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# Compound invariants and embeddings of Cartesian products

by

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**Abstract.** New compound geometric invariants are constructed in order to characterize complemented embeddings of Cartesian products of power series spaces. Bessaga's conjecture is proved for the same class of spaces.

**1. Introduction.** Let $a, b, \widetilde{a}, \widetilde{b}$ be sequences of positive numbers and $E_0(a)$, $E_0(\widetilde{a})$, $E_\infty(b)$, $E_\infty(\widetilde{b})$ finite and infinite power series spaces generated by these sequences. We obtain necessary and sufficient conditions for complemented embedding of $E_0(a) \times E_\infty(b)$ into $E_0(\widetilde{a}) \times E_\infty(\widetilde{b})$ in terms of the sequences $a, b, \widetilde{a}, \widetilde{b}$. Our approach is based on a construction of compound geometric invariants in the spirit of [20]–[22].

As an immediate corollary of our main theorem we get a complete isomorphic classification of Cartesian products of power series spaces, thus solving the problem of finding such a classification by means of geometrical invariants only ([5], Question 2). An alternative approach by using Riesz theory is known in the case when at least one of the Cartesian factors is a Schwartz space [18, 19]. In [4, 5] a complete isomorphic classification was given in the general case by combining both methods: 1) geometrical invariants for the case where at least one of the factors is isomorphic to its hyperplane and 2) Riesz theory methods otherwise. Now by considering some additional invariants we obviate the need of Riesz theory at all.

As another application of our criterion of complemented embedding we prove that each Cartesian product of the kind $E_0(a) \times E_\infty(b)$ satisfies Bessaga's conjecture.

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**2. Preliminaries.** Let us recall that if $(a_{ip})_{i,p\in\mathbb{N}}$ is a matrix of real numbers such that $0 \le a_{ip} \le a_{i,p+1}$, then the *Köthe space* $K(a_{ip})$ is the Fréchet space of all sequences $x = (x_i)$ of scalars such that $|x|_p := \sum_{i\in I} |x_i| a_{ip} < \infty$ for all $p \in \mathbb{N}$, with the topology generated by the system of seminorms $\{|\cdot|_p : p \in \mathbb{N}\}$. The Cartesian product $K(a_{kp}) \times K(b_{kp})$ is naturally isomorphic to $K(c_{ip})$, where $c_{ip} = a_{kp}$ if $i = 2k-1$, $c_{ip} = b_{kp}$ if $i = 2k$. For any sequence $a = (a_k)$ of positive numbers the Köthe spaces

$$E_0(a) = K\left(\exp\left(-\frac{1}{p}a_k\right)\right), \qquad E_\infty(a) = K(\exp(pa_k))$$

are called, respectively, *finite* and *infinite power series spaces*. They are Schwartz spaces if and only if $a_k \to \infty$.

Sequences $a$ and $\widetilde{a}$ of positive numbers are called *weakly equivalent* (we write $a_i \asymp \widetilde{a}_i$) if

$$\exists c > 0: \quad \frac{1}{c}a_i \le \widetilde{a}_i \le ca_i.$$

An increasing sequence $a = (a_i)$ is *shift-stable* if

$$\sup_i a_{i+1}/a_i < \infty.$$

Further, for any set $B$ we denote by $|B|$ the number of elements in $B$ if it is finite and the symbol $\infty$ if $B$ is infinite.

Suppose $X = K(a_{ip}, i \in I)$ and $Y = K(b_{jp}, j \in J)$ are Köthe spaces. An operator $T : X \to Y$ is called *quasi-diagonal* if there exists a function $\varphi : I \to J$ and constants $r_i, i \in I$, such that

$$Te_i = r_i \widetilde{e}_{\varphi(i)}, \qquad i \in I,$$

where $(e_i)$ and $(\widetilde{e}_j)$ are the canonical bases in $X$ and $Y$. We denote by $X \hookrightarrow Y$, $X \overset{c}{\hookrightarrow} Y$, $X \overset{qd}{\hookrightarrow} Y$ and $X \overset{qd}{\simeq} Y$ an embedding, a complemented embedding (i.e. as complemented subspace), a quasi-diagonal embedding and a quasi-diagonal isomorphism, respectively.

The next statement is well known (see, for example, [19]).

LEMMA 1. *If $X$ and $Y$ are Köthe spaces such that $X \overset{qd}{\hookrightarrow} Y$ and $Y \overset{qd}{\hookrightarrow} X$, then $X \overset{qd}{\simeq} Y$.*

Proof. If the quasi-diagonal embeddings $X \overset{qd}{\hookrightarrow} Y$ and $Y \overset{qd}{\hookrightarrow} X$ are defined by $(r_i)$, $\varphi : I \to J$ and $(\varrho_j)$, $\psi : J \to I$, respectively, then by Cantor–Bernstein's theorem there exist complementary subsets $I_1, I_2 \subset I$ and $J_1, J_2 \subset J$ such that $\varphi(I_1) = J_1$ and $\psi(J_2) = I_2$. Putting $Te_i = \gamma_i \widetilde{e}_{g(i)}$, where $\gamma_i = r_i$, $g(i) = \varphi(i)$ for $i \in I_1$ and $\gamma_i = \varrho^{-1}_{\psi^{-1}(i)}$, $g(i) = \psi^{-1}(i)$ for $i \in I_2$, we obtain a quasi-diagonal isomorphism $T$ between $X$ and $Y$.



For a given sequence $a = (a_k)$ of positive numbers consider the following characteristics:

$$m_a(t) = |\{k \in \mathbb{N} : a_k \le t\}|, \qquad \mu_a(\tau, t) = |\{k \in \mathbb{N} : \tau < a_k \le t\}|.$$

LEMMA 2. *If $a = (a_k)$ and $\widetilde{a} = (\widetilde{a}_k)$ are sequences of positive numbers satisfying*

$$\exists c\ \forall t > \tau > 0: \quad \mu_a(\tau, t) \le \mu_{\widetilde{a}}(\tau/c, ct), \tag{1}$$

*then there exists an injection $\varphi : \mathbb{N} \to \mathbb{N}$ such that*

$$\frac{1}{c^2}a_k \le \widetilde{a}_{\varphi(k)} \le c^2 a_k \quad \forall k \in \mathbb{N}. \tag{2}$$

This is proved in [14] by using the Hall–König theorem. An alternative direct proof is given in the survey [21].

