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Continuity of generalized inverses in Banach algebras

by

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Abstract. The main topic of the paper is the continuity of several kinds of generalized inversion of elements in a Banach algebra with identity. We introduce the notion of asymptotic generalized invertibility and completely characterize sequences of elements with this property. Based on this result, we derive continuity criteria which generalize the well known criteria from operator theory.

1. Introduction. Let $\mathfrak{A}$ be an algebra with identity element e over the field $\mathbb{C}$ of complex numbers. An element a of $\mathfrak{A}$ is called *generalized invertible* if there is an element $b \in \mathfrak{A}$ such that

$$aba = a. \tag{I}$$

The element b is not unique in general. In order to force its uniqueness, further conditions have to be imposed. The perhaps most convenient additional conditions are

$$bab = b, \tag{II}$$

$$(ab)^* = ab, \tag{III}$$

$$(ba)^* = ba, \tag{IV}$$

$$ab = ba. \tag{V}$$

Of course, (III) and (IV) make sense for involutive algebras only. One also considers a generalization of (I):

$$a^k ba = a^k \tag{I_k}$$

with some $k \in \mathbb{Z}^+$. Clearly, $(\mathrm{I}) = (\mathrm{I}_1)$.

Elements $b \in \mathfrak{A}$ satisfying (I) and (II) are called (I, II)-inverses or *symmetric inverses* of a. Similarly, (I, II, V)-inverses are called *group inverses*, (I, II, III, IV)-inverses are *Moore–Penrose inverses*, and (I_k, II, V)-inverses are *Drazin inverses* (of degree k). (For a moment, we had to resist the temptation to introduce the notion of *Abel inverse* for the usual *group inverse*.)

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Thus, group inverses are Drazin inverses of degree 1. Furthermore, every generalized invertible element has a symmetric inverse. Indeed, if b is a generalized inverse of a, then bab is a symmetric one:

$$a(bab)a = (aba)ba = aba = a, \qquad (bab)a(bab) = b(ababa)b = bab.$$

It turns out that both Drazin and Moore–Penrose inverses are unique if they exist. This leads to the natural question whether Drazin or Moore–Penrose inversion is a continuous mapping on the set of all Drazin resp. Moore–Penrose invertible elements of $\mathfrak{A}$ (such as usual inversion is continuous on the group of invertible elements). The answer is known to be *no* in general. So the main concern of the present paper is conditions which enforce the continuity.

We have tried to make the paper self-contained; so its first parts can also serve as a brief introduction in and a survey on this topic. For much more detailed accounts on generalized inverses and their applications we refer to the monographs [12] and [14]. Let us also mention the paper [9], which contains a very nice treatment of generalized Drazin inverses, [5] and [6], which deal with Moore–Penrose inverses in C^*-algebras and which, together with developments in numerical analysis, stimulated our efforts in this topic, and [15] and [10] where the continuity of the Drazin inverse is also considered. Further we refer to [7] and [8] where the existence of several kinds of generalized inverses of elements of algebras of matrices over a commutative Banach algebra is studied.

We start with recalling basic algebraic facts about existence and uniqueness of generalized inverses in algebras, and about relations between them. The proofs of these facts are no much longer than the citation of explicit references, and so we add them for completeness, and for the reader's convenience.

Then we turn to properties of generalized inverses which are more related to complete normed algebras. The first one concerns the inverse closedness of subalgebras with respect to generalized inversion, i.e. the question whether an element which belongs to a subalgebra of a large algebra, and which has a generalized inverse in the large algebra, also possesses a generalized inverse in the subalgebra. Secondly, we characterize generalized invertibility of an element in terms of the spectrum of that element.

