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Hochschild cohomology groups of certain algebras of analytic functions with coefficients in one-dimensional bimodules

by

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Abstract. We compute the algebraic and continuous Hochschild cohomology groups of certain Fréchet algebras of analytic functions on a domain U in $\mathbb{C}^n$ with coefficients in one-dimensional bimodules. Among the algebras considered, we focus on $A = A(U)$. For this algebra, our results apply if U is smoothly bounded and strictly pseudoconvex, or if U is a product domain.

1. Introduction. Let $U \subseteq \mathbb{C}^n$ be an open, bounded set, and let $A = A(U)$ be the Banach algebra of analytic functions on U which are continuously extendable to the boundary of U. For each Banach A-bimodule X, the second continuous (respectively, algebraic) Hochschild cohomology group $\mathcal{H}^2(A, X)$ (respectively, $H^2(A, X)$) of A with coefficients in X is defined (see [1] and [7]); there is a natural correspondence between the elements of this group and the equivalence classes of Banach (respectively, algebraic) extensions of A by X. If X is symmetric, then $\mathcal{H}^2(A, X)$ (respectively, $H^2(A, X)$) contains the subgroup $\mathcal{H}^2_s(A, X)$ (respectively, $H^2_s(A, X)$) corresponding to the commutative Banach (respectively, algebraic) extensions of A by X.

The purpose of this note is the computation of these groups for one-dimensional X. It is known ([1, Proposition 4.3]) that $\mathcal{H}^2(A, X)$ and $H^2(A, X)$ vanish unless X is unital and symmetric, and it thus suffices to consider the case where $X = \mathbb{C}$ and the module action is given by a character φ on A, so that $z \cdot f = f \cdot z = \varphi(f)z$ for $f \in A$ and $z \in \mathbb{C}$. We shall confine ourselves to the case where φ is the evaluation at a point $w \in \overline{U}$; we denote the corresponding module by $\mathbb{C}_w$.

Some of our results apply to certain Fréchet algebras A other than $A(U)$ for which there are continuous embeddings $P_n \hookrightarrow A \hookrightarrow \mathcal{O}(U)$, where P_n is the polynomial algebra in n complex variables and $\mathcal{O}(U)$ denotes the

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Fréchet algebra of functions analytic on U; such algebras will be called Fréchet algebras of analytic functions on U.

It is well known that the continuous (respectively, algebraic) Hochschild cohomology groups of A with coefficients in $\mathbb{C}_w$ may be calculated by using projective resolutions of $\mathbb{C}_w$ in the category of Banach or Fréchet (respectively, algebraic) left A-modules. One such resolution is given by the Koszul complex, which will play a major rôle in our calculations.

Hochschild cohomology for Fréchet algebras of analytic functions has been studied by several authors, notably J. L. Taylor [21] and A. Ya. Helemskiĭ [7, 8, 9]. We wish to mention two known results that are directly relevant to the purpose of this note.

In the case where U is a domain of holomorphy and $A = \mathcal{O}(U)$, the embedding $P_n \hookrightarrow A$ is a localization in the sense of [21, Definition 1.2], and the results in [21] yield a complete description of both the algebraic and the continuous Hochschild cohomology of A with coefficients in one-dimensional modules (see §3).

In the case where A is a uniform algebra and the maximal ideal M_φ of A corresponding to a character φ on A admits the decomposition $M_\varphi = I + J$, where I is a Koszul ideal in the sense of [8, p. 226], J is the kernel of a peak set and a certain condition on the interrelation between I and J is satisfied, a special case of a projective resolution of $\mathbb{C}_\varphi$ of the Koszul type exists ([8, Lemma 3.6]). This result is our main source of inspiration in §5, where we use it, in a slightly generalized version, to study the case where A is the algebra $A(U)$ and U is a product domain.

This paper is organized as follows. In §2 we clarify the notation we use and give an account of the notion of a Koszul complex and its basic properties. In §3 and §4 we consider the case where $w \in U$. The results in these two sections apply to a wider class of Fréchet algebras of analytic functions on U and to the groups $H^m(A, \mathbb{C}_w)$ and $\mathcal{H}^m(A, \mathbb{C}_w)$ for arbitrary m. In §3 we show that there are natural embeddings $\mathbb{C}^{\binom{n}{m}} \hookrightarrow H^m(A, \mathbb{C}_w)$ and $\mathbb{C}^{\binom{n}{m}} \hookrightarrow \mathcal{H}^m(A, \mathbb{C}_w)$, and give sufficient conditions for these embeddings to be surjective. In §4 we consider the "symmetric" groups $H^2_{\mathrm{s}}(A, \mathbb{C}_w)$ and $\mathcal{H}^2_{\mathrm{s}}(A, \mathbb{C}_w)$, and show that they vanish in certain cases. In §5 and §6 we discuss the case where $A = A(U)$ and $w \in \partial U$, the boundary of U. We give a partial result that is applicable in the case where the maximal ideal M_w corresponding to w has a decomposition $M_w = I + J$, with I and J ideals of A which satisfy certain conditions; these conditions are similar to, but less restrictive than, those in the aforementioned lemma in [8]. In particular, we obtain a sufficient condition for the vanishing of the "symmetric" groups $H^2_{\mathrm{s}}(A, \mathbb{C}_w)$ and $\mathcal{H}^2_{\mathrm{s}}(A, \mathbb{C}_w)$ which is not too restrictive in the case where U is a product domain. On the other hand, we demonstrate that there are examples where $\mathcal{H}^2_{\mathrm{s}}(A, \mathbb{C}_w)$ is non-trivial. Finally, in §7 we give a summary of the results we have obtained.

2. Preliminaries. Let $U \subseteq \mathbb{C}^N$ be an open set. We write $\mathcal{O}(U)$ for the Fréchet algebra of analytic functions on U. If U is bounded, then we use $A(U)$ to denote the Banach algebra of analytic functions on U which are continuously extendable to ∂U. We say that a subalgebra A of $\mathcal{O}(U)$ is a *Fréchet algebra of analytic functions* on U if A contains the polynomials and A is a Fréchet algebra for a topology which is finer than the compact-open topology on $\mathcal{O}(U)$.

We recall some notation and basic facts used in homology theory. For general background in homological algebra, we refer to [3], [23] and, for the continuous case, to [7] and [20].

Let $\mathbf{K}$ be a subcategory of the category of linear spaces and operators. A *(chain) complex* $\mathcal{F} = (\mathcal{F}, d)$ *in* $\mathbf{K}$ is a sequence of objects and morphisms

$$\mathcal{F}: \quad \ldots \to F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \to \ldots$$

in $\mathbf{K}$ such that $d_n \circ d_{n+1} = 0$ for all n. The elements of $\ker d_n$ (respectively, of $\operatorname{im} d_{n+1}$) are called *$n$-cycles* (respectively, *n-boundaries*). The *homology* of $\mathcal{F}$ at F_n is defined as $H_n(\mathcal{F}) = \ker d_n / \operatorname{im} d_{n+1}$. If $H_n(\mathcal{F}) = 0$ for all n, then $\mathcal{F}$ is called *exact*. The complex $\mathcal{F}$ is *positive* if $F_n = 0$ for all $n < 0$. If $(\mathcal{F}, d)$ and $(\mathcal{F}', d')$ are chain complexes in the category of linear (respectively, Fréchet) spaces, then a *morphism* (respectively, *continuous morphism*) *of chain complexes* of $\mathcal{F}$ into $\mathcal{F}'$ is an indexed set $\alpha = (\alpha_n)$ of linear (respectively, continuous linear) operators $\alpha_n : F_n \to F'_n$ such that $\alpha_{n-1} \circ d_n = d'_n \circ \alpha_n$ for all n.

Let M be an object in $\mathbf{K}$. A *complex over M (in* $\mathbf{K}$*)* is a positive chain complex $\mathcal{F} = (\mathcal{F}, d)$ in $\mathbf{K}$ together with a morphism $\varepsilon : F_0 \to M$ (called an *augmentation*) such that $\varepsilon \circ d_1 = 0$.

We use similar terminology in the case of cochain complexes.

Let A be a commutative, unital algebra. (All algebras considered are complex and associative.) A unital left module over A is termed an *A-module*. We use A-**mod** to denote the category of A-modules and A-module maps. For the A-modules M and N, we write $\operatorname{Hom}_A(M, N)$ for the vector space of all A-module maps from M into N, and we write $M \otimes_A N$ for the tensor product of M and N over A. Note that, since A is commutative, $\operatorname{Hom}_A(M, N)$ and $M \otimes_A N$ are A-modules for the operations

$$(a \cdot \varphi)(m) = a \cdot \varphi(m) \quad \text{and} \quad a \cdot (m \otimes n) = (a \cdot m) \otimes n,$$

where $a \in A$, $m \in M$, $n \in N$ and $\varphi \in \operatorname{Hom}_A(M, N)$. As is customary, we use $\operatorname{Ext}_A(-, N) :$ A-**mod** $\to$ A-**mod** to denote the left-derived cofunctor of the cofunctor $\operatorname{Hom}_A(-, N)$. An A-module P is *projective* (respectively, *flat*) if the functor $\mathrm{Hom}_A(P,-)$ (respectively, $-\otimes_A P$) is exact; that is, if it maps exact complexes into exact complexes.

Let X be an A-module. A complex $\mathcal{F}$ over X with augmentation $\varepsilon : F_0 \to X$ is called a *resolution of* X (*in* A-**mod**) if the complex

$$\mathcal{F} \xrightarrow{\varepsilon} X \to 0 : \qquad \ldots \to F_n \xrightarrow{d_n} F_{n-1} \to \ldots \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} X \to 0$$

is exact. Such a resolution is *projective* (respectively, *flat*) if the F_n are projective (respectively, flat).

Let M be an A-module. We write $M^{\oplus i}$ $(i = 1, 2, \ldots)$ for the direct sum of i copies of M. Furthermore we use ΛM to denote the exterior A-algebra over M, and we write $\wedge$ for the product in ΛM. Recall that $\Lambda M = \bigoplus_{r=0}^{\infty} \Lambda^r M$ is a graded A-algebra, where $\Lambda^r M$ is the rth exterior product of M over A.

Let $\mathbf{a} = (a_1, \ldots, a_n)$ be a finite sequence of elements in A. Then the *Koszul complex* $K(\mathbf{a})$ ([23, p. 111]) is defined; we sometimes write $K(a_1, \ldots, a_n)$ instead of $K(\mathbf{a})$. Recall that the degree p part $K_p(\mathbf{a})$ of $K(\mathbf{a})$ is the pth exterior product $\Lambda^p A^{\oplus n}$ of $A^{\oplus n}$ over A. Let $e_1, \ldots, e_n$ be the canonical basis of $A^{\oplus n}$. Then the set of all elements $e_{i_1} \wedge \ldots \wedge e_{i_p}$, where $1 \le i_1 < \ldots < i_p \le n$, is a basis of $K_p(\mathbf{a})$; in particular, $K_p(\mathbf{a})$ is a free A-module of rank $\binom{n}{p}$. The differential $d_{a,p}$ from $K_p(\mathbf{a})$ to $K_{p-1}(\mathbf{a})$ maps $e_{i_1} \wedge \ldots \wedge e_{i_p}$ to $\sum_{k=1}^{p} (-1)^{k+1} a_{i_k} e_{i_1} \wedge \ldots \wedge \widehat{e}_{i_k} \wedge \ldots \wedge e_{i_p}$; here $\widehat{e}_{i_k}$ signifies that e_{i_k} is omitted in the product. We write $H_*(K(\mathbf{a}))$ for the homology of $K(\mathbf{a})$. Note that, in all cases,

$$H_0(K(\mathbf{a})) = A/\langle a_1, \ldots, a_n\rangle, \tag{2.1}$$

where $\langle a_1, \ldots, a_n\rangle$ is the ideal generated by $a_1, \ldots, a_n$.

Recall that an element a of A is called a *non-zero divisor* if ab is non-zero for each non-zero element b of A. A finite sequence $a_1, \ldots, a_n$ of elements in A is called a *regular sequence on* A ([23, p. 105]) if the equivalence class $\overline{a}_j$ in $A/\langle a_1, \ldots, a_{j-1}\rangle$ is a non-zero divisor for each j. (For $j = 1$ this means that a_1 is a non-zero divisor.) We shall use the following elementary theorem of homological algebra (see [15, 23]).

THEOREM 2.1. *Suppose that* $\mathbf{a} = (a_1, \ldots, a_n)$ *is a regular sequence on a commutative, unital algebra* A. *Then the Koszul complex* $K(\mathbf{a})$ *provides a free resolution of length* n *of* $A/\langle a_1, \ldots, a_n\rangle$.

