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Approximation problems and representations of Hardy spaces in circular domains

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Abstract. We derive various approximation results in the theory of Hardy spaces on circular domains G. Two applications are given, one to operators which admit a nice representation of $H^\infty(G)$, and the other to extremal problems with links to the theory of differential equations.

1. Introduction. Let G be a bounded, finitely-connected domain in $\mathbb{C}$, whose boundary consists of the union of a finite number of disjoint Jordan loops. We know [12, Theorem 2, p. 237] that such a domain is conformally equivalent to a *circular domain* G_1, that is, a domain consisting of the open unit disc from which a finite number of pairwise disjoint closed discs have been removed, and from now on we shall assume that G is itself a circular domain:

$$G = \mathbb{D} \setminus \bigcup_{j=1}^{N} (a_j + r_j\overline{\mathbb{D}}), \tag{1}$$

with the obvious inequalities satisfied by the a_j and r_j for $j = 1, \ldots, N$. We write $D_j = a_j + r_j\mathbb{D}$ for $1 \leq j \leq n$. We now define some additional notation in this context.

Let Γ denote the boundary of G, so that

$$\Gamma = \Gamma_0 \cup \Gamma_1 \cup \ldots \cup \Gamma_N, \tag{2}$$

where $\Gamma_0 = \partial\mathbb{D}$, and, for $j = 1, \ldots, N$, we have $\Gamma_j = a_j + r_j\partial\mathbb{D}$. We give Γ_0 the usual positive orientation, and for $j > 0$ we give Γ_j the negative orientation. We also equip Γ with the usual notion of Lebesgue measure or arc length. For $1 \leq j \leq N$, we write $\Omega_j = \mathbb{C} \setminus \{a_j + r_j\overline{\mathbb{D}}\}$, so that

$$G = \mathbb{D} \cap \bigcap_{j=1}^{N} \Omega_j. \tag{3}$$

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In Section 2, we give a self-contained approach to the theory of Hardy spaces defined on a circular domain, clarifying the relationships between functions on G and their boundary values on ∂G. In particular, we prove versions of the approximation result of Kisliakov [15, 16] using the elementary approach of Chalendar and Esterle [8], valid for Hardy spaces $H^p(G)$ on a circular domain G.

As a first application of these results we give a generalization of the main technical result of [8] to the operators $T \in \mathcal{L}(\mathcal{H})$ which admit an absolutely continuous G-rational normal boundary dilation (the terminology is reviewed below). Moreover, our result leads to a new definition of the class $\mathbb{A}^G_{\aleph_0}$, as mentioned in [8] in the particular case where G is the open unit disc.

As a second application we consider some extremal problems of approximation in Hardy spaces $H^p(G)$ generalizing some results in [4, 3] for Hardy spaces on the disc.

2. Hardy spaces in circular domains. It is our aim in this section to prove versions of the approximation results of Kisliakov [15, 16] based on the elementary approach of Chalendar and Esterle [8], valid for Hardy spaces $H^p(G)$ on a circular domain G. Such spaces were defined by Rudin [19] in terms of functions f such that $|f(z)|^p$ has a *harmonic majorant* on G, that is, a real harmonic function $u(z)$ such that $|f(z)|^p \le u(z)$ on G. Rudin then showed that $H^p(G)$ has a natural decomposition into "elementary" Hardy spaces. We give an independent and largely self-contained approach, using an equivalent definition (similar to one used in [2]) that brings out more clearly the relationships between $H^p(G)$ and $L^p(\partial G)$.

DEFINITION 2.1. Let G be a bounded domain whose boundary consists of a finite number of closed rectifiable Jordan curves. For $1 \le p < \infty$, the space $H^p(\partial G)$ is defined to be the closure in $L^p(\partial G)$ of the set R_G of rational functions whose poles lie in the complement of $\overline{G}$. Moreover, $A(\partial G)$ is defined to be the closure in $C(\partial G)$ of the set R_G.

REMARK 2.1. By a version of Mergelyan's theorem [20, p. 394], $A(\partial G)$ is precisely the set of restrictions to ∂G of functions that are continuous on $\overline{G}$ and holomorphic in G, since uniform convergence on G and ∂G are equivalent by the maximum principle.

In the particular case when G is a circular domain, we define the Hardy space $H^p(G)$ as a sum of elementary spaces. For domains such as the Ω_j, it is clear that we can adapt the usual definitions of $H^p(\mathbb{D})$ and define the Hardy space $H^p_0(\Omega_j)$ of functions analytic in Ω_j, with a zero limit at infinity. A function in $H^p_0(\Omega_j)$ has a natural Laurent expansion

$$f(z) = \sum_{k=1}^{\infty} c_k (z-a_j)^{-k}. \tag{4}$$

DEFINITION 2.2. For $1 \le p \le \infty$ the Hardy space $H^p(G)$ is defined by

$$H^p(G) = H^p(\mathbb{D}) \bigoplus_{j=1}^{N} H^p_0(\Omega_j). \tag{5}$$

In the case $p = \infty$, we can regard $H^\infty(G)$ as a direct sum of spaces $H^\infty(\mathbb{D}) \oplus \bigoplus_{j=1}^N H^\infty_0(\Omega_j)$ (cf. Theorem 7.1 of [10]), and hence we may regard it as a closed subspace $H^\infty(\partial G)$ of $L^\infty(\partial G)$. We now make a similar identification of $H^p(G)$ and $H^p(\partial G)$ for $1 \le p < \infty$.

First let us prove that the above two definitions are equivalent.