After these preparations, we turn our attention to the main topic of the paper: the continuity of several kinds of generalized inverses. The origins of our approach lie in numerical analysis. Motivated by [13], and based on experience from [19] and [16]–[18] (where continuity with respect to the strong operator topology of sequences of Toeplitz matrices is considered), we introduce some kind of asymptotic generalized invertibility of a sequence $(a_n) \subseteq \mathfrak{A}$. This means, for example, that we require $\|a_n b_n a_n - a_n\| \to 0$ in

place of $a_n b_n a_n - a_n = 0$. We prove that a convergent sequence is asymptotically group invertible if and only if its limit is group invertible, and we derive a criterion for asymptotic group invertibility (Theorem 1). This will enable us to obtain a criterion for continuity of group inverses (and therefore also of Drazin and Moore–Penrose inverses) in Banach algebras (Theorem 2) which generalizes the well-known criterion for the continuity of the Moore–Penrose inverses of Fredholm operators in terms of their kernel dimension. These results are then applied to the algebra of bounded linear operators on a Banach space.

Finally, we consider a problem which also has its roots in numerical analysis: the regularization of approximation sequences for ill-posed equations. In particular, we will discuss how to regularize an asymptotically group (or Moore–Penrose) invertible sequence in order to get a sequence which has an exact and *continuous* group (resp. Moore–Penrose) inverse. This regularization is intimately related to the asymptotic splitting of the approximation numbers of the entries of the sequence.

The authors are grateful to the referee for helpful comments and for turning their attention to the papers [15] and [10].

2. Uniqueness and existence. Let us start with the uniqueness of generalized inverses. Symmetric inverses are not unique in general; for example, every matrix $\begin{pmatrix} \alpha & 0 \\ 1 & 0 \end{pmatrix}$ is a symmetric generalized inverse of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. But it turns out that both Drazin and Moore–Penrose inverses are unique if they exist.

LEMMA 1. *An element can have at most one Drazin inverse and at most one Moore–Penrose inverse.*

Proof. Let b and c be Drazin inverses of a of the same degree k. Then

$$\begin{aligned} b = bab = b(ab)^k &= b^{k+1}a^k = b^{k+1}a^k ca \\ &= b^{k+1}a^k(ca)^{k+1} = b^{k+1}a^{2k+1}c^{k+1} \end{aligned}$$

and, analogously, $c = b^{k+1}a^{2k+1}c^{k+1}$, i.e. $b = c$. Similarly, if b and c are Moore–Penrose inverses of a, then

$$\begin{aligned} b = bab = b(ab)^* &= bb^*a^* = bb^*(aca)^* = bb^*a^*(ac)^* \\ &= bb^*a^*ac = bb^*a^*(aca)c = bb^*a^*a(ca)^*c = bb^*a^*aa^*c^*c \end{aligned}$$

and, analogously, $c = bb^*a^*aa^*c^*c$, i.e. $b = c$. ∎

Observe that every Drazin inverse of degree k is also a Drazin inverse of degree $k+1$. Thus, if an element has Drazin inverses of degrees k_1 and k_2, then these Drazin inverses coincide.

COROLLARY 1. *Let $\mathfrak{A}$ be an involutive algebra. If b is the Drazin resp. Moore–Penrose inverse of a, then b^* is the Drazin resp. Moore–Penrose inverse of a^*. Drazin and Moore–Penrose inverses of self-adjoint elements are self-adjoint again.*

Proof. Taking adjoints in (I_k), (II) and (V) yields

$$a^*b^*(a^k)^* = (a^k)^*, \quad b^*a^*b^* = b^*, \quad a^*b^* = b^*a^*.$$

The first and third equality together ensure that $(a^k)^*b^*a^* = (a^k)^*$, i.e. b^* is the Drazin inverse of a^*. Analogously, the other assertions follow. ∎

