Hochschild cohomology groups of certain algebras of analytic functions with coefficients in one-dimensional bimodules

by

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Abstract. We compute the algebraic and continuous Hochschild cohomology groups of certain Fréchet algebras of analytic functions on a domain $U$ in $\mathbb{C}^n$ with coefficients in one-dimensional bimodules. Among the algebras considered, we focus on $A = A(U)$. For this algebra, our results apply if $U$ is smoothly bounded and strictly pseudoconvex, or if $U$ is a product domain.

1. Introduction. Let $U \subseteq \mathbb{C}^n$ be an open, bounded set, and let $A = A(U)$ be the Banach algebra of analytic functions on $U$ which are continuously extendable to the boundary of $U$. For each Banach $A$-bimodule $X$, the second continuous (respectively, algebraic) Hochschild cohomology group $\mathcal{H}^2(A, X)$ (respectively, $H^2(A, X)$) of $A$ with coefficients in $X$ is defined (see [1] and [7]); there is a natural correspondence between the elements of this group and the equivalence classes of Banach (respectively, algebraic) extensions of $A$ by $X$. If $X$ is symmetric, then $\mathcal{H}^2(A, X)$ (respectively, $H^2(A, X)$) contains the subgroup $\mathcal{H}_c^2(A, X)$ (respectively, $H^2_c(A, X)$) corresponding to the commutative Banach (respectively, algebraic) extensions of $A$ by $X$.

The purpose of this note is the computation of these groups for one-dimensional $X$. It is known ([1], Proposition 4.3) that $\mathcal{H}^2(A, X)$ and $H^2(A, X)$ vanish unless $X$ is unital and symmetric, and it thus suffices to consider the case where $X = C$ and the module action is given by a character $\varphi$ on $A$, so that $x \cdot f = f \cdot x = \varphi(f)x$ for $f \in A$ and $x \in C$. We shall confine ourselves to the case where $\varphi$ is the evaluation at a point $w \in U$; we denote the corresponding module by $C_w$.

Some of our results apply to certain Fréchet algebras $A$ other than $A(U)$ for which there are continuous embeddings $P_n \hookrightarrow A \hookrightarrow O(U)$, where $P_n$ is the polynomial algebra in $n$ complex variables and $O(U)$ denotes the
Fréchet algebra of functions analytic on \( U \); such algebras will be called Fréchet algebras of analytic functions on \( U \).

It is well known that the continuous (respectively, algebraic) Hochschild cohomology groups of \( A \) with coefficients in \( C_w \) may be calculated by using projective resolutions of \( C_w \) in the category of Banach or Fréchet (respectively, algebraic) left \( A \)-modules. One such resolution is given by the Koszul complex, which will play a major rôle in our calculations.

Hochschild cohomology for Fréchet algebras of analytic functions has been studied by several authors, notably J. L. Taylor [21] and A. Ya. Helemskiĭ [7, 8, 9]. We wish to mention two known results that are directly relevant to the purpose of this note.

In the case where \( U \) is a domain of holomorphy and \( A = O(U) \), the embedding \( P_n \to A \) is a localization in the sense of [21, Definition 1.2], and the results in [21] yield a complete description of both the algebraic and the continuous Hochschild cohomology of \( A \) with coefficients in one-dimensional modules (see §3).

In the case where \( A \) is a uniform algebra and the maximal ideal \( M_\varphi \) of \( A \) corresponding to a character \( \varphi \) on \( A \) admits the decomposition \( M_\varphi = I + J \), where \( I \) is a Koszul ideal in the sense of [8, p. 226], \( J \) is the kernel of a peak set and a certain condition on the interrelation between \( I \) and \( J \) is satisfied, a special case of a projective resolution of \( C_\varphi \) of the Koszul type exists ([8, Lemma 3.6]). This result is our main source of inspiration in §5, where we use it, in a slightly generalized version, to study the case where \( A \) is the algebra \( A(U) \) and \( U \) is a product domain.

This paper is organized as follows. In §2 we clarify the notation we use and give an account of the notion of a Koszul complex and its basic properties. In §3 and §4 we consider the case where \( w \in U \). The results in these two sections apply to a wider class of Fréchet algebras of analytic functions on \( U \) and to the groups \( H^m(A, C_w) \) and \( H^m(A, C_w) \) for arbitrary \( m \). In §3 we show that there are natural embeddings \( C_w \to H^m(A, C_w) \) and \( C_w \to H^m(A, C_w) \), and give sufficient conditions for these embeddings to be surjective. In §4 we consider the “symmetric” groups \( H^m(A, C_w) \) and \( H^m(A, C_w) \), and show that they vanish in certain cases. In §§5 and §6 we discuss the case where \( A = O(U) \) and \( w \in \partial U \), the boundary of \( U \). We give a partial result that is applicable in the case where the maximal ideal \( M_w \) corresponding to \( w \) has a decomposition \( M_w = I + J \), with \( I \) and \( J \) ideals of \( A \) which satisfy certain conditions; these conditions are similar to, but less restrictive than, those in the aforementioned lemma in [8]. In particular, we obtain a sufficient condition for the vanishing of the “symmetric” groups \( H^m(A, C_w) \) and \( H^m(A, C_w) \), which is not too restrictive in the case where \( U \) is a product domain. On the other hand, we demonstrate that there are examples where \( H^2(A, C_w) \) is non-trivial. Finally, in §7 we give a summary of the results we have obtained.

2. Preliminaries. Let \( U \subseteq C^N \) be an open set. We write \( O(U) \) for the Fréchet algebra of analytic functions on \( U \). If \( U \) is bounded, then we use \( A(U) \) to denote the Banach algebra of analytic functions on \( U \) which are continuously extendable to \( U \). We say that a subalgebra \( A \) of \( O(U) \) is a Fréchet algebra of analytic functions on \( U \) if \( A \) contains the polynomials and \( A \) is a Fréchet algebra for a topology which is finer than the compact-open topology on \( O(U) \).

We recall some notation and basic facts used in homology theory. For general background in homological algebra, we refer to [3], [23] and, for the continuous case, to [7] and [20].

Let \( K \) be a subcategory of the category of linear spaces and operators. A (chain) complex \( F = (F, d) \) in \( K \) is a sequence of objects and morphisms

\[
F : \ldots \to F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \to \ldots
\]

in \( K \) such that \( d_n \circ d_{n+1} = 0 \) for all \( n \). The elements of \( \ker d_n \) (respectively, of \( \operatorname{im} d_{n+1} \)) are called \( n \)-cycles (respectively, \( n \)-boundaries). The homology of \( F \) at \( F_n \) is defined as \( H_n(F) = \ker d_n / \operatorname{im} d_{n+1} \). If \( H_n(F) = 0 \) for all \( n \), then \( F \) is called exact. The complex \( F \) is positive if \( F_n = 0 \) for all \( n < 0 \). If \( (F, d) \) and \( (F', d') \) are chain complexes in the category of linear (respectively, Fréchet) spaces, then a morphism (respectively, continuous morphism) of chain complexes of \( F \) into \( F' \) is an indexed set \( \alpha = (\alpha_n) \) of linear (respectively, continuous linear) operators \( \alpha_n : F_n \to F'_n \) such that \( \alpha_{n-1} \circ d_n = d_n \circ \alpha_n \) for all \( n \).

Let \( M \) be an object in \( K \). A complex over \( M \) (in \( K \)) is a positive chain complex \( F = (F, d) \) in \( K \) together with a morphism \( \varepsilon : F_0 \to M \) (called an augmentation) such that \( \varepsilon \circ d_1 = 0 \).

We use similar terminology in the case of cochain complexes.

Let \( A \) be a commutative, unital algebra. (All algebras considered are complex and associatively.) A unital left module over \( A \) is termed an \( A \)-module. We use \( A \text{-mod} \) to denote the category of \( A \)-modules and \( A \)-module maps. For the \( A \)-modules \( M \) and \( N \), we write \( \operatorname{Hom}_A(M, N) \) for the vector space of all \( A \)-module maps from \( M \) to \( N \), and we write \( M \otimes_A N \) for the tensor product of \( M \) and \( N \) over \( A \). Note that, since \( A \) is commutative, \( \operatorname{Hom}_A(M, N) \) and \( M \otimes_A N \) are \( A \)-modules for the operations

\[
(a \cdot \varphi)(m) = a \cdot \varphi(m) \quad \text{and} \quad a \cdot (m \otimes n) = (a \cdot m) \otimes n,
\]

where \( a \in A \), \( m \in M \), \( n \in N \) and \( \varphi \in \operatorname{Hom}_A(M, N) \). As is customary, we use \( \operatorname{Ext}_A(\cdot, N) : A \text{-mod} \to A \text{-mod} \) to denote the left-derived cofunctor of the cofunctor \( \operatorname{Hom}_A(\cdot, N) \). An \( A \)-module \( P \) is projective (respectively, flat)
if the functor $\text{Hom}_A(P, -)$ (respectively, $- \otimes_A P$) is exact; that is, if it maps exact complexes into exact complexes.

Let $X$ be an $A$-module. A complex $\mathcal{F}$ over $X$ with augmentation $\varepsilon : F_0 \to X$ is called a resolution of $X$ (in $A$-mod) if the complex

$$
\mathcal{F} : 0 \to X \to 0 : \ldots \to F_i \overset{d_i}{\to} F_{i-1} \to \ldots \to d_1 F_0 \overset{\delta}{\to} X \to 0
$$

is exact. Such a resolution is projective (respectively, flat) if the $F_n$ are projective (respectively, flat).

Let $M$ be an $A$-module. We write $M^{\oplus i}$ ($i = 1, 2, \ldots$) for the direct sum of $i$ copies of $M$. Furthermore we use $AM$ to denote the exterior algebra over $M$, and we write $\wedge$ for the product in $AM$. Recall that $AM = \bigoplus_{n=0}^{\infty} A^n M$ is a graded $A$-algebra, where $A^n M$ is the $n$th exterior product of $M$ over $A$.

Let $a = (a_1, \ldots, a_n)$ be a finite sequence of elements in $A$. Then the Koszul complex $K_a$ ([23, p. 111]) is defined; we sometimes write $K(a_1, \ldots, a_n)$ instead of $K(a)$. Recall that the degree $p$ part $K_p(a)$ of $K(a)$ is the $p$th exterior product $A^p A^n M$ over $A$. Let $e_1, \ldots, e_n$ be the canonical basis of $A^n M$. Then the set of all elements $e_{i_1} \wedge \ldots \wedge e_{i_n}$, where $1 \leq i_1 < \ldots < i_n \leq n$, is a basis of $K_p(a)$; in particular, $K_p(a)$ is a free $A$-module of rank $\binom{n}{p}$. The differential $d_a$ from $K_p(a)$ to $K_{p-1}(a)$ maps $e_{i_1} \wedge \ldots \wedge e_{i_p}$ to $\sum_{k=1}^{\binom{n}{p}} (-1)^{k-1} e_{i_1} e_{i_{k+1}} \wedge \ldots \wedge e_{i_k} \wedge \ldots \wedge e_{i_p}$; here $\delta$ signifies that $e_i$ is omitted in the product. We write $H_n(K(a))$ for the homology of $K(a)$. Note that, in all cases,

$$H_0(K(a)) = A/(a_1, \ldots, a_n),$$

where $(a_1, \ldots, a_n)$ is the ideal generated by $a_1, \ldots, a_n$.

Recall that an element $a$ of $A$ is called a non-zero divisor if $ab$ is non-zero for each non-zero element $b$ of $A$. A finite sequence $a_1, \ldots, a_n$ of elements in $A$ is called a regular sequence on $A$ ([23, p. 105]) if the equivalence class $\bar{a}_j$ in $A/(a_1, \ldots, a_{j-1})$ is a non-zero divisor for each $j$. (For $j = 1$ this means that $a_1$ is a non-zero divisor.) We shall use the following elementary theorem of homological algebra (see [15, 23]).

**Theorem 2.1.** Suppose that $a = (a_1, \ldots, a_n)$ is a regular sequence on a commutative, unital algebra $A$. Then the Koszul complex $K(a)$ provides a free resolution of length $n$ of $A/(a_1, \ldots, a_n)$.

Let $X$ be a unital $A$-bimodule. Then the Hochschild complex (see [23, p. 301])

$$0 \to X \overset{\delta}{\to} \text{Hom}_C(A, X) \overset{\delta}{\to} \text{Hom}_C(A \otimes_C A, X) \overset{\delta}{\to} \ldots$$

of $A$ and $X$ is defined. Here $\delta = (\delta^i)_{i \in \mathbb{N}}$ is the coboundary operator. The cohomology groups of this complex are the algebraic Hochschild cohomology groups $H^n(A, X)$ ($n = 0, 1, \ldots$) of $A$ with coefficients in $X$. Following customary notation, we write $Z^n(A, X)$ for $\ker \delta^n$ and $B^n(A, X)$ for the range of $\delta^{n-1}$, respectively. Therefore $H^n(A, X) = Z^n(A, X)/B^n(A, X)$, where $Z^n(A, X)$ (respectively, $B^n(A, X)$) is the space of $n$-cocycles (respectively, of $n$-coboundaries) of $A$ with coefficients in $X$.

