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On Bell's duality theorem for harmonic functions

by

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Abstract. Define $h^\infty(E)$ as the subspace of $C^\infty(\overline{B}, E)$ consisting of all harmonic functions in B, where B is the ball in the n-dimensional Euclidean space and E is any Banach space. Consider also the space $h^{-\infty}(E^*)$ consisting of all harmonic E^*-valued functions g such that $(1-|x|)^m f$ is bounded for some $m > 0$. Then the dual $h^\infty(E)^*$ is represented by $h^{-\infty}(E^*)$ through $\langle f, g\rangle_0 = \lim_{r\to 1}\int_B \langle f(rx), g(x)\rangle\,dx$, $f \in h^{-\infty}(E^*)$, $g \in h^\infty(E)$. This extends the results of S. Bell in the scalar case.

1. Introduction and notation. Denote by $h^\infty(\overline{\Omega})$ the subspace of the Fréchet space $C^\infty(\overline{\Omega})$ consisting of all harmonic functions in Ω, where $\Omega \subset \mathbb{R}^n$ is a bounded open region with smooth boundary. Also consider for every $m \in \mathbb{N}$ the space $h^{-m}(\Omega)$ of all harmonic functions f in $W^{-m}(\Omega)$ and let $h^{-\infty}(\Omega) = \bigcup_{m\in\mathbb{N}} h^{-m}(\Omega)$. In [2], S. Bell constructed for every $m \in \mathbb{N}$ a linear differential operator L^m of order $N(m) = (m/2)(m+3)$ with $C^\infty(\overline{\Omega})$ coefficients mapping the Sobolev space $W^{m+N}(\Omega)$ into $W_0^m(\Omega)$ and such that $PL^m = P$, where P is the orthogonal projection on the space of harmonic functions in $L^2(\Omega)$. Then for all $f \in h^\infty(\overline{\Omega})$ and $g \in h^{-\infty}(\Omega)$ the expression

$$\langle f, g\rangle_0 = \langle L^m f, g\rangle_m, \quad g \in h^{-m}(\Omega), \tag{1.1}$$

is well defined and makes $h^\infty(\overline{\Omega})$ and $h^{-\infty}(\Omega)$ a dual pair. A harmonic function f is in $h^{-\infty}(\Omega)$ if and only if $d(x)^s f(x)$ is bounded for some $s \in \mathbb{N}$, where $d(x)$ is the distance to $\partial\Omega$. We refer the reader to [13–15] for the use of Bell's operators to calculate duals of spaces of harmonic functions.

The purpose of this paper is to calculate the dual space of the Banach-valued version of the space $h^\infty(\overline{\Omega})$ when Ω is the unit ball. We prove that in this case, Bell's result can be extended with no restriction on the Banach space, and moreover, the duality (1.1) can be written as a limit of integrals, without the need of using Bell's operators L^m. In Section 2, we collect some

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basic facts on the Banach-valued Sobolev spaces, including an easy proof of Sobolev's lemma for Banach-valued functions defined in the ball.

We set $\varepsilon(y) = 1 - |y|$ and let $\boldsymbol{N}$ be the set of nonnegative integers. E always denotes a complex Banach space. $L^2(E)$ is the space of Bochner measurable functions f in the unit ball B in $\mathbb{R}^n$ with $\|f(\cdot)\|$ square integrable. $L^2\,\mathrm{Harm}(E)$ is the space of all harmonic functions belonging to $L^2(E)$. The Poisson kernel in the ball is given by

$$P(x,y) = c_n \frac{1-(rR)^2}{(1-2rRx'\cdot y' + r^2R^2)^{n/2}},$$

where $x = Rx'$, $y = ry'$, $R = |x|$ and $r = |y|$. The reproducing kernel for the space $L^2\,\mathrm{Harm} = L^2\,\mathrm{Harm}(\mathbb{C})$ is

$$b(x,y) = \left[\varrho^{1-n}\frac{\partial}{\partial\varrho}\,\varrho^n P(Rx', \varrho^2 y')\right]_{\varrho=\sqrt{r}},$$

that is, the operator $Pf(x) = \int_B b(x,y) f(y)\,dy$ is the orthogonal projection of L^2 onto $L^2\,\mathrm{Harm}$ (see [6]). P also defines a continuous projection of $L^2(E)$ onto $L^2\,\mathrm{Harm}(E)$ for any Banach space E, in fact the integral operator with kernel $|b(x,y)|$ is continuous in L^2 (see for example [5, 10, 16]).