It is well known that every matrix $A \in \mathbb{C}^{N\times N}$ is generalized invertible. On the other hand, the function $a(x) = x$, considered as an element of the algebra $C[0,1]$ of all complex-valued continuous functions on the interval $[0,1]$, is an example of an element which is not generalized invertible (and also not (I_k)-invertible for any k). Thus, our next concern is the existence of generalized inverses. The following lemma shows that the central role is played by the group invertibility. We denote by $\mathrm{Comm}\, a$ the *commutator algebra* $\{b \in \mathfrak{A} : ab = ba\}$ of the element a. Further, we only consider involutive algebras which are *reducing* in the following sense: If $a^*a = 0$ for an element $a \in \mathfrak{A}$, then $a = 0$. Examples of reducing algebras are C^*-algebras and the algebra $L^1(\mathbb{R})$ (with the Fourier convolution as multiplication).

LEMMA 2. (a) *Let $\mathfrak{A}$ be an algebra with identity. An element $a \in \mathfrak{A}$ is Drazin invertible of degree k in $\mathfrak{A}$ if and only if a^k is group invertible in the algebra* $\mathrm{Comm}\, a$.

(b) *Let $\mathfrak{A}$ be an involutive and reducing algebra with identity. An element $a \in \mathfrak{A}$ is Moore–Penrose invertible if and only if a^*a is group invertible.*

Proof. If a is Drazin invertible and b is the Drazin inverse of a of degree k, then b^k is the group inverse of a^k:

$$a^kb^ka^k = a^k(ba)^k = a^k, \quad b^ka^kb^k = (bab)^k = b^k, \quad a^kb^k = b^ka^k,$$

and from $ba = ab$ we conclude that b^k belongs to $\mathrm{Comm}\, a$.

If, conversely, b is the group inverse of a^k in $\mathrm{Comm}\, a$, then ba^{k-1} is the Drazin inverse of degree k of a:

$$a^k(ba^{k-1})a = a^kba^k = a^k, \quad (ba^{k-1})a(ba^{k-1}) = (ba^kb)a^{k-1} = ba^{k-1},$$

and $a(ba^{k-1}) = (ba^{k-1})a$ since $b \in \mathrm{Comm}\, a$.

Let now a be Moore–Penrose invertible and b be the Moore–Penrose inverse of a. Then bb^* is the group inverse of a^*a as one easily checks. If, conversely, b is the group inverse of a^*a, then ba^* is the Moore–Penrose inverse of a: It can be easily verified that ba^* is a (II, III, IV)-inverse of a. Moreover, $a^*aba^*a = a^*a$ implies $a^*a(e - ba^*a) = 0$ and, consequently,

$$(e - ba^*a)a^*a(e - ba^*a) = [a(e - ba^*a)]^*[a(e - ba^*a)] = 0.$$

The reducing property of $\mathfrak{A}$ yields $a(e - ba^*a) = 0$ or $aba^*a = a$, i.e. ba^* is also a (I)-inverse of a. ∎

We shall see in Corollary 5 (Section 4) that, in case $\mathfrak{A}$ is a Banach algebra, the group invertibility of the element a^k in $\mathrm{Comm}\, a$ in assertion (a) can be replaced by its group invertibility in the algebra $\mathfrak{A}$ itself.

One has the following simple criterion for the existence of group inverses.

LEMMA 3. *Let $\mathfrak{A}$ be an algebra with identity e. An element $a \in \mathfrak{A}$ is group invertible if and only if there is an idempotent $p \in \mathfrak{A}$ such that*

$$(1) \qquad a + p \text{ is invertible}, \quad ap = 0, \quad \text{and } ap = pa.$$

If these conditions are satisfied, then the group inverse b of a is given by $b = (a+p)^{-1}(e-p)$, and $p = e - ba$.

Proof. Let a be group invertible and b be the group inverse of a. Then $p := e - ba$ is an idempotent element, and one has $ap = pa$ and $ap = a(e - ba) = 0$. Moreover, $pb = bp = b(e - ab) = 0$ and, thus,

$$(a + p)(b + p) = ab + ap + pb + p = ab + p = e,$$

i.e. $a + p$ is invertible and $(a + p)^{-1} = b + p$.