Let $\varphi : A \to C$ be a character on $A$. Then $C \otimes C$ is a $A$-bimodule for the operations $a \cdot z = z \cdot a = \varphi(a) z$ ($a \in A, z \in C$). We call $C_{\varphi}$ the $A$-bimodule corresponding to $\varphi$. It is known (see [23, Lemma 9.1.9]) that

$$H^n(A, C_{\varphi}) = \text{Ext}^n_A(C, C_{\varphi}) \text{ for } n \geq 0$$

in $A$-mod; here $C_{\varphi}$ is regarded as a left $A$-module on the right-hand side of the equation.

By a Fréchet space we mean a complete metrizable locally convex space; a Fréchet algebra is a complex algebra which is a Fréchet space such that multiplication is jointly continuous. Let $E$ and $F$ be Fréchet spaces. Then we use $E \otimes F$ to denote the algebraic tensor product of $E$ and $F$ endowed with the projective tensor product topology. The completion $E \hat{\otimes} F$ of $E \otimes F$ is the projective tensor product of $E$ and $F$ and it has the usual universal property of the tensor product (cf. [7, II.4]). A continuous linear map $\varphi : E \to F$ is called admissible if ker $\varphi$ is complemented in $E$ and im $\varphi$ is closed and complemented in $F$.

Let $A$ be a commutative, unital Fréchet algebra. A unital left Fréchet module over $A$ is termed a Fréchet $A$-module. We use $A$-Fr-mod to denote the category of Fréchet $A$-modules and continuous $A$-module maps.

Let $M, N$ be Fréchet $A$-modules. We write $\text{Hom}_{A, cont}(M, N)$ for the $A$-module of continuous $A$-module maps from $M$ into $N$. We also write $\text{Ext}_{A, cont}(-, N)$ for the left-derived cofunctor of

$$\text{Hom}_{A, cont}(-, N) : A$-Fr-mod \to A$-mod.$$

A (chain) complex $\mathcal{F}$ in $A$-Fr-mod is called admissible if it splits as a complex of Fréchet spaces. A Fréchet $A$-module $P$ is projective ([7, III.1.13]) if, for every admissible complex $\mathcal{F}$, the complex $\text{Hom}_{A, cont}(P, \mathcal{F})$ is exact.

Let $X$ be a Fréchet $A$-module. A complex $\mathcal{F}$ over $X$ with augmentation $\varepsilon : F_0 \to X$ is called a resolution (in $A$-Fr-mod) if the complex $\mathcal{F} \overset{\varepsilon}{\to} X \to 0$ is admissible. If every module in $\mathcal{F}$ is projective, then such a resolution is called a projective resolution of $X$ (in $A$-Fr-mod).

Let $X$ be a unital Fréchet $2$-bimodule. Then the Hochschild–Kunzowitz complex (see [7, I.3.2])

$$0 \to X \overset{\delta}{\to} \text{Hom}_{C, cont}(A, X) \overset{\delta}{\to} \text{Hom}_{C, cont}(A \hat{\otimes} C, A, X) \overset{\delta}{\to} \ldots$$

of $A$ and $X$ is defined. The cohomology groups of this complex are the continuous Hochschild cohomology groups $H^n(A, X)$ ($n = 0, 1, \ldots$) of $A$.
with coefficients in $X$. We write $\mathcal{Z}^n(A, X)$ for $\ker \partial^n$ and $B^n(A, X)$ for the range of $\delta^{n-1}$, respectively. Hence $\mathcal{H}^n(A, X) = \mathcal{Z}^n(A, X)/B^n(A, X)$, where $\mathcal{Z}^n(A, X)$ (respectively, $B^n(A, X)$) is the space of continuous $n$-cocycles (respectively, of continuous $n$-coboundaries) of $A$ with coefficients in $X$.

There is a natural embedding of $\mathcal{Z}^n(A, X)$ in $\mathcal{Z}^n(A, X)$, and this map induces an obvious comparison map $\iota_n : \mathcal{H}^n(A, X) \to H^n(A, X)$. Note that, since $A$ is commutative, $\mathcal{H}^n(A, X)$ and $H^n(A, X)$ are $A$-modules, and $\iota_n$ is an $A$-module map.

Assume that $X$ is symmetric, i.e. $\alpha \cdot x = x \cdot \alpha$ for $\alpha \in A$ and $x \in X$. Then the symmetric 2-cocycles (respectively, the continuous symmetric 2-cocycles) form a subspace of $\mathcal{Z}^2(A, X)$ (respectively, of $\mathcal{Z}^2(A, X)$) which is denoted by $\mathcal{Z}_s^2(A, X)$ (respectively, by $\mathcal{Z}_c^2(A, X)$). The quotient

$$H_s^n(A, X) = \mathcal{Z}_s^n(A, X)/B^n(A, X)$$

(respectively, $H_c^n(A, X) = \mathcal{Z}_c^n(A, X)/B^n(A, X)$) is the second symmetric (respectively, the second continuous symmetric) Hochschild cohomology group of $A$ with coefficients in $X$ (see [I, 23]).

Let $\varphi : A \to C$ be a continuous character on $A$. Then $C = \mathbb{C}_\varphi$ is a Fréchet $A$-bimodule, and in analogy to (2.2) we see (cf. [7, III.4.12]) that

$$\mathcal{H}^n(A, \mathbb{C}_\varphi) = \text{Ext}_{A, \text{cont}}^n(\mathbb{C}_\varphi, \mathbb{C}_\varphi)$$

for $n \geq 0$

in the category $A$-$\text{mod}$.

Let $A$ be a commutative, unital Banach algebra. Denote by $A$-$\text{Ba-mod}$ the subcategory of $A$-$\text{Fr-mod}$ consisting of unital left Banach $A$-modules. A module $P$ in $A$-$\text{Ba-mod}$ is flat ([7, VII.1.1]) if, for every admissible complex $\mathcal{F}$, the complex $\mathcal{F} \otimes_A P$ is exact; here $\otimes_A$ denotes the tensor product of Banach $A$-modules [cf. [7, II.3]]. A resolution $\mathcal{F} \to X \to 0$ over a module $X$ in $A$-$\text{Ba-mod}$ is flat if every module in $\mathcal{F}$ is flat in $A$-$\text{Ba-mod}$.

According to (2.2) (respectively, (2.3)), we may compute $H^n(A, \mathbb{C}_\varphi)$ (respectively, $\mathcal{H}^n(A, \mathbb{C}_\varphi)$) by using projective resolutions of $\mathbb{C}_\varphi$ in the category $A$-$\text{mod}$ (respectively, $A$-$\text{Fr-mod}$). However, less is needed, as the next elementary lemma shows. For a proof of part (ii) of this lemma, we refer to [10]. Although we do not have an exact reference, the result in part (i) is surely well known to the specialist.

**Lemma 2.2.** Let $A$ be a commutative, unital Banach algebra, and let $\varphi$ be a character on $A$. Then:

(i) The $A$-modules $\text{Ext}_{A, \text{cont}}^n(\mathbb{C}_\varphi, \mathbb{C}_\varphi)$ may be computed by using flat resolutions. That is, if $\mathcal{F} \to \mathbb{C}_\varphi$ is a flat resolution of $\mathbb{C}_\varphi$ in $A$-$\text{mod}$, then

$$\text{Ext}_{A, \text{cont}}^n(\mathbb{C}_\varphi, \mathbb{C}_\varphi) \cong H^n(\text{Hom}_A(\mathcal{F}, \mathbb{C}_\varphi))$$

for $n = 0, 1, \ldots$;

(ii) The $A$-modules $\text{Ext}_{A, \text{cont}}^n(\mathbb{C}_\varphi, \mathbb{C}_\varphi)$ may be computed by using flat resolutions of $\mathbb{C}_\varphi$ in the category $A$-$\text{Ba-mod}$.

**Proof of (i).** For each $A$-module $M$, the algebraic dual $\text{Hom}_A(M, \mathbb{C})$ is an $A$-module for the operation

$$(a \cdot \varphi)(m) = \varphi(a \cdot m) \quad (a \in A, \varphi \in \text{Hom}_A(M, \mathbb{C}), m \in M).$$

It is a basic result of homological algebra (see [3, VI.8.1], for example) that, in the category $A$-$\text{mod}$, $M$ is flat if and only if $\text{Hom}_A(M, \mathbb{C})$ is injective.

Now let $\mathcal{F} \to \mathbb{C}_\varphi$ be a flat resolution of $\mathbb{C}_\varphi$ in $A$-$\text{mod}$, and let $n \geq 0$ be fixed. Since the functor $\text{Hom}_A(\mathbb{C}_\varphi, \mathbb{C})$ is exact, $\text{Hom}_A(\mathcal{F}, \mathbb{C}_\varphi)$ is a resolution of $\mathbb{C}_\varphi$. This resolution is injective on the basis of the results attained in the previous paragraph. Thus $\text{Ext}_{A, \text{cont}}^n(\mathbb{C}_\varphi, \mathbb{C}_\varphi)$ is the nth cohomology of the complex $\mathcal{G} = \text{Hom}_A(\mathbb{C}_\varphi, \text{Hom}_A(\mathcal{F}, \mathbb{C}_\varphi))$. However, it is obvious that the functors $\text{Hom}_A(\mathbb{C}_\varphi, \mathbb{C}_\varphi)$ and $\text{Hom}_A(\mathcal{F}, \mathbb{C}_\varphi)$ are isomorphic. We conclude that

$$\text{Ext}_{A, \text{cont}}^n(\mathbb{C}_\varphi, \mathbb{C}_\varphi) = H^n(\mathcal{G}) \cong H^n(\text{Hom}_A(\mathcal{F}, \mathbb{C}_\varphi))$$

as required. ■

Let $\varphi$ be a continuous character on a commutative, unital Fréchet algebra $A$. We shall be interested in the question of when the comparison map

$$\iota_n : \mathcal{H}^n(A, \mathbb{C}_\varphi) \to H^n(A, \mathbb{C}_\varphi)$$

is injective. In the case where $n = 1$, this is always true; in the case where $n = 2$, we have the following proposition, which is a straightforward generalization of [7, I.1.19] (see also [1, Theorem 2.16]).

**Proposition 2.3.** Let $A$ be a unital, commutative Fréchet algebra, and let $\varphi$ be a continuous character on $A$. Set $M_\varphi = \ker \varphi$. Then the comparison map

$$\iota_2 : H^2(A, \mathbb{C}_\varphi) \to H^2(A, \mathbb{C}_\varphi)$$

is injective if and only if the product map

$$A : M_\varphi \otimes \mathbb{C}_\varphi \to M_\varphi : f \otimes g \mapsto fg$$

is open.

**Proof.** First assume that $A$ is open. Let $\psi$ be a linear functional on $A$ such that $\mu = \delta^2 \psi$ is continuous. Let $\psi_0$ be the restriction of $\psi$ to $M_\varphi$, and let $\mu_0$ be the restriction of $\mu$ to $M_\varphi \otimes \mathbb{C}_\varphi$. Then $\mu_0 = -\psi_0 \circ A$. But $A$ is open and $\mu_0$ is continuous, and so $\psi_0$ is continuous. Let $\psi_1$ be a continuous extension of $\psi_0$ to $A$ such that $\psi_1(1_A) = \mu(1_A \otimes 1_A)$. Then $\mu = \delta^2 \psi_1 \in B^2(A, \mathbb{C}_\varphi)$. This shows that $\mathcal{Z}^2(A, \mathbb{C}_\varphi) \cap B^2(A, \mathbb{C}_\varphi) = B^2(A, \mathbb{C}_\varphi)$, and hence $\iota_2$ is injective.

Now suppose that $A$ is not open. Set $E = M_\varphi^2$. Since $A$ is a Fréchet algebra, $E$ is a metrizable locally convex space. Hence (see [22, Proposition II.36.3]) the topology on $E$ is identical to the Mackey topology. Let $\sigma$ denote the quotient topology on $E$ induced by $A$. Then $\sigma$ is strictly finer than the
original topology on $E$ because $A$ is not open. However, the Mackey topology is the finest locally convex vector space topology which is compatible with the duality between $E$ and $E'$, and so there exists a discontinuous linear functional $\psi$ on $E$ which is continuous with respect to $\sigma$. We may extend $\psi$ to a linear functional on $A$ which we also denote by $\psi$. Set $\mu = \delta_1 \psi$. Then $\mu \in Z^2(A, C_\varphi)$ because $\psi \circ A$ is continuous. Set $\alpha = \mu + B^1(A, C_\varphi)$. Then $\iota_2(\alpha) = 0$, but $\alpha \neq 0$ because $\psi|E$ is discontinuous. Thus $\iota_2$ is not injective.