Let m be a finitely additive function on the Borel sets of B with values in E. Define the 2-*variation* of m as

$$\|m\|_2 = \sup\left(\sum_{A\in\pi}\frac{\|m(A)\|^2}{\lambda(A)}\right)^{1/2},$$

where the supremum is taken over all finite (measurable) partitions π of B and λ is the Lebesgue measure. We denote by $V^2(E)$ the space of measures with $\|m\|_2 < \infty$. All the measures in $V^2(E)$ are countably additive, λ-continuous and have bounded variation. For $f \in L^2(E)$ and $m \in V^2(E^*)$ we can give sense to the duality

$$\langle f, m\rangle = \int_B f\,dm \tag{1.2}$$

proceeding in the obvious way in the case of simple functions. We have $L^2(E)^* = V^2(E^*)$ (isometrically) through (1.2). If such an m has a density g, then $g \in L^2(E^*)$ and we write $\langle f, m\rangle = \int_B \langle f, g\rangle\,dx$ (see the argument in the proof of [3, Proposition 3] and [7, Theorem II.1.4]). The reader is referred to [7, 9] for detailed expositions on vector measures. In [3] the following is proved: For any $m \in V^2(E)$, there exists a positive function $g \in L^2$ such that

$$\Big\|\int_B \phi\,dm\Big\| \le \int_B |\phi(x)| g(x)\,dx$$

for any $\phi \in L^2$. It follows that

$$Pm(x) = \int_B b(x,y)\,dm(y)$$

gives a continuous extension $P : V^2(E) \to L^2\,\mathrm{Harm}(E)$ ([1]).

LEMMA 1.1. *For $f \in L^2(E)$ and $m \in V^2(E^*)$, we have*

$$\int_B Pf(x)\,dm = \int_B \langle f(x), Pm(x)\rangle\,dx.$$

Proof. It suffices to prove this for $f = \phi\otimes e \in C_c^\infty(B)\otimes E$. Since $d\langle e, m\rangle$ has an L^2 density and $b(x,y)$ is bounded on $\operatorname{supp}\phi \times B$,

$$\begin{aligned}\int_B \langle f(x), Pm(x)\rangle\,dx &= \int_B \Big\langle \phi(x)\otimes e, \int_B b(x,y)\,dm(y)\Big\rangle\,dx\\ &= \int_B \phi(x)\int_B b(x,y)\,d\langle e, m(y)\rangle\,dx\\ &= \int_B\Big(\int_B b(x,y)\phi(x)\,dx\Big)\,d\langle e, m(y)\rangle\\ &= \int_B Pf(x)\,dm. \ \blacksquare\end{aligned}$$

Given $g : B \to E$, define $g_r(x) = g(rx)$ for $r \in (0,1)$ and $x \in B$. If $m \in V^2(E)$ and $r \in (0,1)$ we define m_r by $m_r(A) = r^{-n} m(rA)$ for every Borel set in B. We have $m_r \in V^2(E)$, $\|m_r\|_2 = r^{-n/2}\|m\|_2$ and

$$\int_B \phi\,dm_r = \frac{1}{r^n}\int_{rB}\phi\Big(\frac{x}{r}\Big)\,dm$$

for every $\phi \in L^2$.

$\mathcal{D}'(B,E)$ denotes the space of E-valued distributions in B, that is, the space of all continuous linear operators $f : C_c^\infty(B) \to E$. For $f \in \mathcal{D}'(B,E)$, $\partial^\alpha f \in \mathcal{D}'(B,E)$ is defined exactly as in the scalar case. Also $L^1_{\mathrm{loc}}(B,E)$ and $V^2(E)$ are embedded in $\mathcal{D}'(B,E)$.

2. Duality on vector-valued Sobolev spaces. Let m be a nonnegative integer. We denote by $W^m(E)$ the Sobolev space consisting of all distributions $f \in \mathcal{D}'(B,E)$ such that $\partial^\alpha f \in L^2(E)$ for every $|\alpha| \le m$. The norm in $W^m(E)$ is given by

$$\|f\|_m = \|(\partial^\alpha f)_{|\alpha|\le m}\|_{L^2(E)^L},$$

where L is the cardinality of $\{\alpha \in \boldsymbol{N}^n : |\alpha| \le m\}$. Define $W_0^m(E) = \overline{C_c^\infty(B,E)}^{W^m(E)}$ and its dual space $W^{-m}(E^*) = W_0^m(E)^*$, provided with

([1]) We thank Oscar Blasco for this remark.

the dual norm that we denote by $\|\cdot\|_{-m}$. Now we shall use the duality (1.2) to give a representation of $W^{-m}(E^*)$:

As in the scalar case, there exists an embedding $W^m(E) \hookrightarrow L^2(E)^L$ given by $f \mapsto (\partial^\alpha f)_{|\alpha|\le m}$. By (1.2), $(L^2(E)^L)^* = V^2(E^*)^L$, thus every $\Phi \in W^m(E)^*$ may be written as

$$\Phi(f) = \sum_{|\alpha|\le m} \int \partial^\alpha f\, dm_\alpha, \quad f \in W^m(E).$$