Let X be a unital A-bimodule. Then the *Hochschild complex* (see [23, p. 301])

$$0 \to X \xrightarrow{\delta^0} \mathrm{Hom}_{\mathbb{C}}(A, X) \xrightarrow{\delta^1} \mathrm{Hom}_{\mathbb{C}}(A \otimes_{\mathbb{C}} A, X) \xrightarrow{\delta^2} \ldots$$

of A and X is defined. Here $\delta = (\delta^i)_{i\in\mathbb{N}}$ is the *coboundary operator*. The cohomology groups of this complex are the *algebraic Hochschild cohomology groups* $H^n(A, X)$ $(n = 0, 1, \ldots)$ of A with coefficients in X. Following customary notation, we write $Z^n(A, X)$ for $\ker \delta^n$ and $B^n(A, X)$ for the range of δ^{n-1}, respectively. Therefore $H^n(A, X) = Z^n(A, X)/B^n(A, X)$, where $Z^n(A, X)$ (respectively, $B^n(A, X)$) is the space of n-*cocycles* (respectively, of n-*coboundaries*) of A with coefficients in X.

Let $\varphi : A \to \mathbb{C}$ be a character on A. Then $\mathbb{C} = \mathbb{C}_\varphi$ is a A-bimodule for the operations $a \cdot z = z \cdot a = \varphi(a)z$ $(a \in A,\ z \in \mathbb{C})$. We call $\mathbb{C}_\varphi$ the A-bimodule *corresponding to* φ. It is known (see [23, Lemma 9.1.9]) that

$$H^n(A, \mathbb{C}_\varphi) = \mathrm{Ext}_A^n(\mathbb{C}_\varphi, \mathbb{C}_\varphi) \quad \text{for } n \ge 0 \tag{2.2}$$

in A-**mod**; here $\mathbb{C}_\varphi$ is regarded as a left A-module on the right-hand side of the equation.

By a *Fréchet space* we mean a complete metrizable locally convex space; a *Fréchet algebra* is a complex algebra which is a Fréchet space such that multiplication is jointly continuous. Let E and F be Fréchet spaces. Then we use $E \otimes_p F$ to denote the algebraic tensor product of E and F endowed with the projective tensor product topology. The completion $E \mathbin{\widehat{\otimes}_p} F$ of $E \otimes_p F$ is the *projective tensor product* of E and F and it has the usual universal property of the tensor product (cf. [7, II.4]). A continuous linear map $\varphi : E \to F$ is called *admissible* if $\ker \varphi$ is complemented in E and $\operatorname{im} \varphi$ is closed and complemented in F.

Let A be a commutative, unital Fréchet algebra. A unital left Fréchet module over A is termed a *Fréchet A-module*. We use A-**Fr**-**mod** to denote the category of Fréchet A-modules and continuous A-module maps.

Let M, N be Fréchet A-modules. We write $\mathrm{Hom}_{A,\,\mathrm{cont}}(M, N)$ for the A-module of continuous A-module maps from M into N. We also write $\mathrm{Ext}_{A,\,\mathrm{cont}}(-, N)$ for the left-derived cofunctor of

$$\mathrm{Hom}_{A,\,\mathrm{cont}}(-, N) : A\text{-}\mathbf{Fr}\text{-}\mathbf{mod} \to A\text{-}\mathbf{mod}\,.$$

A (*chain*) *complex* $\mathcal{F}$ in A-**Fr**-**mod** is called *admissible* if it splits as a complex of Fréchet spaces. A Fréchet A-module P is *projective* ([7, III.1.13]) if, for every admissible complex $\mathcal{F}$, the complex $\mathrm{Hom}_{A,\,\mathrm{cont}}(P, \mathcal{F})$ is exact.

Let X be a Fréchet A-module. A complex $\mathcal{F}$ over X with augmentation $\varepsilon : F_0 \to X$ is called a *resolution* (*in* A-**Fr**-**mod**) if the complex $\mathcal{F} \xrightarrow{\varepsilon} X \to 0$ over X is admissible. If every module in $\mathcal{F}$ is projective, then such a resolution is called a *projective resolution of* X (*in* A-**Fr**-**mod**).

Let X be a unital Fréchet A-bimodule. Then the *Hochschild–Kamowitz complex* (see [7, I.3.2])

$$0 \to X \xrightarrow{\delta^0} \mathrm{Hom}_{\mathbb{C},\,\mathrm{cont}}(A, X) \xrightarrow{\delta^1} \mathrm{Hom}_{\mathbb{C},\,\mathrm{cont}}(A \mathbin{\widehat{\otimes}_p} A, X) \xrightarrow{\delta^2} \ldots$$

of A and X is defined. The cohomology groups of this complex are the *continuous Hochschild cohomology groups* $\mathcal{H}^n(A, X)$ $(n = 0, 1, \ldots)$ of A

with coefficients in X. We write $\mathcal{Z}^n(A,X)$ for $\ker\delta^n$ and $\mathcal{B}^n(A,X)$ for the range of δ^{n-1}, respectively. Hence $\mathcal{H}^n(A,X)=\mathcal{Z}^n(A,X)/\mathcal{B}^n(A,X)$, where $\mathcal{Z}^n(A,X)$ (respectively, $\mathcal{B}^n(A,X)$) is the space of *continuous n-cocycles* (respectively, of *continuous n-coboundaries*) of A with coefficients in X.

There is a natural embedding of $\mathcal{Z}^n(A,X)$ in $Z^n(A,X)$, and this map induces an obvious *comparison map* $\iota_n:\mathcal{H}^n(A,X)\to H^n(A,X)$. Note that, since A is commutative, $\mathcal{H}^n(A,X)$ and $H^n(A,X)$ are A-modules, and ι_n is an A-module map.

Assume that X is *symmetric*, i.e. $a\cdot x=x\cdot a$ for $a\in A$ and $x\in X$. Then the symmetric 2-cocycles (respectively, the continuous symmetric 2-cocycles) form a subspace of $Z^2(A,X)$ (respectively, of $\mathcal{Z}^2(A,X)$) which is denoted by $Z^2_s(A,X)$ (respectively, by $\mathcal{Z}^2_s(A,X)$). The quotient

$$H^2_s(A,X)=Z^2_s(A,X)/B^2(A,X)$$

(respectively, $\mathcal{H}^2_s(A,X)=\mathcal{Z}^2_s(A,X)/\mathcal{B}^2(A,X)$) is the *second symmetric* (respectively, the *second continuous symmetric*) *Hochschild cohomology group of A with coefficients in X* (see [1, 23]).

Let $\varphi:A\to\mathbb{C}$ be a continuous character on A. Then $\mathbb{C}=\mathbb{C}_\varphi$ is a Fréchet A-bimodule, and in analogy to (2.2) we see (cf. [7, III.4.12]) that

$$\mathcal{H}^n(A,\mathbb{C}_\varphi)=\operatorname{Ext}^n_{A,\,\mathrm{cont}}(\mathbb{C}_\varphi,\mathbb{C}_\varphi)\quad\text{for } n\ge 0 \tag{2.3}$$

in the category A-**mod**.

Let A be a commutative, unital Banach algebra. Denote by A-**Ba-mod** the subcategory of A-**Fr-mod** consisting of unital left Banach A-modules. A module P in A-**Ba-mod** is *flat* ([7, VII.1.1]) if, for every admissible complex $\mathcal{F}$, the complex $\mathcal{F}\mathbin{\widehat{\otimes}}_A P$ is exact; here $\widehat{\otimes}_A$ denotes the tensor product of Banach A-modules (cf. [7, II.3]). A resolution $\mathcal{F}\to X\to 0$ over a module X in A-**Ba-mod** is *flat* if every module in $\mathcal{F}$ is flat in A-**Ba-mod**.

According to (2.2) (respectively, (2.3)), we may compute $H^n(A,\mathbb{C}_\varphi)$ (respectively, $\mathcal{H}^n(A,\mathbb{C}_\varphi)$) by using projective resolutions of $\mathbb{C}_\varphi$ in the category A-**mod** (respectively, A-**Fr-mod**). However, less is needed, as the next elementary lemma shows. For a proof of part (ii) of this lemma, we refer to [10]. Although we do not have an exact reference, the result in part (i) is surely well known to the specialist.

LEMMA 2.2. *Let A be a commutative, unital Banach algebra, and let φ be a character on A. Then:*

(i) *The A-modules $\operatorname{Ext}^*_A(\mathbb{C}_\varphi,\mathbb{C}_\varphi)$ may be computed by using flat resolutions. That is, if $\mathcal{F}\to\mathbb{C}_\varphi$ is a flat resolution of $\mathbb{C}_\varphi$ in A-**mod**, then*

$$\operatorname{Ext}^n_A(\mathbb{C}_\varphi,\mathbb{C}_\varphi)\cong H^n(\operatorname{Hom}_A(\mathcal{F},\mathbb{C}_\varphi))\quad(n=0,1,\ldots);$$

(ii) *The A-modules $\operatorname{Ext}^*_{A,\,\mathrm{cont}}(\mathbb{C}_\varphi,\mathbb{C}_\varphi)$ may be computed by using flat resolutions of $\mathbb{C}_\varphi$ in the category A-**Ba-mod**.*

Proof of (i). For each A-module M, the algebraic dual $\operatorname{Hom}_{\mathbb{C}}(M,\mathbb{C})$ is an A-module for the operation

$$(a\cdot\varphi)(m)=\varphi(a\cdot m)\quad(a\in A,\ \varphi\in\operatorname{Hom}_{\mathbb{C}}(M,\mathbb{C}),\ m\in M).$$

It is a basic result of homological algebra (see [3, VI.5.1], for example) that, in the category A-**mod**, M is flat if and only if $\operatorname{Hom}_{\mathbb{C}}(M,\mathbb{C})$ is injective.

Now let $\mathcal{F}\to\mathbb{C}_\varphi$ be a flat resolution of $\mathbb{C}_\varphi$ in A-**mod**, and let $n\ge 0$ be fixed. Since the functor $\operatorname{Hom}_{\mathbb{C}}(-,\mathbb{C})$ is exact, $\operatorname{Hom}_{\mathbb{C}}(\mathcal{F},\mathbb{C})$ is a resolution of $\mathbb{C}_\varphi$. This resolution is injective on the basis of the results attained in the previous paragraph. Thus $\operatorname{Ext}^n_A(\mathbb{C}_\varphi,\mathbb{C}_\varphi)$ is the nth cohomology of the complex $\mathcal{G}=\operatorname{Hom}_A(\mathbb{C}_\varphi,\operatorname{Hom}_{\mathbb{C}}(\mathcal{F},\mathbb{C}))$. However, it is obvious that the functors $\operatorname{Hom}_A(-,\mathbb{C}_\varphi)$ and $\operatorname{Hom}_A(\mathbb{C}_\varphi,\operatorname{Hom}_{\mathbb{C}}(-,\mathbb{C}))$ are isomorphic. We conclude that

$$\operatorname{Ext}^n_A(\mathbb{C}_\varphi,\mathbb{C}_\varphi)=H^n(\mathcal{G})\cong H^n(\operatorname{Hom}_A(\mathcal{F},\mathbb{C}_\varphi)),$$

as required. ∎

Let φ be a continuous character on a commutative, unital Fréchet algebra A. We shall be interested in the question of when the comparison map

$$\iota_n:\mathcal{H}^n(A,\mathbb{C}_\varphi)\to H^n(A,\mathbb{C}_\varphi)$$

is injective. In the case where $n=1$, this is always true; in the case where $n=2$, we have the following proposition, which is a straightforward generalization of [7, I.1.19] (see also [1, Theorem 2.16]).

PROPOSITION 2.3. *Let A be a unital, commutative Fréchet algebra, and let φ be a continuous character on A. Set $M_\varphi=\ker\varphi$. Then the comparison map*

$$\iota_2:\mathcal{H}^2(A,\mathbb{C}_\varphi)\to H^2(A,\mathbb{C}_\varphi)$$

is injective if and only if the product map

$$\Lambda:M_\varphi\otimes_p M_\varphi\to M^2_\varphi: f\otimes g\mapsto fg$$

is open.

Proof. First assume that Λ is open. Let ψ be a linear functional on A such that $\mu=\delta^1\psi$ is continuous. Let ψ_0 be the restriction of ψ to M^2_φ, and let μ_0 be the restriction of μ to $M_\varphi\otimes_p M_\varphi$. Then $\mu_0=-\psi_0\circ\Lambda$. But Λ is open and μ_0 is continuous, and so ψ_0 is continuous. Let ψ_1 be a continuous extension of ψ_0 to A such that $\psi_1(1_A)=\mu(1_A\otimes 1_A)$. Then $\mu=\delta^1\psi_1\in\mathcal{B}^2(A,\mathbb{C}_\varphi)$. This shows that $\mathcal{Z}^2(A,\mathbb{C}_\varphi)\cap B^2(A,\mathbb{C}_\varphi)=\mathcal{B}^2(A,\mathbb{C}_\varphi)$, and hence ι_2 is injective.