COROLLARY 3. *If $a = (a_k)$ and $\widetilde{a} = (\widetilde{a}_k)$ are sequences of positive numbers satisfying* (1), *then $E_0(a)$ can be quasi-diagonally and isomorphically embedded into $E_0(\widetilde{a})$, and $E_\infty(a)$ can be quasi-diagonally and isomorphically embedded into $E_\infty(\widetilde{a})$.*

Notice that if $a$ is bounded the situation is trivial:

$$E_0(a) \overset{qd}{\simeq} \ell^1, \quad E_\infty(a) \overset{qd}{\simeq} \ell^1.$$

If $X$ is a Fréchet space and $s$ is an integer we denote by $X^{(s)}$ an $s$-codimensional subspace of $X$ if $s \ge 0$ and a product of the kind $X \times L$, where $\dim L = -s$, if $s < 0$.

Let $X, (|\cdot|_p)$ and $Y, (\|\cdot\|_p)$ be Fréchet spaces, and $(x_i)$ and $(y_i)$ sequences of elements of $X$ and $Y$, respectively. The sequences $(x_i)$ and $(y_i)$ are:

- *equivalent* if

$$\forall p\ \exists q, C: \quad |x_i|_p \le C\|y_i\|_q,$$
$$\forall p\ \exists q, C: \quad \|y_i\|_p \le C|x_i|_q;$$

- *quasi-equivalent* if there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ and constants $r_i > 0$ such that the sequences $(x_i)$ and $(r_i y_{\sigma(i)})$ are equivalent;
- *weakly quasi-equivalent* if there exist mappings $\sigma, \nu : \mathbb{N} \to \mathbb{N}$ and constants $r_i, \varrho_i > 0$ such that the sequences $(x_i)$ and $(r_i y_{\sigma(i)})$ are equivalent and so are $(y_i)$ and $(\varrho_i x_{\nu(i)})$.

In general, it is an open problem whether any two bases in a nuclear Fréchet space are quasi-equivalent. The answer is positive for the quite large class of all Fréchet spaces with regular absolute basis.

Dragilev [6] showed that the notion of weak quasi-equivalence is useful for attacking the quasi-equivalence problem and proved that any two bases in a nuclear Fréchet space are weakly quasi-equivalent.

Bessaga [1] generalized the result of Dragilev on weak quasi-equivalence proving that any basis in a complemented subspace of a nuclear Fréchet space with basis is weakly quasi-equivalent to a subsequence of the given basis. He also formulated the following

BESSAGA'S CONJECTURE. *If $X$ is a Köthe space and $E$ is a complemented subspace of $X$ with absolute basis $(x_i)$, then the sequence $(x_i)$ is quasi-equivalent to a subsequence of the canonical basis of $X$.*

Generalizing the results of Dragilev [6] and Bessaga [1] to weak quasi-equivalence in nuclear spaces, Kondakov and Zahariuta [11], [10] (see also [9]) proved the following

PROPOSITION 4. *If $X$ is a Köthe space, then any absolute basis in a complemented subspace of $X$ is weakly quasi-equivalent to a subsequence of the canonical basis of $X$.*

**3. Compound invariants.** Suppose $E$ is a linear space, $U$ and $V$ are absolutely convex sets in $E$ and $\mathcal{E}_V$ is the set of all finite-dimensional subspaces of $E$ which are spanned by elements of $V$. We put

$$\beta(V,U) = \sup\{\dim L : L \in \mathcal{E}_V,\ L \cap U \subset V\}.$$

It is obvious that

$$V \subset \widetilde{V},\ \widetilde{U} \subset U \Rightarrow \beta(V,U) \le \beta(\widetilde{V},\widetilde{U})$$

and of course if $T$ is an injective linear operator defined on $E$, then $\beta(T(V), T(U)) = \beta(V,U)$.

Classical linear topological invariants such as approximative or diametral dimensions (Kolmogorov [8], Pełczyński [16], Bessaga, Pełczyński, Rolewicz [2], Mityagin [12]) can be simply described by the following family of functions, defined in terms of the characteristic $\beta$ (see, for example, [3]):

$$\gamma_E = \gamma_E(\mathcal{U}) = \{\beta(tV,U) : V,U \in \mathcal{U}\},$$

where $\mathcal{U}$ is any fundamental system of neighborhoods of zero in the locally convex space $E$. More precisely, for two locally convex spaces $E$ and $F$ we define the relation $\gamma_E \prec \gamma_F$ as follows:

$$\forall \widetilde{V}\ \exists V\ \forall U\ \exists \widetilde{U}\ \exists c > 0: \quad \beta_E(tV,U) \le \beta_F(ct\widetilde{V},\widetilde{U}),$$

where $V, U$ and $\widetilde{V}, \widetilde{U}$ are taken from any fundamental system $\mathcal{U}$ in $E$ and $\widetilde{\mathcal{U}}$ in $F$, respectively (it is clear that the relation does not depend on the choice of fundamental systems). Now the relation $\gamma_E \approx \gamma_F$ can be defined as $\gamma_E \prec \gamma_F$ together with $\gamma_F \prec \gamma_E$, so that the following statement holds.

PROPOSITION 5. *If $E \simeq F$, then $\gamma_E \approx \gamma_F$.*



Let $E$ be a Köthe space and $A$ be the set of all sequences with positive terms. For any $a = (a_i) \in A$ we define the following weighted norm (it may be unbounded) and weighted ball:

$$\|x\|_a = \sum_i |x_i| a_i, \quad B_a = \{x \in E : \|x\|_a < 1\}.$$

For calculations of invariants it is very convenient that the characteristic $\beta$ can be easily computed for weighted balls.