LEMMA 2.1. *For $1 \le p < \infty$ we can identify $H^p(G)$ isomorphically with $H^p(\partial G)$. Moreover, the sum defining $H^p(G)$ is a topological direct sum, in that the projections onto the summands are bounded independently of p.*

Proof. Let $f \in H^p(G)$. Then we can write $f = f_0 + \sum_{j=1}^N f_j$, where $f_0 \in H^p(\mathbb{D})$ and $f_j \in H^p_0(\Omega_j)$ for $1 \le j \le N$.

By classical results on Hardy spaces, there exist $(N+1)$ sequences $(R_k^{(j)})_{k\ge1}$, $0 \le j \le N$, of rational functions with poles in $\mathbb{C}\setminus\mathbb{D}, D_1, \ldots, D_N$, respectively, satisfying

$$\lim_{k\to\infty} \|f_j - R_k^{(j)}\|_{L^p(\Gamma_j)} = 0 \quad \text{for each } j. \tag{6}$$

Since H^p convergence implies local uniform convergence, we see that $(R_k^{(j)})_{k\ge1}$ converges uniformly to f_j on each Γ_i, for $i \in \{0,1,\ldots,N\}\setminus\{j\}$. It is now clear that the sequence $(\sum_{j=0}^N R_k^{(j)})_{k\ge1}$ converges to f in $L^p(\partial G)$.

Now consider a function $f \in H^p(\partial G)$; we may define functions $f_0, \ldots, f_N$ by

$$f_0(\lambda) = \frac{1}{2i\pi} \int_{\Gamma_0} \frac{f(\xi)}{\xi-\lambda}\, d\xi, \qquad \lambda \in \mathbb{D}, \tag{7}$$

and, for $1 \le j \le N$,

$$f_j(\lambda) = \frac{1}{2i\pi} \int_{\Gamma_j} \frac{f(\xi)}{\xi-\lambda}\, d\xi, \qquad \lambda \in \Omega_j. \tag{8}$$

Suppose first that f is a rational function with poles off $\overline{G}$. Then each f_j $(0 \le j \le N)$ is analytic in $\mathbb{D}$ or Ω_j, as appropriate, and $f = f_0 + \ldots + f_N$ in G, by Cauchy's theorem.

Fix j and consider λ lying on a circle $C_{k,\varepsilon}$ close to Γ_j, say $\lambda \in (1-\varepsilon)\mathbb{T}$ if $j = 0$, and $\lambda \in a_j + (1+\varepsilon)\mathbb{T}$ if $1 \le j \le N$. By construction, if ε is sufficiently small, there is a constant $\delta > 0$ such that $\operatorname{dist}(\lambda, \Gamma_k) \ge \delta$ for all $k \ne j$.

Hence, by Hölder's inequality, if $\lambda \in C_{k,\varepsilon}$ and $j \neq k$, then

$$|f_j(\lambda)| \le \left(\frac{1}{2\pi}\int_{\Gamma_j} |f|^p\right)^{1/p} \left(\frac{1}{2\pi\delta^q}\right)^{1/q}, \tag{9}$$

implying that

$$\Big(\int_{C_{k,\varepsilon}} |f_j(\lambda)|^p\,d\lambda\Big)^{1/p} \le \frac{A_k}{\delta}\|f\|_p, \tag{10}$$

where A_k depends only on Ω_k. Since $f_k = f - \sum_{j\neq k} f_j$, we obtain

$$\|f_k\|_p \le \left(1+\frac{NA}{\delta}\right)\|f\|_p, \tag{11}$$

where $A = \max\{A_0,\dots,A_N\}$.

Now let f be a general function in $H^p(\partial G)$, and let $(R^{(n)})_{n\ge1}$ be a sequence of rational functions with poles off $\overline{G}$, such that $R^{(n)} \to f$ in $L^p(\partial G)$. The corresponding functions $R_j^{(n)}$ converge locally uniformly to functions f_j, by an argument similar to that in (9) above. By Fatou's lemma, $f_0 \in H^p(\mathbb{D})$ and $f_j \in H_0^p(\Omega_j)$ for $1 \le j \le n$, and moreover,

$$\|f_j\|_p \le B \limsup_{n\to\infty} \|R_j^{(n)}\|_p \le B\|f\|_p, \tag{12}$$

where B is a constant depending only on G. ■

We also need to consider the complementary spaces consisting of functions analytic in the unbounded disconnected domain $G' = \mathbb{C}\setminus\overline{G}$.

Definition 2.3. Let G be a circular domain. For $1 \le p < \infty$, the Hardy space $H^p(G')$ is defined to be the space of functions f defined on $G' = \mathbb{C}\setminus\overline{G}$ such that the restriction of f to each component C of G' lies in the usual Hardy space $H^p(C)$ (in the case when C is the unbounded component, this means that $f(1/z)$ is in the usual Hardy space H^p, and implies the existence of a finite limit at infinity). Moreover, we define $A(G')$ as the set of functions continuous on $\overline{G'}$ and analytic in G', again with a limit at infinity. The spaces $H^p(G')$ and $A(G')$ can be regarded as closed subspaces of $L^p(\partial G)$ and $C(\partial G)$, in a natural way.

It is an easy consequence of Mergelyan's theorem that $A(G')$ can be regarded as the closure in $C(\partial G)$ of the set R of rational functions with poles in G, and that, for $1 \le p < \infty$, $H^p(G')$ is the closure of R in $L^p(\partial G)$. We shall use the notation $H_0^p(G')$ for the spaces consisting of those functions f in $H^p(G')$ such that $f(\infty) = 0$.