Since $C_c^\infty(B)\otimes E$ is dense in $W_0^m(E)$, two elements $(m_\alpha), (n_\alpha) \in V^2(E^*)^L$ represent the same Φ in $W_0^m(E)^*$ if and only if

$$\sum_{|\alpha|\le m} \int (\partial^\alpha \phi \otimes e)\, dm_\alpha = \sum_{|\alpha|\le m} \int (\partial^\alpha \phi \otimes e)\, dn_\alpha$$

for every $e \in E$ and $\phi \in C_c^\infty(B)$. This is true if and only if

$$\sum_{|\alpha|\le m} (-1)^{|\alpha|}\partial^\alpha m_\alpha = \sum_{|\alpha|\le m} (-1)^{|\alpha|}\partial^\alpha n_\alpha \tag{2.1}$$

in $\mathcal{D}'(B, E^*)$. Let $\boldsymbol{M}$ be the subspace of $V^2(E^*)^L$ consisting of (m_α) such that $\sum_{|\alpha|\le m}(-1)^{|\alpha|}\partial^\alpha m_\alpha = 0$. Then as in scalar Sobolev spaces we have

THEOREM 2.1. (1) *For every $\Phi \in W^{-m}(E^*)$, there exists $\boldsymbol{n} = (n_\alpha) \in V^2(E^*)^L$ representing Φ such that $\|\Phi\|_{-m} = \|\boldsymbol{n}\|_{V^2(E^*)^L}$.*

(2) *$\boldsymbol{M}$ is closed in $V^2(E^*)^L$ and $W_0^m(E)^* = V^2(E^*)^L/\boldsymbol{M}$, where in particular $\|\Phi\|_{-m} = \min\{\|\boldsymbol{m}\|_{V^2(E^*)^L} : \boldsymbol{m}$ represents $\Phi\}$.*

Proof. The proof is the same as in the scalar case. ∎

REMARK 2.2. (a) $W^{-m}(E^*)$ may be identified with the space of E^*-valued distributions of the form $\sum_{|\alpha|\le m}(-1)^{|\alpha|}\partial^\alpha m_\alpha$ with $(m_\alpha) \in V^2(E^*)^L$, and provided with the norm $\|\cdot\|_{-m}$.

(b) If $\sum_{|\alpha|\le m}(-1)^{|\alpha|}\partial^\alpha m_\alpha$ is represented by a locally integrable function g, then $\langle f, g\rangle_m = \int_B \langle f(x), g(x)\rangle\, dx$ for all $f \in C_c^\infty(B)\otimes E$, where $\langle\cdot,\cdot\rangle_m$ denotes the duality in $W^m(E)$. By density, the same is true for every $f \in W_0^m(E)$, provided $g \in L^2(E^*)$.

(c) If $g = \sum_{|\alpha|\le m}(-1)^{|\alpha|}\partial^\alpha m_\alpha \in W^{-m}(E^*)$ is locally integrable, then an easy calculation shows that $g_r = \sum_{|\alpha|\le m}(-1)^{|\alpha|} r^{-|\alpha|}\partial^\alpha (m_\alpha)_r$. Since $\|(m_\alpha)_r\|_2 = r^{-n/2}\|m_\alpha\|_2$, it follows that $(g_r)_{0<r_0\le r<1}$ is bounded in $W^{-m}(E^*)$. Then using (b) above and the density of $C_c^\infty(B)\otimes E$ in $W_0^m(E)$, we see that $\lim_{r\to 1}\langle f, g_r\rangle_m = \langle f, g\rangle_m$ for every $f \in W_0^m(E)$.

We will need the Banach-valued Sobolev lemma; for completeness we include an easy proof in the case of the ball:

THEOREM 2.3. *For any Banach space E and $k, m \in \boldsymbol{N}$ such that $2m > n$, there exists a continuous embedding*

$$W^{m+k}(E) \to C_b^k(E),$$

where $C_b^k(E)$ is the space of functions $u : B \to E$ such that $\partial^\alpha u$ is bounded and continuous for $|\alpha| \le k$.