Now suppose that Λ is not open. Set $E=M^2_\varphi$. Since A is a Fréchet algebra, E is a metrizable locally convex space. Hence (see [22, Proposition II.36.3]) the topology on E is identical to the Mackey topology. Let σ denote the quotient topology on E induced by Λ. Then σ is strictly finer than the original topology on E because Λ is not open. However, the Mackey topology is the finest locally convex vector space topology which is compatible with the duality between E and E', and so there exists a discontinuous linear functional ψ on E which is continuous with respect to σ. We may extend ψ to a linear functional on A which we also denote by ψ. Set $\mu = \delta^1\psi$. Then $\mu \in \mathcal{Z}^2(A, \mathbb{C}_\varphi)$ because $\psi \circ \Lambda$ is continuous. Set $a = \mu + \mathcal{B}^2(A, \mathbb{C}_\varphi)$. Then $\iota_2(a) = 0$, but $a \neq 0$ because $\psi|E$ is discontinuous. Thus ι_2 is not injective. ∎

COROLLARY 2.4. *In the situation of Proposition* 2.3, *suppose that M_φ is algebraically finitely generated. Then the comparison map $\iota_2 : \mathcal{H}^2(A, \mathbb{C}_\varphi) \to H^2(A, \mathbb{C}_\varphi)$ is injective.*

Proof. Let Λ be the map considered in Proposition 2.3. Set $F = M_\varphi \otimes_p M_\varphi / \ker \Lambda$, and let $\overline{\Lambda} : F \to M_\varphi^2$ be the induced map. Let $\{b_1, \ldots, b_m\}$ be a finite set of generators for M_φ. We define maps $\varrho : M_\varphi^{\oplus m} \to M_\varphi^2$ by $\varrho(a_1, \ldots, a_m) = \sum_{i=1}^m a_i b_i$ and $\tau : M_\varphi^{\oplus m} \to F$ by $\tau(a_1, \ldots, a_m) = \sum_{i=1}^m a_i \otimes b_i + \ker \Lambda$. Then ϱ and τ are continuous surjections, and $\overline{\Lambda} \circ \tau = \varrho$. Clearly, M_φ^2 has finite codimension in A; in addition, it is the continuous image of a Fréchet space. Hence an obvious application of the Open Mapping Theorem gives that M_φ^2 is closed. Therefore, again by the Open Mapping Theorem, ϱ is open. It follows that $\overline{\Lambda}$, and hence also Λ, is open. By Proposition 2.3, ι_2 is injective. ∎

We end this section with the following theorem which we shall need in §3. The main argument in the proof is well known in the literature (see [3, Theorem VIII.4.2], for example). Nonetheless, we provide a short proof here for the sake of completeness.

THEOREM 2.5. *Let φ be a continuous character on a unital, commutative Fréchet algebra A. Let $\mathbf{a} = (a_1, \ldots, a_n)$ be a finite sequence of elements in A, and let $\mathcal{K} = K(\mathbf{a})$ be the corresponding Koszul complex. Suppose that*

$$\langle a_1, \ldots, a_n \rangle = \ker \varphi, \tag{2.4}$$

$$H_j(\mathcal{K}) = \{0\} \quad \textit{for } j > 0. \tag{2.5}$$

Then

$$\dim_{\mathbb{C}}(H^m(A, \mathbb{C}_\varphi)) = \binom{n}{m} \quad (m \geq 0).$$

*Moreover, $\mathcal{K}$ is a complex in the category A-**Fr-mod**; if each differential of $\mathcal{K}$ is admissible, then also*

$$\dim_{\mathbb{C}}(\mathcal{H}^m(A, \mathbb{C}_\varphi)) = \binom{n}{m} \quad (m \geq 0).$$

Proof. Let $m \geq 0$, and let $\mathcal{F}$ denote the complex $\mathrm{Hom}_A(\mathcal{K}, \mathbb{C}_\varphi)$. We see from (2.1), (2.4) and (2.5) that $\mathcal{K}$ is a projective resolution of $\mathbb{C}_\varphi$ in A-**mod**.

Thus, by (2.2), the A-module $H^m(A, \mathbb{C}_\varphi)$ is the mth cohomology $H^m(\mathcal{F})$ of $\mathcal{F}$. However, it follows from (2.4) that each morphism in $\mathcal{F}$ is zero. Hence $H^m(\mathcal{F})$ is the degree m part F_m of $\mathcal{F}$. Since, in addition, $K_m(\mathbf{a})$ is a free A-module of rank $\binom{n}{m}$, we conclude that

$$H^m(A, \mathbb{C}_\varphi) = F_m = \mathrm{Hom}_A(K_m(\mathbf{a}), \mathbb{C}_\varphi) \cong \mathbb{C}^{\oplus\binom{n}{m}}.$$

In particular, we see that $\dim_{\mathbb{C}}(H^m(A, \mathbb{C}_\varphi)) = \binom{n}{m}$, and the first part of the theorem is proved.

Each module in $\mathcal{K}$ is a finite direct sum of copies of A and therefore a projective (even free) Fréchet A-module. Furthermore, it is obvious that each differential of $\mathcal{K}$ is continuous. Thus $\mathcal{K}$ is a complex in A-**Fr-mod** consisting of projective modules. If each differential of $\mathcal{K}$ is admissible, then $\mathcal{K}$ is a projective resolution of $\mathbb{C}_\varphi$ in A-**Fr-mod**, and we may finish the proof by using the same arguments as in the algebraic case; indeed, all we have to do is to substitute (2.3) for (2.2) and $\mathrm{Hom}_{A,\mathrm{cont}}(-,-)$ for $\mathrm{Hom}_A(-,-)$. ∎

3. The calculation of $H^n(A, \mathbb{C}_w)$ and $\mathcal{H}^n(A, \mathbb{C}_w)$ in the case where $w \in U$. Let A be a Fréchet algebra of analytic functions on an open set $U \subseteq \mathbb{C}^N$, and let $w = (w_1, \ldots, w_N) \in U$ be fixed. We wish to determine the A-modules $H^n(A, \mathbb{C}_w)$ and $\mathcal{H}^n(A, \mathbb{C}_w)$, where $\mathbb{C}_w$ is the A-bimodule corresponding to the evaluation at w; we are especially interested in the case where $n = 2$.

Suppose that $N = 2$. Then it is trivial to verify that the map

$$f \otimes g \mapsto \frac{\partial f}{\partial z_1}(w) \frac{\partial g}{\partial z_2}(w)$$

is a continuous cocycle which is not a coboundary, not even in the algebraic sense. Our next proposition generalizes this observation. Namely, we show that, for each $n \leq N$, we have a commuting diagram

$$\begin{tikzcd}
 & \mathcal{H}^n(A, \mathbb{C}_w) \arrow[dd, "\iota_n"] \\
\mathbb{C}_w^{\oplus\binom{N}{n}} = \underbrace{\mathbb{C}_w \oplus \ldots \oplus \mathbb{C}_w}_{\binom{N}{n}\ \text{times}} \arrow[ur, "\alpha_n"] \arrow[dr, "\beta_n"'] & \\
 & H^n(A, \mathbb{C}_w)
\end{tikzcd}$$

where α_n and β_n are A-linear embeddings which are given by explicit formulas, and where ι_n is the comparison map. In fact, we prove a slightly more general result.

PROPOSITION 3.1. *Let A be a Fréchet algebra of analytic functions on an open set $U \subseteq \mathbb{C}^N$. Suppose that φ is a continuous character on A which* *has an extension to a character on $\mathcal{O}(U)$. Let $n \in \{1,\dots,N\}$, and let $\{e_\tau\}$ be a vector space basis of $\mathbb{C}_\varphi^{\oplus\binom{N}{n}}$, where τ runs through the set of all strictly increasing maps from $\{1,\dots,n\}$ to $\{1,\dots,N\}$. Let*

$$\alpha_n : \mathbb{C}_\varphi^{\oplus\binom{N}{n}} \to \mathcal{H}^n(A,\mathbb{C}_\varphi)$$

be the unique linear map which assigns to each e_τ the equivalence class of the continuous n-cocycle given by

$$f_1 \otimes \dots \otimes f_n \mapsto \varphi\Big(\frac{\partial f_1}{\partial z_{\tau(1)}} \cdots \frac{\partial f_n}{\partial z_{\tau(n)}}\Big). \tag{3.1}$$

Furthermore, let $\beta_n = \iota_n \circ \alpha_n$, where ι_n is the comparison map. Then α_n and β_n are A-linear embeddings.

Proof. It is easy to demonstrate that α_n is a well-defined A-linear map. What remains to be shown is that $\beta_n = \iota_n \circ \alpha_n$ is injective. To show this, assume that $\tau_1,\dots,\tau_r$ is a finite sequence of distinct strictly increasing maps from $\{1,\dots,n\}$ to $\{1,\dots,N\}$ such that

$$\beta_n\Big(\sum_{i=1}^r \gamma_i e_{\tau_i}\Big) = \sum_{i=1}^r \gamma_i \beta_n(e_{\tau_i}) = 0 \tag{3.2}$$

for some $\gamma_1,\dots,\gamma_r \in \mathbb{C}$; we have to show that $\gamma_1 = \dots = \gamma_r = 0$. We infer from (3.2) that there exists a linear functional Δ on $A \otimes \dots \otimes A$ ($n-1$ times) such that

$$\sum_{i=1}^r \gamma_i \Theta(e_{\tau_i}) = \delta^{n-1}\Delta, \tag{3.3}$$

where, for each $i \in \{1,\dots,r\}$, the n-cocycle $\Theta(e_{\tau_i})$ is defined as in (3.1) (with τ replaced by τ_i). Let $j \in \{1,\dots,r\}$ be fixed. For each $i \in \{1,\dots,n\}$, we set

$$W_i = Z_{\tau_j(i)} - \varphi(Z_{\tau_j(i)}),$$

where $Z_1,\dots,Z_N$ are the coordinate functions. Certainly, we have $W_i \in A$ and $\varphi(W_i) = 0$ for each i. Let $\mathfrak{S}_n$ be the symmetric group on n symbols. Let $i \in \{1,\dots,r\}$ and let $\sigma \in \mathfrak{S}_n$. We then have

$$\Theta(e_{\tau_i})(W_{\sigma(1)} \otimes \dots \otimes W_{\sigma(n)}) = \begin{cases} 1 & \text{if } i = j \text{ and } \sigma = \mathrm{id}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

We conclude from (3.3) and (3.4) that

$$\delta^{n-1}\Delta(W_1 \otimes \dots \otimes W_n) = \gamma_j \quad \text{and} \quad \delta^{n-1}\Delta(W_{\sigma(1)} \otimes \dots \otimes W_{\sigma(n)}) = 0$$

for every permutation $\sigma \neq \mathrm{id}$. In addition, the functions W_i lie in the kernel of φ. Hence

$$\gamma_j = \sum_{k=1}^{n-1} (-1)^k \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\Delta(W_{\sigma(1)} \otimes \dots \otimes W_{\sigma(k)}W_{\sigma(k+1)} \otimes \dots \otimes W_{\sigma(n)}).$$

However, it is obvious that

$$\sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\Delta(W_{\sigma(1)} \otimes \dots \otimes W_{\sigma(k)}W_{\sigma(k+1)} \otimes \dots \otimes W_{\sigma(n)}) = 0$$

for all $k = 1,\dots,n-1$, and consequently $\gamma_j = 0$. This shows that β_n is injective, as required. ∎

In the remainder of this section we consider, for each $n \le N$, the following two obvious questions concerning the embeddings α_n and β_n in Proposition 3.1.

QUESTION 1. Is $\beta_n : \mathbb{C}_w^{\oplus\binom{N}{n}} \hookrightarrow H^n(A,\mathbb{C}_w)$ an isomorphism?

QUESTION 2. Is $\alpha_n : \mathbb{C}_w^{\oplus\binom{N}{n}} \hookrightarrow \mathcal{H}^n(A,\mathbb{C}_w)$ an isomorphism?

A simple but useful fact pertinent to these questions is stated in the next proposition.

PROPOSITION 3.2. *Suppose that Question 1 can be answered affirmatively, and that the comparison map $\iota_n : \mathcal{H}^n(A,\mathbb{C}_w) \to H^n(A,\mathbb{C}_w)$ is injective. Then we have an affirmative answer to Question 2.*

Proof. This is immediately evident from the fact that $\iota_n \circ \alpha_n = \beta_n$. ∎

NOTATION. In the following, we use $Z_1,\dots,Z_N$ to denote the coordinate functions, and we write $\mathbf{Z} - \mathbf{w}$ for the sequence $(Z_1 - w_1,\dots,Z_N - w_N)$; recall that $K(\mathbf{Z} - \mathbf{w})$ is the Koszul complex for $\mathbf{Z} - \mathbf{w}$.

Our next proposition constitutes the basis for our attempts to answer Questions 1 and 2.

PROPOSITION 3.3. *Let A be a Fréchet algebra of analytic functions on an open set $U \subseteq \mathbb{C}^N$, and let ε_w be the evaluation at a point $w \in U$. Let*

$$\alpha_n : \mathbb{C}_w^{\oplus\binom{N}{n}} \hookrightarrow \mathcal{H}^n(A,\mathbb{C}_w) \quad \text{and} \quad \beta_n : \mathbb{C}_w^{\oplus\binom{N}{n}} \hookrightarrow H^n(A,\mathbb{C}_w)$$

be the A-linear embeddings constructed in Proposition 3.1. Suppose that

$$\langle Z_1 - w_1,\dots,Z_N - w_N\rangle = \ker \varepsilon_w, \tag{3.5}$$

$$H_j(K(\mathbf{Z} - \mathbf{w})) = \{0\} \quad \text{for } j > 0. \tag{3.6}$$

Then β_n is an isomorphism for all n, and α_n is an isomorphism for $n \le 2$.