Now $H^2(\partial G)$ is a closed subspace of $L^2(\partial G)$, and we can define a projection P_+ from $L^2(\partial G)$ onto $H^2(\partial G) \cong H^2(\mathbb{D}) \oplus H_0^2(\Omega_1) \oplus \dots \oplus H_0^2(\Omega_N)$ by

$$(P_+f)(\lambda) = \frac{1}{2i\pi}\int_{\partial G} \frac{f(\xi)}{\xi-\lambda}\,d\xi. \tag{13}$$

It is clear that $P_+f = f$ for $f \in H^2(\partial G)$, by Cauchy's theorem, and $H_0^2(G')$ is a complementary subspace to $H^2(\partial G)$ in $L^2(\partial G)$, since by Mergelyan's theorem $L^2(\partial G)$ is the closure of the set of all rational functions with poles off ∂G. Moreover, by the residue theorem, $P_+f = 0$ if f is rational with poles in G and satisfying $f(\infty) = 0$.

For the approximation result that we shall derive, we need to consider the effect of P_+ on $L^\infty(\partial G)$.

Lemma 2.2. *Let $f \in L^\infty(\partial G)$. Then $P_+f \in H^p(\partial G)$ for each $p \ge 1$. Moreover, $(I-P_+)(f) \in H^p(G')$.*

Proof. Certainly, $f \in L^p(\partial G)$ for all $p \ge 1$. Fix now $p > 1$. Under the identification made in Lemma 2.1, $P_+f = g_0 + \dots + g_N$, where

$$g_0(\lambda) = \frac{1}{2i\pi}\int_{\mathbb{T}} \frac{f(\xi)}{\xi-\lambda}\,d\xi, \tag{14}$$

and, for $1 \le j \le N$,

$$g_j(\lambda) = \frac{1}{2i\pi}\int_{\Gamma_j} \frac{f(\xi)}{\xi-\lambda}\,d\xi. \tag{15}$$

We are now in a position to use the well-known fact (p. 150 of [13]) that the standard Riesz projection $P_+ : L^2(\mathbb{T}) \to H^2(\mathbb{T})$ maps $L^p(\mathbb{T})$ into $H^p(\mathbb{T})$ for $1 < p < \infty$: the integral in (14) is just the Riesz projection from $L^p(\mathbb{T})$ into $H^p(\mathbb{D})$, and the integrals in (15) are Riesz projections from $L^p(\Gamma_j)$ onto $H_0^p(\Omega_j)$.

The result for $p = 1$ follows since $H^2 \subset H^1$.

Consider now $I - P_+$. This is just the Riesz projection from $L^2(\partial G)$ into $H_0^2(G')$, since $P_+f = f$ when f is rational with poles in G', and $P_+f = 0$ when f is rational with poles in G and $f(\infty) = 0$. Hence $I - P_+$ maps L^∞ into $H^p(G')$ for each finite p. ■

Our next result can be deduced (in a version valid for all $p > 2$) using the general interpolation results of Bourgain [7] and Pisier [18], which use in particular the theorem of Jones [14] that $H^p = (H^2, H^\infty)_{\theta,p}$ where $1/p = (1-\theta)/2$, in the sense of interpolation between real Banach spaces. Rather than introduce the machinery of interpolation spaces, we instead deduce the result we need directly from [8], which uses a more elementary approach.

Proposition 2.1. *Let $p > 4$, let $\varepsilon > 0$ be sufficiently small, and let $f \in H^p(\partial G)$. Then there exists a function $g \in H^\infty(\partial G)$ such that*

$$\|f-g\|_2 < \varepsilon\|f\|_p, \tag{16}$$

and

$$\|g\|_\infty < C\varepsilon^{2/(4-p)}\|f\|_p, \tag{17}$$

where C is a constant that depends only on G and p.

Proof. Using Lemma 2.1, we decompose f as a sum $f = f_0 + \ldots + f_N$, where $f_0 \in H^p(\mathbb{D})$, and $f_j \in H_0^p(\Omega_j)$ for $j \geq 1$. By Proposition 2.2 of [8], for each j we can find functions $g_j \in H^\infty(\Omega_j)$ (which may be regarded as closed subspaces of $L^\infty(\Gamma_j)$) such that

$$\|f_j - g_j\|_{L^2(\partial G)} < \frac{\varepsilon}{N+1}\|f_j\|_{L^p(\Gamma_j)}, \tag{18}$$

and

$$\|g_j\|_\infty < C_j \varepsilon^{2/(4-p)}\|f_j\|_{L^p(\Gamma_j)}, \tag{19}$$

the constant C_j being independent of f or f_j. (To do this we find a sufficiently good approximation in $L^2(\Gamma_j)$, and then observe that for functions in $H^2(\Omega_j)$ the norm in $L^2(\partial G)$ is dominated by a constant multiple of the norm in $L^2(\Gamma_j)$.) Hence, writing $g = g_0 + \ldots + g_N$, we have $g \in H^\infty(G)$,

$$\|f - g\|_2 < \varepsilon\|f\|_p, \tag{20}$$

and

$$\|g\|_\infty < C\varepsilon^{2/(4-p)}\|f\|_p, \tag{20}$$

where $C \leq C_0 + \ldots + C_N$. ∎

REMARK 2.2. The fact that a similar result holds for $H^p(G')$ is an immediate consequence of Proposition 2.2 of [8], since it is possible to perform independent approximations on each component of G'.

The following corollary of Proposition 2.1 can also be proved directly and in a stronger form (with $(1/\varepsilon)^\delta$ replaced by $\log(1/\varepsilon)$ in (23)) using a more general result of Kisliakov [15, 16]: these rely on weak type $(1,1)$ estimates for operators given by Calderón–Zygmund kernels. Our approach here is more elementary.