Proof. Notice that the restrictions of functions from $C_c^\infty(\mathbb{R}^n, E)$ are dense in $W^m(E)$: let $u \in W^m(E)$ and $\varepsilon > 0$; then there exists $0 < r < 1$ such that $\|u - u_r\|_m < \varepsilon$. Since $u_r \in W^m(r^{-1}B, E)$, we can assume from the beginning that $u \in W^m(\Omega, E)$ with $\overline{B} \subset \Omega$. Then a standard argument using an approximate identity $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(\varepsilon^{-1}x)$ with a nonnegative $\phi \in C_c^\infty(\mathbb{R}^n)$ shows that $\|\phi_\varepsilon * u - u\|_m \to 0$ as $\varepsilon \to 0$ (see [1, Lemma 3.15]). Hence it suffices to prove the Sobolev lemma for the restriction of $u \in C_c^\infty(\mathbb{R}^n, E)$ to B. Let $e^* \in E^*$ with $\|e^*\| = 1$. Then by the scalar Sobolev lemma

$$|\langle \partial^\alpha u(x), e^*\rangle| \le C\|\langle u(\cdot), e^*\rangle\|_{m+k} \le C\|u\|_{m+k}$$

for $|\alpha| \le k$ and $x \in B$, since $\langle \partial^\alpha u(\cdot), e^*\rangle \in W^m$. This proves that $\|u\|_{C_b^k(E)} \le C\|u\|_{m+k}$.

LEMMA 2.4. *The Bergman projection P maps $W_0^m(E)$ continuously into $W^m(E)$.*

Proof. Let $y \ne 0$ and $\widetilde{y} = |y|^{-2}y$. Then writing

$$(1 - 2rRx'\cdot y' + r^2R^2)^{1/2} = r|x - \widetilde{y}|$$

and

$$1 - x\cdot y = \frac{1}{2}\{(1 - r^2R^2) + (1 - 2rRx'\cdot y' + r^2R^2)\},$$

we see after an easy calculation that for $y \ne 0$,

$$b(x,y) = c_n\left\{-\frac{4r^2R^2}{r^n|x-\widetilde{y}|^n} + \frac{n(1-r^2R^2)^2}{r^{n+2}|x-\widetilde{y}|^{n+2}}\right\}.$$

Using the Leibniz rule and the estimate $1 - rR \le |x - \widetilde{y}|$ on $B \times (B \setminus \frac{1}{2}B)$, it follows that

$$|\partial_x^\alpha b(x,y)| \le C(\alpha)|x - \widetilde{y}|^{-n-|\alpha|} \tag{2.2}$$

in this set. Now, for $\phi \in C_c^\infty(B, E)$ and $|\alpha| \le m$,

$$\|\partial^\alpha P\phi(x)\| \le \int_{\frac{1}{2}B} |\partial_x^\alpha b(x,y)|\cdot\|\phi(y)\|\, dy + C(\alpha)\int_{B\setminus\frac{1}{2}B} \frac{\varepsilon(y)^m}{|x-\widetilde{y}|^{n+|\alpha|}}\cdot\frac{\|\phi(y)\|}{\varepsilon(y)^m}\, dy.$$

Let P_1 and P_2 be the integral operators with kernels $|\partial_x^\alpha b(x,y)|\chi_{\frac{1}{2}B}$ and $\varepsilon(y)^m|x-\widetilde{y}|^{-n-|\alpha|}\chi_{B\setminus\frac{1}{2}B}$. Both operators are bounded in L^2: P_1 has a bounded kernel and P_2 can be treated as in [6, Lemma 3.3]. Finally, the Banach-valued version of Taylor's theorem (see [8]) implies that

$$\|\phi/\varepsilon^m\|_{L^2(E)} \le C\|\phi\|_m.$$

Thus we have $\|P\phi\|_m \le C\|\phi\|_m$. ∎

3. The dual of $h^\infty(E)$

DEFINITION 3.1. Define $h^\infty(E)$ to be the subspace of $C^\infty(\overline{B}, E)$ consisting of harmonic functions in B.

We provide it with the topology of $C^\infty(\overline{B}, E)$, namely, with the system of norms

$$p_m(f) = \sup_{|\alpha|\le m,\, x\in B} \|\partial^\alpha f(x)\|.$$

$h^\infty(E)$ is a Fréchet space with this topology.

DEFINITION 3.2. (a) We define $h^{-m}(E^*)$ as the subspace of $W^{-m}(E^*)$ consisting of harmonic functions.

(b) Denote by $h^{-\infty}(E)$ the space of all harmonic functions $u : B \to E$ such that $\varepsilon(x)^s u(x)$ is a bounded function for some $s \in \mathbb{N}$.

Now we prove that as in the scalar case (see [2, Lemma 2]), we can represent $h^{-\infty}(E^*) = \bigcup_{m\in\mathbb{N}} h^{-m}(E^*)$.