If, conversely, the conditions (1) are satisfied, then $b := (a+p)^{-1}(e-p)$ is the group inverse of a: Indeed,

$$a(a+p)^{-1}(e-p)a = (a+p)(a+p)^{-1}(e-p)a = (e-p)a = a,$$
$$(a+p)^{-1}(e-p)a(a+p)^{-1}(e-p) = (a+p)^{-2}(e-p)(a+p) = (a+p)^{-1}(e-p),$$

and axiom (V) follows from $ap = pa$ immediately. ∎

In other words: a is group invertible if and only if there is an idempotent p which commutes with a such that $a + p = (e-p)a(e-p) + p$ is invertible.

COROLLARY 2. *If $a \in \mathfrak{A}$ is group invertible, then there is exactly one idempotent p satisfying* (1).

Proof. Let p_1 and p_2 satisfy (1) in place of p. Then $(a + p_1)^{-1}(e - p_1)$ and $(e - p_2)(a + p_2)^{-1}$ are group inverses of a by Lemma 3, and hence, $(a+p_1)^{-1}(e-p_1) = (e-p_2)(a+p_2)^{-1}$ by Lemma 1. Thus, $(e-p_1)(a+p_2) = (a + p_1)(e - p_2)$, which implies $p_1 = p_2$. ∎

We shall refer to this idempotent as the *group idempotent* of a. It is obvious from the axioms of group inverses that, if b is the group inverse of a, then a is the group inverse of b, and that the elements a and b have the same group idempotent.

COROLLARY 3. *Let $\mathfrak{A}$ be an involutive algebra with identity. The group idempotent of a self-adjoint group invertible element is self-adjoint.*

Proof. Adjoining the axioms, one sees that a^* is group invertible whenever a is, and that the group idempotent of a^* is the adjoint of the group idempotent of a. Hence, group idempotents of self-adjoint elements are self-adjoint. ∎

Now conditions for the existence of Drazin and Moore–Penrose inverses follow without great effort.

PROPOSITION 1. (a) *Let $\mathfrak{A}$ be an algebra with identity. An element $a \in \mathfrak{A}$ is Drazin invertible of degree k if and only if there is an idempotent $p \in \mathfrak{A}$ such that*

$$a + p \text{ is invertible}, \quad a^k p = 0, \quad \text{and } ap = pa. \tag{2}$$

If (2) is satisfied, then the Drazin inverse of a is $(a+p)^{-1}(e-p)$.

(b) *Let $\mathfrak{A}$ be an involutive and reducing algebra with identity. An element $a \in \mathfrak{A}$ is Moore–Penrose invertible if and only if there is a projection (i.e. a self-adjoint idempotent) $p \in \mathfrak{A}$ such that*

$$a^*a + p \text{ is invertible}, \quad ap = 0, \quad \text{and } a^*ap = pa^*a. \tag{3}$$

If (3) is satisfied, then the Moore–Penrose inverse of a is $(a^*a+p)^{-1}a^*$.

Proof. (a) If a is Drazin invertible of degree k, then a^k is group invertible in $\mathrm{Comm}\, a$, and conversely (Lemma 2). By Lemma 3, the group invertibility of a^k in $\mathrm{Comm}\, a$ is equivalent to the existence of an idempotent $p \in \mathrm{Comm}\, a$ such that

$$a^k + p \text{ is invertible}, \quad a^k p = 0, \quad \text{and } a^k p = pa^k.$$

The latter condition is redundant since even $ap = pa$. We claim that $a^k + p$ is invertible if and only if $a + p$ is invertible.