Suppose, furthermore, that each differential of $K(\mathbf{Z} - \mathbf{w})$ is admissible. Then α_n is an isomorphism for all n.

Proof. This is immediately evident from Theorem 2.5, Propositions 3.1 and 3.2 and Corollary 2.4. ■

We now give two results which show that (3.5) and (3.6) are satisfied in certain cases. The first is an application of Theorem 2.1 in §2.

PROPOSITION 3.4. *Let A be an algebra of analytic functions on an open set $U \subseteq \mathbb{C}^N$, and let $w \in U$. Let $M_k = \{(z_i)_{i=1}^N \in \mathbb{C}^N : z_i = w_i \text{ for } 1 \le i \le k\}$, $1 \le k \le N$. Suppose that, for every $k \in \{1,\dots,N\}$ and every $f \in A$ with $f(U \cap M_k) = \{0\}$, there exist $g_1,\dots,g_k \in A$ with $f = \sum_{i=1}^k (Z_i - w_i)g_i$. Then (3.5) and (3.6) are satisfied.*

Proof. Our hypothesis (for $k = N$) implies that (3.5) is satisfied. By Theorem 2.1, (3.6) will follow once we have shown that the sequence $Z_1 - w_1, \dots, Z_N - w_N$ is a regular sequence on A. To show this, first note that $Z_1 - w_1$ is a non-zero divisor in A. Now let $i \in \{2,\dots,N\}$, and set $J = \langle Z_1 - w_1, \dots, Z_{i-1} - w_{i-1}\rangle$. Suppose that $(Z_i - w_i)f = 0$ modulo J for some $f \in A$. Then there exist $g_1,\dots,g_{i-1} \in A$ such that $(Z_i - w_i)f = \sum_{j=1}^{i-1}(Z_j - w_j)g_j$. This implies that $f(U \cap M_{i-1}) = \{0\}$. Thus, according to our hypothesis, there exist $h_1,\dots,h_{i-1} \in A$ such that $f = \sum_{j=1}^{i-1}(Z_j - w_j)h_j$. Hence $f = 0$ modulo J. This shows that $Z_i - w_i$ is a non-zero divisor modulo J, as required. ■

The second tool we shall use to verify (3.5) and (3.6) is the following deep theorem, which is a special case of a more general result stated in the book by J. Eschmeier and M. Putinar ([5]).

NOTATION. Recall that $\mathrm{Lip}_\alpha(\overline{U})$ is the Banach algebra of functions on $\overline{U}$ which satisfy the usual Lipschitz condition of order α. Furthermore, $C^r(\overline{U})$ denotes the algebra of functions f on U for which the derivatives $D^\alpha(f)$ exist and have a continuous extension to $\overline{U}$ for each multiindex α of order not exceeding r. The algebra $C^r(\overline{U})$ is endowed with the topology of uniform convergence of all derivatives of order not exceeding r. Note that $C^0(\overline{U}) = C(\overline{U})$ and thus $C^0(\overline{U}) \cap \mathcal{O}(U) = A(U)$.

THEOREM 3.5 ([5, Theorem 8.1.1]). *Suppose that $U \subseteq \mathbb{C}^N$ is a bounded, open set. Let B be one of the Fréchet algebras $\mathrm{Lip}_\alpha(\overline{U})$ (for some $0 < \alpha < 1$) or $C^r(\overline{U})$ (for some $0 \le r \le \infty$). Suppose further that, for each $q \in \{1,\dots,N\}$ and each closed $(0,q)$-form f on U with coefficients in B, there exists a $(0,q-1)$-form u on U with coefficients in B such that $\bar\partial u = f$. Then (3.5) and (3.6) are satisfied for $A = \mathcal{O}(U) \cap B$.*

Before we apply these last two results, we give an elementary lemma which we shall use in the case where U is a product domain.

LEMMA 3.6. *Let $G \subseteq \mathbb{C}$, $H \subseteq \mathbb{C}^k$ be bounded, open sets containing the origin, and let $f \in A(G \times H)$ be such that $f(0,w) = 0$ for all $w \in \overline{H}$. Then there exists $g \in A(G \times H)$ such that $f(z,w) = zg(z,w)$ for all $z \in \overline{G}$ and $w \in \overline{H}$.*

Proof. First note that, for every $w \in \overline{H}$, the function $z \mapsto f(z,w)$ is analytic on G; indeed, this follows readily from the uniform continuity of f and the fact that w is the limit of a sequence of elements in H. We consider the map

$$g(z,w) = \begin{cases} z^{-1}f(z,w) & \text{if } z \in \overline{G}\setminus\{0\} \text{ and } w \in \overline{H}, \\ \frac{\partial}{\partial z}f(0,w) & \text{if } z = 0 \text{ and } w \in \overline{H}. \end{cases}$$

Then $f(z,w) = zg(z,w)$ for $z \in \overline{G}$ and $w \in \overline{H}$, and g is analytic on $G \times H$. We need to show that g is continuous on $\overline{G} \times \overline{H}$. Certainly, g is continuous on $\overline{G}\setminus\{0\} \times \overline{H}$ and on $\{0\} \times \overline{H}$. Now let $w \in \overline{H}$, and suppose that (z_n, w_n) is a sequence in $\overline{G}\setminus\{0\} \times \overline{H}$ which converges to $(0,w)$. Choose $r > 0$ such that $\Delta(0,r) = \{z \in \mathbb{C} : |z| < r\} \subseteq G$. We may assume that $z_n \in \Delta(0, r/2)$ for all n. Then

$$g(z_n, w_n) = \frac{f(z_n,w_n)}{z_n} = \sum_{k=0}^\infty \frac{1}{(k+1)!}\frac{\partial^{k+1} f}{\partial z^{k+1}}(0,w_n)z_n^k$$

for all n. It follows from Cauchy's estimate that

$$\left|\frac{1}{(k+1)!}\frac{\partial^{k+1} f}{\partial z^{k+1}}(0,w_n)z_n^k\right| \le \frac{1}{r2^k}\|f\|_\infty$$

for all n and k. We may therefore apply Lebesgue's Dominated Convergence Theorem to deduce that

$$\lim_{n\to\infty} g(z_n,w_n) = \sum_{k=0}^\infty \lim_{n\to\infty}\frac{1}{(k+1)!}\frac{\partial^{k+1} f}{\partial z^{k+1}}(0,w_n)z_n^k = \frac{\partial f}{\partial z}(0,w) = g(0,w),$$

and so g is continuous at $(0,w)$. ■

We can now state the main result of this section, where we apply the above theorems and propositions to certain classes of Fréchet algebras of analytic functions.

THEOREM 3.7. *Let A be a Fréchet algebra of analytic functions on an open set $U \subseteq \mathbb{C}^N$. Let $w \in U$, and let*

$$\alpha_n : \mathbb{C}_w^{\oplus\binom{N}{n}} \hookrightarrow \mathcal{H}^n(A, \mathbb{C}_w) \quad \text{and} \quad \beta_n : \mathbb{C}_w^{\oplus\binom{N}{n}} \hookrightarrow H^n(A, \mathbb{C}_w)$$

be the A-linear embeddings constructed in Proposition 3.1. Consider the following cases:

(i) *U is pseudoconvex, and $A = \mathcal{O}(U)$.*
(ii) *U is a bounded product domain, and $A = A(U)$.*

(iii) *U is bounded and strictly pseudoconvex with C^2-boundary, and $A = \mathcal{O}(U) \cap B$, where B is one of the Banach algebras $\mathrm{Lip}_\alpha(\overline{U})$ (for some $0 < \alpha < 1$) or $C^r(\overline{U})$ (for some $0 \le r \le \infty$).*

In the case (i) *both α_n and β_n are isomorphisms for all n. In the cases* (ii) *and* (iii), *β_n is an isomorphism for all n, and α_n is an isomorphism if $n \le 2$.*

Proof. Suppose that (i) holds. Then [21, Proposition 4.3] asserts that the embedding $P_N \hookrightarrow A$, where P_N denotes the polynomial algebra in N complex variables, is a localization in the sense of [21, Definition 1.2]. Thus (by [21, Proposition 1.7]) $\mathcal{H}^n(A, \mathbb{C}_w) \cong \mathcal{H}^n(P_N, \mathbb{C}_w)$. However, it follows from [21, Proposition 4.5] that the vector space dimension of this latter space is equal to $\binom{N}{n}$. We conclude that α_n is an isomorphism for all n. The assertion about β_n follows from [18, Theorem 4.1] and Proposition 3.4.

Suppose that (ii) is true, so that $U = G_1 \times \ldots \times G_N$ for some bounded, open subsets G_i of the complex plane. We claim that the condition in Proposition 3.4 is satisfied for $A = A(U)$. Indeed, choose $k \in \{1, \ldots, N\}$ and $f \in A(U)$, and suppose that $f(M_k \cap U) = \{0\}$, where $M_k = \{(z_i)_{i=1}^N \in \mathbb{C}^N : z_j = w_j \ (1 \le j \le k)\}$. Then $f = \sum_{i=1}^k f_i$, where $f_1, \ldots, f_k \in A(U)$ are defined by

$$f_i(z_1, \ldots, z_N) = f(w_1, \ldots, w_{i-1}, z_i, \ldots, z_N) - f(w_1, \ldots, w_i, z_{i+1}, \ldots, z_N)$$

for $1 \le i \le k$ and $(z_j)_{j=1}^N \in U$. For each $i \in \{1, \ldots, k\}$, we have

$$f_i(z_1, \ldots, z_{i-1}, w_i, z_{i+1}, \ldots, z_N) = 0 \quad (z_j \in G_j,\ j \in \{1, \ldots, N\} \setminus \{i\}).$$

Therefore we see from Lemma 3.6 that there exist $g_1, \ldots, g_k \in A(U)$ such that $f_i = (Z_i - w_i)g_i$ for $1 \le i \le k$. It follows that $f = \sum_{i=1}^k (Z_i - w_i)g_i$, as required.

Finally, we consider case (iii), so that U is bounded and strictly pseudoconvex with a C^2-boundary. Then it is known ([12, Theorem 2.6.1]) that, for every $q \in \{1, \ldots, N\}$, every $r \ge 0$ and every closed $(0,q)$-form f on U with coefficients in $C^r(\overline{U})$, there exists a $(0, q-1)$-form u on U such that $\bar{\partial} u = f$ and such that each coefficient function h of u satisfies $h \in C^r(U)$ and $D^\gamma(h) \in \mathrm{Lip}_\alpha(U)$ for each $\alpha \in (0,1)$ and each multiindex γ of order not exceeding r. But this certainly implies that the coefficients of u are functions in $C^r(\overline{U}) \cap \mathrm{Lip}_\alpha(\overline{U})$ for each $\alpha \in (0,1)$, and so the condition in Theorem 3.5 is satisfied. Application of this theorem together with Proposition 3.3 yields the desired result. ■

4. The calculation of $H_s^2(A, \mathbb{C}_w)$ and $\mathcal{H}_s^2(A, \mathbb{C}_w)$ in the case where $w \in U$. As in the previous section, let A be a Fréchet algebra of analytic functions on an open set $U \subseteq \mathbb{C}^N$, and let $w \in U$ be fixed. In this section, we

demonstrate how the Koszul complex can be used to obtain a sufficient condition on A for the vanishing of the symmetric second Hochschild cohomolgy groups

$$\mathcal{H}_s^2(A, \mathbb{C}_w) \quad \text{and} \quad H_s^2(A, \mathbb{C}_w).$$

We begin with a simple lemma.

LEMMA 4.1. *Let A be a unital, commutative algebra, and let φ be a character on A. Suppose $\ker\varphi = \langle a_1, \ldots, a_m \rangle$ is algebraically finitely generated. Let $\mu \in Z^2(A, \mathbb{C}_\varphi)$. Then μ is a coboundary if and only if $\sum_{i=1}^m \mu(b_i \otimes a_i) = 0$ for all $b_1, \ldots, b_m \in \ker\varphi$ such that $\sum_{i=1}^m b_i a_i = 0$.*

Proof. It is obvious that the condition in the lemma is necessary for μ to be a coboundary. Conversely, suppose that the condition is satisfied, and let $M = \ker\varphi$. Then there is a linear functional λ on M^2 such that $\lambda(\sum_{i=1}^m b_i a_i) = -\sum_{i=1}^m \mu(b_i \otimes a_i)$ for all $b_1, \ldots, b_m \in M$. We may extend λ to a linear functional on A such that $\lambda(1_A) = 1$. Then $\mu = \delta^1\lambda \in B^2(A, \mathbb{C}_\varphi)$, as required. ■

Suppose that A satisfies the conditions (3.5) and (3.6) in Proposition 3.3. Let $\mu \in Z_s^2(A, \mathbb{C}_w)$. According to Proposition 3.3, there exist a linear map $S : A \to \mathbb{C}_w$, distinct pairs $(i_1, j_1), \ldots, (i_n, j_n)$ with $1 \le i_k < j_k \le N$ $(1 \le k \le n)$ and complex numbers $\alpha_1, \ldots, \alpha_n$ such that

$$\mu(f \otimes g) = S(fg) + \sum_{k=1}^n \alpha_k \frac{\partial f}{\partial z_{i_k}}(w) \frac{\partial g}{\partial z_{j_k}}(w)$$

for all $f, g \in A$ which vanish at w. But μ is symmetric, and so it follows that

$$\alpha_k = \mu((Z_{i_k} - w_{i_k}) \otimes (Z_{j_k} - w_{j_k})) - \mu((Z_{j_k} - w_{j_k}) \otimes (Z_{i_k} - w_{i_k})) = 0$$

for $1 \le k \le n$. We conclude that $\mu \in B^2(A, \mathbb{C}_w)$, and therefore $H_s^2(A, \mathbb{C}_w) = \{0\}$.