LEMMA 3.3. *Let $s > n$. Then there exist $C_1, C_2 > 0$ such that for any harmonic $u : B \to E^*$,*

$$C_1\|u\|_{-s-n} \le \sup_{x\in B} \varepsilon(x)^s\|u(x)\|_{E^*} \le C_2\|u\|_{-s+n}.$$

Proof. Let $\phi \in C_c^\infty(B, E)$. By the Sobolev lemma and Taylor's theorem (see [8]),

$$\|\phi(x)\| \le C \sup_{\substack{y\in\mathbb{R}^n\\ |\alpha|=s}} \|\partial^\alpha\phi(y)\|\varepsilon(x)^s \le C\|\phi\|_{n+s}\varepsilon(x)^s.$$

Then

$$\int_B |\langle\phi, u\rangle|\,dx = \int_B |\langle\phi(x)\varepsilon(x)^{-s}, u(x)\varepsilon(x)^s\rangle|\,dx \le C\|\phi\|_{n+s} \sup_{x\in B} \varepsilon(x)^s\|u(x)\|_{E^*},$$

that is,

$$\|u\|_{-s-n} \le C \sup_{x\in B} \varepsilon(x)^s\|u(x)\|_{E^*}.$$

The other inequality follows as in the scalar case, if we bear in mind that the mean value theorem for harmonic functions is valid for Banach-valued functions. ∎

PROPOSITION 3.4. *$h^{-m}(E^*)$ is weak*-closed (and hence closed) in $W^{-m}(E^*)$.*

Proof. Let (g_λ) be a net in $h^{-m}(E^*)$ converging pointwise to $T \in W^{-m}(E^*)$. For fixed $e \in E$, $e \circ g_\lambda$ converges to $e \circ T \in \mathcal{D}'(B)$. Since $e \circ g_\lambda$ is harmonic for every λ, we have $\Delta(e \circ T) = 0$, hence $e \circ T$ is represented by a harmonic function g_e. Let $\Theta \in C_c^\infty(B)$ be a radial function with $\int_B \Theta(x)\,dx = 1$. For every $x \in B$, define $\Theta_x(y) = \varepsilon^{-n}\Theta((y-x)/\varepsilon)$, where $\varepsilon = (1-|x|)/2$. Define $g(x) = T(\Theta_x)$. Since the mapping $x \mapsto \Theta_x$ is C^∞, it follows that $g \in C^\infty(B, E^*)$. We have

$$\langle e, g(x)\rangle = \langle e, T(\Theta_x)\rangle = \langle e\circ T, \Theta_x\rangle = \int_B g_e(y)\Theta_x(y)\,dy$$
$$= g_e(x) = e \circ T(x),$$

whence it follows at once that $T = g$ and g is weak*-harmonic. Since g is also C^∞, it is harmonic and $T \in h^{-m}(E^*)$. ∎

COROLLARY 3.5. $h^{-\infty}(E^*) = \bigcup_{m\in\mathbb{N}} h^{-m}(E^*)$.

THEOREM 3.6. (a) *For every $g \in h^{-m}(E^*)$, the limit*

$$\langle u, g\rangle_0 = \lim_{r\to1} \int_B u(x)g_r(x)\,dx \tag{3.1}$$

exists and defines a continuous linear functional on $h^\infty(E)$. Moreover, $\langle u, g\rangle_0 = \langle L^s u, g\rangle_s$ for every $s \ge m$ (the product in the integrand should be interpreted as the duality $\langle E, E^*\rangle$).

(b) *Every $T \in h^\infty(E)^*$ can be represented as in (3.1) by a unique $g \in h^{-\infty}(E^*)$.*

Proof. Let $g \in h^{-m}(E^*)$ and $s \ge m$. Consider the operators L^s mentioned in the first section. Then L^s maps continuously $C^\infty(\overline{B}, E)$ into $W_0^s(E)$ when we equip $C^\infty(\overline{B}, E)$ with the norm $\|\ \|_{s+N(s)}$, hence $PL^s f = Pf$ also holds for smooth Banach-valued functions on $\overline{B}$. Then by Lemma 1.1 and Remark 2.2 we get

$$\int_B u(x)g_r(x)\,dx = \int_B Pu(x)g_r(x)\,dx = \int_B PL^s u(x)g_r(x)\,dx$$
$$= \int_B L^s u(x)Pg_r(x)\,dx = \langle L^s u, g_r\rangle_s \to \langle L^s u, g\rangle_s \quad \text{as } r \to 1.$$

Since

$$|\langle u, g\rangle_0| \le \|L^s u\|_m\|g\|_{-m} \le C\|u\|_{m+N(m)}\|g\|_{-m} \le Cp_{m+N(m)}(u)\|g\|_{-m},$$

the functional $\langle\cdot, g\rangle_0$ is continuous and the proof of (a) is complete.