**Corollary 2.4.** In the situation of Proposition 2.3, suppose that $M_\varphi$ is algebraically finitely generated. Then the comparison map $\iota_2 : H^2(A, C_\varphi) \to H^2(A, C_\varphi)$ is injective.

**Proof.** Let $A$ be the map considered in Proposition 2.3. Set $F = M_\varphi \otimes \varphi \otimes M_\varphi / \ker \lambda$, and let $\tilde{A} : F \to M_\varphi^2$ be the induced map. Let $\{b_1, \ldots, b_m\}$ be a finite set of generators for $M_\varphi$. Define maps $g : M_\varphi^m \to M_\varphi^2$ by $g(a_1, \ldots, a_m) = \sum_{i=1}^m a_i b_i$ and $\tau : M_\varphi^m \to F$ by $\tau(a_1, \ldots, a_m) = \sum_{i=1}^m a_i \otimes b_i + \ker \lambda$. Then $g$ and $\tau$ are continuous surjections, and $\tilde{A} \circ \tau = g$. Clearly, $M_\varphi^2$ has finite codimension in $A$; in addition, it is the continuous image of a Fréchet space. Hence an obvious application of the Open Mapping Theorem gives that $M_\varphi^2$ is closed. Therefore, again by the Open Mapping Theorem, $g$ is open. It follows that $\tilde{A}$, and hence also $A$, is open. By Proposition 2.3, $\iota_2$ is injective.

We end this section with the following theorem which we shall need in §3. The main argument in the proof is well known in the literature (see [3, Theorem VIII-4.2], for example). Nonetheless, we provide a short proof here for the sake of completeness.

**Theorem 2.5.** Let $\varphi$ be a continuous character on a unital, commutative Fréchet algebra $A$. Let $a = (a_1, \ldots, a_n)$ be a finite sequence of elements in $A$, and let $K = K(a)$ be the corresponding Koszul complex. Suppose that 

$$
(2.4) \quad \langle a_1, \ldots, a_n \rangle = \ker \varphi, \\
(2.5) \quad H_j(K) = \{0\} \text{ for } j > 0.
$$

Then 

$$
\dim_C(H^m(A, C_\varphi)) = \binom{n}{m} \quad (m \geq 0).
$$

Moreover, $K$ is a complex in the category $A$-$\text{Fr-mod}$; if each differential of $K$ is admissible, then also 

$$
\dim_C(H^m(A, C_\varphi)) = \binom{n}{m} \quad (m \geq 0).
$$

**Proof.** Let $m \geq 0$, and let $F$ denote the complex $\text{Hom}_A(K, C_\varphi)$. We see from (2.1), (2.4) and (2.5) that $K$ is a projective resolution of $C_\varphi$ in $A$-$\text{mod}$. Thus, by (2.2), the $A$-module $H^m(A, C_\varphi)$ is the $m$th cohomology $H^m(F)$ of $F$. However, it follows from (2.4) that each morphism in $F$ is zero. Hence $H^m(F)$ is the degree $m$ part $F_m$ of $F$. Since, in addition, $K_m(a)$ is a free $A$-module of rank $\binom{n}{m}$, we conclude that 

$$
H^m(A, C_\varphi) = F_m = \text{Hom}_A(K_m(a), C_\varphi) \cong \mathbb{C}^{\binom{n}{m}}.
$$

In particular, we see that $\dim_C(H^m(A, C_\varphi)) = \binom{n}{m}$, and the first part of the theorem is proved.

Each module in $K$ is a finite direct sum of copies of $C_\varphi$ and therefore a projective (even free) Fréchet $A$-module. Furthermore, it is obvious that each differential of $K$ is continuous. Thus $K$ is a complex in $A$-$\text{Fr-mod}$ consisting of projective modules. If each differential of $K$ is admissible, then $K$ is a projective resolution of $C_\varphi$ in $A$-$\text{Fr-mod}$, and we may finish the proof by using the same arguments as in the algebraic case; indeed, all we have to do is to substitute (2.3) for (2.2) and $\text{Hom}_{A, \text{cont}}(-, -)$ for $\text{Hom}_{A}(-, -)$.

3. The calculation of $H^n(A, C_w)$ and $H^n(A, C_w)$ in the case where $w \in U$. Let $A$ be a Fréchet algebra of analytic functions on an open set $U \subseteq \mathbb{C}^N$, and let $w = (w_1, \ldots, w_N) \in U$ be fixed. We wish to determine the $A$-modules $H^n(A, C_w)$ and $H^n(A, C_w)$, where $C_w$ is the $A$-bimodule corresponding to the evaluation at $w$; we are especially interested in the case where $n = 2$.

Suppose that $N = 2$. Then it is trivial to verify that the map 

$$
f \otimes g \mapsto \frac{\partial f}{\partial z_1}(w) \frac{\partial g}{\partial z_2}(w)
$$

is a continuous cocycle which is not a coboundary, not even in the algebraic sense. Our next proposition generalizes this observation. Namely, we show that, for each $n \leq N$, we have a commuting diagram

$$
\begin{array}{cccc}
\mathbb{C}^{\otimes \binom{n}{m}} & \xrightarrow{\alpha_n} & H^n(A, C_w) \\
\downarrow{\beta_n} & & \downarrow{\iota_n} \\
C_w \oplus \cdots \oplus C_w & \xrightarrow{\otimes^{\binom{n}{m} \times}} & H^n(A, C_w)
\end{array}
$$

where $\alpha_n$ and $\beta_n$ are $A$-linear embeddings which are given by explicit formulas, and where $\iota_n$ is the comparison map. In fact, we prove a slightly more general result.

**Proposition 3.1.** Let $A$ be a Fréchet algebra of analytic functions on an open set $U \subseteq \mathbb{C}^N$. Suppose that $\varphi$ is a continuous character on $A$ which
has an extension to a character on $O(U)$. Let $n \in \{1, \ldots, N\}$, and let $\{e_r\}$ be a vector space basis of $C^\otimes_{\varphi}^{(n)}$, where $\tau$ runs through the set of all strictly increasing maps from $\{1, \ldots, n\}$ to $\{1, \ldots, N\}$. Let

$$\alpha_n : C^\otimes_{\varphi}^{(n)} \to H^n(A, C_\varphi)$$

be the unique linear map which assigns to each $e_r$ the equivalence class of the continuous $n$-cocycle given by

$$f_1 \otimes \ldots \otimes f_n \mapsto \varphi \left( \frac{\partial f_1}{\partial \tau_1} \ldots \frac{\partial f_n}{\partial \tau_n} \right).$$

Furthermore, let $\beta_n = \iota_n \circ \alpha_n$, where $\iota_n$ is the comparison map. Then $\alpha_n$ and $\beta_n$ are $A$-linear embeddings.

**Proof.** It is easy to demonstrate that $\alpha_n$ is a well-defined $A$-linear map. What remains to be shown is that $\beta_n = \iota_n \circ \alpha_n$ is injective. To show this, assume that $\tau_1, \ldots, \tau_r$ is a finite sequence of distinct strictly increasing maps from $\{1, \ldots, n\}$ to $\{1, \ldots, N\}$ such that

$$\beta_n \left( \sum_{i=1}^r \gamma_i e_{\tau_i} \right) = \sum_{i=1}^r \gamma_i \beta_n(e_{\tau_i}) = 0$$

for some $\gamma_1, \ldots, \gamma_r \in C^\otimes_{\varphi}$, we have to show that $\gamma_1 = \ldots = \gamma_r = 0$. We infer from (3.2) that there exists a linear functional $\Delta$ on $A \otimes \ldots \otimes A$ ($n-1$ times) such that

$$\sum_{i=1}^r \gamma_i \Theta(e_{\tau_i}) = \delta^{n-1} \Delta,$$

where, for each $i \in \{1, \ldots, r\}$, the $n$-cocycle $\Theta(e_{\tau_i})$ is defined as in (3.1) (with $\tau$ replaced by $\tau_i$). Let $j \in \{1, \ldots, r\}$ be fixed. For each $i \in \{1, \ldots, n\}$, we set

$$W_i = Z_{\gamma_i} - \varphi(Z_{\gamma_i}),$$

where $Z_1, \ldots, Z_N$ are the coordinate functions. Certainly, we have $W_i \in A$ and $\varphi(W_i) = 0$ for each $i$. Let $S_n$ be the symmetric group on $n$ symbols.

Let $i \in \{1, \ldots, r\}$ and let $\sigma \in S_n$. We then have

$$\Theta(e_{\tau_i})(W_{\sigma(1)} \otimes \ldots \otimes W_{\sigma(n)}) = \begin{cases} 1 & \text{if } i = j \text{ and } \sigma = \text{id}, \\ 0 & \text{otherwise}. \end{cases}$$

We conclude from (3.3) and (3.4) that

$$\delta^{n-1} \Delta(W_1 \otimes \ldots \otimes W_n) = \gamma_j \text{ and } \delta^{n-1} \Delta(W_{\sigma(1)} \otimes \ldots \otimes W_{\sigma(n)}) = 0$$

for every permutation $\sigma \neq \text{id}$. In addition, the functions $W_i$ lie in the kernel of $\varphi$. Hence

$$\gamma_j = \sum_{k=1}^{n-1} (-1)^k \sum_{\sigma \in S_n} \text{sgn}(\sigma) \Delta(W_{\sigma(1)} \otimes \ldots \otimes W_{\sigma(k)}) W_{\sigma(k+1)} \otimes \ldots \otimes W_{\sigma(n)}).$$

However, it is obvious that

$$\sum_{\sigma \in S_n} \text{sgn}(\sigma) \Delta(W_{\sigma(1)} \otimes \ldots \otimes W_{\sigma(k)}) W_{\sigma(k+1)} \otimes \ldots \otimes W_{\sigma(n)}) = 0$$

for all $k = 1, \ldots, n-1$, and consequently $\gamma_j = 0$. This shows that $\beta_n$ is injective, as required. 

In the remainder of this section we consider, for each $n \leq N$, the following two obvious questions concerning the embeddings $\alpha_n$ and $\beta_n$ in Proposition 3.1.

**Question 1.** Is $\beta_n : C^\otimes_{\varphi}^{(n)} \to H^n(A, C_\varphi)$ an isomorphism?

**Question 2.** Is $\alpha_n : C^\otimes_{\varphi}^{(n)} \to H^n(A, C_\varphi)$ an isomorphism?

A simple but useful fact pertinent to these questions is stated in the next proposition.

**Proposition 3.2.** Suppose that Question 1 can be answered affirmatively, and that the comparison map $\iota_n : H^n(A, C_\varphi) \to H^n(A, C_\varphi)$ is injective. Then we have an affirmative answer to Question 2.

**Proof.** This is immediately evident from the fact that $\iota_n \circ \alpha_n = \beta_n$. 

**Notation.** In the following, we use $Z_1, \ldots, Z_N$ to denote the coordinate functions, and we write $Z - w$ for the sequence $(Z_1 - w_1, \ldots, Z_N - w_N)$; recall that $K(Z - w)$ is the Koszul complex for $Z - w$.

Our next proposition constitutes the basis for our attempts to answer Questions 1 and 2.

**Proposition 3.3.** Let $A$ be a Fréchet algebra of analytic functions on an open set $U \subseteq \mathbb{C}^N$, and let $e_w$ be the evaluation at a point $w \in U$. Let

$$\alpha_n : C^\otimes_{\varphi}^{(n)} \to H^n(A, C_\varphi)$$

and $\beta_n : C^\otimes_{\varphi}^{(n)} \to H^n(A, C_\varphi)$

be the $A$-linear embeddings constructed in Proposition 3.1. Suppose that

$$\begin{align*}
(Z_1 - w_1, \ldots, Z_N - w_N) &= \text{ker } e_w, \\
H_j(K(Z - w)) &= \{0\} \quad \text{for } j > 0.
\end{align*}$$

Then $\beta_n$ is an isomorphism for all $n$, and $\alpha_n$ is an isomorphism for $n \leq 2$.

Suppose, furthermore, that each differential of $K(Z - w)$ is admissible. Then $\alpha_n$ is an isomorphism for all $n$. 