LEMMA 6. *If $a, b \in A$, then*

$$\beta(B_a, B_b) = |\{i : a_i/b_i \le 1\}|.$$

Proof. Put

$$J = \{i : a_i \le b_i\}, \quad Px = \sum_{i \in J} x_i e_i \quad \text{for } x = \sum_{i=1}^{\infty} x_i e_i,$$

and let $M$ be the linear span of $\{e_i : i \in J\}$. Then, obviously, $\|x\|_a \le \|x\|_b$ for $x \in M$. Hence $M \cap B_b \subset B_a$ and $\beta(B_a, B_b) \ge \dim M = |J|$.

Conversely, suppose $L$ is a finite-dimensional subspace in $X$ satisfying $L \cap B_b \subset B_a$ (i.e. $\|x\|_a \le \|x\|_b$) for all $x \in L$. If $\dim L > |J|$ then there exists $x = \sum_{i=1}^{\infty} x_i e_i \in L$, $x \ne 0$, such that $Px = 0$. But then $x_i = 0$ for $i \in J$ and there exists $i \notin J$ such that $x_i \ne 0$. Since $a_i > b_i$ for $i \notin J$, we have $\|x\|_a > \|x\|_b$, which is a contradiction. Hence $\beta(B_a, B_b) = |J|$.

Let us describe some geometrical constructions on pairs of absolutely convex sets; these constructions will be used later as elementary blocks to produce appropriate compound invariants. For a given couple of absolutely convex sets $U, V$ in $E$ we consider $U \cap V$ and $\operatorname{conv}(U \cup V)$, which are obviously invariant with respect to any linear bijection. For weighted balls $U = B_a$, $V = B_b$, $a, b \in A$, we have the following relations:

$$B_{a \vee b} \subset B_a \cap B_b \subset 2B_{a \vee b}, \quad \operatorname{conv}(B_a \cup B_b) = B_{a \wedge b}, \tag{3}$$

where

$$a \wedge b = (\min\{a_i, b_i\}), \quad a \vee b = (\max\{a_i, b_i\}).$$

These relations will be very useful for calculation of compound invariants.

Another simple construction can be obtained by power interpolation. For a given pair of balls $B_{a^{(0)}}, B_{a^{(1)}}$, $a^{(0)}, a^{(1)} \in A$, we consider an $\alpha$-interpolation ball $B_{a^{(0)}}^{1-\alpha} B_{a^{(1)}}^{\alpha} := B_{a^{(\alpha)}}$ with $a_i^{(\alpha)} = (a_i^{(0)})^{1-\alpha}(a_i^{(1)})^{\alpha}$, $-\infty < \alpha < \infty$. We have the following simple fact.

LEMMA 7. *Suppose $E, \widetilde{E}$ are Köthe spaces, $T : E \to \widetilde{E}$ is a linear operator, and $B_{a^{(0)}}, B_{a^{(1)}}$ and $\widetilde{B}_{\widetilde{a}^{(0)}}, \widetilde{B}_{\widetilde{a}^{(1)}}$ are two pairs of balls in $E$ and $\widetilde{E}$, respectively. If*

$$T(B_{a^{(0)}}) \subset \widetilde{B}_{\widetilde{a}^{(0)}}, \quad T(B_{a^{(1)}}) \subset \widetilde{B}_{\widetilde{a}^{(1)}},$$

*then for any* $\alpha \in (0,1)$ *we have*

$$T(B^{1-\alpha}_{a^{(0)}} B^{\alpha}_{a^{(1)}}) \subset \widetilde{B}^{1-\alpha}_{\widetilde{a}^{(0)}} \widetilde{B}^{\alpha}_{\widetilde{a}^{(1)}}.$$

Proof. Let $(e_i)$ and $(\widetilde{e}_i)$ be the canonical bases in $E$ and $\widetilde{E}$, respectively. Put

$$Te_i = \sum_j t_{ij}\widetilde{e}_j, \quad i = 1,2,\ldots;$$

then since $\|Tx\|_{\widetilde{a}^{(\nu)}} \le \|x\|_{a^{(\nu)}}$, $\nu = 0,1$, we have, for any $i$,

$$\|Te_i\|_{\widetilde{a}^{(\nu)}} = \sum_j |t_{ij}|\widetilde{a}_j^{(\nu)} \le \|e_i\|_{a^{(\nu)}} = a_i^{(\nu)}, \qquad \nu = 0,1.$$

Therefore by the Hölder inequality it follows that

$$\begin{aligned}\|Te_i\|_{\widetilde{a}^{(\alpha)}} &= \sum_j |t_{ij}|(\widetilde{a}_j^{(0)})^{1-\alpha}(\widetilde{a}_j^{(1)})^{\alpha} \\ &\le \Big(\sum_j |t_{ij}|\widetilde{a}_j^{(0)}\Big)^{1-\alpha}\Big(\sum_j |t_{ij}|\widetilde{a}_j^{(1)}\Big)^{\alpha} \le a_i^{(\alpha)}.\end{aligned}$$

Hence,

$$\|Tx\|_{\widetilde{a}^{(\alpha)}} \le \sum_i |x_i|\cdot\|Te_i\|_{\widetilde{a}^{(\alpha)}} \le \sum_i |x_i| a_i^{(\alpha)} = \|x\|_{a^{(\alpha)}}.$$

If $E = K(a_{ip})$ and $U_p = \{x \in E : |x|_p = \sum_i |x_i|a_{ip} < 1\}$, $p = 1,2,\ldots$, are the corresponding unit balls, then $U_p = B_{a_p}$, where $a_p = (a_{ip})$. Further, we write $U_p^{\alpha}U_q^{1-\alpha}$ instead of $B^{\alpha}_{a_p}B^{1-\alpha}_{a_q}$.