To prove (b), let $T \in h^\infty(E)^*$. Then the continuity of T and Sobolev's lemma imply that there exist $C > 0$ and $m \in \mathbb{N}$ such that

$$|\langle \phi, T\rangle| \le C\|\phi\|_m, \quad \phi \in h^\infty(E).$$

By Hahn–Banach's theorem we can extend T continuously to $W^m(E)$. Hence there exists $(m_\alpha)_{|\alpha|\le m} \in V^2(E^*)^L$ such that

$$T(f) = \sum_{|\alpha|\le m} \int \partial^\alpha f\, dm_\alpha, \quad f \in W^m(E).$$

Define, for $0 < r \le 1$,

$$g^{(r)} = P\Big(\sum_{|\alpha|\le m} (-1)^{|\alpha|}\partial^\alpha L^s(Pm_\alpha)_r\Big).$$

Then $\{g^{(r)}\} \subset L^2\operatorname{Harm}(E^*)$ and we claim that it is a bounded set in $W^{-m}(E^*)$:

Let $\phi \in C_c^\infty(B)\otimes E$. Since $PL^m = P$ on $C_c^\infty(B)\otimes E$ and $L^m(Pm_\alpha)_r$ vanishes to order $m-1$ on the unit sphere, we have

$$\begin{aligned}\Big|\int_B \phi(x)g^{(r)}(x)\,dx\Big| &= \Big|\int_B P\phi(x)\sum_{|\alpha|\le m}(-1)^{|\alpha|}\partial^\alpha L^m(Pm_\alpha)_r(x)\,dx\Big|\\
&= \Big|\int_B \sum_{|\alpha|\le m} \partial^\alpha P\phi(x) L^m(Pm_\alpha)_r(x)\,dx\Big|\\
&= \Big|\int_B \sum_{|\alpha|\le m} \partial^\alpha P\phi(x)(Pm_\alpha)_r(x)\,dx\Big|\\
&\le C\|\phi\|_m\|(Pm_\alpha)\|_{L^2(E^*)^L} \quad \text{(Lemma 2.4)}\\
&\le C\|\phi\|_m\|(m_\alpha)\|_{V^2(E^*)^L}.\end{aligned}$$

Hence $g^{(r)}$ is bounded and $\lim_{r\to1}\int_B \phi(x)g^{(r)}(x)\,dx$ converges pointwise to $\int_B\sum_{|\alpha|\le m}\partial^\alpha P\phi(x)(Pm_\alpha)(x)\,dx$ on $C_c^\infty(B)\otimes E$. Then by Proposition 3.4, $g^{(r)}$ converges to Φ in $h^{-m}(E^*)$ in the weak*-topology, where

$$\Phi(u) \equiv \int_B \sum_{|\alpha|\le m} \partial^\alpha Pu(x)(Pm_\alpha)(x)\,dx, \quad u \in W_0^m(E),$$

and Φ is represented by a harmonic function g. Finally, for any $u \in h^\infty(E)$,

$$\begin{aligned}\int_B L^s u(x)g^{(r)}(x)\,dx &= \int_B u(x)g^{(r)}(x)\,dx\\
&= \int_B u(x)\sum_{|\alpha|\le m}(-1)^{|\alpha|}\partial^\alpha L^m(Pm_\alpha)_r(x)\,dx\\
&= \int_B \sum_{|\alpha|\le m}\partial^\alpha u(x)(Pm_\alpha)_r(x)\,dx.\end{aligned}$$

Taking the limit as $r \to 1$, we obtain

$$\begin{aligned}\langle u, g\rangle_0 = \langle L^m u, g\rangle_m &= \int_B \sum_{|\alpha|\le m} \partial^\alpha u(x)(Pm_\alpha)(x)\,dx\\
&= \int_B \sum_{|\alpha|\le m}\partial^\alpha u(x)\,dm_\alpha(x) = \langle u, T\rangle.\end{aligned}$$

The uniqueness of g can be proved as in the scalar case [2].

REMARK 3.7. (a) For smooth bounded regions Ω, P is a continuous projection in the scalar $W^m(\Omega)$. The proof is based on the fact that $P = I - \Delta G\Delta$, where $G(v)$ is the solution operator of $\Delta^2\phi = v$, with $\phi = d\phi/dn = 0$ on $\partial\Omega$. Lemma 2.4 is a weak version of this result for vector-valued functions in the case of the ball, which is proved directly, since an explicit formula for the kernel is available in this case. The proof of Theorem 3.6 would be the same for any star shaped region for which Lemma 2.4 holds true. In [17], Straube proves that the duality of $h^\infty(\Omega)$ and $h^{-\infty}(\Omega)$ may be written as $\lim_{\varepsilon\to0}\int_{D_\varepsilon} fg\,dx$, where $f \in h^{-\infty}(\Omega)$, $g \in h^\infty(\Omega)$, and $D_\varepsilon = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) < \varepsilon\}$. Here the existence of the boundary value (distribution) of f plays a role in the proof. This opens the possibility of calculating the dual of $h^\infty(\Omega, E)$ for general regions Ω.