Proof. This is immediately evident from Theorem 2.5, Propositions 3.1 and 3.2 and Corollary 2.4. ■

We now give two results which show that (3.5) and (3.6) are satisfied in certain cases. The first is an application of Theorem 2.1 in §2.

**Proposition 3.4.** Let \( A \) be an algebra of analytic functions on an open set \( U \subseteq \mathbb{C}^N \), and let \( w \in U \). Let \( M_k = \{(z_1)_{i=1}^{k} \in \mathbb{C}^N : z_i = w_i \text{ for } 1 \leq i \leq k\} \), \( 1 \leq k \leq N \). Suppose that, for every \( k \in \{1, \ldots, N\} \) and every \( f \in A \) with \( f(U \cap M_k) = \{0\} \), there exist \( g_1, \ldots, g_k \in A \) with \( f = \sum_{i=1}^{k}(z_i - w_i)g_i \). Then (3.5) and (3.6) are satisfied.

Proof. Our hypothesis (for \( k = N \)) implies that (3.5) is satisfied. By Theorem 2.1, (3.6) will follow once we have shown that the sequence \( Z_1 - w_1, \ldots, Z_N - w_N \) is a regular sequence on \( A \). To show this, first note that \( Z_i - w_i \) is a non-zero divisor in \( A \). Now let \( i \in \{2, \ldots, N\} \), and set \( J = (Z_1 - w_1, \ldots, Z_{i-1} - w_{i-1}) \). Suppose that \( (Z_i - w_i)f = 0 \) modulo \( J \) for some \( f \in A \). Then there exist \( g_1, \ldots, g_{i-1} \in A \) such that \( (Z_i - w_i)f = \sum_{j=1}^{i-1}(Z_j - w_j)g_j \). This implies that \( f(U \cap M_{i-1}) = \{0\} \). Thus, according to our hypothesis, there exist \( h_1, \ldots, h_{i-1} \in A \) such that \( f = \sum_{j=1}^{i-1}(Z_j - w_j)h_j \). Hence \( f = 0 \) modulo \( J \). This shows that \( Z_i - w_i \) is a non-zero divisor modulo \( J \), as required. ■

The second tool we shall use to verify (3.5) and (3.6) is the following deep theorem, which is a special case of a more general result stated in the book by J. Eschmeier and M. Putinar ([5]).

Notation. Recall that \( \text{Lip}_\alpha(U) \) is the Banach algebra of functions on \( U \) which satisfy the usual Lipschitz condition of order \( \alpha \). Furthermore, \( C^r(U) \) denotes the algebra of functions \( f \) on \( U \) for which the derivatives \( D^nf \) exist and have a continuous extension to \( U \) for each multiindex \( \alpha \) of order not exceeding \( r \). The algebra \( C^r(U) \) is endowed with the topology of uniform convergence of all derivatives of order not exceeding \( r \). Note that \( C^0(U) = C(U) \) and thus \( C^0(U) \cap \mathcal{O}(U) = A(U) \).

**Theorem 3.5 ([5, Theorem 8.1.1]).** Suppose that \( U \subseteq \mathbb{C}^N \) is a bounded, open set. Let \( B \) be one of the Fréchet algebras \( \text{Lip}_\alpha(U) \) (for some \( 0 < \alpha < 1 \)) or \( C^r(U) \) (for some \( 0 \leq r \leq \infty \)). Suppose further that, for each \( g \in \{1, \ldots, N\} \) and each closed, \( g \)-form \( f \) on \( U \) with coefficients in \( B \), there exists a (0, \(-1\)-form \( w \) on \( U \) with coefficients in \( B \) such that \( \partial w = f \). Then (3.5) and (3.6) are satisfied for \( A = \mathcal{O}(U) \cap B \).

Before we apply these last two results, we give an elementary lemma which we shall use in the case where \( U \) is a product domain.

**Lemma 3.6.** Let \( G \subseteq \mathbb{C} \), \( H \subseteq \mathbb{C}^k \) be bounded, open sets containing the origin, and let \( f \in A(G \times H) \) be such that \( f(0, w) = 0 \) for all \( w \in \bar{H} \). Then there exists \( g \in A(G \times H) \) such that \( f(z, w) = zg(z, w) \) for all \( z \in \bar{G} \) and \( w \in \bar{H} \).

Proof. First note that, for every \( w \in \bar{H} \), the function \( z \mapsto f(z, w) \) is analytic on \( G \); indeed, this follows readily from the uniform continuity of \( f \) and the fact that \( w \) is the limit of a sequence of elements in \( H \). We consider the map

\[
\begin{align*}
g(z, w) &= \begin{cases} \frac{1}{\partial_z f(0, w)} & \text{if } z \in \bar{G} \setminus \{0\} \quad \text{and} \quad w \in \bar{H}, \\
\begin{array}{l}
s \mapsto f(0, w) & \text{if } s = 0 \quad \text{and} \quad w \in \bar{H}.
\end{array}
\end{cases}
\end{align*}
\]

Then \( f(z, w) = zg(z, w) \) for \( z \in \bar{G} \) and \( w \in \bar{H} \), and \( g \) is analytic on \( G \times H \). We need to show that \( g \) is continuous on \( \bar{G} \times \bar{H} \). Certainly, \( g \) is continuous on \( \bar{G} \setminus \{0\} \times \bar{H} \) and on \( \{0\} \times \bar{H} \). Now let \( w \in \bar{H} \), and suppose that \( (w_n, w_n) \) is a sequence in \( \bar{G} \setminus \{0\} \times \bar{H} \) which converges to \( (0, w) \). Choose \( r > 0 \) such that \( \Delta(0, r) = \{z \in \mathbb{C} : |z| < r\} \subseteq G \). We may assume that \( w_n \in \Delta(0, r/2) \) for all \( n \). Then

\[
g(z_n, w_n) = \frac{f(z_n, w_n)}{z_n} = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \frac{\partial^{k+1}f}{\partial z^{k+1}}(0, w_n)z_n^k
\]

for all \( n \). It follows from Cauchy's estimate that

\[
\left| \frac{1}{(k+1)!} \frac{\partial^{k+1}f}{\partial z^{k+1}}(0, w_n)z_n^k \right| \leq \frac{1}{r^{2k}} ||f||_\infty
\]

for all \( n \) and \( k \). We may therefore apply Lebesgue's Dominated Convergence Theorem to deduce that

\[
\lim_{n \to \infty} g(z_n, w_n) = \lim_{n \to \infty} \frac{1}{(k+1)!} \frac{\partial^{k+1}f}{\partial z^{k+1}}(0, w_n)z_n^k = \frac{\partial f}{\partial z}(0, w) = f(0, w),
\]

and so \( g \) is continuous at \( (0, w) \). ■

We can now state the main result of this section, where we apply the above theorems and propositions to certain classes of Fréchet algebras of analytic functions.

**Theorem 3.7.** Let \( A \) be a Fréchet algebra of analytic functions on an open set \( U \subseteq \mathbb{C}^N \). Let \( w \in U \), and let

\[
\alpha_w : \mathcal{H}^{\alpha}(U) \to \mathcal{H}^{\alpha}(A, \mathcal{C}_w) \quad \text{and} \quad \beta_w : \mathcal{H}^{\alpha}(U) \to \mathcal{H}^{\alpha}(A, \mathcal{C}_w)
\]

be the A-linear embeddings constructed in Proposition 3.1. Consider the following cases:

(i) \( U \) is pseudoconvex, and \( A = \mathcal{O}(U) \).
(ii) \( U \) is a bounded product domain, and \( A = A(U) \).
(iii) $U$ is bounded and strictly pseudoconvex with $C^2$-boundary, and $A = O(U) \cap B$, where $B$ is one of the Banach algebras $\text{Lip}_\alpha(U)$ (for some $0 < \alpha < 1$) or $C^r(U)$ (for some $0 \leq r \leq \infty$).

In the case (i) both $\alpha_n$ and $\beta_n$ are isomorphisms for all $n$. In the cases (ii) and (iii), $\beta_n$ is an isomorphism for all $n$, and $\alpha_n$ is an isomorphism if $n \leq 2$.

**Proof.** Suppose that (i) holds. Then [21, Proposition 4.3] asserts that the embedding $P_N \hookrightarrow A$, where $P_N$ denotes the polynomial algebra in $N$ complex variables, is a localization in the sense of [21, Definition 1.2]. Thus (by [21, Proposition 1.7]) $\mathcal{H}^N(A, C_\omega) \cong \mathcal{H}^N(P_N, C_\omega)$. However, it follows from [21, Proposition 4.5] that the vector space dimension of this latter space is equal to $(\binom{N}{k})$. We conclude that $\alpha_n$ is an isomorphism for all $n$. The assertion about $\beta_n$ follows from [18, Theorem 4.1] and Proposition 3.4.

Suppose that (ii) is true, so that $U = G_1 \times \ldots \times G_N$ for some bounded, open subsets $G_i$ of the complex plane. We claim that the condition in Proposition 3.4 is satisfied for $A = A(U)$. Indeed, choose $k \in \{1, \ldots, N\}$ and $f \in A(U)$, and suppose that $f(M_k \cap U) = \{0\}$, where $M_k = \{z_I \mid 1 \leq I \leq k\}$. Then $f = f_1 \cdots f_N$, where $f_1, \ldots, f_N \in A(U)$ are defined by

$$f_i(z_1, \ldots, z_N) = f_1(z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_N) - f_1(z_1, \ldots, z_{i-1}, w_i, z_{i+1}, \ldots, z_N)$$

for $1 \leq i \leq k$ and $\{z_j \mid j \in \{1, \ldots, N\} \setminus \{i\}\}$. The $f_i$ are in $A(U)$ such that $f_i(G_j, j \in \{1, \ldots, N\} \setminus \{i\}) = 0$. Therefore we see from Lemma 3.6 that there exist $g_1, \ldots, g_i \in A(U)$ such that $f_i = (Z_i - w_i)g_i$ for $1 \leq i \leq k$. It follows that $f = \sum_{i=1}^k (Z_i - w_i)g_i$, as required.

Finally, we consider case (iii), so that $U$ is bounded and strictly pseudoconvex with a $C^2$-boundary. Then it is known ([12, Theorem 2.6.1]) that, for every $g \in \{1, \ldots, N\}$, every $r \geq 0$ and every closed $(0, q)$-form $f$ on $U$ with coefficients in $C^r(U)$, there exists a $(0, q - 1)$-form $u$ on $U$ such that $\partial u = f$ and each coefficient function $h$ of $u$ satisfies $h \in C^r(U)$ and $\partial h \in \text{Lip}_\alpha(U)$ for each $\alpha \in (0, 1)$ and each multindex $\gamma$ of order not exceeding $r$. But this certainly implies that the coefficients of $u$ are functions in $C^r(U) \cap \text{Lip}_\alpha(U)$ for each $\alpha \in (0, 1)$, and so the condition in Theorem 3.5 is satisfied. Application of this theorem together with Proposition 3.3 yields the desired result. \[
\]

**4. The calculation of $\mathcal{H}^2_p(A, C_\omega)$ and $\mathcal{H}^2_w(A, C_\omega)$ in the case where $w \in U$.** As in the previous section, let $A$ be a Fréchet algebra of analytic functions on an open set $U \subseteq C^N$, and let $w \in U$ be fixed. In this section, we demonstrate how the Koszul complex can be used to obtain a sufficient condition on $A$ for the vanishing of the symmetric second Hochschild homology groups.

$$\mathcal{H}^2_p(A, C_\omega) \text{ and } \mathcal{H}^2_w(A, C_\omega).$$

We begin with a simple lemma.

**Lemma 4.1.** Let $A$ be a unital, commutative algebra, and let $\varphi$ be a character on $A$. Suppose that $\ker \varphi = (a_1, \ldots, a_m)$ is algebraically finitely generated. Let $\mu \in B^2(A, C_\varphi)$. Then $\mu$ is a coboundary if and only if $\sum_{i=1}^m \mu(b_i \otimes a_i) = 0$ for all $b_1, \ldots, b_m \in \ker \varphi$ such that $\sum_{i=1}^m b_i a_i = 0$.

