \mapsto a^{-1/2}xa^{1/2}$ is an isometric isomorphism of $(\mathcal{A}, \| \|, *)$ onto $(\mathcal{A}, \| \|_a, \#_a)$. In this setting, $\mathcal{A}$ can also be represented by the inclusion map in $L(\mathcal{H}, \langle , \rangle_a)$.

Note that the map $a \mapsto \langle , \rangle_a \mapsto \#_a$ is not one-to-one, since (10) says that if $a \in \mathbb{C}I$ then $\#_a = *$. If we regard this map in G^+ with values in the set of involutions of $\mathcal{A}$, then two elements $a, b \in G^+$ with $a = bz$ for z in the center of $\mathcal{A}$,

(11) $$\mathcal{Z}(\mathcal{A}) = \{z \in \mathcal{A} : zc = cz \text{ for all } c \in \mathcal{A}\},$$

produce the same involution $\#_a$.

2.2. Recall the properties of the conditional expectation induced by a fixed projection $p \in P$. Note that the set $\mathcal{A}_p$ of elements of $\mathcal{A}$ which commute with p is the C^*-subalgebra of $\mathcal{A}$ of diagonal matrices in terms of the representation (5). We denote by $E_p : \mathcal{A} \to \mathcal{A}_p \subset \mathcal{A}$ the conditional expectation defined by compressing to the diagonal:

$$E_p(a) = pap + (1-p)a(1-p) = \begin{pmatrix} pap & 0 \\ 0 & (1-p)a(1-p) \end{pmatrix} \quad a \in \mathcal{A}.$$

This expectation has the following well-known properties ([28], Chapter 2): for all $a \in \mathcal{A}$,

1. $E_p(bac) = bE_p(a)c$ for all $b, c \in \mathcal{A}_p$.
2. $E_p(a^*) = E_p(a)^*$.
3. If $b \le a$ then $E_p(b) \le E_p(a)$. In particular, $E_p(G^+) \subset G^+$.
4. $\|E_p(a)\| \le \|a\|$.
5. If $0 \le a$, then $2E_p(a) \ge a$.

3. Idempotents with the same range. The main purpose of this section is to describe, for a fixed $p \in P$, the map which sends each $a \in G^+$ to the unique $q \in Q_p$ which is $\#_a$-selfadjoint. This problem was posed and solved in [20] when $\mathcal{A} = L(\mathcal{H})$. Here we use 2×2 matrix arguments to give very short proofs of the results of [20]. Moreover, we generalize these results and apply them to understand some aspects of the geometry of the space Q.

Let us fix the notations. For each $a \in G^+$ denote by

(12) $$\mathcal{S}_a = \mathcal{S}_a(\mathcal{A}) = \{b \in \mathcal{A} : b^{\#_a} = b\}$$

the set of $\#_a$-selfadjoint elements of $\mathcal{A}$.

3.1. Definition. Let $\mathcal{A}$ be a C^*-algebra and $p \in P$ a fixed projection of $\mathcal{A}$. We consider the map $\varphi_p : G^+ \to Q_p$ given by

$$\varphi_p(a) = \text{the unique } q \in Q_p \cap \mathcal{S}_a, \qquad a \in G^+.$$

Note that the existence and uniqueness of such a q follow from (1) applied to the C^*-algebra $\mathcal{A}$ with the star $\#_a$.

3.2. Proposition. *Let $\mathcal{A}$ be a C^*-algebra and $p \in P$. Then, for all $a \in G^+(\mathcal{A})$,*

(13) $$\varphi_p(a) = pE_p(a)^{-1}a,$$

where E_p is the conditional expectation defined in 2.2. In particular,

$$\|\varphi_p(a)\| \le 2\|a\|\,\|a^{-1}\|.$$

Proof. Suppose that, in matrix form, we have

$$a = \begin{pmatrix} a_1 & a_2 \\ a_2^* & a_3 \end{pmatrix} \quad \text{and then} \quad E_p(a) = \begin{pmatrix} a_1 & 0 \\ 0 & a_3 \end{pmatrix}.$$

Since $\varphi_p(a) \in Q_p$, by (7) there exists $x \in p\mathcal{A}(1-p)$ such that $\varphi_p(a) = p+x$. On the other hand, by (10), $p+x \in \mathcal{S}_a$ if and only if $a^{-1}(p+x)^*a = p+x$, i.e. $(p+x^*)a = a(p+x)$. In matrix form,

$$(p+x^*)a = \begin{pmatrix} 1 & 0 \\ x^* & 0 \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ a_2^* & a_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ x^*a_1 & x^*a_2 \end{pmatrix} \quad \text{and}$$
$$a(p+x) = \begin{pmatrix} a_1 & a_2 \\ a_2^* & a_3 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_1 & a_1x \\ a_2^* & a_2^*x \end{pmatrix}.$$

Then $(p+x^*)a = a(p+x)$ if and only if $a_2 = a_1x$. Note that $a \in G^+(\mathcal{A})$ implies $a_1 \in G^+(p\mathcal{A}p)$. Then

(14) $$\varphi_p(a) = \begin{pmatrix} 1 & a_1^{-1}a_2 \\ 0 & 0 \end{pmatrix},$$

and now formula (13) can be proved by easy computations. Finally, since $2E_p(a) \ge a$, we deduce that $E_p(a)^{-1} \le 2a^{-1}$ and the inequality $\|\varphi_p(a)\| \le 2\|a\|\,\|a^{-1}\|$ follows easily. ■

3.3. Remark. There is a way to describe φ_p in terms of (2) with the star $\#_a$. In this sense we obtain, for $p \in P$ and $a \in G^+$,

$$\varphi_p(a) = p(1+p-a^{-1}pa)^{-1} = p(a+ap-pa)^{-1}a.$$

Clearly, $a+ap-pa = E_p(a)+2a_2^*$ and one gets (13), since $p(a+ap-pa)^{-1} = pE_p(a)^{-1}$. However, it seems difficult to obtain bounds for $\|\varphi_p(a)\|$ by using this approach.