(b) In the the proof of Theorem 3.6 it is shown that $h^m(E)^*$ can be identified with $h^{-m}(E^*)$ for any Banach space E. Here $h^m(E)$ is the space of all harmonic functions in $W^m(E)$. This might be the same for every smooth region Ω, as in [13] for scalar spaces.

4. Some tensor product considerations. Given a locally convex space F we denote by F_b^* its topological dual equipped with the topology of uniform convergence on bounded subsets of F.

From Bell's paper [2] and having in mind that $h^\infty(\overline{B})$ is a nuclear space ($C^\infty(\partial B)$ is nuclear [11, Ch. II, p. 55] and the Poisson integral transform is an isomorphism of $C^\infty(\partial B)$ onto $h^\infty(\overline{B})$), it follows that the mapping $\Phi(g) = \langle\cdot, g\rangle_0$ is a topological isomorphism of $h^{-\infty}(B)$ onto $h^\infty(\overline{B})_b^*$, when $h^{-\infty}(B)$ is the inductive limit of the Banach spaces $h^{-m}(B)$. We claim that it is also true in the vector-valued case.

PROPOSITION 4.1. *The mapping Φ above is a topological isomorphism of $h^{-\infty}(E^*)$ onto $h^\infty(E)_b^*$, when $h^{-\infty}(E^*)$ is the inductive limit of the Banach spaces $h^{-m}(E^*)$.*

Proof. The proof is a consequence of the following considerations:

(1) $h^\infty(\overline{B})\otimes E$ is dense in $h^\infty(E)$.

(2) The projective topology, the ε-topology and the $h^\infty(E)$-topology coincide on $h^\infty(\overline{B})\otimes E$:

(1) follows from Theorem 3.6 and Hahn–Banach's theorem. Due to the nuclearity of $h^\infty(\overline{B})$, to prove (2) it suffices to show that the identity

$$h^\infty(\overline{B}) \otimes_{h^\infty(E)} E \to h^\infty(\overline{B}) \otimes_\varepsilon E$$

is continuous. Let $(q_m)_{m=0}^\infty$ be the norms defining the topology of $h^\infty(\overline{B})$, and denote by $b_{q_m}^\circ$ the absolute polar in $h^\infty(\overline{B})^*$ of the ball $\{u : q_m(u) \le 1\}$. Then

$$(q_m \otimes_\varepsilon \|\cdot\|_E)\Big(\sum_{i=1}^k u_i \otimes e_i\Big) = \sup_{T\in b_{q_m}^\circ} \Big\|\sum_{i=1}^k T(u_i)e_i\Big\|_E.$$

By Grothendieck's localization theorem (see [12, I, p. 225, (5)]), we can find $s \ge 1$ such that $\Phi^{-1}(b_{q_m}^\circ) \subset h^{-s}(B)$ and $\sup\{\|g\|_{-s} : g \in \Phi^{-1}(b_{q_m}^\circ)\} < \infty$. Moreover, if $g = \Phi^{-1}(T) = \sum_{|\alpha|\le s}(-1)^{|\alpha|}\partial^\alpha v_\alpha$ with $v_\alpha \in L^2$, then

$$\Big\|\sum_{i=1}^k T(u_i)e_i\Big\|_E = \Big\|\sum_{i=1}^k \langle u_i, g\rangle_0 e_i\Big\|_E = \Big\|\sum_{i=1}^k \langle L^s u_i, g\rangle e_i\Big\|_E$$
$$= \Big\|\sum_{i=1}^k \sum_{|\alpha|\le s} \int_B \partial^\alpha L^s u_i(x) v_\alpha(x) e_i\,dx\Big\|_E$$
$$\le \Big(\sum_{|\alpha|\le s}\|v_\alpha\|_2^2\Big)^{1/2}\Big(\sum_{|\alpha|\le s}\int_B \Big\|\sum_{i=1}^k \partial^\alpha L^s u_i(x)e_i\Big\|_E^2\,dx\Big)^{1/2},$$

thus

$$\Big\|\sum_{i=1}^k T(u_i)e_i\Big\|_E \le \|g\|_{-s}\Big(\sum_{|\alpha|\le s}\int_B \Big\|\sum_{i=1}^k \partial^\alpha L^s u_i(x)e_i\Big\|_E^2\,dx\Big)^{1/2}$$

so that

$$(q_m \otimes_\varepsilon \|\cdot\|_E)\Big(\sum_{i=1}^k u_i \otimes e_i\Big) \le C\Big\|L^s\Big(\sum_{i=1}^k u_i \otimes e_i\Big)\Big\|_s \le C\,p_{s+N}\Big(\sum_{i=1}^k u_i \otimes e_i\Big),$$