**Proof.** It is obvious that the condition in the lemma is necessary for $\mu$ to be a coboundary. Conversely, suppose that the condition is satisfied, and let $M = \ker \varphi$. Then there is a linear functional $\lambda$ on $M$ such that $\lambda(\sum_{i=1}^m b_i a_i) = -\sum_{i=1}^m \mu(b_i \otimes a_i)$ for all $b_1, \ldots, b_m \in M$. We may extend $\lambda$ to a linear functional on $A$ such that $\lambda(1_A) = 1$. Then $\mu = \delta \lambda \in B^2(A, C_\varphi)$, as required. \[
\]

Suppose that $A$ satisfies the conditions (3.5) and (3.6) in Proposition 3.3. Let $\mu \in Z^2_p(A, C_\varphi)$. According to Proposition 3.3, there exist a linear map $S : A \rightarrow C_\omega$, distinct pairs $(i_1, j_1), \ldots, (i_n, j_n)$ with $1 \leq i_k < j_k \leq N$ ($1 \leq k \leq n$) and complex numbers $a_1, \ldots, a_n$ such that

$$\mu(f \otimes g) = S(fg) + \sum_{k=1}^n \alpha_k \frac{\partial f}{\partial z_{i_k}}(w) \frac{\partial g}{\partial z_{j_k}}(w)$$

for all $f, g \in A$ which vanish at $w$. But $\mu$ is symmetric, and so it follows that $\alpha_k = \mu((z_{i_k} - w_{i_k}) \otimes (z_{j_k} - w_{j_k})) - \mu((z_{j_k} - w_{j_k}) \otimes (z_{i_k} - w_{i_k})) = 0$ for $1 \leq k \leq n$. We conclude that $\mu \in B^2(A, C_\omega)$, and therefore $\mathcal{H}^2_p(A, C_\omega) = \{0\}$.

However, the next proposition, which is valid for general commutative Fréchet algebras, shows that Proposition 3.3 is not needed here; this result follows from an elementary calculation based on the definition of the Koszul complex, and we may weaken the homological condition on $K(Z \cdot w)$.

**Proposition 4.2.** Let $A$ be a commutative, unital Fréchet algebra, and let $\varphi$ be a continuous character on $A$. Suppose that $\ker \varphi = (a_1, \ldots, a_n)$ is algebraically finitely generated, and that the first homology of the Koszul complex $K$ associated with $a_1, \ldots, a_n$ is zero. Then $\mathcal{H}^2_p(A, C_\varphi) = \mathcal{H}^2_w(A, C_\varphi) = \{0\}$.

**Proof.** Let $\mu \in Z^2_p(A, C_\varphi)$, and let $b_1, \ldots, b_n \in A$ be such that $\sum_{i=1}^n a_i b_i = 0$. Since $H_1(K) = 0$, there exist $c_j \in A$, $1 \leq i < j \leq n$, such that

$$\sum_{i=1}^n a_i b_i = 0$$

$\Rightarrow$ $\sum_{i=1}^n a_i b_i = 0$. Since $H_1(K) = 0$, there exist $c_j \in A$, $1 \leq i < j \leq n$, such that
\[ b_i = \sum_{j=1}^{i-1} a_j c_{ji} - \sum_{j=i+1}^{n} a_j c_{ij} \quad (1 \leq i \leq n).\]

Thus
\[
\sum_{i=1}^{n} \mu(a_i \otimes b_i) = \sum_{i=1}^{n} \mu\left( a_i \otimes \left( \sum_{j=1}^{i-1} a_j c_{ji} - \sum_{j=i+1}^{n} a_j c_{ij} \right) \right)
= \sum_{i=1}^{n} \sum_{j=1}^{i-1} \mu(a_i \otimes a_j c_{ji}) - \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mu(a_i \otimes a_j c_{ij})
= \sum_{i=1}^{n} \sum_{j=1}^{i-1} \mu(a_i \otimes a_j c_{ji}) - \sum_{i=1}^{n} \sum_{j=1}^{i} \mu(a_i \otimes a_j c_{ij}) = 0,
\]
where the second equality from below is valid since
\[ \mu(a_i \otimes a_j c_{ji}) = \mu(a_i c_{ji} \otimes a_j) = \mu(a_i \otimes a_j c_{ji}) \]
for all \( i < j \). The result now follows from Lemma 4.1 and Corollary 2.4. \( \blacksquare \)

**Corollary 4.3.** Let \( A \) be a Fréchet algebra of analytic functions on an open set \( U \subseteq \mathbb{C}^N \), and let \( w \in U \). Then, in each of cases (i)--(iii) considered in Theorem 3.7, both \( H^2_{\mathcal{O}}(A, \mathbb{C}^w) \) and \( H^2_{\mathcal{A}}(A, \mathbb{C}^w) \) are trivial.

**Proof.** We have shown in the proof of Theorem 3.7 that, in each case, the conditions (3.5) and (3.6) are satisfied, and therefore Proposition 4.2 may be applied. \( \blacksquare \)

We finish this section with a further elementary result for the case \( A = A(U) \). Here we can prove that \( H^2_{\mathcal{O}}(A(U), \mathbb{C}^w) \) is trivial, but we do not know whether the same is true for \( H^2_{\mathcal{A}}(A(U), \mathbb{C}^w) \).

**Proposition 4.4.** Let \( U \subseteq \mathbb{C}^N \) be an open, bounded and geometrically convex set with \( C^2 \) boundary. Then \( H^2_{\mathcal{O}}(A(U), \mathbb{C}^w) = \{0\} \) for every \( w \in U \).

**Proof.** We may assume that \( w = 0 \). Let \( A = A(U) \), and let \( M = \{ f \in A : f(w) = 0 \} \). According to [11, Theorem 1], \( M \) is algebraically generated by the coordinate functions \( Z_1, \ldots, Z_N \). Let \( \mu \in Z^2(\mathcal{O}, \mathbb{C}^w) \), and let \( f_1, \ldots, f_N \in A \) be such that \( \sum_{i=1}^{N} Z_i f_i = 0 \). For each \( f \in A \) and each \( r \in (0, 1) \), we set \( U_r = U + r^{-1}(1-r)U \), and we define \( f^{(r)} \in \mathcal{O}(U_r) \) by \( f^{(r)}(z) = f(rz) \) for \( z \in U_r \). Note that \( U_r \) is a convex, open set which contains \( U \). For each \( r \in (0, 1) \) we define \( \mu_r \in Z^2(\mathcal{O}(U_r), \mathbb{C}^w) \) as
\[
\mu_r(f \otimes g) = \mu(f(U) \otimes g(\overline{U})) \quad (f, g \in \mathcal{O}(U_r)).
\]

Now let \( r \in (0, 1) \) be fixed. We see clearly that \( \sum_{i=1}^{N} Z_i f^{(r)} = 0 \). Furthermore, since \( U_r \) is geometrically convex, it is pseudoconvex. We may therefore conclude from Lemma 4.1 and Corollary 4.3 that
\[
\sum_{i=1}^{N} \mu(Z_i \otimes f^{(r)}(\overline{U})) = \sum_{i=1}^{N} \mu_r(Z_i \otimes f^{(r)}) = 0.
\]
If \( f \) is any function in \( A \), then, since \( f \) is uniformly continuous, \( f^{(r)}|\overline{U} \to f \) uniformly on \( \overline{U} \) as \( r \to 1 \). Therefore (4.1) and the continuity of \( \mu \) imply that \( \sum_{i=1}^{N} \mu(Z_i \otimes f_i) = 0 \). The result now follows from Corollary 2.4 and Lemma 4.1. \( \blacksquare \)

**Example.** Let \( U = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^4 < 1\} \). Then \( U \) is geometrically convex and smoothly bounded, but not strictly pseudoconvex. Hence there are simple examples of sets \( U \) that satisfy the conditions of Proposition 4.4, but to which we cannot apply Corollary 4.3.

5. The calculation of \( H^n(A(U), \mathbb{C}^w) \) and \( \mathcal{H}^n(A(U), \mathbb{C}^w) \) in the case

where \( w \in \partial U \). Let \( A \) be the algebra \( A(U) \) for some bounded, open set \( U \subseteq \mathbb{C}^N \). In this section, we investigate the spaces \( H^n(A, \mathbb{C}^w) \) and \( \mathcal{H}^n(A, \mathbb{C}^w) \) where \( w \) is a point in the boundary \( \partial U \) of \( U \).

It is well known (see [2, p. 101], for example) that the maximal ideal \( M_w \) of \( A \) associated with \( w \) has a bounded approximate identity if and only if \( w \) is a peak point for \( A \) (i.e. there is \( f \in A \) such that \( f(w) = 1 \) and \( |f(z)| < 1 \) for \( z \in \overline{U} \setminus \{w\} \)). Our first proposition shows that this is the "easy" case in our situation.

The part in the proposition which concerns \( \mathcal{H}^n(A, \mathbb{C}^w) \) is certainly well known (see [14, Proposition 1.5]); however, we provide a proof for both the continuous and the algebraic situations because the argument is valid in both cases.

**Proposition 5.1.** Let \( A \) be a unital, commutative Banach algebra, and let \( \varphi \) be a character on \( A \). Suppose that \( M = \ker \varphi \) has a bounded approximate identity. Then
\[
\mathcal{H}^n(A, \mathbb{C}^w) = H^n(A, \mathbb{C}^w) = \{0\} \quad (n \geq 1).
\]

**Proof.** Since \( M \) has a bounded approximate identity, it is flat in \( A_{\text{Ba-mod}} \) ([? VIII.1.5]) and in \( A_{\text{mod}} \) ([24, Theorem B]). Therefore
\[
0 \to M \to A \xrightarrow{\mathcal{I}} \mathcal{C}_w,
\]
where \( \mathcal{I} \) is the inclusion map, is a flat resolution of \( \mathcal{C}_w \) in both categories. Thus Lemma 2.2 implies that (5.1) is true for every \( n \geq 2 \). Furthermore, \( M = M^2 \) by Cohen’s factorization theorem, and hence (5.1) is also true for \( n = 1 \). \( \blacksquare \)
COROLLARY 5.2. Let $U \subseteq \mathbb{C}^N$ be a bounded, strictly pseudoconvex domain with $\partial^2$-boundary. Then

$$\mathcal{H}^0(A(U), C_w) = H^n(A(U), C_w) = \{0\} \quad (w \in \partial U, \ n \geq 1).$$

Proof. It is known ([18, VI.1.14]) that every boundary point of $U$ is a peak point for $A(U)$ in this case. ■

Simple examples show that the maximal ideal $M_w$ may fail to have a bounded approximate identity and that, consequently, Proposition 5.1 is not applicable. For instance, consider the case where $A = A(U)$ is the polydisk algebra. Then $w = (w_i)_i \in \partial U$ if $|w_i| = 1$ for some $1 \leq i \leq N$, and $w$ is a peak point if $|w_i| = 1$ for all $1 \leq i \leq N$, which in turn is equivalent to $w$ lying in the Shilov boundary $\partial A$ of $A$. However, we always have the decomposition

$$(5.2) \quad M_w = (Z_{i_1} - w_{i_1}, \ldots, Z_{i_k} - w_{i_k}) + \mathfrak{J},$$

where $\{i_1, \ldots, i_k\}$ is the set of indices $j$ such that $|w_j| < 1$, and $J$ is the kernel of the peak set $\Delta = \{(z_i) \in \mathbb{D}^N : z_j = w_j \text{ for } j \notin \{i_1, \ldots, i_k\}\}$. Decompositions of this kind have been studied by A. Ya. Helemskii (cf. [8]). We follow his approach here to show that a description of the spaces $\mathcal{H}^0(A(C_w))$ and $H^n(A, C_w)$ can be given whenever we have a decomposition of $M_w$ of the type given in (5.2).

DEFINITION. We say that a sequence $a_1, \ldots, a_k$ of elements of a Banach algebra $A$ is a strongly regular sequence on $A$ if $a_1, \ldots, a_k$ is a regular sequence on $A$ in the algebraic sense and, for each $i \in [1, \ldots, k]$, the ideal $(a_1, \ldots, a_i)$ generated by $a_1, \ldots, a_i$ is closed and complemented in $A$.

Remark. Our notion of a strongly regular sequence on $A$ corresponds to the notion of Kosei ideals used in [8].

Before we proceed, let us introduce the following two basic notions from homological algebra.

NOTATION. (i) Let $A$ be a commutative, unital algebra. Let $(\mathcal{F}, \varphi)$ and $(\mathcal{G}, \psi)$ be chain complexes of $A$-modules. Then the tensor product chain complex $\mathcal{F} \otimes_A \mathcal{G}$ of $\mathcal{F}$ and $\mathcal{G}$ is defined (cf. [23, 2.7.1]). Recall that the degree $n$ part $(\mathcal{F} \otimes_A \mathcal{G})_n$ of $\mathcal{F} \otimes_A \mathcal{G}$ is $\bigoplus_{i+j=n} F_i \otimes_A G_j$. On $F_i \otimes_A G_j$ (with $i+j = n$), the differential $d_n$ from $(\mathcal{F} \otimes_A \mathcal{G})_n$ into $(\mathcal{F} \otimes_A \mathcal{G})_{n-1}$ is the zero map to $F_i \otimes_A G_{j-1}$ unless $i = s$ or $j = t$. From $F_i \otimes_A G_j$ to $F_{i-1} \otimes_A G_j$ it is $\varphi_i \otimes \text{id}$, and from $F_i \otimes_A G_j$ to $F_i \otimes A G_{j-1}$ it is $(-1)^j \text{id} \otimes \psi_j$. As is customary, we identify an $A$-module $M$ with the complex

$$\cdots \to 0 \to M \to 0 \to \cdots$$

concentrated in degree 0; in particular, the tensor product chain complex $\mathcal{F} \otimes_A M$ is defined for each $A$-module $M$.