3.4. Consider the space G^+ as an open subset of $\mathcal{S} = \mathcal{S}(\mathcal{A}) = \mathcal{S}_1(\mathcal{A})$, the closed real subspace of selfadjoint elements of $\mathcal{A}$. Then the map $\varphi_p : G^+ \to \mathcal{A}$ is real analytic. Indeed, if $h \in \mathcal{S}$ and $\|h\| < 1$, then

(15) $$\varphi_p(1+h) = p(1+E_p(h))^{-1}(1+h) = p\sum_{n=0}^{\infty}(-1)^n E_p(h)^n(1+h),$$

and this formula is clearly real analytic near 1. Further computations starting from (15) give the more explicit formula

(16) $$\varphi_p(1+h) = p + \sum_{n=1}^{\infty}(-1)^{n-1}(ph)^n(1-p),$$

again for all $h \in \mathcal{S}$ with $\|h\| < 1$. These computations are very similar to those appearing in the proof of Theorem 5.1 of [20]. We include them for the sake of completeness. By (15),

$$\begin{aligned}\varphi_p(1+h) &= \sum_{n=0}^{\infty}(-1)^n(php)^n(p+ph)\\ &= \sum_{n=0}^{\infty}(-1)^n(php)^n + \sum_{n=0}^{\infty}(-1)^n(ph)^{n+1}\\ &= p + \sum_{n=1}^{\infty}(-1)^n(ph)^n p + \sum_{n=1}^{\infty}(-1)^{n-1}(ph)^n\\ &= p + \sum_{n=1}^{\infty}(-1)^{n-1}(ph)^n(1-p).\end{aligned}$$

As a consequence (see also Theorem 3.1 of [20]) the tangent map $(T\varphi_p)_1 : \mathcal{S} \to \mathcal{A}$ is given by

$$(T\varphi_p)_1(X) = pX(1-p) \quad \text{for } X \in \mathcal{S}. \tag{17}$$

Actually, by Proposition 2.1, Q_p is an affine manifold parallel to the closed subspace $p\mathcal{A}(1-p)$, which can also be regarded as its "tangent" space. In this sense $(T\varphi_p)_1$ is just the natural compression of $\mathcal{S}$ onto $p\mathcal{A}(1-p)$.

Note that formulas (15) and (16) do not depend on the selected star in $\mathcal{A}$. Using this fact, formula (16) can be generalized to a power series around each $a \in G^+$ by using (16) with the star $\#_a$ at $q = \varphi_p(a)$. Indeed, note that for every $b \in G^+$, $\langle , \rangle_b = (\langle , \rangle_a)_{a^{-1}b}$ is induced from $\langle , \rangle_a$ by $a^{-1}b$, which is a-positive. If $h \in \mathcal{S}$ and $\|h\| < \|a^{-1}\|^{-1}$, then $a + h \in G^+$, $\|a^{-1}h\|_a = \|a^{-1/2}ha^{-1/2}\| \le \|h\|\,\|a^{-1}\| < 1$ and

$$\varphi_p(a+h) = \varphi_q(1+a^{-1}h) = q + \sum_{n=1}^{\infty}(-1)^{n-1}(qa^{-1}h)^n(1-q), \tag{18}$$

showing the real analyticity of φ_p in G^+ and also giving the way to compute the tangent map $(T\varphi_p)_a$ at every $a \in G^+$.

Formulas (17), (18) and their consequence, the real analyticity of φ_p for $\mathcal{A} = L(\mathcal{H})$, are the main results of [20]. Here we generalize these results to an arbitrary C^*-algebra $\mathcal{A}$. In the following section, we explore some of their interesting geometrical interpretations and applications.

4. Differential geometry of Q. The space Q of all idempotents of a C^*-algebra (or, more generally, of a Banach algebra) has a rich topological and geometrical structure, studied for example in [17], [29], [11], [24], [7] and [8].

We recall some facts on the structure of Q as a closed submanifold of $\mathcal{A}$. The reader is referred to [7] and [8] for details. The tangent space of Q at q

is naturally identified with

$$\begin{aligned}\{X \in \mathcal{A} : qX + Xq = X\} &= \{X \in \mathcal{A} : qXq = (1-q)X(1-q) = 0\}\\ &= q\mathcal{A}(1-q) \oplus (1-q)\mathcal{A}q.\end{aligned} \tag{19}$$

In terms of the matrix representation induced by q,

$$T(Q)_q = \left\{ \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \in \mathcal{A} \right\} \tag{20}$$

The set P is a real submanifold of Q. The tangent space $(TP)_p$ at $p \in P$ is

$$\{X \in \mathcal{A} : pX + Xp = X,\ X^* = X\},$$

which in terms of the matrix representation induced by p is

$$T(P)_p = \left\{ \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix} \in \mathcal{A} \right\} = T(Q)_p \cap \mathcal{S}. \tag{21}$$

The space Q (resp. P) is a discrete union of homogeneous spaces of G (resp. $\mathcal{U}_{\mathcal{A}}$) by means of the natural action

$$G \times Q \to Q, \qquad (g, q) \mapsto gqg^{-1} \tag{22}$$

(resp. $\mathcal{U}_{\mathcal{A}} \times \mathcal{P} \to \mathcal{P}$, $(u, p) \mapsto upu^*$).