Applying the characteristic $\beta$ to some synthetic neighborhoods obtained from the given neighborhoods as output of some multiparameter constructions (composed by using the elementary constructions considered above), we approach what we call compound invariants. In particular, we can get the following simplest two-parameter invariants [22–24]:

$$\beta(t^{-1}U_p \cap \tau U_r, U_q), \qquad \beta(U_q, \operatorname{conv}(t^{-1}U_p \cup \tau U_r)),$$

or some more complicated invariants, involving also power interpolation constructions, for example:

$$\beta(U_p^{\alpha}U_r^{1-\alpha} \cap t^{-1}U_p \cap \tau U_r, U_q).$$

This method, suggested in [22–24], inputs the new more geometric and properly invariant content to the method of invariant characteristics for Köthe spaces [21, 22], which was a natural development of Mityagin's results for non-Montel power series spaces. In the proofs of Theorems 8 and 10 below we shall use some multiparameter characteristics of this kind and show their invariance.



**4. Embedding of power series spaces.** In [13], [14] (see also [15]) B. S. Mityagin obtained a criterion for isomorphism of non-Schwarzian $\ell^2$-power series spaces. He proved the necessity of his criterion by analyzing the spectral properties of the operators that generate Hilbert scales corresponding to the given power series spaces.

Here we consider a criterion for embedding of power series spaces, which is a modification of Mityagin's results. We prove the necessity by using an appropriate compound invariant. The sufficiency follows from Lemma 2.

THEOREM 8. *Let* $a = (a_i)$ *and* $\widetilde{a} = (\widetilde{a}_i)$ *be sequences of positive numbers such that* $a_i \ge 1$ *and* $\widetilde{a}_i \ge 1$. *Suppose* $X = E_0(a)$ (*or* $X = E_\infty(a)$) *and* $Y = E_0(a)$ (*or* $Y = E_\infty(a)$, *respectively*). *The following conditions are equivalent*:

(i) $X \hookrightarrow Y$;

(ii) *there exists* $C > 0$ *such that for* $t > \tau > 0$ *we have* $\mu_a(\tau,t) \le \mu_{\widetilde{a}}(\tau/C, Ct)$;

(iii) *there exists an injection* $\sigma : \mathbb{N} \to \mathbb{N}$ *such that*

$$\exists C: \quad 1/C^2 \le \widetilde{a}_{\sigma(i)}/a_i \le C^2.$$

(iv) $X \overset{qd}{\hookrightarrow} Y$.

Proof. It is obvious that (iii)⇒(iv), (iv)⇒(i), and by Lemma 2 we have (ii)⇒(iii). Now we prove that (i)⇒(ii).

Since the proof is the same for finite power series spaces, only the case of infinite type power series spaces is considered. For convenience we write $V \prec W$ if $V \subset \operatorname{const} W$. Suppose that $T : E_\infty(a) \hookrightarrow E_\infty(\widetilde{a})$ is an embedding. Let $(U_p)$ and $(V_p)$ be the systems of unit balls in $E_\infty(a)$ and $E_\infty(\widetilde{a})$, respectively. Put $W_p = V_p \cap R(T)$, where $R(T)$ denotes the range of $T$. Choose indices

$$p_2 < p < q < q_1 < r_2 < r$$

so that

$$W_{p_2} \succ T(U_p) \succ T(U_q) \succ W_{q_1} \succ W_{r_2} \succ T(U_r).$$

Then from the elementary properties of the characteristic $\beta$ it follows that for some constant $C > 0$,

$$\begin{aligned}\beta(e^{-\tau}U_p \cap e^tU_r, U_q) &= \beta(e^{-\tau}T(U_p) \cap e^tT(U_r), T(U_q)) \\ &\le \beta(C(e^{-\tau}W_{p_2} \cap e^tW_{r_2}), W_{q_1}) \\ &\le \beta(C(e^{-\tau}V_{p_2} \cap e^tV_{r_2}), V_{q_1}).\end{aligned}$$

Using (3) and Lemma 6 we estimate both sides of this inequality, from below and above, respectively, and obtain

$$\begin{aligned}&\left|\left\{i : \frac{\max\{\exp(\tau + pa_i), \exp(-t + ra_i)\}}{\exp(qa_i)} \le 1\right\}\right| \\ &\qquad\le \left|\left\{i : \frac{\max\{\exp(\tau + p_2b_i), \exp(-t + r_2b_i)\}}{\exp(q_1b_i)} \le 2C\right\}\right|.\end{aligned}$$

This inequality is equivalent to

$$|\{i : e^{\tau+(p-q)a_i} \le 1,\ e^{-t+(r-q)a_i} \le 1\}| \le |\{i : e^{\tau+(p_2-q_1)b_i} \le 2C,\ e^{-t+(r_2-q_1)b_i} \le 2C\}|.$$

Taking logarithms we obtain

$$\left|\left\{i : \frac{\tau}{q-p} \le a_i \le \frac{t}{r-q}\right\}\right| \le \left|\left\{i : \frac{\tau - \log 2C}{q_1 - p_2} \le b_i \le \frac{t+\log 2C}{r_2 - q_1}\right\}\right|.$$

Hence,

$$\exists M\ \forall t > \tau > 0 : \quad |\{i : \tau \le a_i \le t\}| \le |\{i : \tau/M - M \le \widetilde{a}_i \le Mt + M\}|.$$

From this, condition (iii) follows immediately. Indeed, if $C > M(M+1)$, then either $\tau/M - M < 1$, so $\tau/C < 1$, or $\tau/M - M \ge 1$, so $\tau/C < \tau/M - M$. Therefore, taking into account that $\widetilde{a}_i \ge 1$, we obtain

$$\{i : \tau/M - M \le \widetilde{a}_i \le Mt + M\} \subset \{i : \tau/C < \widetilde{a}_i \le Ct\},$$

which completes the proof.