However, the next proposition, which is valid for general commutative Fréchet algebras, shows that Proposition 3.3 is not needed here; this result follows from an elementary calculation based on the definition of the Koszul complex, and we may weaken the homological condition on $K(\mathbf{Z} - \mathbf{w})$.

PROPOSITION 4.2. *Let A be a commutative, unital Fréchet algebra, and let φ be a continuous character on A. Suppose that $\ker\varphi = \langle a_1, \ldots, a_n \rangle$ is algebraically finitely generated, and that the first homology of the Koszul complex $\mathcal{K}$ associated with $a_1, \ldots, a_n$ is zero. Then $H_s^2(A, \mathbb{C}_\varphi) = \mathcal{H}_s^2(A, \mathbb{C}_\varphi) = \{0\}$.*

Proof. Let $\mu \in Z_s^2(A, \mathbb{C}_\varphi)$, and let $b_1, \ldots, b_n \in A$ be such that $\sum_{i=1}^n a_i b_i = 0$. Since $H_1(\mathcal{K}) = 0$, there exist $c_{ij} \in A$, $1 \le i < j \le n$, such that

$$b_i = \sum_{j=1}^{i-1} a_j c_{ji} - \sum_{j=i+1}^{n} a_j c_{ij} \quad (1 \le i \le n).$$

Thus

$$\begin{aligned}\sum_{i=1}^{n} \mu(a_i \otimes b_i) &= \sum_{i=1}^{n} \mu\Big(a_i \otimes \Big(\sum_{j=1}^{i-1} a_j c_{ji} - \sum_{j=i+1}^{n} a_j c_{ij}\Big)\Big) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{i-1} \mu(a_i \otimes a_j c_{ji}) - \sum_{i=1}^{n}\sum_{j=i+1}^{n} \mu(a_i \otimes a_j c_{ij}) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{i-1} \mu(a_j \otimes a_i c_{ji}) - \sum_{i=1}^{n}\sum_{j=i+1}^{n} \mu(a_i \otimes a_j c_{ij}) = 0,\end{aligned}$$

where the second equality from below is valid since

$$\mu(a_i \otimes a_j c_{ji}) = \mu(a_i c_{ji} \otimes a_j) = \mu(a_j \otimes a_i c_{ji})$$

for all $i < j$. The result now follows from Lemma 4.1 and Corollary 2.4. ∎

COROLLARY 4.3. *Let A be a Fréchet algebra of analytic functions on an open set $U \subseteq \mathbb{C}^N$, and let $w \in U$. Then, in each of cases* (i)–(iii) *considered in Theorem* 3.7, *both $H_s^2(A, \mathbb{C}_w)$ and $\mathcal{H}_s^2(A, \mathbb{C}_w)$ are trivial.*

Proof. We have shown in the proof of Theorem 3.7 that, in each case, the conditions (3.5) and (3.6) are satisfied, and therefore Proposition 4.2 may be applied. ∎

We finish this section with a further elementary result for the case $A = A(U)$. Here we can prove that $\mathcal{H}_s^2(A(U), \mathbb{C}_w)$ is trivial, but we do not know whether the same is true for $H_s^2(A(U), \mathbb{C}_w)$.

PROPOSITION 4.4. *Let $U \subseteq \mathbb{C}^N$ be an open, bounded and geometrically convex set with C^2-boundary. Then $\mathcal{H}_s^2(A(U), \mathbb{C}_w) = \{0\}$ for every $w \in U$.*

Proof. We may assume that $w = 0$. Let $A = A(U)$, and let $M = \{f \in A : f(w) = 0\}$. According to [11, Theorem 1], M is algebraically generated by the coordinate functions $Z_1, \ldots, Z_N$. Let $\mu \in \mathcal{Z}_s^2(A, \mathbb{C}_0)$, and let $f_1, \ldots, f_N \in A$ be such that $\sum_{i=1}^N Z_i f_i = 0$. For each $f \in A$ and each $r \in (0,1)$, we set $U_r = U + r^{-1}(1-r)U$, and we define $f^{(r)} \in \mathcal{O}(U_r)$ by $f^{(r)}(z) = f(rz)$ for $z \in U_r$. Note that U_r is a convex, open set which contains $\overline{U}$. For each $r \in (0,1)$ we define $\mu_r \in Z_s^2(\mathcal{O}(U_r), \mathbb{C}_0)$ as

$$\mu_r(f \otimes g) = \mu(f|\overline{U} \otimes g|\overline{U}) \quad (f, g \in \mathcal{O}(U_r)).$$

Now let $r \in (0,1)$ be fixed. We see clearly that $\sum_{i=1}^N Z_i f_i^{(r)} = 0$. Furthermore, since U_r is geometrically convex, it is pseudoconvex. We may therefore

conclude from Lemma 4.1 and Corollary 4.3 that

$$\sum_{i=1}^{N} \mu(Z_i \otimes f_i^{(r)}|\overline{U}) = \sum_{i=1}^{N} \mu_r(Z_i \otimes f_i^{(r)}) = 0. \tag{4.1}$$

If f is any function in A, then, since f is uniformly continuous, $f^{(r)}|\overline{U} \to f$ uniformly on $\overline{U}$ as $r \to 1$. Therefore (4.1) and the continuity of μ imply that $\sum_{i=1}^N \mu(Z_i \otimes f_i) = 0$. The result now follows from Corollary 2.4 and Lemma 4.1. ∎

EXAMPLE. Let $U = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^4 < 1\}$. Then U is geometrically convex and smoothly bounded, but not strictly pseudoconvex. Hence there are simple examples of sets U that satisfy the conditions of Proposition 4.4, but to which we cannot apply Corollary 4.3.

5. The calculation of $H^n(A(U), \mathbb{C}_w)$ and $\mathcal{H}^n(A(U), \mathbb{C}_w)$ in the case where $w \in \partial U$. Let A be the algebra $A(U)$ for some bounded, open set $U \subseteq \mathbb{C}^N$. In this section, we investigate the spaces $H^n(A, \mathbb{C}_w)$ and $\mathcal{H}^n(A, \mathbb{C}_w)$ where w is a point in the boundary ∂U of U.

It is well known (see [2, p. 101], for example) that the maximal ideal M_w of A associated with w has a bounded approximate identity if and only if w is a *peak point* for A (i.e. there is $f \in A$ such that $f(w) = 1$ and $|f(z)| < 1$ for $z \in \overline{U} \setminus \{w\}$). Our first proposition shows that this is the "easy" case in our situation.

The part in the proposition which concerns $\mathcal{H}^n(A, \mathbb{C}_w)$ is certainly well known (see [14, Proposition 1.5]); however, we provide a proof for both the continuous and the algebraic situations because the argument is valid in both cases.

PROPOSITION 5.1. *Let A be a unital, commutative Banach algebra, and let φ be a character on A. Suppose that $M = \ker\varphi$ has a bounded approximate identity. Then*

$$\mathcal{H}^n(A, \mathbb{C}_\varphi) = H^n(A, \mathbb{C}_\varphi) = \{0\} \quad (n \ge 1). \tag{5.1}$$

Proof. Since M has a bounded approximate identity, it is flat in **A-Ba-mod** ([7, VIII.1.5]) and in **A-mod** ([24, Theorem B]). Therefore

$$0 \to M \xrightarrow{\iota} A \xrightarrow{\varphi} \mathbb{C}_\varphi,$$

where ι is the inclusion map, is a flat resolution of $\mathbb{C}_\varphi$ in both categories. Thus Lemma 2.2 implies that (5.1) is true for every $n \ge 2$. Furthermore, $M = M^2$ by Cohen's factorization theorem, and hence (5.1) is also true for $n = 1$. ∎

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COROLLARY 5.2. *Let $U \subseteq \mathbb{C}^N$ be a bounded, strictly pseudoconvex domain with C^2-boundary. Then*

$$\mathcal{H}^n(A(U), \mathbb{C}_w) = H^n(A(U), \mathbb{C}_w) = \{0\} \quad (w \in \partial U,\ n \geq 1).$$

Proof. It is known ([18, VI.1.14]) that every boundary point of U is a peak point for $A(U)$ in this case. ∎

Simple examples show that the maximal ideal M_w may fail to have a bounded approximate identity and that, consequently, Proposition 5.1 is not applicable. For instance, consider the case where $A = A(U)$ is the polydisc algebra. Then $w = (w_i)_{i=1}^N \in \partial U$ if $|w_i| = 1$ for *some* $1 \leq i \leq N$, and w is a peak point if $|w_i| = 1$ for *all* $1 \leq i \leq N$, which in turn is equivalent to w lying in the Shilov boundary ∂A of A. However, we always have the decomposition

$$M_w = \langle Z_{i_1} - w_{i_1}, \ldots, Z_{i_k} - w_{i_k} \rangle + J, \tag{5.2}$$

where $\{i_1, \ldots, i_k\}$ is the set of indices j such that $|w_j| < 1$, and J is the kernel of the peak set $\Delta = \{(z_i)_i \in \overline{\mathbb{D}}^N : z_j = w_j \text{ for } j \notin \{i_1, \ldots, i_k\}\}$. Decompositions of this kind have been studied by A. Ya. Helemskiĭ (cf. [8]). We follow his approach here to show that a description of the spaces $\mathcal{H}^n(A, \mathbb{C}_w)$ and $H^n(A, \mathbb{C}_w)$ can be given whenever we have a decomposition of M_w of the type given in (5.2).

DEFINITION. We say that a sequence $a_1, \ldots, a_k$ of elements of a Banach algebra A is a *strongly regular sequence on* A if $a_1, \ldots, a_k$ is a regular sequence on A in the algebraic sense and, for each $i \in \{1, \ldots, k\}$, the ideal $\langle a_1, \ldots, a_i \rangle$ generated by $a_1, \ldots, a_i$ is closed and complemented in A.

REMARK. Our notion of a strongly regular sequence on A corresponds to the notion of *Koszul ideals* used in [8].

Before we proceed, let us introduce the following two basic notions from homological algebra.

NOTATION. (i) Let A be a commutative, unital algebra. Let $(\mathcal{F}, \varphi)$ and $(\mathcal{G}, \psi)$ be chain complexes of A-modules. Then the *tensor product chain complex* $\mathcal{F} \otimes_A \mathcal{G}$ of $\mathcal{F}$ and $\mathcal{G}$ is defined (cf. [23, 2.7.1]). Recall that the degree n part $(\mathcal{F} \otimes_A \mathcal{G})_n$ of $\mathcal{F} \otimes_A \mathcal{G}$ is $\bigoplus_{i+j=n} F_i \otimes_A G_j$. On $F_i \otimes_A G_j$ (with $i + j = n$), the differential d_n from $(\mathcal{F} \otimes_A \mathcal{G})_n$ into $(\mathcal{F} \otimes_A \mathcal{G})_{n-1}$ is the zero map to $F_s \otimes_A G_t$ unless $i = s$ or $j = t$. From $F_i \otimes_A G_j$ to $F_{i-1} \otimes_A G_j$ it is $\varphi_i \otimes \mathrm{id}$, and from $F_i \otimes_A G_j$ to $F_i \otimes_A G_{j-1}$ it is $(-1)^i \mathrm{id} \otimes \psi_j$. As is customary, we identify an A-module M with the complex

$$\ldots \to 0 \to M \to 0 \to \ldots$$

concentrated in degree 0; in particular, the tensor product chain complex $\mathcal{F} \otimes_A M$ is defined for each A-module M.

For a definition of the tensor product of chain complexes in the continuous case, we refer to [7, II.5.25].

(ii) Let $\alpha : (\mathcal{F}, \varphi) \to (\mathcal{G}, \psi)$ be a morphism of chain complexes in the category of linear spaces. Then (cf. [23, 1.5.1]) the *mapping cone* of α is the chain complex whose degree n part is $F_{n-1} \oplus G_n$; the differential is given by

$$d_n(a, b) = (-\varphi_{n-1}(a), \psi_n(b) - \alpha(a)).$$

REMARK. The notion of a "mapping cone" is a generalization of the topological mapping cone of a simplicial map (see [23, p. 19]).