There is a natural connection on Q (resp. P) which induces a linear connection in the tangent bundle TQ (resp. TP). The geodesics of this connection, i.e. the curves γ such that the covariant derivative of $\dot\gamma$ vanishes, can be computed. For $X \in (TQ)_p$ (resp. $(TP)_p$), the unique geodesic γ with $\gamma(0) = p$ and $\dot\gamma(0) = X$ is given by

$$\gamma(t) = e^{tX'}pe^{-tX'},$$

where $X' = [X, p] = Xp - pX$. Thus, the exponential map $\exp_p : T(Q)_p \to Q$ is given by

$$\exp_p(X) = e^{X'}pe^{-X'} \quad \text{for } X \in T(Q)_p. \tag{23}$$

4.1. Proposition. *The inverse of the affine bijective map*

$$\Gamma : Q_p \to p\mathcal{A}(1-p), \qquad \Gamma(q) = q - p,$$

of (8) *is the restriction of the exponential map at p to the closed subspace $p\mathcal{A}(1-p) \subset T(Q)_p$. That is, for $x \in p\mathcal{A}(1-p)$,* $\exp_p(x) = p + x \in Q_p$.

Proof. Let $x \in p\mathcal{A}(1-p)$. Then

$$\begin{aligned}\exp_p(x) &= \exp_p\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\\ &= \exp\begin{pmatrix} 0 & -x \\ 0 & 0 \end{pmatrix} p \exp\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \qquad \text{by (23)}\\ &= \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} p \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} = p + x. \ \blacksquare\end{aligned}$$

4.2. Remark. The map φ_p of 3.1 can also be described using Proposition 4.1. In fact, consider the real analytic map

$$u_p : G^+ \to G(\mathcal{A}), \quad u_p(a) = \exp(-pE_p(a)^{-1}a(1-p)), \quad a \in G^+.$$

Then, by 4.1, $\varphi_p(a) = u_p(a)pu_p(a)^{-1}$. This is an explicit formula for an invertible element which conjugates p with $\varphi_p(a)$. This can be a useful tool for lifting curves of idempotents to curves of invertible elements of $\mathcal{A}$.

Now we consider the map φ_p by letting p vary in P:

$$\varphi : P \times G^+ \to Q, \quad \varphi(p,a) = \varphi_p(a) = pE_p(a)^{-1}a, \tag{24}$$

for $p \in P$, $a \in G^+$. Consider also the map $\phi : Q \to P$ given by (1):

$$\phi(q) = qq^*(1-(q-q^*)^2)^{-1} \quad \text{for } q \in Q. \tag{25}$$

This map ϕ assigns to any $q \in Q$ the unique $p \in P$ with the same range as q.

4.3. Proposition. *The map $\varphi : P \times G^+ \to Q$ is a C^∞ fibration. For $q \in Q$, let $p = \phi(q)$ and $x = q - p \in p\mathcal{A}(1-p)$. Then the fibre of q is*

$$\varphi^{-1}(q) = \left\{ \left(p, \begin{pmatrix} a_1 & a_1 x \\ x^* a_1 & a_3 \end{pmatrix}\right) : 0 < a_1 \text{ and } x^* a_1 x < a_3 \right\}, \tag{26}$$

where the inequalities are considered in $p\mathcal{A}p$ and $(1-p)\mathcal{A}(1-p)$, respectively. Moreover, the fibration φ splits by means of the C^∞ global cross section

$$s : Q \to P \times G^+, \quad s(q) = (\phi(q), |2q-1|), \tag{27}$$

for $q \in Q$, where $|z| = (z^*z)^{1/2}$.

Proof. First we verify (26). Fix $q \in Q$. The only possible first coordinate of every pair in $\varphi^{-1}(q)$ must be $p = \phi(q)$, since it is the unique projection in P with the same range as q.

Given

$$a = \begin{pmatrix} a_1 & a_2 \\ a_2^* & a_3 \end{pmatrix} \in G^+,$$

we know by (14) that $\varphi(p,a) = q$ if and only if $a_2 = a_1(q-p) = a_1 x$. Then

$$a = \begin{pmatrix} a_1 & a_1 x \\ x^* a_1 & a_3 \end{pmatrix}.$$

The inequalities $x^* a_1 x < a_3$ in $(1-p)\mathcal{A}(1-p)$ and $a_1 > 0$ in $p\mathcal{A}p$ are easily seen to be equivalent to the fact that

$$\begin{pmatrix} a_1 & a_1 x \\ x^* a_1 & a_3 \end{pmatrix} \in G^+.$$

This shows (26).

Set $\varepsilon = 2q-1$. It is clear that $\varepsilon^2 = 1$, i.e. ε is a symmetry. Consider its polar decomposition $\varepsilon = \varrho\lambda$, where $\lambda = |\varepsilon| \in G^+$ and ϱ is a unitary element of $\mathcal{A}$. From the uniqueness of the polar decomposition it follows that $\varrho = \varrho^* = \varrho^{-1}$, i.e. ϱ is a unitary selfadjoint symmetry. Then, since $q = (\varepsilon+1)/2$,

$$\lambda^{-1}q^*\lambda = \lambda^{-1}\frac{\varepsilon^*+1}{2}\lambda = \lambda^{-1}\frac{\lambda\varrho+1}{2}\lambda = \frac{\varrho\lambda+1}{2} = \frac{\varepsilon+1}{2} = q.$$

Therefore $q \in \mathcal{S}_\lambda(\mathcal{A})$ and $\varphi(p,\lambda) = \varphi(s(q)) = q$, proving that s is a cross section of φ. ∎