COROLLARY 9 (Mityagin's criterion for isomorphism of power series spaces). *Let $a = (a_i)$ and $\widetilde{a} = (\widetilde{a}_i)$ be sequences of positive numbers such that $a_i \ge 1$ and $\widetilde{a}_i \ge 1$. The following conditions are equivalent*:

(i) $E_0(a) \simeq E_0(\widetilde{a})$ (*respectively* $E_\infty(a) \simeq E_\infty(\widetilde{a})$);

(ii) *there exists $C > 0$ such that for $t > \tau > 0$ we have*

$$\mu_a(\tau, t) \le \mu_{\widetilde{a}}(\tau/C, Ct), \qquad \mu_{\widetilde{a}}(\tau, t) \le \mu_a(\tau/C, Ct);$$

(iii) *there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that*

$$\exists C : \quad 1/C^2 \le \widetilde{a}_{\sigma(i)}/a_i \le C^2.$$

(iv) $E_0(a) \overset{qd}{\simeq} E_0(\widetilde{a})$ (*respectively* $E_\infty(a) \overset{qd}{\simeq} E_\infty(\widetilde{a})$).

**5. Main result.** In [4], [5] a complete isomorphic classification of the Cartesian products of the kind $E_0^1(a) \times E_\infty^1(b)$ is obtained by combining two methods: the method of compound invariants (following [17]) and the method of Fredholm operator theory (following [18, 19]. The first method also works in the case where the Cartesian factors are non-Montel spaces (when the second method just fails), but if both Cartesian factors are Montel spaces some important information is lost: by this method it was only shown that the relation

$$E_0(a) \times E_\infty(b) \simeq E_0(\widetilde{a}) \times E_\infty(\widetilde{b})$$

implies

$$E_0(a) \simeq E_0(\widetilde{a})^{(s_1)}, \quad E_\infty(b) \simeq E_\infty(\widetilde{b})^{(s_2)} \tag{4}$$



with some integer $s_1$, $s_2$; here, following [19], for every locally convex space $X$ and an integer $s$ we use the notation $X^{(s)}$ for any $s$-codimensional closed subspace of $X$ if $s \ge 0$ or any product of the kind $X \times L$, where $\dim L = -s$, if $s < 0$. In contrast, the Fredholm operator method gives more [19] in this case: the possibility to choose $s_1$, $s_2$ so that $s_1 + s_2 = 0$ (this fact is based on the stability of operator index under strictly singular perturbations).

The question "How to get this precise information by using invariants only?" (stated in [5], Question 2) is solved here by considering an additional compound invariant. Moreover, we have the following complete characterization of complemented embeddings of Cartesian products $E_0(a) \times E_\infty(b)$ by means of invariants only.

THEOREM 10. *Let $X = E_0(a) \times E_\infty(b)$ and $Y = E_0(\widetilde{a}) \times E_\infty(\widetilde{b})$. Then the following statements are equivalent*:

(i) $X \overset{c}{\hookrightarrow} Y$;

(ii) *there exist $C > 0$ and $\tau_0 > 0$ such that for $\tau_0 < \tau \le t$ we have*

$$\mu_a(\tau, t) \le \mu_{\widetilde{a}}(\tau/C, Ct), \tag{5}$$

$$\mu_b(\tau, t) \le \mu_{\widetilde{b}}(\tau/C, Ct), \tag{6}$$

$$m_a(t) + m_b(\tau) \le m_{\widetilde{a}}(Ct) + m_{\widetilde{b}}(C\tau); \tag{7}$$

(iii) $X \overset{qd}{\hookrightarrow} Y$.

Proof. The Cartesian products $E_0(a) \times E_\infty(b)$ and $E_0(\widetilde{a}) \times E_\infty(\widetilde{b})$ are naturally isomorphic to the Köthe spaces $E = K(c_{ip})$ and $F = K(d_{ip})$, where

$$c_{ip} = \begin{cases} \exp(-a_k/p), & i = 2k-1, \\ \exp(pb_k), & i = 2k, \end{cases} \qquad d_{ip} = \begin{cases} \exp(-\widetilde{a}_k/p), & i = 2k-1, \\ \exp(p\widetilde{b}_k), & i = 2k. \end{cases}$$

Thus it is sufficient to prove the theorem for $E$, $F$ instead of $X$, $Y$.

First we prove that (i)⇒(ii). Suppose $T : E \overset{c}{\hookrightarrow} F$ is a complemented embedding. Let $Z \subset F$ be a complementary subspace to $T(E)$, that is, $F = T(E) \oplus Z$. We denote by $(U_p)$ and $(V_p)$ the systems of unit balls in $E$ and $F$, respectively. Put $W_p = V_p \cap Z$ and choose indices

$$p_2 < p < q < q_1 < r_2 < r < s < s_1, \qquad 2p < q,\ 2q_1 < r_2,$$

so that

$$V_{p_2} \succ T(U_p) \oplus W_p \succ T(U_q) \oplus W_q \succ V_{q_1} \succ V_{r_2} \succ T(U_r) \oplus W_r \succ T(U_s) \oplus W_s \succ V_{s_1}.$$

Then from the elementary properties of $\beta$ and Lemma 7 it follows that for some constant $c > 0$,

(8) $$\beta(U_p^{1/2}U_r^{1/2}\cap e^{-\tau}U_p\cap e^tU_r,U_q)\le\beta(cV_{p_2}^{1/2}V_{r_2}^{1/2}\cap e^{-\tau}V_{p_2}\cap e^tV_{r_2},V_{q_1}),$$

(9) $$\beta(U_r,\operatorname{conv}(U_q^{1/2}U_s^{1/2}\cup e^{-t}U_q\cup e^{\tau}U_s))\le\beta(cV_{r_2},\operatorname{conv}(V_{q_1}^{1/2}V_{s_1}^{1/2}\cup e^{-t}V_{q_1}\cup e^{\tau}V_{s_1})),$$

(10) $$\beta(\operatorname{conv}((U_p^{1/2}U_r^{1/2}\cap e^tU_r)\cup e^{\tau}U_r),U_q)\le\beta(\operatorname{conv}((V_{p_2}^{1/2}V_{r_2}^{1/2}\cap e^tV_{r_2})\cup e^{\tau}V_{r_2}),V_{q_1}).$$