For a definition of the tensor product of chain complexes in the continuous case, we refer to [7, II.5.25].

(ii) Let $\alpha : (\mathcal{F}, \varphi) \to (\mathcal{G}, \psi)$ be a morphism of chain complexes in the category of linear spaces. Then (cf. [23, 1.5.1]) the mapping cone of $\alpha$ is the chain complex whose degree $n$ part is $F_{n-1} \oplus G_n$; the differential is given by

$$d_n(a, b) = (-\varphi_{n-1}(a), \psi_n(b) - \alpha(a)).$$

Remark. The notion of a “mapping cone” is a generalization of the topological mapping cone of a simplicial map (see [23, p. 19]).

The following result is implicitly contained in [8] (cf. also [7, Lemma V.1.2]). However, we provide a brief proof for the convenience of the reader.

LEMMA 5.3. Let $\alpha : (\mathcal{F}, \varphi) \to (\mathcal{G}, \psi)$ be a continuous morphism of chain complexes in the category of Fréchet spaces. Suppose that, for some $n$, $\varphi_n$ and $\psi_{n+1}$ are admissible, and $H_n(\mathcal{F})$ or $H_n(\mathcal{G})$ is zero. Then the $(n+1)$th differential of the mapping cone of $\alpha$ is admissible.

Proof. Set $\varphi = \varphi_n$ and $\psi = \psi_{n+1}$. It follows from our hypothesis that there are continuous linear maps $\varphi' : F_{n-1} \to F_n$ and $\psi' : G_n \to G_{n+1}$ such that $\varphi = \varphi \circ \varphi' \circ \varphi$ and $\psi = \psi \circ \psi' \circ \psi$. Let $\varrho$ denote the $(n+1)$th differential of the mapping cone of $\alpha$. For $a \in F_{n-1}$ and $b \in G_n$, we define

$$\varrho'(a, b) = (-\varrho'(a), \psi'(b) - (\psi' \circ \alpha_n \circ \varrho)(a)).$$

Then $\varrho' : F_{n-1} \oplus G_n \to F_n \oplus G_{n+1}$ is a continuous linear map. A straightforward computation shows that $\varrho = \varrho \circ \varrho' \circ \varrho - \tau$, where

$$\tau(a, b) = (0, ((\text{id} \otimes \varrho) \circ \varrho_n)(\text{id} \otimes \varphi_n \circ \psi_n))(a) \quad (a \in F_n, \ b \in G_{n+1}).$$

Since $H_n(\mathcal{F}) = 0$ or $H_n(\mathcal{G}) = 0$, $\tau$ is the zero operator. Hence $\varrho = \varrho \circ \varrho' \circ \varrho$. We conclude that $\varrho$ is admissible, as required. ■

We can now state the main result of this section.

THEOREM 5.4. Let $A$ be a commutative, unital Banach algebra, and let $M_\varphi$ be the kernel of a character $\varphi$ on $A$. Suppose that $M_\varphi = I + J$, where $I$ and $J$ are ideals of $A$ and $I$ is algebraically finitely generated by $a_1, \ldots, a_k$.

(i) $J$ is flat in $A$-mod,

(ii) $a_1, \ldots, a_k$ is a regular sequence on $A$, and

(iii) $IJ = I \cap J$,

then

$$H^n(A, C_\varphi) \cong \text{Hom}_{C}(J/(M_\varphi J), C)^{\mathbb{Z}^k}_{(n-1)^\mathbb{Z}^k} \cong C_{\varrho}^n(\mathbb{Z}) \quad (n \geq 0).$$

If, furthermore,

(iv) $J$ is closed, and flat in $A$-Ba-mod,
(v) $a_1,\ldots,a_k$ is a strongly regular sequence on $A$, and
(vi) $(a_1,\ldots,a_1)J$ is closed and complemented in $J$ $(i = 1,\ldots,k)$,
then
\[ H^n(A, C_\varphi) \cong \text{Hom}_{C,\text{cont}}(J/M_{\varphi}J, C) \otimes \omega_{\varphi}(*) \oplus C_\varphi(\cdot) \quad (n \geq 0). \]

Proof. Suppose first that (i)–(iii) are satisfied. Let $\mathcal{K}$ be the Koszul complex for the sequence $a_1,\ldots,a_k$. By tensoring the inclusion $J \hookrightarrow A$ with $\mathcal{K}$, we obtain the map
\[ \alpha : \mathcal{K} \otimes_A J \rightarrow \mathcal{K} \otimes_A A = \mathcal{K} \]
of chain complexes; note that $\mathcal{K} \otimes_A J$ is the Koszul complex for the pair $(J,a)$, where $a = (a_1,\ldots,a_k)$ is regarded as the $k$-tuple of operators on $J$ of multiplication by the $a_i$'s (cf. [21, p. 210]). In particular, we see that $\mathcal{K} \otimes_A J$ coincides with the continuous tensor product of $\mathcal{K}$ and $J$ as defined in [7].

Let $\mathcal{F}$ be the mapping cone of $\alpha$. By [23, 1.5.2 and 1.5.3], there is a long exact sequence
\[ \cdots \rightarrow H_n(\mathcal{K}) \rightarrow H_n(\mathcal{F}) \rightarrow H_{n-1}(\mathcal{K} \otimes_A J) \rightarrow \delta \rightarrow H_{n-1}(\mathcal{K}) \rightarrow \cdots \]
in $A\text{-mod}$, where $\delta$ is the map induced on homology by $\alpha$. By Theorem 2.1, condition (ii) implies that $H_n(\mathcal{K}) = \{0\}$ for $n \geq 1$. Hence $H_n(\mathcal{K} \otimes_A J) = \{0\}$ also applies for $n \geq 1$ because $J$ is flat. It follows that $H_n(\mathcal{F}) = \{0\}$ for $n \geq 2$, and that there is an exact sequence
\[ 0 \rightarrow H_1(\mathcal{F}) \rightarrow H_0(\mathcal{K} \otimes_A J) \rightarrow H_0(\mathcal{K}) \rightarrow H_0(\mathcal{F}) \rightarrow 0. \]
By [23, 4.5.2], $H_0(\mathcal{K} \otimes_A J) = J/JI$ and $H_0(\mathcal{K}) = A/I$. Since $\delta$ is induced on homology by $\alpha$, it is the map
\[ J/II \rightarrow A/I : a + JJ \rightarrow a + I. \]
The kernel of this map is $J \cap I/II$, which is the zero space by condition (iii). Hence $\delta$ is injective, and we conclude from (5.3) that $H_1(\mathcal{F}) = \{0\}$ and
\[ H_0(\mathcal{F}) = (A/I)/(I + J/I) = A/(I + J) = A/M_\varphi = C_\varphi, \]
where we have made use of the fact that $M_\varphi = I + J$ in the second-to-last equality.

We have shown that $H_n(\mathcal{F}) = \{0\}$ for $n \geq 1$ and $H_0(\mathcal{F}) = C_\varphi$. Thus $\mathcal{F}$ is a resolution of $C_\varphi$ in $A\text{-mod}$. By the definition of a mapping cone, the degree $n$ part of $\mathcal{F}$ is
\[ F_n = (\mathcal{K} \otimes_A J)n-1 \oplus K_n = (K_{n-1} \otimes_A J) \oplus K_n. \]
But $J, K_{n-1}$ and $K_n$ are flat $A$-modules, and therefore $F_n$ is flat. Consequently, $\mathcal{F}$ is a flat resolution of $C_\varphi$, and by Lemma 2.2 and (2.3) we see that
\[ H^n(A, C_\varphi) \cong H^n(\text{Hom}_A(\mathcal{F}, C_\varphi)) \quad (n \geq 0). \]

The differential $d_n$ from $F_n$ into $F_{n-1}$ is the map which sends a pair $(a \otimes_A b, c)$ (where $a \in K_{n-1}, b \in J$ and $c \in K_n$) to
\[ (-\varphi_{n-1}(a) \otimes_A b, \varphi_n(c) - b \cdot a), \]
where $\phi = (\varphi_n)$ is the differential of $\mathcal{K}$. We can easily verify that, for every $n \geq 0$, each $\varphi \in \text{Hom}_A(F_n, C_\varphi)$ vanishes on the image of $d_n$. However, this only means that $\text{Hom}_A(\mathcal{F}, C_\varphi)$ is a complex with zero morphisms. Thus $H^n(\text{Hom}_A(\mathcal{F}, C_\varphi))$ is the degree $n$ part of $\text{Hom}_A(\mathcal{F}, C_\varphi)$, which is $\text{Hom}_A(F_n, C_\varphi)$. Hence we see from (5.4) and (5.5) that, for each $n \geq 0$,
\[ H^n(A, C_\varphi) \cong \text{Hom}_A(K_{n-1} \otimes_A J, C_\varphi) \otimes \text{Hom}_A(K_n, C_\varphi) \]
\[ \cong \text{Hom}_A(J, C_\varphi)(\cdot, n) \otimes C_\varphi(\cdot). \]

It is obvious that $\text{Hom}_A(J, C_\varphi)$ is isomorphic to $\text{Hom}_C(J/(M_{\varphi}J), C)$, however, and we have therefore proved the first part of the theorem.

Now suppose, furthermore, that the conditions (iv)–(vi) are satisfied. We claim that $\mathcal{F}$ is a flat resolution of $C_\varphi$ in $A\text{-Ba-mod}$. Indeed, since $J$ is closed and flat in $A\text{-Ba-mod}$, $\mathcal{F}$ is a complex in $A\text{-Ba-mod}$ consisting of flat modules. Moreover, we already know that $H_0(\mathcal{F}) = C_\varphi$ and $H_0(\mathcal{F}) = \{0\}$ for $n \geq 1$. Hence it remains to be shown that each differential $d_n$ of $\mathcal{F}$ is admissible. We infer from (v) (respectively, (vi)) and [21, Proposition 4.1] that every differential of the complex $\mathcal{K}$ (respectively, $\mathcal{K} \otimes_A J$) is admissible, and that $H_n(\mathcal{K})$ (respectively, $H_n(\mathcal{K} \otimes_A J)$) is zero for all $n \geq 1$. We conclude from Lemma 5.3 that $d_n$ is admissible for all $n \geq 2$. Moreover, $d_1$ is admissible because $H_1(\mathcal{F}) = 0$ and $d_1 = M_\varphi$ is of finite codimension in $F_0 = A$. Thus $d_n$ is admissible for all $n$, as required.

It follows from Lemma 2.2 and (2.3) that
\[ H^n(A, C_\varphi) \cong H^n(\text{Hom}_A(\mathcal{F}, C_\varphi)) \quad (n \geq 0). \]

The second part of the theorem may now be proved by a computation which is completely analogous to (5.7).
Corollary 5.5. Let $A$ be a commutative, unital Banach algebra, and let $M_φ$ be the kernel of a character $φ$ on $A$. Suppose that $M_φ = I + J$, where $I$ and $J$ are ideals of $A$ and $I$ is algebraically finitely generated by $a_1, \ldots, a_k$. If

(i) $J$ has a bounded approximate identity,
(ii) $a_1, \ldots, a_k$ is a strongly regular sequence on $A_i$ and
(iii) $\langle a_1, \ldots, a_i \rangle \cap J$ is complemented in $J$ ($i = 1, \ldots, k$),

then

$$H^n(A, C_φ) \cong H^n(A, C_φ) \cong C_φ^{\otimes n}(k) \quad (n \geq 0).$$

Proof. We have already observed that (i) implies that $J$ is flat in both $A\text{-}\text{mod}$ and $A\text{-}\text{Ba-mod}$. Moreover, Cohen's Factorization Theorem asserts that $J = J^2 \subseteq M_φ J$ and $\langle a_1, \ldots, a_i \rangle J = \langle a_1, \ldots, a_i \rangle \cap J$ for $1 \leq i \leq k$. Hence the result follows from Theorem 5.4.