4.4. The space P is the selfadjoint part of the space Q. But each $a \in G^+$ induces the star $\#_a$ and therefore another submanifold of Q of $\#_a$-selfadjoint idempotents. Let $a \in G^+$ and denote the $\#_a$-selfadjoint part of Q by

$$P_a = P_a(\mathcal{A}) = \{q \in Q : q^{\#_a} = q\}. \tag{28}$$

We are going to relate the manifolds P and P_a. There is an obvious way of mapping P onto P_a, namely $p \mapsto a^{-1/2}pa^{1/2}$. Its tangent map is the restriction of the isometric isomorphism $X \mapsto a^{-1/2}Xa^{1/2}$ from $\mathcal{S}$ onto $\mathcal{S}_a$ mentioned in Section 2. We now study some less obvious maps between P and P_a.

For a fixed $a \in G^+$, consider the map

$$\varphi_a : P \to P_a, \quad \varphi_a(p) = \varphi(p,a), \quad p \in P. \tag{29}$$

Then φ_a is a diffeomorphism between the submanifolds P and P_a of Q, and φ_a^{-1} is just the map ϕ of (25) restricted to P_a. The problem which naturally arises is the study of the tangent map of φ_a in order to compare different P_a, $a \in G^+$.

The tangent space $(TP_a)_q$ for $q \in P_a$ can be described as in (21),

$$(TP_a)_q = \left\{ Y = \begin{pmatrix} 0 & y \\ y^{\#_a} & 0 \end{pmatrix} \in \mathcal{A} \right\} = T(Q)_q \cap \mathcal{S}_a, \tag{30}$$

where the matricial representations are in terms of q. Therefore any $Y \in (TP_a)_q$ is characterized by its $1,2$ entry $y = qY$:

$$Y = y + y^{\#_a} = qY + Yq. \tag{31}$$

4.5. Proposition. *Let $p \in P$, $a \in G^+$ and*

$$X = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in (TP)_p.$$

Set $q = \varphi_a(p) \in P_a$. Then, in terms of p,

$$q(T\varphi_a)_p(X) = \begin{pmatrix} 0 & a_1^{-1}x(a_3 - a_2^* a_1^{-1} a_2) \\ 0 & 0 \end{pmatrix} = y.$$

Therefore $(T\varphi_a)_p(X) = y + y^{\#_a}$ and $\|(T\varphi_a)_p(X)\|_a = \|y\|_a$.

Proof. We have the formula of Proposition 3.2,

$$\varphi_a(p) = \varphi_p(a) = pE_p(a)^{-1}a = p(pap + (1-p)a(1-p))^{-1}a.$$

By the standard method of taking a smooth curve γ in P such that $\gamma(0) = p$ and $\dot\gamma(0) = X$, one gets

$$(T\varphi_a)_p(X) = [X - pE_p(a)^{-1}(Xap + paX - Xa(1-p) - (1-p)aX)]E_p(a)^{-1}a.$$

Since p and $E_p(a)$ commute, $pE_p(a)^{-1}(1-p)aX = 0$. In matrix form in terms of p, by direct computation it follows that

$$\begin{aligned}(T\varphi_a)_p(X) &= \left[\begin{pmatrix} 0 & x \\ x^* & 0\end{pmatrix} - \begin{pmatrix} a_1^{-1} & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix} xa_2^* + a_2x^* & a_1x - xa_3 \\ 0 & 0\end{pmatrix}\right] \\ &\quad \cdot \begin{pmatrix} 1 & a_1^{-1}a_2 \\ a_3^{-1}a_2^* & 1\end{pmatrix} \\ &= \begin{pmatrix} -a_1^{-1}xa_2^* - a_1^{-1}a_2x^* & a_1^{-1}xa_3 \\ x^* & 0\end{pmatrix}\begin{pmatrix} 1 & a_1^{-1}a_2 \\ a_3^{-1}a_2^* & 1\end{pmatrix} \\ &= \begin{pmatrix} -a_1^{-1}a_2x^* & a_1^{-1}(xa_3 - xa_2^*a_1^{-1}a_2 - a_2x^*a_1^{-1}a_2) \\ x^* & x^*a_1^{-1}a_2\end{pmatrix}.\end{aligned}$$

Multiplying by $q = \varphi(p,a)$, from (14), one obtains

$$\begin{aligned} q(T\varphi_a)_p(X) &= \begin{pmatrix} 1 & a_1^{-1}a_2 \\ 0 & 0\end{pmatrix}\begin{pmatrix} -a_1^{-1}a_2x^* & a_1^{-1}(xa_3 - xa_2^*a_1^{-1}a_2 - a_2x^*a_1^{-1}a_2) \\ x^* & x^*a_1^{-1}a_2\end{pmatrix} \\ &= \begin{pmatrix} 0 & a_1^{-1}x(a_3 - a_2^*a_1^{-1}a_2) \\ 0 & 0\end{pmatrix} = y,\end{aligned}$$

as desired. The fact that $\|y\|_a = \|Y\|_a$ is clear by regarding them as elements of $(\mathcal{A}, \#_a)$ and using (30).

5. The polar decomposition. In this section it is convenient to identify Q with the set of symmetries (or reflections) $\{\varepsilon \in \mathcal{A} : \varepsilon^2 = 1\}$ and P with the set of selfadjoint symmetries $\{\varrho \in \mathcal{A} : \varrho = \varrho^* = \varrho^{-1}\}$ by means of the affine map $x \mapsto 2x - 1$.