We show that (8), (9), (10) imply (5), (6), (7), respectively. Estimating the left-hand side of (8) from below and the right-hand side from above by using (3), the elementary properties of $\beta$ and Lemma 6 we obtain

$$\left|\left\{i:\frac{\max(c_{ip}^{1/2}c_{ir}^{1/2},e^{\tau}c_{ip},e^{-t}c_{ir})}{c_{iq}}\le 1\right\}\right|\le\left|\left\{i:\frac{\max(d_{ip_2}^{1/2}d_{ir_2}^{1/2},e^{\tau}d_{ip_2},e^{-t}d_{ir_2})}{d_{iq_1}}\le 4c\right\}\right|.$$

It follows that

(11) $$\left|\left\{i:\frac{c_{ip}^{1/2}c_{ir}^{1/2}}{c_{iq}}\le 1,\ \frac{e^{\tau}c_{ip}}{c_{iq}}\le 1,\ \frac{e^{-t}c_{ir}}{c_{iq}}\le 1\right\}\right|\le\left|\left\{i:\frac{d_{ip_2}^{1/2}d_{ir_2}^{1/2}}{d_{iq_1}}\le 4c,\ \frac{e^{\tau}d_{ip_2}}{d_{iq_1}}\le 4c,\ \frac{e^{-t}d_{ir_2}}{d_{iq_1}}\le 4c\right\}\right|.$$

The first inequality on the left-hand side of (11) is $c_{ip}^{1/2}c_{ir}^{1/2}\le c_{iq}$. For the even indices $i=2k$ it is equivalent to $(p+r-2q)b_k\le 0$, which is impossible because $r>2q$. For $i=2k-1$ it is equivalent to $(2/q-1/p-1/r)a_k\le 0$, which is always true because $q>2p$. Therefore, the left-hand side of (11) equals

(12) $$\left|\left\{k:\frac{\tau}{1/p-1/q}\le a_k\le\frac{t}{1/q-1/r}\right\}\right|.$$

Consider now the right-hand side of (11). The first inequality there is $d_{ip_2}^{1/2}d_{ir_2}^{1/2}\le 4cd_{iq_1}$. For $i=2k$ it is equivalent to

$$\widetilde{b}_k\le\tau_1:=(2\log 4c)/(p_2+r_2-2q_1).$$

In this case the other two inequalities imply

$$\frac{\tau-\log 4c}{q_1-p_2}\le\widetilde{b}_k\le\frac{t+\log 4c}{r_2-q_1},$$

therefore for $\tau>\tau_2:=\tau_1(q_1-p_2)+\log 4c$ the triple of inequalities on the right-hand side of (11) does not hold for even indices.



For $i=2k-1$ the first inequality on the right-hand side of (11) is equivalent to $(2/q_1-1/p_2-1/r_2)\widetilde{a}_k\le 2\log 4c$, which is always true because $q_1>2p_2$ (we can assume without loss of generality that $c>1$). Therefore the right-hand side of (11) equals for $\tau>\tau_2$ the expression

(13) $$\left|\left\{k:\frac{\tau-\log 4c}{1/p_2-1/q_1}\le\widetilde{a}_k\le\frac{t+\log 2c}{1/q_1-1/r_2}\right\}\right|.$$

Since for $\tau>\tau_2$ the expression (12) is less than (13), there exists a constant $C>0$ and a $\tau_0>\tau_2$ such that (5) holds. In an analogous way (9) implies (6).

Finally, we prove that (10) implies (7). Estimating as above both sides of (10) we obtain

$$\left|\left\{i:\frac{\min(\max(c_{ip}^{1/2}c_{ir}^{1/2},e^{-t}c_{ir}),e^{-\tau}c_{ir})}{c_{iq}}\le 1\right\}\right|\le\left|\left\{i:\frac{\min(\max(d_{ip_2}^{1/2}d_{ir_2}^{1/2},e^{-t}d_{ir_2}),e^{-\tau}d_{ir_2})}{d_{iq_1}}\le 2c\right\}\right|,$$

which is equivalent to

(14) $$\left|\left\{i:\frac{c_{ip}^{1/2}c_{ir}^{1/2}}{c_{iq}}\le 1,\ \frac{e^{-t}c_{ir}}{c_{iq}}\le 1\right\}\cup\left\{i:\frac{e^{-\tau}c_{ir}}{c_{iq}}\le 1\right\}\right|\le\left|\left\{i:\frac{d_{ip_2}^{1/2}d_{ir_2}^{1/2}}{d_{iq_1}}\le 2c,\ \frac{e^{-t}d_{ir_2}}{d_{iq_1}}\le 2c\right\}\cup\left\{i:\frac{e^{-\tau}d_{ir_2}}{d_{iq_1}}\le 2c\right\}\right|.$$

Since the inequality $c_{ip}^{1/2}c_{ir}^{1/2}\le c_{iq}$ holds only for the odd indices $i=2k-1$ the left-hand side of (14) equals, for $t>\tau$,

(15) $$\left|\left\{k:a_k\le\frac{t}{1/q-1/r}\right\}\right|+\left|\left\{k:b_k\le\frac{\tau}{r-q}\right\}\right|.$$

The first inequality on the right-hand side of (14) is $d_{ip_2}^{1/2}d_{ir_2}^{1/2}\le 2cd_{iq_1}$. It holds for each odd index $i=2k-1$, and for $i=2k$ it is equivalent to $\widetilde{b}_k\le\tau_3:=2(\log 2c)/(p_2+r_2-2q_1)$. Therefore for $\tau>(r_2-q_1)\tau_3$ the right-hand side of (14) equals

(16) $$\left|\left\{k:a_k\le\frac{t+\log 2c}{1/q_1-1/r_2}\right\}\right|+\left|\left\{k:b_k\le\frac{\tau+\log 2c}{r_2-q_1}\right\}\right|,$$

so, obviously, there exist $C>0$ and $\tau_0>0$ such that (7) holds.

Now we show that (ii)⇒(iii).