COROLLARY 2.1. *Let $\varepsilon > 0$ be sufficiently small, let $\delta > 0$, and let $f \in L^\infty(\partial G)$. Then there exist $g^+ \in H^\infty(G)$ and $g^- \in H^\infty(G')$ such that*

$$\|f - (g^+ + g^-)\|_2 < \varepsilon\|f\|_\infty, \tag{22}$$

and

$$\|g^+\|_\infty + \|g^-\|_\infty < C(1/\varepsilon)^\delta\|f\|_\infty, \tag{23}$$

where C is a constant that depends only on G and δ.

Proof. Using Lemma 2.2, we write

$$f^+ = P_+ f \in \bigcap_{p\geq 1} H^p(\partial G), \tag{24}$$

and

$$f^- = (I - P_+)f \in \bigcap_{p\geq 1} H^p(G'). \tag{25}$$

Let $p = (2/\delta) + 4$. By Proposition 2.1 and Remark 2.2, there exist functions $g^+ \in H^\infty(\partial G)$ and $g^- \in H^\infty(G')$ such that

$$\|f - (g^+ + g^-)\|_2 < \varepsilon\|f\|_p \leq \varepsilon\|f\|_\infty, \tag{26}$$

and

$$\|g^+\|_\infty + \|g^-\|_\infty < C\varepsilon^{2/(4-p)}\|f\|_p \leq C(1/\varepsilon)^\delta\|f\|_\infty. \ \blacksquare \tag{27}$$

3. Applications to operator theory. Let $\mathcal{H}$ be a separable, infinite-dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$.

First consider G a (nonempty) bounded domain in $\mathbb{C}$. If the spectrum $\sigma(T)$ of $T \in \mathcal{L}(\mathcal{H})$ satisfies $\sigma(T) \subset G^-$ (the closure of G), we denote by $\mathcal{A}_T^G$ the rational dual algebra generated by T, $\mathcal{A}_T^G = \{r(T) : r \in R_G\}^{-w*}$ (recall that R_G is defined in Definition 2.1), and we say that G is a *spectral set for T* if

$$\|f(T)\| \leq \sup\{|f(\lambda)| : \lambda \in G\}, \quad f \in R_G.$$

In the case where T admits a finitely connected domain G whose boundary consists of a finite union of disjoint Jordan loops, by virtue of [11], we may assume without loss of generality with respect to properties we consider that T admits a G-rational normal boundary dilation, that is, there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$, a normal operator $N \in \mathcal{L}(\mathcal{K})$ such that $\sigma(N) \subset \partial G$ and for any $r \in R_G$ and any $x \in \mathcal{H}$, $r(T)x = P_{\mathcal{H}} r(N)x$. Recall that the existence of normal boundary dilations for the case of an annulus was proved by Agler [1].

We denote by $\mathcal{D}N(G)$ the class of operators admitting G-rational normal boundary dilation. Furthermore we know that such a domain G is conformally equivalent to a circular domain G_1 via a conformal map ϕ which, in fact, satisfies $\phi \in R_G{}^-$ and $\phi^{-1} \in R_{G_1}{}^-$. Under these conditions if, for $T \in \mathcal{D}N(G)$, we set $T_1 = \phi(T)$, it is straightforward to check that $\mathcal{A}_T^G = \mathcal{A}_{T_1}^{G_1}$, $T_1 \in \mathcal{D}N(G_1)$.

Thus, from now on, we assume that G is a circular domain, as described above in (1).

Any normal operator N satisfying $\sigma(N) \subset \Gamma$ has a canonical decomposition $N = N_a \oplus N_s$ where N_a is an absolutely continuous normal operator (i.e. has a scalar spectral measure absolutely continuous with respect to Lebesgue measure) and N_s is a singular normal operator (i.e. has a scalar spectral measure singular with respect to Lebesgue measure). We say that an operator $T \in \mathcal{D}N(G)$ is absolutely continuous if it admits an absolutely continuous boundary dilation N and we denote by $\mathcal{D}N_a(G)$ the class of such T.

Let β_0 denote the position function (i.e., $\beta_0(z) = z$) in any of the spaces we consider. Any absolutely continuous normal operator $N \in \mathcal{L}(\mathcal{K})$ is (unitarily equivalent to) a part of the operator M_{β_0} of multiplication by β_0 on some space $L^2(\Gamma,\mathcal{D})$ of $\mathcal{D}$-valued functions on the Borel set Γ ($\mathcal{D}$ being a Hilbert space).

For $x,y\in\mathcal{K}$, one defines the function $x\overset{T}{\cdot}y$ by $x\overset{T}{\cdot}y(\xi)=(x(\xi),y(\xi))_{\mathcal{D}}$. Thus for $T\in\mathcal{D}N_{\rm a}(G)$ we can define a sesquilinear map $\mathcal{H}\times\mathcal{H}\ni(x,y)\mapsto x\overset{T}{\cdot}y\in L^1(\Gamma)$ which does not depend on any choice of the normal boundary dilation [9, p. 347]. We can also define an $L^\infty(\Gamma)$-functional calculus Ψ_T for T by $\Psi_T(f)=f(T)$ where:

$$\langle f,x\overset{T}{\cdot}y\rangle=(f(T)x,y),\quad f\in L^\infty(\Gamma).$$

This functional calculus extends the $H^\infty(\Gamma)$-functional calculus (see [9]) Φ_T for T defined by $\Phi_T(h)=h(T)$ where

$$h(T)x=P_{\mathcal{H}}h(N)x,\quad h\in H^\infty(\Gamma),\ x\in\mathcal{H}.$$

Clearly, Φ_T coincides with the Nagy–Foiaş functional calculus in the case where T is an absolutely continuous contraction. Though N is not necessarily unique, this functional calculus is uniquely defined as shown by Theorem 2.3 in [9].