Recall that every invertible element c of a unital C^*-algebra admits polar decompositions $c = \varrho_1\lambda_1 = \lambda_2\varrho_2$ with $\lambda_1, \lambda_2 \in G^+$ and $\varrho_1, \varrho_2 \in \mathcal{U}_{\mathcal{A}}$. Moreover,

$$\lambda_1 = |c|, \quad \lambda_2 = |c^*| \quad \text{and} \quad \varrho_1 = \varrho_2 = |c^*|^{-1}c = c|c|^{-1}.$$

In particular, if ε is a symmetry, its polar decompositions are $\varepsilon = |\varepsilon^*|\varrho = \varrho|\varepsilon|$ and

$$\varrho = \varrho^* = \varrho^{-1} \in P. \tag{32}$$

This remark defines the retraction

$$\Omega : Q \to P, \quad \Omega(\varepsilon) = \varrho. \tag{33}$$

The map Ω has been studied from a differential geometric viewpoint in [7]. If $\varepsilon \in Q$, it is easy to show that $|\varepsilon^*| = |\varepsilon|^{-1}$ and $|\varepsilon^*|^{1/2}\varrho = \varrho|\varepsilon^*|^{-1/2}$ (see [7]). This section is devoted to studying, for each $a \in G^+$, the restriction

$$\Omega_a = \Omega|_{P_a} : P_a \to P. \tag{34}$$

Observe that, with the identification mentioned above, $P_a = Q \cap \mathcal{S}_a = Q \cap \mathcal{U}_a$, where $\mathcal{U}_a = \{u \in G : u^{-1} = u^{\#_a}\}$ is the group of $\#_a$-unitary elements of $\mathcal{A}$.

5.1. Proposition. *For every $a \in G^+$ the map $\Omega_a : P_a \to P$ of (34) is a diffeomorphism.*

Proof. By the remarks above, for every $\varepsilon \in Q$,

$$\Omega_a(\varepsilon) = \varrho = |\varepsilon|\varepsilon, \tag{35}$$

which is clearly a C^∞ map.

Set $b = a^{1/2}$ and consider, for a fixed $\varrho \in P$, the polar decomposition of $b\varrho b$ given by $b\varrho b = w|b\varrho b|$, with $w \in \mathcal{U}_{\mathcal{A}}$. Since $b\varrho b$ is invertible and selfadjoint by (32), it is easy to prove (see [9]) that

$$w = w^* = w^{-1} \in P, \quad wb\varrho b = b\varrho bw, \quad wb\varrho b = |b\varrho b| \in G^+.$$

Let $\varepsilon = b^{-1}wb$. It is clear by the construction that $\varepsilon \in P_a$. Also, $\varepsilon\varrho = \lambda > 0$, since

$$b\varepsilon\varrho b = wb\varrho b = |b\varrho b| \in G^+.$$

Therefore the polar decomposition of ε must be $\varepsilon = \lambda\varrho$. So $\lambda = |\varepsilon^*|$ and $\Omega_a(\varepsilon) = \varrho$. Hence

$$\Omega_a^{-1}(\varrho) = a^{-1/2}\,(a^{1/2}\varrho a^{1/2}|a^{1/2}\varrho a^{1/2}|^{-1})\,a^{1/2}, \tag{36}$$

which is also a C^∞ map, showing that Ω_a is a diffeomorphism. ∎

5.2. Remark. The fibres of the retraction Ω over each $p \in P$ are in some sense "orthogonal" to P. In order to explain this remark, consider the algebra $\mathcal{A} = M_n(\mathbb{C})$ of all $n \times n$ matrices with complex entries. Then $M_n(\mathbb{C})$ has a natural scalar product given by $\langle X, Y\rangle = \operatorname{tr}(Y^*X)$. It is easy to prove that for every $p \in P$, $(TP)_p$ is orthogonal to $(T\Omega^{-1}(P))_p$. The same result holds in every C^*-algebra with a trace τ. Then the map $a \mapsto \Omega_a(\varrho)^{-1}$ of (36) can be considered as the "normal" movement which produces $\#_a$-selfadjoint projections for every $a \in G^+$.

On the other hand, the map φ_p of (14), which was also studied in [20], gives another way to get $\#_a$-selfadjoint projections for every $a \in G^+$. In terms of the geometry of Q this way is, in the above sense, an oblique movement. A related movement is to take, for each $a \in G^+$, an $\#_a$-selfadjoint projection q' with $\ker q' = \ker p$.

Combining, for a fixed $a \in G^+$, the maps φ_a of (29) and Ω_a of (34), one obtains a C^∞ movement of the space P. The following proposition describes this movement explicitly.

5.3. PROPOSITION. *Let $a \in G^+$. Then the map $\Omega_a \circ \varphi_a : P \to P$ is a diffeomorphism of P. For $p \in P$, let $\varphi_a(p) = q = p + x$ and $\varepsilon = 2q - 1$. In terms of p, we have $x = a_1^{-1}a_2$ if*

$$a = \begin{pmatrix} a_1 & a_2 \\ a_2^* & a_3 \end{pmatrix}$$

and

$$\Omega_a \circ \varphi_a(p) = \begin{pmatrix} 1+xx^* & 0 \\ 0 & 1+x^*x \end{pmatrix}^{-1/2} \begin{pmatrix} 1 & x \\ x^* & -1 \end{pmatrix} = [qq^* + (1-q)^*(1-q)]^{-1/2}(q+q^*-1). \tag{37}$$