Take any $t_0 \geq \tau_0$ and define

$$M_1 := \{k \in \mathbb{N} : a_k > t_0\}, \qquad \widetilde{M}_1 := \{k \in \mathbb{N} : \widetilde{a}_k > t_0/C\},$$
$$M_2 := \{k \in \mathbb{N} : b_k > t_0\}, \qquad \widetilde{M}_2 := \{k \in \mathbb{N} : \widetilde{b}_k > t_0/C\},$$
$$L_1 := \{i = 2k-1 : k \in M_1\}, \qquad \widetilde{L}_1 := \{i = 2k-1 : k \in \widetilde{M}_1\},$$
$$L_2 := \{i = 2k : k \in M_2\}, \qquad \widetilde{L}_2 := \{i = 2k : k \in \widetilde{M}_2\},$$
$$L_3 := \mathbb{N} \setminus (L_1 \cup L_2), \qquad \widetilde{L}_3 := \mathbb{N} \setminus (\widetilde{L}_1 \cup \widetilde{L}_2).$$

By Lemma 2 there exist injections $\sigma_\nu : M_\nu \to \widetilde{M}_\nu$, $\nu = 1, 2$, such that

$$\text{(17)} \qquad \begin{aligned} a_k/C^2 &\leq \widetilde{a}_{\sigma_1(k)} \leq C^2 a_k, \quad k \in M_1, \\ b_k/C^2 &\leq \widetilde{b}_{\sigma_2(k)} \leq C^2 b_k, \quad k \in M_2. \end{aligned}$$

Assuming that (ii) is true, we are going to construct a permutational isomorphic embedding $T : E \to F$ in the form

$$\text{(18)} \qquad Te_i = \widetilde{e}_{\sigma(i)},$$

where $\sigma : \mathbb{N} \to \mathbb{N}$ is an injection that will be constructed by using the injections $\sigma_\nu$, and $\{e_i\}$, $\{\widetilde{e}_i\}$ are the canonical bases in $E$, $F$, respectively. First we define the injection

$$\text{(19)} \qquad \sigma'(i) = \begin{cases} 2\sigma_1\left(\dfrac{i+1}{2}\right) - 1 & \text{if } i \in L_1, \\ 2\sigma_2\left(\dfrac{i}{2}\right) & \text{if } i \in L_2, \end{cases}$$

acting from $L_1 \cup L_2$ into $\widetilde{L}_1 \cup \widetilde{L}_2$. Now we have to consider separately the following two cases:

($\alpha$) $Y$ is non-Montel, that is, at least one of the sequences $\widetilde{a}$, $\widetilde{b}$ does not tend to $\infty$;

($\beta$) $Y$ is Montel.

Consider the case ($\alpha$). Choose $t_0$ so large that at least one of the sets $\mathbb{N} \setminus \widetilde{M}_\nu$, $\nu = 1, 2$, is infinite. Then $\widetilde{L}_3$ is infinite and we can extend (19) to some injection $\sigma : \mathbb{N} \to \mathbb{N}$ by mapping $L_3$ into $\widetilde{L}_3$. It is easy to check that the corresponding operator (18) is an isomorphic embedding.

Consider the case ($\beta$). From (7) it follows that the sequences $a$ and $b$ also tend to $\infty$, so $l := |L_3| < \infty$. We can assume, without loss of generality, that $a, b, \widetilde{a}, \widetilde{b}$ are non-decreasing (if not, one can reorder them). Moreover, we can assume that the injections $\sigma_1$ and $\sigma_2$ are increasing (if not, one can modify them to be increasing preserving the relation (17)). If $\widetilde{l} := |\mathbb{N} \setminus \sigma'(L_1 \cup L_2)| \geq l$, then, mapping the set $L_3$ into $\mathbb{N} \setminus \sigma'(L_1 \cup L_2)$ we obtain an extension $\sigma$ of $\sigma'$ such that the corresponding operator (18) is an isomorphic embedding.



In the case where $\widetilde{l} < l$ and at least one of the sequences $\widetilde{a}$ and $\widetilde{b}$ is shift-stable, we can easily modify $\sigma_1$ or $\sigma_2$ so as to have $l \leq \widetilde{l}$ (e.g. if $\widetilde{a}$ is a shift-stable sequence then we can interchange $\sigma_1(k)$ with $l + \sigma_1(k)$).

Finally, we consider the case when $\widetilde{l} < l$ and neither $\widetilde{a}$ nor $\widetilde{b}$ is shift-stable. Then from (17) it follows that also $a$ and $b$ are not shift-stable. Since $\widetilde{l} < \infty$ there exist $l'_\nu \in \widetilde{M}_\nu$, $\nu = 1, 2$, such that

$$\sigma_\nu(k+1) = \sigma_\nu(k) + 1, \quad k \geq l'_\nu, \ \nu = 1, 2.$$

So, there exist integers $s_1$, $s_2$ such that

$$\text{(20)} \qquad \sigma_\nu(k) = s_\nu + k, \quad k \geq l'_\nu, \ \nu = 1, 2.$$

From (17) we get $a_k \asymp \widetilde{a}_{s_1+k}$ and $b_k \asymp \widetilde{b}_{s_2+k}$. Therefore

$$E_0(a) \times E_\infty(b) \overset{qd}{\simeq} E_0(\widetilde{a})^{(s_1)} \times E_\infty(\widetilde{b})^{(s_2)} \overset{qd}{\simeq} (E_0(\widetilde{a}) \times E_\infty(\widetilde{b}))^{(s_1+s_2)}.$$

Thus it remains to prove that

$$\text{(21)} \qquad s_1 + s_2 \geq 0.$$

Since $a$ and $b$ are not shift-stable, there exist $k_v \geq l'_\nu$, $\nu = 1, 2$, such that

$$C^4 a_{k_1} < a_{k_1+1}, \quad C^4 b_{k_2} < b_{k_2+1}.$$

Therefore, using (17) and (20), for $\tau = Ca_{k_1}$ and $t = Cb_{k_2}$ we obtain

$$m_a(\tau) = k_1, \quad m_b(t) = k_2, \quad m_{\widetilde{a}}(C\tau) = k_1 + s_1, \quad m_{\widetilde{b}}(Ct) = k_2 + s_2.$$

Now (7) yields (21). The theorem is proved.