The following result is implicitly contained in [8] (cf. also [7, Lemma V.1.2]). However, we provide a brief proof for the convenience of the reader.

LEMMA 5.3. *Let $\alpha : (\mathcal{F}, \varphi) \to (\mathcal{G}, \psi)$ be a continuous morphism of chain complexes in the category of Fréchet spaces. Suppose that, for some n, φ_n and ψ_{n+1} are admissible, and $H_n(\mathcal{F})$ or $H_n(\mathcal{G})$ is zero. Then the $(n+1)$th differential of the mapping cone of α is admissible.*

Proof. Set $\varphi = \varphi_n$ and $\psi = \psi_{n+1}$. It follows from our hypothesis that there are continuous linear maps $\varphi' : F_{n-1} \to F_n$ and $\psi' : G_n \to G_{n+1}$ such that $\varphi = \varphi \circ \varphi' \circ \varphi$ and $\psi = \psi \circ \psi' \circ \psi$. Let ϱ denote the $(n+1)$th differential of the mapping cone of α. For $a \in F_{n-1}$ and $b \in G_n$, we define

$$\varrho'(a, b) = (-\varphi'(a), \psi'(b) - (\psi' \circ \alpha_n \circ \varphi')(a)).$$

Then $\varrho' : F_{n-1} \oplus G_n \to F_n \oplus G_{n+1}$ is a continuous linear map. A straightforward computation shows that $\varrho = \varrho \circ \varrho' \circ \varrho - \tau$, where

$$\tau(a, b) = (0, ((\mathrm{id}_{G_n} - \psi \circ \psi')\alpha_n(\mathrm{id}_{F_n} - \varphi' \circ \varphi))(a)) \quad (a \in F_n,\ b \in G_{n+1}).$$

Since $H_n(\mathcal{F}) = 0$ or $H_n(\mathcal{G}) = 0$, τ is the zero operator. Hence $\varrho = \varrho \circ \varrho' \circ \varrho$. We conclude that ϱ is admissible, as required. ∎

We can now state the main result of this section.

THEOREM 5.4. *Let A be a commutative, unital Banach algebra, and let M_φ be the kernel of a character φ on A. Suppose that $M_\varphi = I + J$, where I and J are ideals of A and I is algebraically finitely generated by $a_1, \ldots, a_k$. If*

(i) *J is flat in* A-**mod**,

(ii) *$a_1, \ldots, a_k$ is a regular sequence on A, and*

(iii) *$IJ = I \cap J$,*

then

$$H^n(A, \mathbb{C}_\varphi) \cong \mathrm{Hom}_{\mathbb{C}}(J/(M_\varphi J), \mathbb{C})^{\oplus \binom{k}{n-1}} \oplus \mathbb{C}_\varphi^{\oplus \binom{k}{n}} \quad (n \geq 0).$$

If, furthermore,

(iv) *J is closed, and flat in* A-**Ba**-**mod**,

(v) $a_1, \ldots, a_k$ *is a strongly regular sequence on* A, *and*
(vi) $\langle a_1, \ldots, a_i \rangle J$ *is closed and complemented in* J $(i = 1, \ldots, k)$,

then

$$\mathcal{H}^n(A, \mathbb{C}_\varphi) \cong \mathrm{Hom}_{\mathbb{C},\,\mathrm{cont}}(J/\overline{M_\varphi J}, \mathbb{C})^{\oplus\binom{k}{n-1}} \oplus \mathbb{C}_\varphi^{\oplus\binom{k}{n}} \quad (n \geq 0).$$

Proof. Suppose first that (i)–(iii) are satisfied. Let $\mathcal{K}$ be the Koszul complex for the sequence $a_1, \ldots, a_k$. By tensoring the inclusion $J \hookrightarrow A$ with $\mathcal{K}$, we obtain the map

$$\alpha : \mathcal{K} \otimes_A J \to \mathcal{K} \otimes_A A = \mathcal{K}$$

of chain complexes; note that $\mathcal{K} \otimes_A J$ is the Koszul complex for the pair $(J, \mathbf{a})$, where $\mathbf{a} = (a_1, \ldots, a_k)$ is regarded as the k-tuple of operators on J of multiplication by the a_i's (cf. [21, p. 210]). In particular, we see that $\mathcal{K} \otimes_A J$ coincides with the continuous tensor product of $\mathcal{K}$ and J as defined in [7].

Let $\mathcal{F}$ be the mapping cone of α. By [23, 1.5.2 and 1.5.3], there is a long exact sequence

$$\ldots \to H_n(\mathcal{K}) \to H_n(\mathcal{F}) \to H_{n-1}(\mathcal{K} \otimes_A J) \xrightarrow{\delta} H_{n-1}(\mathcal{K}) \to \ldots$$

in A-**mod**, where δ is the map induced on homology by α. By Theorem 2.1, condition (ii) implies that $H_n(\mathcal{K}) = \{0\}$ for $n \geq 1$. Hence $H_n(\mathcal{K} \otimes_A J) = \{0\}$ also applies for $n \geq 1$ because J is flat. It follows that $H_n(\mathcal{F}) = \{0\}$ for $n \geq 2$, and that there is an exact sequence

$$0 \to H_1(\mathcal{F}) \to H_0(\mathcal{K} \otimes_A J) \xrightarrow{\delta} H_0(\mathcal{K}) \to H_0(\mathcal{F}) \to 0. \tag{5.3}$$

By [23, 4.5.2], $H_0(\mathcal{K} \otimes_A J) = J/JI$ and $H_0(\mathcal{K}) = A/I$. Since δ is induced on homology by α, it is the map

$$J/JI \to A/I : a + JI \to a + I.$$

The kernel of this map is $J \cap I/JI$, which is the zero space by condition (iii). Hence δ is injective, and we conclude from (5.3) that $H_1(\mathcal{F}) = \{0\}$ and

$$H_0(\mathcal{F}) = (A/I)/((I+J)/I) = A/(I+J) = A/M_\varphi = \mathbb{C}_\varphi,$$

where we have made use of the fact that $M_\varphi = I + J$ in the second-to-last equality.

We have shown that $H_n(\mathcal{F}) = \{0\}$ for $n \geq 1$ and $H_0(\mathcal{F}) = \mathbb{C}_\varphi$. Thus $\mathcal{F}$ is a resolution of $\mathbb{C}_\varphi$ in A-**mod**. By the definition of a mapping cone, the degree n part of $\mathcal{F}$ is

$$F_n = (\mathcal{K} \otimes_A J)_{n-1} \oplus K_n = (K_{n-1} \otimes_A J) \oplus K_n. \tag{5.4}$$

But J, K_{n-1} and K_n are flat A-modules, and therefore F_n is flat. Consequently, $\mathcal{F}$ is a flat resolution of $\mathbb{C}_\varphi$, and by Lemma 2.2 and (2.2) we see that

$$H^n(A, \mathbb{C}_\varphi) \cong H^n(\mathrm{Hom}_A(\mathcal{F}, \mathbb{C}_\varphi)) \quad (n \geq 0). \tag{5.5}$$

The differential d_n from F_n into F_{n-1} is the map which sends a pair $(a \otimes_A b, c)$ (where $a \in K_{n-1}$, $b \in J$ and $c \in K_n$) to

$$(-\phi_{n-1}(a) \otimes_A b, \phi_n(c) - b \cdot a), \tag{5.6}$$

where $\phi = (\phi_n)$ is the differential of $\mathcal{K}$. We can easily verify that, for every $n \geq 0$, each $\varrho \in \mathrm{Hom}_A(F_n, \mathbb{C}_\varphi)$ vanishes on the image of d_{n+1}. However, this only means that $\mathrm{Hom}_A(\mathcal{F}, \mathbb{C}_\varphi)$ is a complex with zero morphisms. Thus $H^n(\mathrm{Hom}_A(\mathcal{F}, \mathbb{C}_\varphi))$ is the degree n part of $\mathrm{Hom}_A(\mathcal{F}, \mathbb{C}_\varphi)$, which is $\mathrm{Hom}_A(F_n, \mathbb{C}_\varphi)$. Hence we see from (5.4) and (5.5) that, for each $n \geq 0$,

$$\begin{aligned} H^n(A, \mathbb{C}_\varphi) &\cong \mathrm{Hom}_A(K_{n-1} \otimes_A J, \mathbb{C}_\varphi) \oplus \mathrm{Hom}_A(K_n, \mathbb{C}_\varphi) \\ &\cong \mathrm{Hom}_A(J, \mathbb{C}_\varphi)^{\oplus\binom{k}{n-1}} \oplus \mathbb{C}_\varphi^{\oplus\binom{k}{n}}. \end{aligned} \tag{5.7}$$

It is obvious that $\mathrm{Hom}_A(J, \mathbb{C}_\varphi)$ is isomorphic to $\mathrm{Hom}_{\mathbb{C}}(J/(M_\varphi J), \mathbb{C})$, however, and we have therefore proved the first part of the theorem.

Now suppose, furthermore, that the conditions (iv)–(vi) are satisfied. We claim that $\mathcal{F}$ is a flat resolution of $\mathbb{C}_\varphi$ in A-**Ba-mod**. Indeed, since J is closed and flat in A-**Ba-mod**, $\mathcal{F}$ is a complex in A-**Ba-mod** consisting of flat modules. Moreover, we already know that $H_0(\mathcal{F}) = \mathbb{C}_\varphi$ and $H_n(\mathcal{F}) = 0$ for $n \geq 1$. Hence it remains to be shown that each differential d_n of $\mathcal{F}$ is admissible. We infer from (v) (respectively, (vi)) and [21, Proposition 4.1] that every differential of the complex $\mathcal{K}$ (respectively, $\mathcal{K} \otimes_A J$) is admissible, and that $H_n(\mathcal{K})$ (respectively, $H_n(\mathcal{K} \otimes_A J)$) is zero for all $n \geq 1$. We conclude from Lemma 5.3 that d_n is admissible for all $n \geq 2$. Moreover, d_1 is admissible because $H_1(\mathcal{F}) = 0$ and $\mathrm{im}\, d_1 = M_\varphi$ is of finite codimension in $F_0 = A$. Thus d_n is admissible for all n, as required.

It follows from Lemma 2.2 and (2.3) that

$$\mathcal{H}^n(A, \mathbb{C}_\varphi) \cong H^n(\mathrm{Hom}_{A,\,\mathrm{cont}}(\mathcal{F}, \mathbb{C}_\varphi)) \quad (n \geq 0).$$

The second part of the theorem may now be proved by a computation which is completely analogous to (5.7). ■

Remarks. (i) The resolution $\mathcal{F}$ in the proof of the theorem coincides with the resolution which is studied in [8] with the aim of determining $\mathrm{dh}_A \mathbb{C}_\varphi$, the projective homological dimension of $\mathbb{C}_\varphi$ in A-**Ba-mod**. It is shown in [8] that, under the assumption that A is a uniform algebra, J is the kernel of a peak set and conditions (ii) and (iii) of Corollary 5.5 below are satisfied, $\mathcal{F}$ is a *projective* resolution of $\mathbb{C}_\varphi$ in A-**Ba-mod**.

(ii) In the situation of the theorem, $\dim(J/(M_\varphi J)) \leq 1$. Indeed, let ϕ and ψ be linear functionals on J which vanish on $M_\varphi J$. There then exists a linear functional ϱ on $J \otimes_A J$ which sends $a \otimes_A b$ to $\phi(a)\psi(b)$. The flatness of J implies that there is a linear functional θ on A such that $\theta(ab) = \phi(a)\psi(b)$ for all $a, b \in A$. It follows that $\phi(a)\psi(b) = \phi(b)\psi(a)$ for all $a, b \in A$, and thus ϕ and ψ are collinear, as required.