Proof. In matrix form,

$$\varepsilon = 2\varphi_a(p) - 1 = \begin{pmatrix} 1 & 2x \\ 0 & -1 \end{pmatrix}$$

so that

$$\varepsilon^*\varepsilon = \begin{pmatrix} 1 & 0 \\ 2x^* & -1 \end{pmatrix}\begin{pmatrix} 1 & 2x \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2x \\ 2x^* & 4x^*x+1 \end{pmatrix} = |\varepsilon|^2.$$

On the other hand, by (27), $q \in P_{|\varepsilon|}(\mathcal{A})$. Therefore, by (14),

$$|\varepsilon| = \begin{pmatrix} b & bx \\ x^*b & c \end{pmatrix}$$

with b, c positive. Straightforward computations show that

$$b = (1+xx^*)^{-1/2} \quad \text{and} \quad c^2 = 4x^*x + 1 - x^*(1+xx^*)^{-1}x.$$

Since $x^*(1+xx^*) = (1+x^*x)x^*$, we obtain

$$c^2 = 4x^*x + 1 - (1+x^*x)^{-1}x^*x = 4x^*x + (1+x^*x)^{-1} = (1+x^*x)^{-1}(4(x^*x)^2 + 4x^*x + 1) = (1+x^*x)^{-1}(2x^*x+1)^2.$$

Then $c = (1+x^*x)^{-1/2}(2x^*x+1)$ and

$$|\varepsilon| = \begin{pmatrix} 1+xx^* & 0 \\ 0 & 1+x^*x \end{pmatrix}^{-1/2} \begin{pmatrix} 1 & x \\ x^* & 2x^*x+1 \end{pmatrix}. \tag{38}$$

Now the two formulas of (37) follow by easy matrix computations. ∎

5.4. REMARK. It is interesting to observe that the factor $q + q^* - 1$ of (37) has been characterized by Buckholtz [4] as the inverse of $P_{R(q)} - P_{\ker q}$.

A natural question about these movements is the following: for $p \in P$, how far can $\Omega_a \circ \varphi_a(p)$ be from p? In order to answer this question we consider the orbit

$$\mathcal{O}_p := \{r \in P : \Omega_a \circ \varphi_a(p) = r \text{ for some } a \in G^+\}. \tag{39}$$

The next result is a metric characterization of $\mathcal{O}_p$ based on some results about the "unit disk" of the projective space of $\mathcal{A}$ defined by p (see [2]).

5.5. PROPOSITION. *Let $p \in P$. Then*

$$\mathcal{O}_p = \{r \in P : \|r - p\| < \sqrt{2}/2\}.$$

Proof. Fix $a \in G^+$. Let $q = \varphi_a(p)$, $\varepsilon = 2q - 1$ and $r = \Omega_a \circ \varphi_a(p)$. By (27), r is also obtained if we replace a by $|\varepsilon|$, since $\varphi_a(p) = q = \varphi_{|\varepsilon|}(p)$ and $r = \Omega_a(q) = \Omega(q) = \Omega_{|\varepsilon|}(q)$. Note that $|\varepsilon|$ is positive and ϱ-unitary, i.e. unitary for the signed inner product $\langle\,,\rangle_\varrho$ given by $\varrho = |\varepsilon|\varepsilon = 2r - 1$. Indeed, by (32), $|\varepsilon|^{\#_\varrho} = \varrho^{-1}|\varepsilon|\varrho = |\varepsilon|^{-1}$.

Since $\varepsilon = \varrho|\varepsilon| = |\varepsilon|^{-1/2}\varrho|\varepsilon|^{1/2}$, also $q = |\varepsilon|^{-1/2}r|\varepsilon|^{1/2}$. In [2] it is shown that the square root of a ϱ-unitary is also ϱ-unitary. Then $|\varepsilon|^{-1/2}$ is ϱ-unitary. It is also shown in [2] that

$$\|r - P_{R(\lambda r \lambda^{-1})}\| < \sqrt{2}/2$$

for all positive ϱ-unitary λ. Note that $p = P_{R(q)}$ and then we must have $\|p - r\| < \sqrt{2}/2$. In Proposition 6.13 of [2] it is shown that for all $r \in P$ such that $\|r - p\| < \sqrt{2}/2$, there exists a positive $(2r-1)$-unitary λ such that $p = P_{R(\lambda r \lambda^{-1})}$. In this case $r = \Omega_\lambda \circ \varphi_\lambda(p) \in \mathcal{O}_p$. ∎

6. New short geodesics.

Lengths of geodesics in P have been studied in [25], [3], [23] and [2]. It has been proved that if $p, r \in P$ and $\|p - r\| < 1$, then there exists a unique geodesic of P joining them which has minimal length. On the other hand, the fibres $\Omega^{-1}(P)$ are geodesically complete and the geodesic joining $q_1, q_2 \in \Omega^{-1}(P)$ is a shortest curve in Q (see [8]). This final section is devoted to showing the existence of "short oblique geodesics", i.e. geodesics which are contained neither in P nor in the fibres.

More precisely, the idea of the present section is to use the different stars $\#_a$ for $a \in G^+$ in order to find short curves between pairs of nonselfadjoint idempotents of $\mathcal{A}$. Basically, we want to characterize those pairs $q, r \in Q$ such that there exist $a \in G^+$ with $q, r \in P_a$. If q and r remain close in P_a, they can be joined by a short curve in the space P_a.