From $a^k p = 0$ and $ap = pa$ we conclude that

$$a^k + p = a^k(e-p) + p = ((e-p)a(e-p)+p)^k.$$

Thus, $a^k + p$ is invertible if and only if $(e-p)a(e-p)+p$ is invertible. Further we have

$$a + p = (e-p)a(e-p) + pap + p =: c + r \tag{4}$$

where $c := (e-p)a(e-p)+p$, and where $r := pap$ is nilpotent: $r^k = (pap)^k = a^k p = 0$. Our claim follows from the following observation:

If c and r are commuting elements of an algebra with identity e, and if r is nilpotent, then c is invertible if and only if $c + r$ is invertible.

Indeed, if r is nilpotent, then so is $-r$, and thus it is sufficient to verify only one direction. Let c be invertible and $r^k = 0$. Then $e + c^{-1}r$ is invertible:

$$\begin{aligned}(e - c^{-1}r + c^{-2}r^2 - \ldots + (-1)^{k-1}c^{-(k-1)}r^{k-1})(e + c^{-1}r) \\ = e + (-1)^{k-1}c^{-k}r^k = e,\end{aligned}$$

and, hence, $c + r = c(e + c^{-1}r)$ is invertible. Thus, $(e-p)a(e-p)+p = c$ is invertible if and only if $a + p = c + r$ is invertible.

The formula of the Drazin inverse can be verified straightforwardly.

(b) If a is Moore–Penrose invertible, then a^*a is group invertible, and conversely (Lemma 2). By Lemma 3, group invertibility of a^*a is equivalent to the existence of an idempotent p such that

$$a^*a + p \text{ is invertible}, \quad a^*ap = 0, \quad \text{and } a^*ap = pa^*a.$$

The second condition implies $(ap)^*(ap) = 0$ whence via the reducing property it follows that $ap = 0$. Moreover, p is self-adjoint by Corollary 3. Thus, conditions (3) are necessary and sufficient for the Moore–Penrose invertibility of a.

Finally, we know from Lemmas 2 and 3 that the Moore–Penrose inverse of a is $(a^*a+p)^{-1}(e-p)a^*$, which coincides with $(a^*a+p)^{-1}a^*$ since $pa^* = (ap)^* = 0$. ∎

3. Spectral characterizations. Beginning with this section, we suppose $\mathfrak{A}$ to be a complex Banach algebra with identity e. Further, let $\sigma_{\mathfrak{A}}(a)$ or $\sigma(a)$ denote the spectrum of an element a in $\mathfrak{A}$.

LEMMA 4. *Let $\mathfrak{A}$ be a Banach algebra with identity, and let $a \in \mathfrak{A}$ be group invertible. Then either $0 \notin \sigma(a)$ (i.e. a is invertible), or $0 \in \sigma(a)$, and 0 is an isolated point of the spectrum.*

Proof. If p stands for the group idempotent of a, then the assertion follows easily from $a = (e-p)a(e-p)$ and from the invertibility of $(e-p)a(e-p)+p$ by Lemma 3. It is also easy to give a more direct proof: If $a = 0$, then the assertion is correct. If $a \neq 0$, then the group inverse b of a is also non-zero. Let $|\lambda| \in (0, \|b\|^{-1})$. Then $e - \lambda b$ is invertible (Neumann series), and one easily checks that $(e-\lambda b)^{-1}b - \lambda^{-1}(e - ba)$ is the inverse of $a - \lambda e$. ∎

This spectral condition is not sufficient in general, as the following lemma indicates.

LEMMA 5. *Let $\mathfrak{A}$ be a Banach algebra with identity, and let a be in the radical* Rad $\mathfrak{A}$ *of $\mathfrak{A}$. Then a is (I_k)-invertible if and only if $a^k = 0$. In particular, the only generalized invertible element in* Rad $\mathfrak{A}$ *is the zero element.*

Proof. If $a^k = 0$, then every element of $\mathfrak{A}$ is a (I_k)-inverse of a. If, conversely, b is a (I_k)-inverse of a, then $a^k ba = a^k$ and so $a^k(e - ba) = 0$. The element $e - ba$ is invertible if a is in the radical. Thus, $a^k = 0$. ∎

The following proposition singles out a class of Banach algebras for which the spectral characterization from Lemma 4 is also sufficient. Recall that a Banach algebra is *semisimple* if its radical consists of the zero element only.