COROLLARY 5.5. *Let A be a commutative, unital Banach algebra, and let M_φ be the kernel of a character φ on A. Suppose that $M_\varphi = I + J$, where I and J are ideals of A and I is algebraically finitely generated by $a_1, \ldots, a_k$. If*

(i) *J has a bounded approximate identity,*
(ii) *$a_1, \ldots, a_k$ is a strongly regular sequence on A, and*
(iii) *$\langle a_1, \ldots, a_i \rangle \cap J$ is complemented in J $(i = 1, \ldots, k)$,*

then

$$H^n(A, \mathbb{C}_\varphi) \cong \mathcal{H}^n(A, \mathbb{C}_\varphi) \cong \mathbb{C}_\varphi^{\oplus\binom{k}{n}} \quad (n \geq 0).$$

Proof. We have already observed that (i) implies that J is flat in both **A-mod** and **A-Ba-mod**. Moreover, Cohen's Factorization Theorem asserts that $J = J^2 \subseteq M_\varphi J$ and $\langle a_1, \ldots, a_i \rangle J = \langle a_1, \ldots, a_i \rangle \cap J$ for $1 \leq i \leq k$. Hence the result follows from Theorem 5.4. ■

We shall apply Corollary 5.5 in the case where $A = A(U)$ and $U = U_1 \times \ldots \times U_N \subseteq \mathbb{C}^N$ is a product domain. Then for $w = (w_i)_{i=1}^N \in \partial U$ there is a partition (F, G) of $\{1, \ldots, N\}$ such that $w_r \in U_r$ for $r \in F$ and $w_s \in \partial U_s$ for $s \in G$, and the results in §3 show that, for $k = \operatorname{card}(F)$, there are explicit embeddings

$$\alpha_n : \mathbb{C}_w^{\oplus\binom{k}{n}} \hookrightarrow \mathcal{H}^n(A, \mathbb{C}_w) \quad \text{and} \quad \beta_n : \mathbb{C}_w^{\oplus\binom{k}{n}} \hookrightarrow H^n(A, \mathbb{C}_w) \qquad (n = 0, 1, \ldots)$$

such that the diagram

$$\begin{tikzcd} & \mathcal{H}^n(A, \mathbb{C}_w) \arrow[d, "\iota_n"] \\ \mathbb{C}_w^{\oplus\binom{k}{n}} \arrow[ur, "\alpha_n"] \arrow[r, "\beta_n"'] & H^n(A, \mathbb{C}_w) \end{tikzcd}$$

commutes, where ι_n is the comparison map. We conclude this section with a corollary which shows that α_n and β_n are isomorphisms in many cases.

COROLLARY 5.6. *Let A be the algebra $A(U)$ for some bounded product domain $U = U_1 \times \ldots \times U_N \subseteq \mathbb{C}^N$, and let $w = (w_i)_{i=1}^N \in \partial U$. Suppose that, for each $i \in \{1, \ldots, N\}$ such that $w_i \in \partial U_i$, w_i is a peak point for $A(U_i)$. Then*

$$H^n(A, \mathbb{C}_w) \cong \mathcal{H}^n(A, \mathbb{C}_w) \cong \mathbb{C}_w^{\oplus\binom{k}{n}} \quad (n = 0, 1, \ldots),$$

where $k = \operatorname{card}\{i \in \{1, \ldots, N\} : w_i \in U_i\}$.

Proof. We may suppose that $w_i \in U_i$ for $1 \leq i \leq k$ and that $w_j \in \partial U_j$ for $k < j \leq N$. Set $f_i = Z_i - w_i$ $(1 \leq i \leq k)$, and let J be the set of $f \in A$ such that $f(z_1, \ldots, z_k, w_{k+1}, \ldots, w_N) = 0$ for all $z_j \in \overline{U}_j$, $1 \leq j \leq k$. We claim that conditions (i)–(iii) of Corollary 5.5 are satisfied. Set $I_i = \langle f_1, \ldots, f_i \rangle$ $(1 \leq i \leq k)$. We see from Lemma 3.6 that, for each $i \in \{1, \ldots, k\}$, I_i is the kernel of the set $\{w_1\} \times \ldots \times \{w_i\} \times \overline{U}_{i+1} \times \ldots \times \overline{U}_N$. It follows that I_i is closed, and that $M_w = I_k + J$. Our hypothesis concerning w implies that J is the kernel of a peak set; therefore (i) is satisfied. We have already shown (see the proof of Theorem 3.7) that $f_1, \ldots, f_k$ is a regular sequence on A. Moreover, we note (as was done, in the case where A is the polydisc algebra, in [8, p. 231]) that, for each $i \in \{1, \ldots, k\}$, the set M_i of functions not dependent on $z_1, \ldots, z_i$ is a Banach space complement for I_i in A, and that $M_i \cap J$ is a Banach space complement for $I_i \cap J$ in J. Hence (ii) and (iii) are also satisfied. The result now follows from Corollary 5.5. ■

6. The calculation of $H_s^2(A(U), \mathbb{C}_w)$ and $\mathcal{H}_s^2(A(U), \mathbb{C}_w)$ in the case where $w \in \partial U$. As in the previous section, let A be the algebra $A(U)$ for some bounded, open set $U \subseteq \mathbb{C}^N$, and let $w \in \partial U$. In this section, we consider the symmetric Hochschild groups $H_s^2(A, \mathbb{C}_w)$ and $\mathcal{H}_s^2(A, \mathbb{C}_w)$.

Suppose that J is a closed ideal of A which is contained in M_w. Then $\mathbb{C}_w$ is a Banach J-bimodule, as well as a Banach A/J-bimodule, in a canonical way; thus we may consider the spaces $\mathcal{H}_s^2(A/J, \mathbb{C}_w)$, $\mathcal{H}_s^2(A, \mathbb{C}_w)$, and $\mathcal{H}_s^2(J, \mathbb{C}_w)$ and their algebraic counterparts. The next elementary proposition gives us some information on how these spaces are related to one another.

PROPOSITION 6.1. *Let φ be a character on a commutative, unital Banach algebra A. Suppose that J is a closed ideal of A contained in $\ker\varphi$. Set*

$$K = \{\mu \in Z_s^2(A, \mathbb{C}_\varphi) : \mu(J \otimes_{\mathbb{C}} J) = \{0\}\},$$
$$L = \{T \in \operatorname{Hom}_{\mathbb{C}}(A, \mathbb{C}) : T|J \in \operatorname{Hom}_A(J, \mathbb{C}_\varphi)\}.$$

Moreover, let $\mathcal{K}$ (respectively, $\mathcal{L}$) be the set of elements of K (respectively, L) which are continuous. Then there is a commuting diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_A(J, \mathbb{C}_\varphi) \arrow[r] & H^1(J, \mathbb{C}_\varphi) \arrow[r] & K/\delta^1(L) \arrow[r] & H_s^2(A, \mathbb{C}_\varphi) \arrow[r] & H_s^2(J, \mathbb{C}_\varphi) \\ 0 \arrow[r] & \operatorname{Hom}_{A,\mathrm{cont}}(J, \mathbb{C}_\varphi) \arrow[r] \arrow[u, "\sigma_1"] & \mathcal{H}^1(J, \mathbb{C}_\varphi) \arrow[r] \arrow[u, "\sigma_2"] & \mathcal{K}/\delta^1(\mathcal{L}) \arrow[r] \arrow[u, "\sigma_3"] & \mathcal{H}_s^2(A, \mathbb{C}_\varphi) \arrow[r] \arrow[u, "\sigma_4"] & \mathcal{H}_s^2(J, \mathbb{C}_\varphi) \arrow[u, "\sigma_5"] \end{tikzcd}$$

where σ_1 and σ_2 are inclusion maps, σ_3 is the map which sends a coset $k+\delta^1(\mathcal{L})$ to $k+\delta^1(L)$, and σ_4 and σ_5 are the comparison maps for $\mathcal{H}_s^2(A, \mathbb{C}_\varphi)$ and $\mathcal{H}_s^2(J, \mathbb{C}_\varphi)$, respectively.

Suppose further that $J^2 = J$. Then there is a commuting diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & H^2_s(A/J,\mathbb{C}_\varphi) \arrow[r] & H^2_s(A,\mathbb{C}_\varphi) \arrow[r] & H^2_s(J,\mathbb{C}_\varphi) \\ 0 \arrow[r] & \mathcal{H}^2_s(A/J,\mathbb{C}_\varphi) \arrow[r] \arrow[u] & \mathcal{H}^2_s(A,\mathbb{C}_\varphi) \arrow[r] \arrow[u] & \mathcal{H}^2_s(J,\mathbb{C}_\varphi) \arrow[u] \end{tikzcd}$$

where the vertical maps are the respective comparison maps.

Proof. We first construct linear maps $\varrho_1,\dots,\varrho_4$ such that the sequence

$$(6.1)\qquad 0\to \operatorname{Hom}_A(J,\mathbb{C}_\varphi)\xrightarrow{\varrho_1} H^1(J,\mathbb{C}_\varphi)\xrightarrow{\varrho_2} K/\delta^1(L) \xrightarrow{\varrho_3} H^2_s(A,\mathbb{C}_\varphi)\xrightarrow{\varrho_4} H^2_s(J,\mathbb{C}_\varphi)$$

is exact. We choose ϱ_1 to be the inclusion map, and we define

$$\varrho_3(\mu+\delta^1(L))=\mu+B^2(A,\mathbb{C}_\varphi)\qquad (\mu\in K),$$
$$\varrho_4(\mu+B^2(A,\mathbb{C}_\varphi))=\mu|(J\otimes_{\mathbb{C}} J)+B^2(J,\mathbb{C}_\varphi)\qquad (\mu\in Z^2_s(A,\mathbb{C}_\varphi)).$$

For ϱ_2 we choose the map which sends $\varphi\in H^1(J,\mathbb{C}_\varphi)$ to $\delta^1\phi+\delta^1(L)$, where ϕ is an extension of φ to a linear functional on A; it is obvious that ϱ_2 is well defined. It is then trivial to verify that the sequence (6.1) is exact.

Analogously, we may choose linear maps $\tau_1,\dots,\tau_4$ such that the second row of the diagram in the first part of the theorem is exact; we note, however, that the Hahn–Banach Theorem is required for the definition of τ_2. The commutativity of the whole diagram is then immediately obvious.

Let $\kappa : A\to A/J$ be the quotient map, and let $\kappa\otimes\kappa : A\otimes_{\mathbb{C}} A\to (A/J)\otimes_{\mathbb{C}}(A/J)$ denote the linear map which sends $a\otimes b$ into $\kappa(a)\otimes\kappa(b)$. We define

$$\theta_1(\mu+B^2(A/J,\mathbb{C}_\varphi))=\mu\circ(\kappa\otimes\kappa)+\delta^1(L)\qquad (\mu\in Z^2_s(A/J,\mathbb{C}_\varphi)),$$
$$\theta_2(\mu+\mathcal{B}^2(A/J,\mathbb{C}_\varphi))=\mu\circ(\kappa\otimes\kappa)+\delta^1(\mathcal{L})\qquad (\mu\in \mathcal{Z}^2_s(A/J,\mathbb{C}_\varphi)).$$

We then have the commuting diagram

$$(6.2)\qquad \begin{tikzcd} H^2_s(A/J,\mathbb{C}_\varphi) \arrow[r, "\theta_1"] & K/\delta^1(L) \\ \mathcal{H}^2_s(A/J,\mathbb{C}_\varphi) \arrow[r, "\theta_2"] \arrow[u, "\tau"] & \mathcal{K}/\delta^1(\mathcal{L}), \arrow[u, "\sigma_3"'] \end{tikzcd}$$

where τ is the comparison map and σ_3 is defined as in the statement of the theorem.

Now suppose that $J=J^2$. Then clearly both $H^1(A,\mathbb{C}_\varphi)$ and $\mathcal{H}^1(A,\mathbb{C}_\varphi)$ are trivial, and it can easily be verified that the maps θ_1 and θ_2 in (6.2) are isomorphisms. Hence the second part of the theorem follows from the first. ■

Remarks. (i) The proof of this result carries over to the more general case where $\mathbb{C}_\varphi$ is replaced by a finite-dimensional, symmetric Banach A-bimodule E which is annihilated by J. Moreover, there is a similar result for the Hochschild groups $\mathcal{H}^2(A/J,E)$, $\mathcal{H}^2(A,E)$, and $\mathcal{H}^2(J,E)$ and their algebraic counterparts in the "non-commutative" situation where A is a Banach algebra, J is a closed ideal of A, and E is a finite-dimensional Banach A-bimodule which is annihilated by J.

(ii) Results analogous to Proposition 6.1 can be found in the theory of group cohomology (see [23, 6.8.3]) and in the theory of Lie algebra cohomolgy (see [23, 7.5.3]). In these two cases, the result may be obtained by an application of Grothendieck's Spectral Sequence Theorem, and it is thus conceivable that this is also true for Proposition 6.1.

Our next theorem, which is true for general Banach algebras, shows that the symmetric Hochschild groups of A with coefficients in $\mathbb{C}_w$ vanish in the case where M_w is decomposable into a part which is finitely algebraically generated and a part which is, intuitively speaking, "near" to having a bounded approximate identity. Note that the conditions on M_w that we need are similar to the conditions on M_w in Theorem 5.4.

Theorem 6.2. *Let φ be a character on a commutative, unital Banach algebra A, and let $M_\varphi=\ker\varphi$. Suppose that M_φ admits the decomposition $M_\varphi=I+J$, where I and J are ideals of A and I is algebraically finitely generated by $a_1,\dots,a_k$. Suppose, furthermore, that*

(i) $J=\overline{J}=J^2$,
(ii) $H^2_s(J,\mathbb{C}_\varphi)=\mathcal{H}^2_s(J,\mathbb{C}_\varphi)=\{0\}$,
(iii) $a_1,\dots,a_k$ *is a regular sequence on A, and*
(iv) $\langle a_1,\dots,a_j\rangle\cap J=\langle a_1,\dots,a_j\rangle J$ $(j=1,\dots,k)$.

Then $H^2_s(A,\mathbb{C}_\varphi)=\mathcal{H}^2_s(A,\mathbb{C}_\varphi)=\{0\}$.