The first problem is that the positive a need not be unique. This can be fixed up in the following manner:

6.1. LEMMA. *Suppose that $a \in G^+$ and $p, r \in P \cap P_a$. Then $\|p - r\| = \|p - r\|_a$ and, if $\|p - r\| < 1$, the short geodesics which join them in P and P_a are the same and have the same length.*

Proof. Note that $P \cap P_a$ is the space of projections commuting with a. Let $\mathcal{B} = \{a\}' \cap \mathcal{A}$, the relative commutant of a in $\mathcal{A}$. Since $a = a^*$, $\mathcal{B}$ is a

C^*-algebra. Moreover, $P \cap P_a = P(\mathcal{B})$. Now, since $\|p - r\| < 1$, p and q can be joined by the unique short geodesic γ along $P(\mathcal{B})$ (see [25] or [2]) and γ is also a geodesic both for P and P_a. The length of γ is computed in the three algebras in terms of the *norm* of the corresponding tangent vector X. But since $X \in \mathcal{B}$, its norm is the same with the two scalar products involved. ∎

We now give a characterization of pairs of close idempotents $p, q \in Q$ such that $p, q \in P_a$ for some $a \in G^+$. The characterization is done in terms of a tangent vector $X \in T(Q)_p$ such that $q = e^X p e^{-X}$. First we give a slight improvement of the way of obtaining such X which appears in 2) of [25]:

6.2. PROPOSITION. *Let $p \in P$ and $q \in Q$ with $\|p-q\| < 1$. Let $\varepsilon = 2q-1$, $\varrho = 2p-1$,*

$$v_1 = \frac{\varepsilon\varrho + 1}{2} = qp + (1-q)(1-p), \qquad v_2 = \frac{\varrho\varepsilon + 1}{2} = pq + (1-p)(1-q).$$

Then $\|v_1 - 1\| = \|v_2 - 1\| = \|p-q\| < 1$ and

$$X = (\mathrm{Id} - E_p)(\log v_1) = \tfrac{1}{2}(\log v_1 - \log v_2) \tag{40}$$

satisfies $X \in T(Q)_p$ (i.e. $pXp = (1-p)X(1-p) = 0$) and $q = e^X p e^{-X}$.

Proof. Note that $\varrho = \varrho^* = \varrho^{-1} \in P$. Then

$$\|v_1 - 1\| = \left\|\frac{\varepsilon\varrho - 1}{2}\right\| = \frac{1}{2}\|(\varepsilon - \varrho)\varrho\| = \|q-p\| < 1,$$

and similarly for v_2. Let $X_i = \log v_i$ for $i = 1, 2$. Since $v_1\varrho = \varrho v_2$, and each X_i is obtained as a power series in v_i, we also obtain $X_1\varrho = \varrho X_2$. Then, if $X = \frac{1}{2}(X_1 - X_2)$, we have $X\varrho = -\varrho X$, and so $X \in T(Q)_p$.

Note also that $v_1\varrho = \varepsilon v_1$ and $\|v_i - 1\| < 1$ for $i = 1, 2$. So $v_1 p v_1^{-1} = q$. Easy calculations show that v_1 and v_2 commute. As before, this implies that X_1 and X_2 commute. Then

$$w = v_1 v_2 = v_2 v_1 = e^{X_1 + X_2} = \left(\frac{\varepsilon + \varrho}{2}\right)^2$$

commutes with v_1, v_2, ϱ, ε, p and q. Set

$$w^{-1/2} = e^{-(X_1+X_2)/2}.$$

Since $(X_1 + X_2)\varrho = \varrho(X_1 + X_2)$, $w^{-1/2}$ commutes with ϱ. Note that

$$X = \frac{X_1 - X_2}{2} = X_1 - \frac{X_1 + X_2}{2}. \tag{41}$$

This implies $e^X = e^{X_1} w^{-1/2} = v_1 w^{-1/2}$ and therefore

$$e^X p e^{-X} = v_1 p v_1^{-1} = q.$$

Finally, since X has zeros in its diagonal and $(X_1 + X_2)/2$ is diagonal in terms of p, we deduce from (41) that $X = (\mathrm{Id} - E_p)(X_1)$ and the proof is complete. ∎

6.3. PROPOSITION. *Let $p \in P$ and $q \in Q$ be such that $\|p - q\| < 1$. Let*

$$X = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \in T(P)_p$$

be as in (40) such that $e^X p e^{-X} = q$. Then the following are equivalent:

(i) *There exists $a \in G^+$ such that $p, q \in P_a$.*
(ii) *There exists $a \in G^+$ such that $pa = ap$ and $X^{\#_a} = -X$.*
(iii) *There exist $b \in G^+(p\mathcal{A}p)$ and $c \in G^+((1-p)\mathcal{A}(1-p))$ such that*

$$y = -cx^*b.$$

Proof. Condition (ii) can be written as

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \quad \text{and} \quad a^{-1}X^*a = -X.$$

In matrix form

$$\begin{aligned} a^{-1}X^*a &= \begin{pmatrix} a_1^{-1} & 0 \\ 0 & a_2^{-1} \end{pmatrix} \begin{pmatrix} 0 & y^* \\ x^* & 0 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & a_1^{-1}y^*a_2 \\ a_2^{-1}x^*a_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -x \\ -y & 0 \end{pmatrix}, \end{aligned}$$

which is clearly equivalent to condition (iii).

Condition (i) holds if $X^{\#_a} = -X$, since in that case e^X is $\#_a$-unitary and then $q \in P_a$. In order to prove the converse, we consider $\#_a$ instead of * and so condition (i) means that $p, q \in P$. Then, with the notations of 6.2, we have $v_2 = v_1^*$ and

$$X_2 = \log v_2 = \log v_1^* = X_1^* \Rightarrow X^* = \frac{X_2 - X_1}{2} = -X,$$

showing (ii).