PROPOSITION 2. *Let $\mathfrak{A}$ be a semisimple commutative Banach algebra with identity. An element $a \in \mathfrak{A}$ is group invertible if and only if $0 \notin \sigma(a)$ or if 0 is an isolated point of $\sigma(a)$.*

Proof. The necessity has been shown in Lemma 4. For the proof of the sufficiency, we restrict ourselves to the case where 0 belongs to $\sigma(a)$, but is an isolated point of this set (otherwise a is invertible).

Let X be the maximal ideal space of $\mathfrak{A}$, provided with its Gelfand topology, and let $\widehat{a}$ stand for the Gelfand transform of $a \in \mathfrak{A}$. Choose $\varepsilon > 0$ such that

$$\{\lambda \in \mathbb{C} : |\lambda| \leq \varepsilon\} \cap \sigma(a) = \{0\},$$

and define $U := \{x \in X : \widehat{a}(x) = 0\}$ and $V := X \setminus V$. Clearly, U and V are disjoint, $U \cup V = X$, U is closed and V is open in X. Since $\widehat{a}(X) = \sigma(a)$, the set U coincides with $\{x \in X : |\widehat{a}(x)| < \varepsilon\}$ and is thus also open. Accordingly, V is closed. Consequently,

$$\widehat{p}(x) = \begin{cases} 1 & \text{if } x \in U, \\ 0 & \text{if } x \in V, \end{cases}$$

defines a continuous function on X such that

$$\text{(5)} \qquad \widehat{a} + \widehat{p} \text{ is invertible in } C(X) \text{ and } \widehat{a}\widehat{p} = 0.$$

Now recall the following result by Shilov (for a proof see [3], Chapter I, 4.10, Proposition 12).

SHILOV'S IDEMPOTENT THEOREM. *Let $\mathfrak{A}$ be a commutative Banach algebra with identity, and let X denote its maximal ideal space. Suppose further there is a continuous function $\widehat{p}$ on X taking only the values 0 and 1. Then there exists an element p in $\mathfrak{A}$ with Gelfand transform $\widehat{p}$, and this element is unique.*

Shilov's theorem yields the existence of an element $p \in \mathfrak{A}$ with Gelfand transform $\widehat{p}$, and (5) implies that $a + p$ is invertible and ap is in the radical of $\mathfrak{A}$. But the radical of $\mathfrak{A}$ is trivial by assumption, hence, $ap = 0$. By Lemma 3, a is group invertible. ∎

Let us mention in this connection that group invertibility is, in contrast to common invertibility, no longer a *local* property. Whereas a function $a \in C(X)$ is invertible if and only if $a(x)$ is invertible for all $x \in X$, this is no longer true for group invertibility, as the example $X = [0,1]$ and $a(x) = x$ demonstrates.

LEMMA 6. (a) *Let $\mathfrak{A}$ be a semisimple commutative Banach algebra with identity. An element $a \in \mathfrak{A}$ is Drazin invertible if and only if $0 \notin \sigma(a)$ or if 0 is an isolated point of $\sigma(a)$.*

(b) *Let $\mathfrak{A}$ be a C^*-algebra with identity. An element $a \in \mathfrak{A}$ is Moore–Penrose invertible if and only if $0 \notin \sigma(a^*a)$ or if 0 is an isolated point of $\sigma(a^*a)$.*

Proof. (a) An element a is Drazin invertible of degree k if and only if a^k is group invertible in $\operatorname{Comm} a$ (Lemma 2). Since $\operatorname{Comm} a = \mathfrak{A}$, this is equivalent to

$$0 \notin \sigma_{\mathfrak{A}}(a^k) \text{ or } 0 \text{ is an isolated point of } \sigma_{\mathfrak{A}}(a^k)$$

(Proposition 2). As $\sigma(a^k) = (\sigma(a))^k$, this is equivalent to the assertion.