We can write

$$L^2(\Gamma,\mathcal{D})=L^2(\Gamma_0,\mathcal{D})\oplus\ldots\oplus L^2(\Gamma_N,\mathcal{D}),$$

where $L^2(\Gamma_j,\mathcal{D})$ is the subspace of functions in $L^2(\Gamma,\mathcal{D})$ which vanish on $\Gamma\backslash\Gamma_j$, $j=0,\ldots,N$. We denote by P_j the orthogonal projection of $L^2(\Gamma,\mathcal{D})$ onto $L^2(\Gamma_j,\mathcal{D})$ and by L_j the subset of functions in $L^2(\Gamma,\mathcal{D})$ which are essentially bounded on Γ_j for $j=0,\ldots,N$.

In order to generalize one of the main results of [8], first recall a density result.

PROPOSITION 3.1 (Proposition 2 of [21]). *For given $j=0,\ldots,N$, $L_j\cap\mathcal{H}$ is dense in $\mathcal{H}$.*

LEMMA 3.1. *Let $T\in\mathcal{D}N_{\rm a}(G)$. For each $j=0,\ldots,N$, given $f_j\in L^\infty(\Gamma_j)$ and $x\in L_j\cap\mathcal{H}$, we have*

$$\|f_j(T)x\|\le 2\pi\|f_j\|_2\|x|_{\Gamma_j}\|_\infty.$$

Proof. We have for $y\in\mathcal{H}$,

$$|(f_j(T)x,y)|=|\langle f_j,x\overset{T}{\cdot}y\rangle|=\Big|\int_{\Gamma_j}f_j(\xi)(x(\xi),y(\xi))_{\mathcal{D}}|d\xi|\Big|$$
$$\le 2\pi r_j\|f_j\|_2\|x|_{\Gamma_j}\|_\infty\|y\|_2\le 2\pi\|f_j\|_2\|x|_{\Gamma_j}\|_\infty\|y\|_2$$

The lemma follows. ∎

The next lemma is an immediate consequence of the Cauchy–Schwarz inequality.

LEMMA 3.2. *Let $T\in\mathcal{D}N_{\rm a}(G)$ and let x,y be two functions in $L^2(\Gamma,\mathcal{D})$. Then*

$$\|x\overset{T}{\cdot}y\|_1\le C_G\|x\|_2\|y\|_2,$$

where C_G is a positive constant which only depends on G.

We denote by $\mathcal{P}_*(\Gamma)$ the predual of $H^\infty(\Gamma)$: $\mathcal{P}_*(\Gamma)=L^1(\Gamma)/^\perp H^\infty(\Gamma)$ and, for $f\in L^1(\Gamma)$, by $[f]$ the image of f in $\mathcal{P}_*(\Gamma)$ via the quotient map.

The following result generalizes Theorem 3.2 of [8].

THEOREM 3.1. *Let $T\in\mathcal{D}N_{\rm a}(G)$ and let $(x_n)_{n\ge1}$ be a sequence of elements of $\mathcal{H}$. The following assertions are equivalent:*

(i) $\lim_{n\to\infty}\|x_n\overset{T}{\cdot}w\|_1=0$, $w\in\mathcal{H}$,

(ii) $\lim_{n\to\infty}(\|[x_n\overset{T}{\cdot}w]\|+\|[w\overset{T}{\cdot}x_n]\|)=0$, $w\in\mathcal{H}$.

Proof. It is clear that we only have to prove that if $\|[x_n\overset{T}{\cdot}w]\|+\|[w\overset{T}{\cdot}x_n]\|\to0$ for every $w\in\mathcal{H}$, then $\lim_{n\to\infty}\|x_n\overset{T}{\cdot}w\|_1=0$ for every $w\in\mathcal{H}$.

Assume that the sequence $(x_n)_{n\ge1}$ satisfies condition (ii) and let $w\in\mathcal{H}$. We have

$$|(x_n,w)|=|\langle1,[x_n\overset{T}{\cdot}w]\rangle|\le\|[x_n\overset{T}{\cdot}w]\|,$$

and so the sequence $(x_n)_{n\ge1}$ tends weakly to 0. Assume that

$$\limsup_{n\to\infty}\|x_n\overset{T}{\cdot}w\|_1>0.$$

Without loss of generality, we may suppose that for $n\ge1$, $\|x_n\overset{T}{\cdot}w\|_1\ge\tau>0$, $\|x_n\|_2\le1$, $\|w\|_\infty\le1$. Our aim is now to find a contradiction.

We have

$$\|x_n\overset{T}{\cdot}w\|_1=|\langle\varphi_n,x_n\overset{T}{\cdot}w\rangle|,$$

where $\varphi_n\in L^\infty(\Gamma)$, $\|\varphi_n\|_\infty=1$. Write $\varphi_n=\sum_{j=0}^N\varphi_n^j$ where $\varphi_n^j\in L^\infty(\Gamma_j)$ for $j=0,\ldots,N$. So we obtain

$$\|x_n\overset{T}{\cdot}w\|_1\le\sum_{j=0}^N|\langle\varphi_n^j,x_n\overset{T}{\cdot}w\rangle|=\sum_{j=0}^N\int_{\Gamma_j}\varphi_n^j(\xi)(x_n(\xi),w(\xi))_{\mathcal{D}}\,|d\xi|$$
$$=\sum_{j=0}^N\int_{\Gamma_j}\varphi_n^j(\xi)(P_jx_n(\xi),w(\xi))_{\mathcal{D}}\,|d\xi|\le\sum_{j=0}^N\|P_jx_n\overset{T}{\cdot}w\|_1.$$