6.4. REMARK. Let us call a direction (i.e. tangent vector) in Q *good* if it is the direction of a short geodesic. Proposition 6.3 provides a way to obtain good directions. Other good directions occur in the spaces $p\mathcal{A}(1-p)$ and $(1-p)\mathcal{A}p$, determined by the affine spaces of projections with the same range (Q_p) or the same kernel as p, where the straight lines can be considered as short geodesics.

Other good directions can be found looking at pairs $p, q \in Q$ such that, for some $a \in G^+$, $\Omega^a(q) = p$, where Ω^a means the retraction of (33), considering in $\mathcal{A}$ the star $\#_a$. These pairs can be characterized in a very similar way to Proposition 6.3. In fact, in condition (iii) (with the same notations),

$y = -bx^*c$ should be replaced by $y = bx^*c$. These directions are indeed good because it is known [8] that along the fibres of each Ω^a there are short geodesics that join any pair of elements (not only close pairs).

References

[1] S. N. Afriat, *Orthogonal and oblique projections and the characteristics of pairs of vector spaces*, Proc. Cambridge Philos. Soc. 53 (1957), 800–816.

[2] E. Andruchow, G. Corach and D. Stojanoff, *Projective spaces for C^*-algebras*, Integral Equations Operator Theory, to appear.

[3] L. G. Brown, *The rectifiable metric on the set of closed subspaces of Hilbert space*, Trans. Amer. Math. Soc. 337 (1993), 279–289.

[4] D. Buckholtz, *Inverting the difference of Hilbert space projections*, Amer. Math. Monthly 104 (1997), 60–61.

[5] R. R. Coifman and M. A. M. Murray, *Uniform analyticity of orthogonal projections*, Trans. Amer. Math. Soc. 312 (1989), 779–817.

[6] G. Corach, *Operator inequalities, geodesics and interpolation*, in: Functional Analysis and Operator Theory, Banach Center Publ. 30, Inst. Math., Polish Acad. Sci., Warszawa, 1994, 101–115.

[7] G. Corach, H. Porta and L. Recht, *Differential geometry of systems of projections in Banach algebras*, Pacific J. Math. 140 (1990), 209–228.

[8] —, —, —, *The geometry of spaces of projections in C^*-algebras*, Adv. Math. 101 (1993), 59–77.

[9] —, —, —, *The geometry of spaces of selfadjoint invertible elements of a C^*-algebra*, Integral Equations Operator Theory 16 (1993), 771–794.

[10] J. Dieudonné, *Quasi-hermitian operators*, in: Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960), Pergamon Press, Oxford, 1961, 115–122.

[11] W. Gerisch, *Idempotents, their Hermitian components, and subspaces in position p of a Hilbert space*, Math. Nachr. 115 (1984), 283–303.

[12] A. S. Householder and J. A. Carpenter, *The singular values of involutory and of idempotent matrices*, Numer. Math. 5 (1963), 234–237.

[13] N. Kerzman and E. M. Stein, *The Szegő kernel in terms of Cauchy–Fantappiè kernels*, Duke Math. J. 45 (1978), 197–224.

[14] —, —, *The Cauchy kernel, the Szegő kernel, and the Riemann mapping function*, Math. Ann. 236 (1978), 85–93.

[15] Z. V. Kovarik, *Similarity and interpolation between projectors*, Acta Sci. Math. (Szeged) 39 (1977), 341–351.

[16] P. D. Lax, *Symmetrizable linear transformations*, Comm. Pure Appl. Math. 7 (1954), 633–647.

[17] V. J. Mizel and M. M. Rao, *Nonsymmetric projections in Hilbert space*, Pacific J. Math. 12 (1962), 343–357.

[18] A. Odzijewicz, *On reproducing kernels and quantization of states*, Comm. Math. Phys. 114 (1988), 577–597.

[19] —, *Coherent states and geometric quantization*, ibid. 150 (1992), 385–413.

[20] Z. Pasternak-Winiarski, *On the dependence of the orthogonal projector on deformations of the scalar product*, Studia Math. 128 (1998), 1–17.

[21] —, *On the dependence of the reproducing kernel on the weight of integration*, J. Funct. Anal. 94 (1990), 110–134.

[22] Z. Pasternak-Winiarski, *Bergman spaces and kernels for holomorphic vector bundles*, Demonstratio Math. 30 (1997), 199–214.

[23] N. C. Phillips, *The rectifiable metric on the space of projections in a C^*-algebra*, Internat. J. Math. 3 (1992), 679–698.

[24] H. Porta and L. Recht, *Spaces of projections in Banach algebras*, Acta Cient. Venezolana 39 (1987), 408–426.

[25] —, —, *Minimality of geodesics in Grassmann manifolds*, Proc. Amer. Math. Soc. 100 (1987), 464–466.

[26] —, —, *Variational and convexity properties of families of involutions*, Integral Equations Operator Theory 21 (1995), 243–253.

[27] V. Pták, *Extremal operators and oblique projections*, Časopis Pěst. Mat. 110 (1985), 343–350.

[28] Ş. Strătilă, *Modular Theory in Operator Algebras*, Editura Academiei, Bucharest, 1981.

[29] J. Zemánek, *Idempotents in Banach algebras*, Bull. London Math. Soc. 11 (1979), 177–183.

Instituto de Ciencias
Univ. Nac de Gral. Sarmiento
Roca 850, (1663) San Miguel
Pcia. de Buenos Aires, Argentina
E-mail: eandruch@percanta.ungs.edu.ar

Instituto Argentino de Matemática
Saavedra 15 3er piso
(1083) Buenos Aires, Argentina
E-mail: gcorach@mate.dm.uba.ar
demetrio@mate.dm.uba.ar

Received September 25, 1998 (4184)