which proves the required continuity. As a consequence, it follows that $h^\infty(E)$ is a distinguished Fréchet space, that is, $h^\infty(E)_{\mathrm b}^*$ is barreled or equivalently it is ultrabornological (see [12, I, p. 400, (3)]). Now we are in a position to conclude the proof of the proposition. Fix $m \ge 1$ and let A be a bounded subset of $h^\infty(E)$. Then for every $g \in h^{-m}(E^*)$,

$$\sup_{u\in A}|\langle u, g\rangle_0| = \sup_{u\in A}|\langle L^m u, g\rangle_m| \le C\sup\|u\|_{m+N}\,\|g\|_{-m}$$
$$\le C\sup p_{m+N}(u)\,\|g\|_{-m} \le C\|g\|_{-m}.$$

Since m was arbitrarily chosen, this proves the continuity of $h^{-\infty}(E^*)$ into $h^\infty(E)_{\mathrm b}^*$, and since $h^\infty(E)_{\mathrm b}^*$ is ultrabornological, we conclude from Grothendieck's closed graph theorem (see [12, II, p. 44, (6)]) that Φ is a topological isomorphism.

REMARK 4.2. Using Bell's results [2] and the theory of tensor products and nuclear spaces we have the following chain of natural isomorphisms:

$$h^\infty(E)_{\mathrm b}^* \simeq h^\infty(\overline{B})_{\mathrm b}^* \mathbin{\widehat{\otimes}} E^* = h^{-\infty}(B)^* \mathbin{\widehat{\otimes}} E^* = (\operatorname{ind}(h^{-m}(B)^* \mathbin{\widehat{\otimes}}_\pi E^*))^\wedge.$$

It remains open to prove directly, without using Theorem 3.6, that $(\operatorname{ind}(h^{-m}(B)^* \mathbin{\widehat{\otimes}}_\pi E^*))^\wedge = \operatorname{ind} h^{-m}(E^*)$. Notice that in general $h^{-m}(E^*)$ is not topologically isomorphic to $h^{-m}(B)^* \mathbin{\widehat{\otimes}}_\pi E^*$, for example, if $E = L^2$, then $h^{-m}(E^*) \simeq L^2$ but $h^{-m}(B)^* \mathbin{\widehat{\otimes}}_\pi E^* \simeq L^2 \mathbin{\widehat{\otimes}}_\pi L^2$ which is not reflexive.

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Geometry of oblique projections

by

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Abstract. Let $\mathcal{A}$ be a unital C^*-algebra. Denote by P the space of selfadjoint projections of $\mathcal{A}$. We study the relationship between P and the spaces of projections P_a determined by the different involutions $\#_a$ induced by positive invertible elements $a \in \mathcal{A}$. The maps $\varphi_p : P \to P_a$ sending p to the unique $q \in P_a$ with the same range as p and $\Omega_a : P_a \to P$ sending q to the unitary part of the polar decomposition of the symmetry $2q-1$ are shown to be diffeomorphisms. We characterize the pairs of idempotents $q, r \in \mathcal{A}$ with $\|q-r\| < 1$ such that there exists a positive element $a \in \mathcal{A}$ satisfying $q, r \in P_a$. In this case q and r can be joined by a unique short geodesic along the space of idempotents Q of $\mathcal{A}$.

1. Introduction. Let $\mathcal{H}$ be a Hilbert space with scalar product $\langle\,,\rangle$. For every bounded positive invertible operator $a : \mathcal{H} \to \mathcal{H}$ consider the scalar product $\langle\,,\rangle_a$ given by

$$\langle \xi, \eta\rangle_a = \langle \xi, \eta\rangle, \quad \xi, \eta \in \mathcal{H}.$$

It is clear that $\langle\,,\rangle_a$ induces a norm equivalent to the norm induced by $\langle\,,\rangle$. With respect to the scalar product $\langle\,,\rangle_a$, the adjoint of a bounded linear operator $x : \mathcal{H} \to \mathcal{H}$ is

$$x^{\#_a} = a^{-1}x^*a.$$

Thus, x is $\#_a$-selfadjoint if and only if

$$ax = x^*a.$$

Given a closed subspace S of $\mathcal{H}$, denote by $p = P_S$ the orthogonal projection from $\mathcal{H}$ onto S and, for any positive operator a, denote by $\varphi_p(a)$ the unique $\#_a$-selfadjoint projection with range S. In a recent paper, Z. Pasternak-Winiarski [20] proves the analyticity of the map $a \mapsto \varphi_p(a)$ and calculates its Taylor expansion. This study is relevant for understanding reproducing kernels of Hilbert spaces of holomorphic L^2 sections of complex vector bundles and the way they change when the measures and hermitian

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