For any $\alpha>0$ and for each $j=0,\ldots,N$, by Proposition 3.1, we can choose $w_j\in L_j\cap\mathcal{H}$ such that $\|w-w_j\|_2\le\alpha$. Since obviously

$$\|P_jx_n\overset{T}{\cdot}w\|_1\le\|P_jx_n\overset{T}{\cdot}(w-w_j)\|_1+\|P_jx_n\overset{T}{\cdot}w_j\|_1,$$

by Lemma 3.2 we get

$$\|x_n \overset{T}{\cdot} w\|_1 \le (N+1)\alpha C_G + \sum_{j=0}^{N} \|P_j x_n \overset{T}{\cdot} w_j\|_1,$$

where $w_j \in L_j \cap \mathcal{H}$. Now we fix $\alpha > 0$ such that $(N+1)\alpha C_G < \tau/3$ and we denote by $M(\alpha)$ a majorant of the finite sequence $(\|w_j|_{\Gamma_j}\|_\infty)_{0\le j\le N}$.

Since (ii) holds, in particular, for $j = 0, \ldots, N$, we have

$$\lim_{n\to\infty} (\|[P_j x_n \overset{T}{\cdot} w_j]\| + \|[w_j \overset{T}{\cdot} P_j x_n]\|) = 0.$$

In the case where $j = 0$, by Theorem 3.2 of [8], we obtain

$$\lim_{n\to\infty} \|P_0 x_n \overset{T}{\cdot} w_0\|_1 = 0,$$

which means in particular that if n is large enough then $\|P_0 x_n \overset{T}{\cdot} w_0\|_1 \le \tau/(3(N+1))$.

Now we fix $j \in \{1, \ldots, N\}$ and consider $f_n \in L^\infty(\Gamma_j)$ such that

$$\|P_j x_n \overset{T}{\cdot} w_j\|_1 = |\langle f_n, P_j x_n \overset{T}{\cdot} w_j\rangle| \quad \text{with } \|f_n\|_\infty = 1.$$

For any $\varepsilon \in (0, 1/2]$, applying Corollary 2.1, there exists a function $g_n = g_n^+ + g_n^- \in H^\infty(\Omega_j) + H^\infty(D_j)$ satisfying

$$\|g_n - f_n\|_2 \le \varepsilon, \qquad \|g_n^+\|_\infty + \|g_n^-\|_\infty \le C/\sqrt{\varepsilon}.$$

Then we write:

$$\begin{aligned}|\langle f_n, P_j x_n \overset{T}{\cdot} w_j\rangle| &\le |\langle f_n - g_n, P_j x_n \overset{T}{\cdot} w_j\rangle| \\ &\quad + |\langle g_n^+, P_j x_n \overset{T}{\cdot} w_j\rangle| + |\langle g_n^-, P_j x_n \overset{T}{\cdot} w_j\rangle|.\end{aligned}$$

Using Lemma 3.1 we get

$$|\langle f_n - g_n, P_j x_n \overset{T}{\cdot} w_j\rangle| \le 2\pi \|f_n - g_n\|_2 \|w_j|_{\Gamma_j}\|_\infty \le 2\pi\varepsilon M(\alpha).$$

Moreover, since it is easy to check that $\overline{g_n^-} \in H^\infty(\Omega_j)$ and since

$$|\langle g_n^-, P_j x_n \overset{T}{\cdot} w_j\rangle| = |\langle \overline{g_n^-}, w_j \overset{T}{\cdot} P_j x_n\rangle|,$$

we obtain

$$|\langle g_n^+, P_j x_n \overset{T}{\cdot} w_j\rangle| + |\langle g_n^-, P_j x_n \overset{T}{\cdot} w_j\rangle| \le \frac{C}{\sqrt{\varepsilon}} (\|[P_j x_n \overset{T}{\cdot} w_j]\| + \|[w_j \overset{T}{\cdot} P_j x_n]\|).$$

Then, clearly, if we fix ε sufficiently small and then take n large enough, we have $\|P_j x_n \overset{T}{\cdot} w_j\|_1 \le \tau/(3(N+1))$. Thus we get $\|x_n \overset{T}{\cdot} w\|_1 \le 2\tau/3$, contradicting the hypothesis $\|x_n \overset{T}{\cdot} w\|_1 \ge \tau$. ■

DEFINITION 3.1. For a circular domain G, the class $\mathbb{A}^G$ is the subclass of $\mathcal{D}N_a(G)$ consisting of those $T \in \mathcal{L}(\mathcal{H})$ for which Φ_T is isometric. The class $\mathbb{A}^G_{\aleph_0}$ consists of those $T \in \mathbb{A}^G$ such that, for any infinite array $(f_{i,j})_{i,j\ge1}$ of functions in $L^1(\Gamma)$, there exist some sequences $(x_i)_{i\ge1}$ and $(y_j)_{j\ge1}$ in $\mathcal{H}$ such that

$$[f_{i,j}] = [x_i \overset{T}{\cdot} y_j], \quad i, j \ge 1.$$

Also recall that we say that an operator T is in the class $C^G_{0\cdot}$ (resp. $C^G_{\cdot 0}$) if $\sigma(T) \subset G^-$ and if $\lim_{m\to\infty} \|T^m x\| = 0 = \lim_{m\to\infty} r_j^m \|(a_j - T)^{-m} x\|$ (resp. $\lim_{m\to\infty} \|T^{*m} x\| = 0 = \lim_{m\to\infty} r_j^m \|(\overline{a_j} - T^*)^{-m} x\|$) for all $x \in \mathcal{H}$ and $j = 1, \ldots, N$. As in the contraction case, we set $C^G_{00} = C^G_{0\cdot} \cap C^G_{\cdot 0}$.