As a corollary we get the following criterion for isomorphism of Cartesian products of power series spaces.

COROLLARY 11. *Let $X = E_0(a) \times E_\infty(b)$ and $Y = E_0(\widetilde{a}) \times E_\infty(\widetilde{b})$. Then the following statements are equivalent:*

(i) $X \simeq Y$;

(ii) *there exist $C > 0$ and $\tau_0 > 0$ such that for $\tau_0 < \tau \leq t$ we have*

$$\text{(22)} \qquad \mu_a(\tau, t) \leq \mu_{\widetilde{a}}(\tau/C, Ct), \quad \mu_{\widetilde{a}}(\tau, t) \leq \mu_a(\tau/C, Ct),$$

$$\text{(23)} \qquad \mu_b(\tau, t) \leq \mu_{\widetilde{b}}(\tau/C, Ct), \quad \mu_{\widetilde{b}}(\tau, t) \leq \mu_b(\tau/C, Ct),$$

$$\text{(24)} \qquad \begin{aligned} m_a(t) + m_b(\tau) &\leq m_{\widetilde{a}}(Ct) + m_{\widetilde{b}}(C\tau), \\ m_{\widetilde{a}}(t) + m_{\widetilde{b}}(\tau) &\leq m_a(Ct) + m_b(C\tau); \end{aligned}$$

(iii) $X \overset{qd}{\simeq} Y$.

COROLLARY 12. *Bessaga's conjecture is true for any Cartesian product of the kind $E_0(a) \times E_\infty(b)$.*

Proof. Let $F \subset E_0(a) \times E_\infty(b)$ be a complemented subspace with absolute basis $(f_m)$. By Proposition 4 the basis $(f_m)$ is weakly quasi-equivalent to a part of the canonical basis of $E_0(a) \times E_\infty(b)$. Therefore we have

$$F \overset{qd}{\simeq} E_0(a^*) \times E_\infty(b^*)$$

for some sequences $a^*, b^*$ (obtained by repeating some of the terms of $a, b$, respectively). Then the statement follows from Theorem 10.

## References


[1] C. Bessaga, *Some remarks on Dragilev's theorem*, Studia Math. 31 (1968), 307–318.

[2] C. Bessaga, A. Pełczyński and S. Rolewicz, *On diametral approximative dimension and linear homogeneity of F-spaces*, Bull. Acad. Polon. Sci. 9 (1961), 677–683.

[3] P. B. Djakov, *A short proof of the theorem on quasi-equivalence of regular bases*, Studia Math. 55 (1975), 269–271.

[4] P. B. Djakov, M. Yurdakul and V. P. Zahariuta, *On Cartesian products of Köthe spaces*, Bull. Polish Acad. Sci. 43 (1996), 113–117.

[5] —, —, —, *Isomorphic classification of Cartesian products of power series spaces*, Michigan Math. J. 43 (1996), 221–229.

[6] M. M. Dragilev, *On regular bases in nuclear spaces*, Mat. Sb. 68 (1965), 153–173 (in Russian).

[7] —, *Bases in Köthe Spaces*, Rostov-na-Donu, 1983 (in Russian).

[8] A. N. Kolmogorov, *On the linear dimension of topological vector spaces*, Dokl. Akad. Nauk SSSR 120 (1958), 239–241 (in Russian).

[9] V. P. Kondakov, *Problems of Geometry of Nonnormable Spaces*, Rostov State University, Rostov-na-Donu, 1983 (in Russian).

[10] —, *On structure of unconditional bases in some Köthe spaces*, Studia Math. 76 (1983), 137–151 (in Russian).

[11] V. P. Kondakov and V. P. Zahariuta, *On weak equivalence of bases in Köthe spaces*, Izv. Severo-Kavkaz. Nauchn. Tsentra Vyssh. Shkoly Ser. Estestv. Nauk 4 (1982), 110–115 (in Russian).

[12] B. S. Mityagin, *Approximative dimension and bases in nuclear spaces*, Uspekhi Mat. Nauk 16 (1961), no. 4, 63–132 (in Russian).

[13] —, *Sur l'équivalence des bases inconditionnelles dans les échelles de Hilbert*, C. R. Acad. Sci. Paris 269 (1969), 426–428.

[14] —, *Equivalence of bases in Hilbert scales*, Studia Math. 37 (1970), 111–137 (in Russian).

[15] —, *Non-Schwarzian power series spaces*, Math. Z. 182 (1983), 303–310.

[16] A. Pełczyński, *On the approximation of S-spaces by finite-dimensional spaces*, Bull. Acad. Polon. Sci. 5 (1957), 879–881.

[17] M. Yurdakul and V. P. Zahariuta, *Linear topological invariants and isomorphic classification of Cartesian products of locally convex spaces*, Turkish J. Math. 19 (1995), 37–47.

[18] V. P. Zahariuta, *On isomorphisms of Cartesian products of linear topological spaces*, Funktsional. Anal. i Prilozhen. 4 (1970), no. 2, 87–88 (in Russian).





[19] V. P. Zahariuta, *On the isomorphism of Cartesian products of locally convex spaces*, Studia Math. 46 (1973), 201–221.

[20] —, *Linear topological invariants and isomorphisms of spaces of analytic functions*, in: Matem. Analiz i ego Prilozhen. Rostov Univ., Vol. 2 (1970), 3–13, Vol. 3 (1971), 176–180 (in Russian).

[21] —, *Generalized Mityagin invariants and a continuum of pairwise nonisomorphic spaces of analytic functions*, Funktsional. Anal. i Prilozhen. 11 (1977), no. 3, 24–30 (in Russian).

[22] —, *Synthetic diameters and linear topological invariants*, in: School on the Theory of Operators in Function Spaces (abstracts of reports), Minsk, 1978, 51–52 (in Russian).

[23] —, *Linear Topological Invariants and Their Application to Generalized Power Spaces*, Rostov Univ., 1979 (in Russian).

[24] —, *Linear topological invariants and their application to isomorphic classification of generalized power spaces*, Turkish J. Math. 2 (1996), 237–289.



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