We shall apply Corollary 5.5 in the case where $A = A(U)$ and $U = U_1 \times \cdots \times U_N \subseteq C^N$ is a product domain. Then for $w = (w_i)_{i=1}^N \in \partial U$ there is a partition $(F, G)$ of $\{1, \ldots, N\}$ such that $w_i \in U_i$ for $r \in F$ and $w_s \in \partial U_s$ for $s \in G$, and the results in §3 show that, for $k = \text{card}(F)$, there are explicit embeddings

$$\alpha_n : C_φ^{\otimes n}(k) \hookrightarrow H^n(A, C_φ) \quad \text{and} \quad \beta_n : C_φ^{\otimes n}(k) \hookrightarrow H^n(A, C_φ) \quad (n = 0, 1, \ldots)$$

such that the diagram

$$\begin{array}{ccc}
C_φ^{\otimes n}(k) & \xrightarrow{\alpha_n} & H^n(A, C_φ) \\
\downarrow{\beta_n} & & \downarrow{\iota_n} \\
H^n(A, C_φ) & & 
\end{array}$$

commutes, where $\iota_n$ is the comparison map. We conclude this section with a corollary which shows that $\alpha_n$ and $\beta_n$ are isomorphisms in many cases.

Corollary 5.6. Let $A$ be the algebra $A(U)$ for some bounded product domain $U = U_1 \times \cdots \times U_N \subseteq C^N$, and let $w = (w_i)_{i=1}^N \in \partial U$. Suppose that, for each $i \in \{1, \ldots, N\}$ such that $w_i \in \partial U_i$, $w_i$ is a peak point for $A(U)$. Then

$$H^n(A, C_φ) \cong H^n(A, C_φ) \cong C_φ^{\otimes n}(k) \quad (n = 0, 1, \ldots),$$

where $k = \text{card}(\{1, \ldots, N\} : w_i \in U_i)$.

Proof. We may suppose that $w_i \in U_i$ for $1 \leq i \leq k$ and that $w_j \in \partial U_j$ for $k < j \leq N$. Set $f_i = \tilde{z}_i - w_i$ ($1 \leq i < k$), and let $J$ be the set of $f \in A$ such that $f(z_1, \ldots, z_k, w_{k+1}, \ldots, w_N) = 0$ for all $z_j \in \tilde{U}_j$, $1 \leq j \leq k$. We claim that conditions (i)-(iii) of Corollary 5.5 are satisfied. Set $I_i = \langle f_1, \ldots, f_i \rangle (1 \leq i \leq k)$. We see from Lemma 3.6 that, for each $i \in \{1, \ldots, k\}$, $I_i$ is the kernel of the set $\{w_1 \times \cdots \times w_i \times \prod_{j=i+1}^N U_j \}$. It follows that $I_i$ is closed, and that $M_φ = I_k + J$. Our hypothesis concerning $w$ implies that $J$ is the kernel of a peak set; therefore (i) is satisfied. We have already shown (see the proof of Theorem 3.7) that $f_1, \ldots, f_k$ is a regular sequence on $A$. Moreover, we note (as was done, in the case where $A$ is the polydisc algebra, in [8, p. 231]) that, for each $i \in \{1, \ldots, k\}$, the set $M_i$ of functions not dependent on $z_1, \ldots, z_i$ is a Banach space complement for $I_i$ in $A$, and that $M_i \cap J$ is a Banach space complement for $I_i \cap J$ in $J$. Hence (ii) and (iii) are also satisfied. The result now follows from Corollary 5.5.

6. The calculation of $H^2(A(U), C_φ)$ and $H^2(A(U), C_φ)$ in the case where $w \in \partial U$. As in the previous section, let $A$ be the algebra $A(U)$ for some bounded, open set $U \subseteq C^N$, and let $w \in \partial U$. In this section, we consider the symmetric Hochschild groups $H^2(A, C_φ)$ and $H^2(A, C_φ)$.

Suppose that $J$ is a closed ideal of $A$ which is contained in $M_φ$. Then $C_φ$ is a Banach $A$-bimodule, as well as a Banach $A/J$-bimodule, in a canonical way; thus we may consider the spaces $H^2(A/J, C_φ)$, $H^2(A, C_φ)$, and $H^2(J, C_φ)$ and their algebraic counterparts. The next elementary proposition gives us some information on how these spaces are related to one another.

Proposition 6.1. Let $φ$ be a character on a commutative, unital Banach algebra $A$. Suppose that $J$ is a closed ideal of $A$ contained in $\ker φ$. Set

$$K = \{μ ∈ Z^2_φ(A, C_φ) : μ(J \otimes C(J)) = 0\},$$

$$L = \{μ ∈ H^2_φ(A, C_φ) : μ(J) \in H^2(A, J(J))\}.$$

Moreover, let $K$ (respectively, $L$) be the set of elements of $K$ (respectively, $L$) which are continuous. Then there is a commuting diagram with exact rows

$$0 \rightarrow \text{Hom}_A(J, C_φ) \rightarrow H^1(J, C_φ) \rightarrow \text{Hom}_A(J, C_φ) \rightarrow \text{Hom}_A(J, C_φ) \rightarrow \text{Hom}_A(J, C_φ) \rightarrow 0.$$

Moreover, let $κ$ (respectively, $L$) be the set of elements of $K$ (respectively, $L$) which are continuous. Then there is a commuting diagram with exact rows

$$0 \rightarrow \text{Hom}_A(J, C_φ) \rightarrow H^1(J, C_φ) \rightarrow \text{Hom}_A(J, C_φ) \rightarrow \text{Hom}_A(J, C_φ) \rightarrow \text{Hom}_A(J, C_φ) \rightarrow 0.$$
0 \longrightarrow H^0_\mathcal{A}(A/J, \mathcal{C}_\varphi) \longrightarrow H^0_\mathcal{A}(A, \mathcal{C}_\varphi) \longrightarrow H^0_\mathcal{A}(J, \mathcal{C}_\varphi) \\

0 \longrightarrow \mathcal{H}^0_\mathcal{A}(A/J, \mathcal{C}_\varphi) \longrightarrow \mathcal{H}^0_\mathcal{A}(A, \mathcal{C}_\varphi) \longrightarrow \mathcal{H}^0_\mathcal{A}(J, \mathcal{C}_\varphi)

where the vertical maps are the respective comparison maps.

Proof. We first construct linear maps \( \varrho_1, \ldots, \varrho_4 \) such that the sequence

(6.1) \quad 0 \longrightarrow \text{Hom}_\mathcal{A}(J, \mathcal{C}_\varphi) \overset{\varrho_1}{\longrightarrow} H^1(J, \mathcal{C}_\varphi) \overset{\varrho_2}{\longrightarrow} K/\delta^1(L) \overset{\varrho_3}{\longrightarrow} H^0_\mathcal{A}(A, \mathcal{C}_\varphi) \overset{\varrho_4}{\longrightarrow} H^0_\mathcal{A}(J, \mathcal{C}_\varphi) 

is exact. We choose \( \varrho_1 \) to be the inclusion map, and we define

\( \varrho_3(\mu + \delta^1(L)) = \mu + B^2(A, \mathcal{C}_\varphi), \quad (\mu \in K), \)

\( \varrho_4(\mu + B^2(A, \mathcal{C}_\varphi)) = \mu(J \otimes \mathcal{C} J) + B^2(J, \mathcal{C}_\varphi) = \mu \in Z^2_\mathcal{A}(A, \mathcal{C}_\varphi). \)

For \( \varrho_2 \) we choose the map which sends \( \varphi \in H^1(J, \mathcal{C}_\varphi) \) to \( \delta^1 \phi + \delta^1(L) \), where \( \phi \) is an extension of \( \varphi \) to a linear functional on \( A \); it is obvious that \( \varrho_2 \) is well defined. It is then trivial to verify that the sequence (6.1) is exact.

Analogously, we may choose linear maps \( \tau_1, \ldots, \tau_4 \) such that the second row of the diagram in the first part of the theorem is exact; we note, however, that the Hahn–Banach Theorem is required for the definition of \( \tau_2 \). The commutativity of the whole diagram is then immediately obvious.

Let \( \kappa : A \to A/J \) be the quotient map, and let \( \kappa \otimes \kappa : A \otimes A \to (A/J) \otimes (A/J) \) denote the linear map which sends \( a \otimes b \) into \( \kappa(a) \otimes \kappa(b) \). We define

\( \theta_1(\mu + B^2(\kappa(J, \mathcal{C}_\varphi))) = \mu \circ (\kappa \otimes \kappa) + \delta^1(L), \quad (\mu \in Z^2_\mathcal{A}(A/J, \mathcal{C}_\varphi)), \)

\( \theta_2(\mu + B^2(\kappa(J, \mathcal{C}_\varphi))) = \mu \circ (\kappa \otimes \kappa) + \delta^1(L), \quad (\mu \in Z^2_\mathcal{A}(A/J, \mathcal{C}_\varphi)). \)

We then have the commuting diagram

\[
\begin{array}{ccc}
H^0_\mathcal{A}(A/J, \mathcal{C}_\varphi) & \overset{\theta_1}{\longrightarrow} & K/\delta^1(L) \\
\tau & \uparrow & \\
\mathcal{H}^0_\mathcal{A}(A/J, \mathcal{C}_\varphi) & \overset{\theta_2}{\longrightarrow} & K/\delta^1(L)
\end{array}
\]

where \( \tau \) is the comparison map and \( \sigma_3 \) is defined as in the statement of the theorem.

Now suppose that \( J = J^2 \). Then clearly both \( H^1(A, \mathcal{C}_\varphi) \) and \( \mathcal{H}^1(A, \mathcal{C}_\varphi) \) are isomorphisms. Hence the second part of the theorem follows from the first. \( \blacksquare \)

Remarks. (i) The proof of this result carries over to the more general case where \( \mathcal{C}_\varphi \) is replaced by a finite-dimensional, symmetric Banach \( A \)-bimodule \( E \) which is annihilated by \( J \). Moreover, there is a similar result for the Hochschild groups \( \mathcal{H}^2(A/J, E), \mathcal{H}^2(A, E), \) and \( \mathcal{H}^2(J, E) \) and their algebraic counterparts in the “non-commutative” situation where \( A \) is a Banach algebra, \( J \) is a closed ideal of \( A \), and \( E \) is a finite-dimensional Banach \( A \)-bimodule which is annihilated by \( J \).

(ii) Results analogous to Proposition 6.1 can be found in the theory of group cohomology (see [23, 6.8.3]) and in the theory of Lie algebra cohomology (see [23, 7.5.3]). In these two cases, the result may be obtained by an application of Grothendieck’s Spectral Sequence Theorem, and it is thus conceivable that this is also true for Proposition 6.1.

Our next theorem, which is true for general Banach algebras, shows that the symmetric Hochschild groups of \( A \) with coefficients in \( \mathcal{C}_\varphi \) vanish in the case where \( M_n \) is decomposable into a part which is finitely algebraically generated and a part which is, intuitively speaking, “near” to having a bounded approximate identity. Note that the conditions on \( M_n \) that we need are similar to the conditions on \( M_n \) in Theorem 5.4.

Theorem 6.2. Let \( \varphi \) be a character on a commutative, unital Banach algebra \( A \), and let \( \mathcal{M}_\varphi = \ker \varphi \). Suppose that \( \mathcal{M}_\varphi \) admits the decomposition \( \mathcal{M}_\varphi = I + J \), where \( I \) and \( J \) are ideals of \( A \) and \( I \) is algebraically finitely generated by \( a_1, \ldots, a_k \). Suppose, furthermore, that

(i) \( J = J^2 \),

(ii) \( \mathcal{H}^2(I, \mathcal{C}_\varphi) = \mathcal{H}^2(J, \mathcal{C}_\varphi) = \{0\}, \)

(iii) \( a_1, \ldots, a_k \) is a regular sequence on \( A \), and

(iv) \( (a_1, \ldots, a_j) \cap J = (a_1, \ldots, a_j)J \) for \( j = 1, \ldots, k \).

Then \( \mathcal{H}^2(A, \mathcal{C}_\varphi) = \mathcal{H}^2(A, \mathcal{C}_\varphi) = \{0\} \).