The next result generalizes Corollary 4.4 of [8] and is an immediate consequence of the present Theorem 3.1.

COROLLARY 3.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be in the class $\mathbb{A}^G \cap C^G_{00}$ or more generally in the class $\mathbb{A}^G_{\aleph_0}$. Then, for any infinite array of functions in $L^1(\Gamma)$, there exist bounded sequences $(x_i)_{i\ge1}$ and $(y_j)_{j\ge1}$ in $\mathcal{H}$ such that*

$$f_{i,j} = x_i \overset{T}{\cdot} y_j, \quad i, j \ge 1.$$

Proof. By Proposition 3 of [21], since $T \in \mathbb{A}^G$, we know that if $f \in L^1(\Gamma)$, there exist in $\mathcal{H}$ two sequences $(u_n)_{n\ge1}$ and $(v_n)_{n\ge1}$ which converge to 0 in the weak topology and

$$\begin{cases} \lim_{n\to\infty} \|f - u_n \overset{T}{\cdot} v_n\|_1 = 0, \\ \|u_n\| \le \|f\|_1^{1/2} \quad \text{and} \quad \|v_n\| \le \|f\|_1^{1/2}, \quad n \ge 1. \end{cases}$$

By Lemma 3.5 of [10], if $T \in C^G_{00}$, for any sequence $(x_n)_{n\ge1}$ of $\mathcal{H}$ which tends weakly to 0, we have:

$$\lim_{n\to\infty} (\|[x_n \overset{T}{\cdot} w]\| + \|[w \overset{T}{\cdot} x_n]\|) = 0, \quad w \in \mathcal{H}.$$

Applying Theorem 3.1 with $(x_n)_{n\ge1} = (u_n)_{n\ge1}$ and $(x_n)_{n\ge1} = (v_n)_{n\ge1}$, we get

$$\lim_{n\to\infty} (\|u_n \overset{T}{\cdot} w\| + \|v_n \overset{T}{\cdot} w\|) = 0, \quad w \in \mathcal{H}.$$

As in [8], the desired conclusion then follows from a corollary of Proposition 7.2 in [5] (see also Theorem B in [8]). If we suppose that $T \in \mathbb{A}^G_{\aleph_0}$, the desired conclusion is an immediate consequence of Theorem 3.1 in [6] which implies that any operator in $\mathbb{A}^G_{\aleph_0}$ has a compression in $\mathbb{A}^G \cap C^G_{00}$. ■

4. Extremal problems. In this section we work with function spaces on an annulus G bounded by the circles $\Gamma_0 = \mathbb{T} = \partial\mathbb{D}$ and $\Gamma_1 = a + r\mathbb{T}$, where a and r are fixed with $a \in \mathbb{D}$ and $0 < r < 1 - |a|$. The intention is to study approximation of functions on Γ_0 by functions which have an analytic extension to G. By a slight abuse of notation we shall write formulae such as $\|g|_{\Gamma_0}\|_{L^p(\Gamma_0)}$ as $\|g\|_{L^p(\Gamma_0)}$ when $g \in L^p(\partial G)$.

PROBLEM 4.1. *Let $1 \le p \le \infty$ be fixed, let $f \in L^p(\Gamma_0)$ and $M > 0$. Write*

$$\mathcal{B} = \{g \in H^p(\partial G) : \|g\|_{L^p(\Gamma_1)} \le M\}. \tag{28}$$

Find $g \in \mathcal{B}$ such that

$$\|f - g\|_{L^p(\Gamma_0)} = \inf\{\|f - h\|_{L^p(\Gamma_0)} : h \in \mathcal{B}\}. \tag{29}$$

The same problem is to be considered with each $L^p(\Gamma_j)$ replaced by $C(\Gamma_j)$, and $H^p(\partial G)$ replaced by $A(\partial G)$.

4.1. *Density results.* We begin with a few density results which can be seen as a counterpart of results in [4, 3]. Recall that $A(\partial G)$ and $H^p(\partial G)$ are closed subspaces of $C(\Gamma_0 \cup \Gamma_1)$ and $L^p(\Gamma_0 \cup \Gamma_1)$ respectively, and they are proper subspaces as they do not contain functions such as $1/(z - z_0)$, where z_0 is any point in G.

THEOREM 4.1. *Let G be an annulus bounded by two circles Γ_0 and Γ_1 specified as above. Then*

(i) *$A(\partial G)|_{\Gamma_0}$ is dense in $C(\Gamma_0)$.*
(ii) *For $1 \le p < \infty$, $H^p(\partial G)|_{\Gamma_0}$ is dense in $L^p(\Gamma_0)$.*
(iii) *$H^\infty(\partial G)|_{\Gamma_0}$ is not dense in $L^\infty(\Gamma_0)$.*

Proof. (i) Any function in $C(\Gamma_0)$ can be uniformly approximated by trigonometric polynomials $\sum_{n=-N}^{N} c_n z^n$, and hence by rational functions $\sum_{n=-N}^{N} c_n (z-a)^n/(1-\overline{a}z)^n$. Such functions lie in $A(\partial G)$ (restricted to Γ_0).

(ii) Recalling that $C(\Gamma_0)$ is dense in $L^p(\Gamma_0)$, and using (i), we obtain the required result.

(iii) Any function f in $H^\infty(\partial G)|_{\Gamma_0}$ can be decomposed into the sum of a function in $H^\infty(\mathbb{D})$ and a function in $H_0^\infty(\Omega_1)$, so that $f(z) = f_1(z) + \overline{f_2(z)}$, where $f_1, f_2 \in H^\infty$. However, it is known [13, p. 151] that $H^\infty + \overline{H^\infty}$ is not dense in $L^\infty(\Gamma_0)$. ∎

The above results tell us that any continuous function on Γ_0 is the uniform limit of a sequence of restrictions of functions in $A(\partial G)$, and that any L^p function is likewise approximable by $H^p(\partial G)$ functions, at least if p is finite. Our next result shows that this approximation is, in general, only possible at the expense of divergent behaviour on Γ_1.