Proof. By Proposition 6.1, Corollary 2.4 and Proposition 4.2, it suffices to show that the sequence $a_1+J,\dots,a_k+J$ is regular on A/J. To do this, let $i\in\{1,\dots,k\}$ be fixed. Let I be the ideal which is algebraically generated by $a_1,\dots,a_{i-1}$, and let $b\in A$ be such that

$$a_ib+J\in\langle a_1+J,\dots,a_{i-1}+J\rangle=I+J.$$

Then it follows from (iv) that there are $b_1,\dots,b_{i-1}\in A$ and $b_i\in J$ such that

$$a_i(b+b_i)+\sum_{j=1}^{i-1}b_ja_j=0.$$

But now (iii) implies that $b+b_i\in I$. Thus $b+J\in\langle a_1+J,\dots,a_{i-1}+J\rangle$, as required. ■

Corollary 6.3. *Suppose that $U=U_1\times\ldots\times U_N\subseteq\mathbb{C}^N$ is a bounded product domain, and that $w=(w_i)_{i=1}^N\in\partial U$ where, for each i such that*

$w_i \in \partial U_i$, w_i *is a peak point for* $A(U_i)$. *Then* $H^2_s(A(U), \mathbb{C}_w) = \mathcal{H}^2_s(A(U), \mathbb{C}_w) = \{0\}$.

Proof. We have shown, in the proof of Corollary 5.5, that M_w admits a decomposition $M_w = I + J$ such that conditions (i)–(iv) in Theorem 6.2 are satisfied. ∎

In the remainder of this section, we consider the case where $N = 1$ and consequently U is an open, bounded set in the complex plane. We demonstrate that there are examples where $\mathcal{H}^2_s(A, \mathbb{C}_w) \neq \{0\}$ in this situation; in particular, this shows that the assertion made in Corollary 6.3 does not hold for arbitrary product domains.

REMARK. Recall that A is the set of functions in $C(\overline{U})$ which are holomorphic on U. Thus A contains $A(\overline{U})$, the set of functions in $C(\overline{U})$ which are holomorphic on the interior $\mathrm{int}(\overline{U})$ of $\overline{U}$. Of course, $A = A(\overline{U})$ if $U = \mathrm{int}(\overline{U})$, or, more generally, if $\mathrm{int}(\overline{U}) \setminus U$ is analytically negligible. There are cases, however, where $A \neq A(\overline{U})$. The following example demonstrating this was communicated to me by Professor Heinz König. Let $F \subseteq [0, 1]$ be the Cantor set, and let $f : [0, 1] \to [0, 1]$ be the Cantor function. Then the function $g(z) = f(\Re z)$ is holomorphic on $(0, 1) \setminus F \times (0, 1)$ and continuous on $\overline{U} = [0, 1] \times [0, 1]$; however, g is not holomorphic on the interior of $\overline{U}$.

We start with an elementary lemma, which is implicitly contained in [17]. Before we can state our lemma, we need to introduce some additional notation.

NOTATION. Let φ be a character on a commutative Banach algebra A. Recall that the elements of $Z^1(A, \mathbb{C}_\varphi)$ are called *point derivations*. Thus a linear functional d on A is a point derivation if

$$d(ab) = \varphi(a)d(b) + d(a)\varphi(b) \quad (a, b \in A).$$

A *point derivation of order* n at φ is a sequence $(d_k)_{k=0}^n$ of linear functionals on A, with $d_0 = \varphi$, satisfying

$$d_k(ab) = \sum_{j=0}^{k} d_j(a)d_{k-j}(b) \quad (a, b \in A,\ k \in \{0, \ldots, n\}).$$

An *infinite order point derivation* at φ is a sequence $(d_k)_{k=0}^\infty$ such that, for each $n \in \mathbb{N}$, $(d_k)_{k=0}^n$ is a point derivation of order n at φ. A point derivation (d_k) of some order at φ is *degenerate* if $d_1 = 0$, and *continuous* if each linear functional d_n in the sequence is continuous.

LEMMA 6.4. *Let A be a commutative, unital Banach algebra, and let φ be a character on A. Suppose that $\mathcal{H}^2_s(A, \mathbb{C}_\varphi) = \{0\}$. Then for each continuous point derivation d at φ there exists a continuous infinite order point derivation $(d_k)_{k=0}^\infty$ at φ such that $d_1 = d$.*

Proof. Let $n \in \mathbb{N}$, and suppose that $(d_k)_{k=0}^n$ is a continuous point derivation of order n at φ such that $d_1 = d$. Clearly, it suffices to construct a continuous functional d_{n+1} on A such that $(d_k)_{k=0}^{n+1}$ is a continuous point derivation at φ of order $n + 1$. To this end, we consider the continuous functional

$$\mu : A \widehat{\otimes}_p A \to \mathbb{C} : a \otimes b \mapsto -\sum_{k=1}^{n} d_k(a)d_{n+1-k}(b).$$

It is easy to verify that

$$\mu(ab \otimes c) = \sum_{\substack{1 \le i,j,k \le n \\ i+j+k=n+1}} d_i(a)d_j(b)d_k(c) = \mu(a \otimes bc)$$

for all $a, b, c \in \ker\varphi$, and it follows that $\mu \in \mathcal{Z}^2_s(A, \mathbb{C}_\varphi)$. By our hypothesis, there exists a continuous linear functional d_{n+1} such that

$$\varphi(a)d_{n+1}(b) - d_{n+1}(ab) + \varphi(b)d_{n+1}(a) = -\sum_{k=1}^{n} d_k(a)d_{n+1-k}(b)$$

for all $a, b \in A$; however, this means that $(d_k)_{k=0}^{n+1}$ is a continuous point derivation at φ of order $n + 1$, as required. ∎

In [6, Theorem 3.7], a simple example is given for a compact set X in the complex plane such that $R(X)$ (the uniform closure in $C(X)$ of rational functions with poles off X) admits a continuous point derivation at 0, but also such that there is no continuous non-degenerate second order point derivation of $R(X)$ at 0. It is not difficult to see that the example in [6] has the properties that X is the closure of its interior, and that $R(X) = A(X)$. Hence $R(X) = A(U)$, where U denotes the interior of X, and the following theorem is thus a consequence of Lemma 6.4 and the result in [6].

THEOREM 6.5. *There exists an open, bounded set $U \subseteq \mathbb{C}$ such that, for some $w \in \partial U$, $\mathcal{H}^2_s(A(U), \mathbb{C}_w) \neq \{0\}$.*

We see from this example and from Lemma 2.2 that the maximal ideal M_w corresponding to a point $w \in \partial U$ need not be flat in A-**Ba-mod**. In fact, we have the following characterization of flat maximal ideals in the one-dimensional case.

THEOREM 6.6. *Let $U \subseteq \mathbb{C}$ be an open, bounded set; let $w \in \overline{U}$ and let M_w be the maximal ideal in $A = A(U)$ corresponding to w. Then the following are equivalent:*

(i) *M_w is flat in A-**mod**;*
(ii) *$H^2(A, \mathbb{C}_w) = \{0\}$;*
(iii) *$\dim M_w/M_w^2 \le 1$;*
(iv) *M_w is projective in A-**Ba-mod**;*
(v) *M_w is flat in A-**Ba-mod**;*

(vi) $\mathcal{H}^2(A, \mathbb{C}_w) = \{0\}$;

(vii) *w is a peak point for A, or there exists an open neighbourhood V of w such that each $f \in A$ is analytic on V.*

Proof. (i)⇒(ii) and (v)⇒(vi). These follow from Lemma 2.2.

(ii)⇒(iii). This follows from the simple fact that, for two linear functionals ϕ and ψ on A which vanish on $M_w^2 \oplus \mathbb{C}1_A$, the cocycle $f \otimes g \mapsto \phi(f)\psi(g)$ is a coboundary if and only if ϕ and ψ are collinear. (See [17, Proposition 3] for an analogous argument in the continuous case.)

(iii)⇒(vii). In the case where $M_w = M_w^2$, w is isolated in the norm topology on $\overline{U}$ induced by A. Hence (see [2, Corollary 3.3.10 and p. 205]) w is a peak point for A. In the case where $M_w \neq M_w^2$, we see from [4, Theorem 3.1] that all powers of M_w are closed, and that $\dim M_w^{k+1}/M_w^k = 1$ for every $k \geq 1$. By the main theorem in [19], therefore, w is the center of a one-dimensional analytic disc. Using the fact that the maximal ideal space of A is $\overline{U}$ (see [2, Theorem 3.5.7]), we conclude that there exists an open neighbourhood V of w such that each $f \in A$ is analytic on V.

(iv)⇒(v). This follows from the general theory (cf. [7]).

(vi)⇒(vii). In the case where $M_w = \overline{M_w^2}$, the product map π from $M_w \widehat{\otimes}_p M_w$ into M_w (which maps an elementary tensor $f \otimes g$ into fg) is onto by [17, Proposition 1]. It follows that w is isolated in the norm topology on $\overline{U}$ induced by A. Hence, as above, w is a peak point for A. In the case where $M_w \neq \overline{M_w^2}$, w is the center of a one-dimensional analytic disc by [17, Theorem 2].

(vii)⇒(iv) and (vii)⇒(i). In the case where w is a peak point, (iv) is true by [8, Theorem 1]; moreover, M_w has a bounded approximate identity in this case, and therefore (i) holds by the result in [24]. In the case where, for some open neighbourhood V of w, $f|V$ is analytic for each $f \in A$, M_w is a principal ideal. Then (i) is easily verified, and (iv) is true by [16, Theorem 1]. ∎

REMARK. It would be interesting to know whether the conditions in Theorem 6.6 are equivalent to the vanishing of the "symmetric" group $H_s^2(A, \mathbb{C}_w)$, or of $\mathcal{H}_s^2(A, \mathbb{C}_w)$.

7. Summary.

The results attained in the previous sections allow us to prove the following general theorem about the calculation of the Hochschild groups for $A = A(U)$, and about the splitting of extensions of this algebra.

THEOREM 7.1. *Let $U \subseteq \mathbb{C}^N$ be an open, bounded set, and let $A = A(U)$. Suppose that either*

(i) *U is a strictly pseudoconvex domain with C^2-boundary, or*

(ii) *$U = U_1 \times \ldots \times U_N$ is a product domain and, for every $1 \leq i \leq N$, each $w \in \partial U_i$ is a peak point for $A(U_i)$.*

Then:

(a) *Let $w \in U$. For each $\mu \in Z^2(A, \mathbb{C}_w)$, there exists a linear functional S on A such that $\mu - \delta^1 S$ is a linear combination of 2-cocycles of the form*

$$f \otimes g \mapsto \frac{\partial f}{\partial z_i}(w)\frac{\partial g}{\partial z_j}(w),$$

where $1 \leq i < j \leq N$. In the case where μ is continuous, S may be chosen to be continuous. In particular, $H^2(A, \mathbb{C}_w) = \mathcal{H}^2(A, \mathbb{C}_w) = \{0\}$ in the case where $N = 1$.

(b) *Let $w = (w_i)_{i=1}^N \in \partial U$. In the case (i), $H^2(A, \mathbb{C}_w) = \mathcal{H}^2(A, \mathbb{C}_w) = \{0\}$. In case (ii),*

$$H^2(A, \mathbb{C}_w) \cong \mathcal{H}^2(A, \mathbb{C}_w) \cong \mathbb{C}_w^{\oplus\binom{k}{2}},$$

where $k = \operatorname{card}\{i : w_i \in U_i\}$.

(c) *For each $w \in \overline{U}$, $H_s^2(A, \mathbb{C}_w) = \mathcal{H}_s^2(A, \mathbb{C}_w) = \{0\}$.*

(d) *The following statements are equivalent:*

- *Each finite-dimensional Banach algebra extension of A splits strongly.*
- *Each finite-dimensional algebraic extension of A splits algebraically.*
- *$N = 1$.*

(e) *In case (i), each commutative, finite-dimensional Banach algebra (respectively, algebraic) extension of A splits strongly (respectively, algebraically). The same is true in case (ii) if the maximal ideal space of A is $\overline{U}$.*

Proof. Assertion (a) follows from Theorem 3.7, (b) follows from Corollary 5.2 and Corollary 5.6, and (c) is a consequence of Corollary 4.3, Corollary 5.2, and Corollary 6.3. In case (i), the maximal ideal space of $A(U)$ is $\overline{U}$ (see [18, Theorem VII.2.1] and [13, Theorem 7.2.10]). We have already observed that this is also true, for arbitrary U, in the case where $N = 1$. Hence we see, from [1, Theorem 4.4], that (d) (respectively, (e)) follows from (a) and (b) (respectively, (c)). ∎

The partial positive result in part (e) of the theorem notwithstanding we have the following counterexample.

THEOREM 7.2. *There exists an open, bounded set $U \subseteq \mathbb{C}$ such that $A(U)$ admits a one-dimensional, commutative Banach algebra extension which does not split strongly.*

Proof. This follows from Theorem 6.5 and [1, Theorem 4.4]. ∎

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