Proof. By Proposition 6.1, Corollary 2.4 and Proposition 4.2, it suffices to show that the sequence \( a_1 + J, \ldots, a_k + J \) is regular on \( A/J \). To do this, let \( i \in \{1, \ldots, k\} \) be fixed. Let \( I \) be the ideal which is algebraically generated by \( a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k \), and let \( b \in A \) be such that

\[
a_i(a_i + J) \in (a_1 + J, \ldots, a_{i-1} + J).
\]

Then it follows from (iv) that there are \( b_1, \ldots, b_{i-1} \in A \) and \( b_i \in J \) such that

\[
a_i(b + b_i) = \sum_{j=1}^{i-1} b_ia_j = 0.
\]

But now (iii) implies that \( b + b_i \in I \). Thus \( b + J \in (a_1 + J, \ldots, a_{i-1} + J) \), as required. \( \blacksquare \)

Corollary 6.3. Suppose that \( U = U_1 \times \cdots \times U_N \subseteq C^N \) is a bounded product domain, and that \( w = (w_i)_{i=1}^N \in \partial U \) where, for each \( i \) such that
\[ w_i \in \partial U_i, w_j \text{ is a peak point for } A(U_i). \text{ Then } H^*_P(A(U), \mathbb{C}_w) = H^*_P(A(U), \mathbb{C}_w) = \{0\}. \]

**Proof.** We have shown, in the proof of Corollary 5.5, that \( M_w \) admits a decomposition \( M_w = I + J \) such that conditions (i)--(iv) in Theorem 6.2 are satisfied. \( \blacksquare \)

In the remainder of this section, we consider the case where \( N = 1 \) and consequently \( U \) is an open, bounded set in the complex plane. We demonstrate that there are examples where \( H^*_P(A, \mathbb{C}_w) \neq \{0\} \) in this situation; in particular, this shows that the assertion made in Corollary 6.3 does not hold for arbitrary product domains.

**Remark.** Recall that \( A \) is the set of functions in \( C(\overline{U}) \) which are holomorphic on \( U \). Thus \( A \) contains \( A(\overline{U}) \), the set of functions in \( C(\overline{U}) \) which are holomorphic on the interior \( \text{int}(\overline{U}) \) of \( U \). Of course, \( A = A(\overline{U}) \) if \( U = \text{int}(\overline{U}) \), or, more generally, if \( \text{int}(\overline{U}) \setminus U \) is analytically negligible. There are cases, however, where \( A \neq A(\overline{U}) \). The following example demonstrating this was communicated to me by Professor Heinz König. Let \( F \subseteq [0, 1] \) be the Cantor set, and let \( f : [0, 1] \to [0, 1] \) be the Cantor function. Then the function \( g(z) = f(rz) \) is holomorphic on \( (0, 1) \setminus F \) and continuous on \( \overline{U} = [0, 1] \times [0, 1] \); however, \( g \) is not holomorphic on the interior of \( U \).

We start with an elementary lemma, which is implicitly contained in [17]. Before we can state our lemma, we need to introduce some additional notation.

**Notation.** Let \( \phi \) be a character on a commutative Banach algebra \( A \). Recall that the elements of \( Z^1(A, \mathbb{C}_\phi) \) are called point derivations. Thus a linear functional \( d \) on \( A \) is a point derivation if
\[
d(ab) = \phi(a)d(b) + d(a)\phi(b) \quad (a, b \in A).
\]
A point derivation of order \( n \) at \( \phi \) is a sequence \( (d_k)_{k=0}^n \) of linear functionals on \( A \), with \( d_0 = \phi \), satisfying
\[
d_k(ab) = \sum_{j=0}^k d_j(a)d_{k-j}(b) \quad (a, b \in A, k \in \{0, \ldots, n\}).
\]
An infinite order point derivation at \( \phi \) is a sequence \( (d_k)_{k=0}^\infty \) such that, for each \( n \in \mathbb{N}, (d_k)_{k=0}^n \) is a point derivation of order \( n \) at \( \phi \). A point derivation \( (d_k) \) of some order at \( \phi \) is degenerate if \( d_1 = 0 \), and continuous if each linear functional \( d_k \) in the sequence is continuous.

**Lemma 6.4.** Let \( A \) be a commutative, unital Banach algebra, and let \( \phi \) be a character on \( A \). Suppose that \( H^*_P(A, \mathbb{C}_\phi) = \{0\} \). Then for each infinite order point derivation \( d \) at \( \phi \), there exists a continuous infinite order point derivation \( (d_k)_{k=0}^\infty \) at \( \phi \) such that \( d_1 = d \).

**Proof.** Let \( n \in \mathbb{N} \), and suppose that \( (d_k)_{k=0}^n \) is a continuous point derivation of order \( n \) at \( \phi \) such that \( d_1 = d \). Clearly, it suffices to construct a continuous functional \( d_{n+1} \) on \( A \) such that \( (d_k)_{k=0}^{n+1} \) is a continuous point derivation at \( \phi \) of order \( n+1 \). To this end, we consider the continuous functional
\[
\mu : A \to \mathbb{C} : a \mapsto \sum_{k=1}^n d_k(a)d_{n+1-k}(b).
\]
It is easy to verify that
\[
\mu(ab \otimes c) = \sum_{1 \leq i, j \leq n} d_i(a)d_j(b)d_k(c) = (a \otimes bc)
\]
for all \( a, b, c \in \ker \phi \), and it follows that \( \mu \in Z^1(A, \mathbb{C}_\phi) \). By our hypothesis, there exists a continuous linear functional \( d_{n+1} \) such that
\[
\varphi(a)d_{n+1}(b) - d_{n+1}(ab) + \varphi(b)d_{n+1}(a) = -\sum_{k=1}^n d_k(a)d_{n+1-k}(b)
\]
for all \( a, b \in A \); however, this means that \( (d_k)_{k=0}^{n+1} \) is a continuous point derivation at \( \phi \) of order \( n+1 \), as required. \( \blacksquare \)

In [5, Theorem 3.7], a simple example is given for a compact set \( X \) in the complex plane such that \( R(X) \) (the uniform closure in \( C(X) \) of rational functions with poles off \( X \)) admits a continuous point derivation at 0, but also such that there is no continuous non-degenerate second order point derivation of \( R(X) \) at 0. It is not difficult to see that the example in [6] has the properties that \( X \) is the closure of its interior, and that \( R(X) = A(X) \). Hence \( R(X) = A(U) \), where \( U \) denotes the interior of \( X \), and the following theorem is thus a consequence of Lemma 6.4 and the result in [6].

**Theorem 6.5.** There exists an open, bounded set \( U \subseteq \mathbb{C} \) such that, for some \( w \in \partial U, H^*_P(A(U), \mathbb{C}_w) \neq \{0\} \).

We see from this example and from Lemma 2.2 that the maximal ideal \( M_w \) corresponding to a point \( w \in \partial U \) need not be flat in \( A\text{-Ba-mod} \). In fact, we have the following characterization of flat maximal ideals in the one-dimensional case.

**Theorem 6.6.** Let \( U \subseteq \mathbb{C} \) be an open, bounded set; let \( w \in \partial U \) and let \( M_w \) be the maximal ideal in \( A = A(U) \) corresponding to \( w \). Then the following are equivalent:

1. \( M_w \) is flat in \( A\text{-mod} \);
2. \( H^2(A, \mathbb{C}_w) = \{0\} \);
3. \( \dim M_w/M_w^2 \leq 1 \);
4. \( M_w \) is projective in \( A\text{-Ba-mod} \);
5. \( M_w \) is flat in \( A\text{-Ba-mod} \);

w_i \in \partial U_i, w_j \text{ is a peak point for } A(U_i). Then \( H^*_P(A(U), \mathbb{C}_w) = H^*_P(A(U), \mathbb{C}_w) = \{0\} \).
(vi) $\mathcal{H}^2(A, C_w) = \{0\};$

(vii) $w$ is a peak point for $A$, or there exists an open neighbourhood $V$ of $w$ such that each $f \in A$ is analytic on $V$.

Proof. (i)$\Rightarrow$(ii) and (v)$\Rightarrow$(vi). These follow from Lemma 2.2.

(ii)$\Rightarrow$(iii). This follows from the simple fact that, for two linear functionals $\phi$ and $\psi$ on $A$ which vanish on $M^2_w \oplus C_A$, the cocycle $f \otimes g \mapsto \phi(f)\psi(g)$ is a coboundary if and only if $\phi$ and $\psi$ are collinear. (See [17, Proposition 3] for an analogous argument in the continuous case.)

(iii)$\Rightarrow$(vii). In the case where $M_w = M^2_w$, $w$ is isolated in the norm topology on $\overline{U}$ induced by $A$. Hence (see [2, Corollary 3.3.10 and p. 205]) $w$ is a peak point for $A$. In the case where $M_w \neq M^2_w$, we see from [4, Theorem 3.1] that all powers of $M_w$ are closed, and that $\text{dim} M^{k+1}_w/M^k_w = 1$ for every $k \geq 1$. By the main theorem in [19], therefore, $w$ is the center of a one-dimensional analytic disc. Using the fact that the maximal ideal space of $A$ is $\overline{U}$ (see [2, Theorem 3.5.7]), we conclude that there exists an open neighbourhood $V$ of $w$ such that each $f \in A$ is analytic on $V$.

(iv)$\Rightarrow$(v). This follows from the general theory (cf. [7]).

(vi)$\Rightarrow$(vii). In the case where $M_w = M^2_w$, the product map $\pi$ from $M_w \otimes M_w$ into $M_w$ (which maps an elementary tensor $f \otimes g$ into $fg$) is onto by [17, Proposition 1]. It follows that $w$ is isolated in the norm topology on $\overline{U}$ induced by $A$. Hence, as above, $w$ is a peak point for $A$. In the case where $M_w \neq M^2_w$, $w$ is the center of a one-dimensional analytic disc by [17, Theorem 2].

(vii)$\Rightarrow$(iv) and (vii)$\Rightarrow$(i). In the case where $w$ is a peak point, (iv) is true by [8, Theorem 1]; moreover, $M_w$ has a bounded approximate identity in this case, and therefore (i) holds by the result in [24]. In the case where, for some open neighbourhood $V$ of $w$, $f|V$ is analytic for each $f \in A$, $M_w$ is a principal ideal. Then (i) is easily verified, and (iv) is true by [16, Theorem 1].

Remark. It would be interesting to know whether the conditions in Theorem 6.6 are equivalent to the vanishing of the “symmetric” group $H^2_s(A, C_w)$, or of $H^2(A, C_w)$.

7. Summary. The results attained in the previous sections allow us to prove the following general theorem about the calculation of the Hochschild groups for $A = A(U)$, and about the splitting of extensions of this algebra.

**Theorem 7.1.** Let $U \subseteq \mathbb{C}^N$ be an open, bounded set, and let $A = A(U)$. Suppose that either

(i) $U$ is a strictly pseudoconvex domain with $C^2$-boundary, or

(ii) $U = U_1 \times \ldots \times U_N$ is a product domain and, for every $1 \leq i \leq N$, each $w \in \partial U_i$ is a peak point for $A(U_i)$.

Then:

(a) Let $w \in U$. For each $\mu \in \mathbb{Z}^2(A, C_w)$, there exists a linear functional $S$ on $A$ such that $\mu - S(f)$ is a linear combination of 2-cocycles of the form

$$f \otimes g \mapsto \frac{\partial f}{\partial z_i}(w) \frac{\partial g}{\partial z_j}(w),$$

where $1 \leq i < j \leq N$. In the case where $\mu$ is continuous, $S$ may be chosen to be continuous. In particular, $H^2(A, C_w) = H^2(A, C_w) = \{0\}$ in the case where $N = 1$.

(b) Let $w = (w_1, \ldots, w_N) \in \partial U$. In the case (i), $H^2(A, C_w) = H^2(A, C_w) = \{0\}$. In case (ii),

$$H^2(A, C_w) \cong H^2(A, C_w) \cong C^\omega(\overline{U}),$$

where $k = \text{card}\{i : w_i \in U_i\}$.

(c) For each $w \in \overline{U}$, $H^2_s(A, C_w) = H^2_s(A, C_w) = \{0\}$.

(d) The following statements are equivalent:

- Each finite-dimensional Banach algebra extension of $A$ splits strongly.
- Each finite-dimensional algebraic extension of $A$ splits algebraically.
- $N = 1$.

(e) In case (i), each commutative, finite-dimensional Banach algebra (respectively, algebraic) extension of $A$ splits strongly (respectively, algebraically). The same is true in case (ii) if the maximal ideal space of $A$ is $\overline{U}$.

Proof. Assertion (a) follows from Theorem 3.7, (b) follows from Corollary 5.2 and Corollary 5.6, and (c) is a consequence of Corollary 4.3, Corollary 5.2, and Corollary 6.3. In case (i), the maximal ideal space of $A(U)$ is $\overline{U}$ (see [18, Theorem VII.2.1] and [13, Theorem 7.2.10]). We have already observed that this is also true, for arbitrary $U$, in the case where $N = 1$. Hence we see, from [1, Theorem 4.4], that (d) (respectively, (e)) follows from (a) and (b) (respectively, (c)).

The partial positive result in part (e) of the theorem notwithstanding we have the following counterexample.

**Theorem 7.2.** There exists an open, bounded set $U \subseteq \mathbb{C}$ such that $A(U)$ admits a one-dimensional, commutative Banach algebra extension which does not split strongly.

Proof. This follows from Theorem 6.5 and [1, Theorem 4.4].
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References


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