PROPOSITION 4.1. *Suppose that $1 < p \le \infty$ and that (g_n) is a sequence of $H^p(\partial G)$ functions such that $(g_n)|_{\Gamma_0}$ converges to f in $L^p(\Gamma_0)$ norm. If f is not the restriction of an $H^p(\partial G)$ function, then $\lim_{n\to\infty} \|g_n\|_{L^p(\Gamma_1)} = \infty$.*

Proof. Suppose to the contrary that $(\|g_n\|_{L^p(\Gamma_1)})$ has a bounded subsequence. Then, by passing to a further subsequence if necessary, we may suppose that $(g_n)|_{\Gamma_1}$ converges in the weak-* topology to a function f_1 in $H^p(\partial G)$. Now f is necessarily the restriction of f_1, since $g_n \to f$ in $L^p(\Gamma_0)$ norm, and $g_n \to f_1|_{\Gamma_0}$ in the weak-* topology. This establishes the result. ∎

4.2. *Solution for $p = 2$.* Solution of Problem 4.1 is particularly straightforward in the important case $p = 2$, because we can employ the following result.

LEMMA 4.1. *Let $\Gamma_0 = \mathbb{T}$ and $\Gamma_1 = a + r\mathbb{T}$, where $a \in \mathbb{D}$ and $0 < r < 1 - |a|$. Let*

$$w = \frac{z - b}{1 - \overline{b}z}, \tag{30}$$

where $|a - b| < r$, be a conformal bijection that maps G onto an annulus G_s of the form $\{s < |w| < 1\}$, where $0 < s < 1$. Then the sequence of functions $(\phi_n)_{n\in\mathbb{Z}}$ defined by

$$\phi_n(z) = \frac{(z - b)^n}{(1 - \overline{b}z)^{n+1}}, \quad n \in \mathbb{Z}, \tag{31}$$

is an orthogonal basis of $H^2(\partial G)$.

Proof. We begin with the easy observation that $(w^n)_{n\in\mathbb{Z}}$ is an orthogonal (not orthonormal!) basis of $H^2(\partial G_s)$ in the case when G is the annulus bounded by circles of radius 1 and s. The orthogonality is easy to check, and the completeness of the system follows because $H^2(\partial G_s)$ decomposes into a sum of elementary Hardy spaces.

The derivative of the mapping (30) is $w'(z) = (1 - |b|^2)/(1 - \overline{b}z)^2$. Note that we have an isometry T between $H^2(\partial G_s)$ and $H^2(\partial G)$, given by

$$(Tf)(z) = f(w(z))\{w'(z)\}^{1/2}, \tag{32}$$

as is easily seen by changing the variable of integration. When $f(w) = w^n$ then $(Tf)(z)$ is just a constant multiple of $\phi_n(z)$, as required. ∎

We now give the solution to the bounded extremal problem in its simplest form, where we take $G = G_s$, and normalize the boundary circles $\mathbb{T}$ and $s\mathbb{T}$ so that they both have measure 1. The solutions for more complicated annuli, and different normalizations, are easily deduced from the one we present, by using the orthogonal basis (ϕ_n) defined above.

In the case we consider, the following functions are easily seen to form an orthonormal basis of $H^2(\partial G)$:

$$e_n(z) = \frac{z^n}{\sqrt{1 + s^{2n}}}, \quad n \in \mathbb{Z}. \tag{33}$$

THEOREM 4.2. *Let $M > 0$, and let $f(z) = \sum_{k=-\infty}^{\infty} c_k z^k$ be a function in $L^2(\mathbb{T})$ that does not extend to a function in $H^2(\partial G)$ satisfying $\|f\|_{L^2(\Gamma_1)} \le M$. Then the solution to Problem 4.1 for $p = 2$ is given by*

$g(z) = \sum_{k=-\infty}^{\infty} d_k z^k$, *where*

(34) $$d_k = \frac{c_k}{1+\lambda s^{2k}}, \quad k \in \mathbb{Z},$$

and $\lambda > 0$ *satisfies*

(35) $$\sum_{k=-\infty}^{\infty} \frac{|c_k|^2 s^{2k}}{(1+\lambda s^{2k})^2} = M^2.$$

Proof. It is necessary to solve the optimization problem

(36) $$\inf \sum_{k=-\infty}^{\infty} |c_k - d_k|^2 \quad \text{subject to} \quad \sum_{k=-\infty}^{\infty} |d_k|^2 s^{2k} \le M^2,$$

given that $\sum_{k=-\infty}^{\infty} |c_k|^2 s^{2k} > M^2$, and the sum may even be divergent. Let us begin with the case when (c_k) is real and positive for each k. Then the variational equation

(37) $$\sum_{n=-\infty}^{\infty} (c_n - d_n)^2 + \lambda\Big(\sum_{n=-\infty}^{\infty} d_n^2 s^{2n} - M^2\Big),$$

when partially differentiated with respect to d_k and set to zero, yields (34), and it is clear that there is a unique positive value of λ satisfying (35). This also shows that the constraint must be saturated, i.e., $\|g\|_{L^2(\Gamma_1)} = M$. The case when (c_k) is not necessarily real and positive follows easily. ∎

We remark that some of the ideas of this section, including a link with the theory of Toeplitz operators, are being followed up in [17]. Such approximation problems have applications to questions of system identification and Dirichlet–Neumann problems.

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