(b) By Lemma 2, the Moore–Penrose invertibility of a is equivalent to the group invertibility of a^*a which, in turn, is equivalent to the existence of a projection $p \in \mathfrak{A}$ such that

$$\text{(6)} \qquad a^*a + p \text{ is invertible, } a^*ap = 0, \text{ and } a^*ap = pa^*a$$

(Lemma 3 and Corollary 3). Let $\mathfrak{B}$ denote the smallest closed subalgebra of $\mathfrak{A}$ which contains e, a^*a and p. Then $\mathfrak{B}$ is a commutative *-subalgebra of $\mathfrak{A}$, and (6) entails the group invertibility of a^*a in $\mathfrak{B}$. But $\mathfrak{B}$ is—as every C^*-algebra—semisimple. Thus, by Proposition 2, the group invertibility of a^*a in $\mathfrak{B}$ is equivalent to

$$0 \notin \sigma_{\mathfrak{B}}(a^*a) \text{ or } 0 \text{ is an isolated point of } \sigma_{\mathfrak{B}}(a^*a).$$

Since $\sigma_{\mathfrak{B}}(b) = \sigma_{\mathfrak{A}}(b)$ for all $b \in \mathfrak{B}$ (inverse closedness), this yields the assertion. ∎

COROLLARY 4. *Every Drazin invertible element of a commutative semisimple Banach algebra with identity is group invertible.*

4. Inverse closedness.

4. Inverse closedness. We start with an example showing that subalgebras $\mathfrak{B}$ of an algebra $\mathfrak{A}$ which are inverse closed with respect to common invertibility are not necessarily inverse closed with respect to generalized invertibility.

EXAMPLE. Let $\mathfrak{A} = \mathbb{C}^{2\times 2}$, and let $\mathfrak{B} \subseteq \mathfrak{A}$ stand for the algebra of all matrices

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{with } \alpha, \beta \in \mathbb{C}.$$

The algebra $\mathfrak{B}$ is inverse closed in $\mathfrak{A}$ with respect to common invertibility: If $\left(\begin{smallmatrix} \alpha & \beta \\ 0 & \alpha \end{smallmatrix}\right)$ is invertible in $\mathfrak{A}$, then its inverse $\frac{1}{\alpha^2}\left(\begin{smallmatrix} \alpha & -\beta \\ 0 & \alpha \end{smallmatrix}\right)$ belongs to $\mathfrak{B}$. But $\mathfrak{B}$ is not inverse closed with respect to generalized invertibility: The matrix $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ is a generalized inverse of $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ in $\mathfrak{A}$, but $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ has no generalized inverse in $\mathfrak{B}$. Indeed, this matrix lies in the radical of $\mathfrak{B}$, and the only generalized invertible matrix in the radical is the zero matrix (Lemma 5). ∎

Fortunately, this situation changes drastically if we consider group invertibility. To see this, let us introduce a few more notations. Let again $\mathfrak{A}$ be a Banach algebra with identity e, and let $a \in \mathfrak{A}$. By $\mathfrak{A}[a]$ we denote the smallest closed subalgebra of $\mathfrak{A}$ which contains e, a and all elements $(a-\lambda e)^{-1}$ with $\lambda \notin \sigma_{\mathfrak{A}}(a)$.

LEMMA 7. *Let $\mathfrak{A}$ be a Banach algebra with identity e, and let $a \in \mathfrak{A}$.*

(a) *The algebra $\mathfrak{A}[a]$ is the smallest closed and inverse closed subalgebra of $\mathfrak{A}$ which contains e and a.*

(b) *The maximal ideal space of $\mathfrak{A}[a]$ is homeomorphic to $\sigma_{\mathfrak{A}}(a)$, and the homeomorphism sends a character φ into $\varphi(a)$.*

For the proof, see [3], Chapter I, Sections 1.4 and 3.3.

PROPOSITION 3. *Let $\mathfrak{A}$ be a Banach algebra with identity e, and let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$ which contains e and which is inverse closed in $\mathfrak{A}$ with respect to the common invertibility. Then $\mathfrak{B}$ is inverse closed in $\mathfrak{A}$ with respect to group invertibility.*

Proof. Let a be group invertible in $\mathfrak{A}$, and let $p \in \mathfrak{A}$ denote its group idempotent. We claim that $p \in \mathfrak{A}[a]$. Indeed, if n is sufficiently large, then the elements $a-(1/n)e$ are invertible in $\mathfrak{A}$ (by Lemma 4) and, hence, in $\mathfrak{A}[a]$. Thus, $a(a-(1/n)e)^{-1} \in \mathfrak{A}[a]$ for all sufficiently large n. Now we have (recall that $ap = pa = 0$)

$$\begin{aligned} a\left(a-\frac{1}{n}e\right)^{-1} &= a\left((e-p)a(e-p)-\frac{1}{n}(e-p)-\frac{1}{n}p\right)^{-1} \\ &= a\left((e-p)-\frac{1}{n}p\right)^{-1}\left((e-p)a(e-p)+p-\frac{1}{n}(e-p)\right)^{-1} \\ &= a\Big((e-p)-np\Big)\left((e-p)a(e-p)+p-\frac{1}{n}(e-p)\right)^{-1} \\ &= a\left((e-p)a(e-p)+p-\frac{1}{n}(e-p)\right)^{-1}. \end{aligned}$$

The element $(e-p)a(e-p)+p = a+p$ is invertible by Lemma 3, and the continuity of the mapping $x \mapsto x^{-1}$ entails that

$$\lim_{n\to\infty}\left((e-p)a(e-p)+p-\frac{1}{n}(e-p)\right)^{-1} = (a+p)^{-1}.$$

Hence,

$$\lim_{n\to\infty} a\left(a-\frac{1}{n}e\right)^{-1} = a(a+p)^{-1} = e-p,$$

whence it follows that $e-p$ and p belong to $\mathfrak{A}[a]$. But then $a+p \in \mathfrak{A}[a]$ and, by inverse closedness, $(a+p)^{-1} \in \mathfrak{A}[a]$. Thus, the group inverse $(a+p)^{-1}(e-p)$ of A belongs to $\mathfrak{A}[a]$ and, in particular, to every closed and inverse closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$. ∎

As a consequence of the inverse closedness with respect to group invertibility we mention a specification of Lemma 2(a) to the Banach case.

COROLLARY 5. *Let $\mathfrak{A}$ be a Banach algebra with identity. An element $a \in \mathfrak{A}$ is Drazin invertible of degree k in $\mathfrak{A}$ if and only if a^k is group invertible in $\mathfrak{A}$.*

Proof. The necessity of the group invertibility of a^k in $\mathfrak{A}$ is a consequence of Lemma 2. What we have to prove for its sufficiency is, again by Lemma 2, that group invertibility of a^k in $\mathfrak{A}$ implies group invertibility of a^k in $\operatorname{Comm} a$. Since a^k belongs to $\operatorname{Comm} a$, and $\operatorname{Comm} a$ is inverse closed in $\mathfrak{A}$ with respect to common invertibility, this follows immediately from Proposition 3. ∎

COROLLARY 6. (a) *Let $\mathfrak{A}$ be a Banach algebra with identity e, and let $\mathfrak{B}$ be a closed and inverse closed subalgebra of $\mathfrak{A}$ with $e \in \mathfrak{B}$. Then $\mathfrak{B}$ is inverse closed in $\mathfrak{A}$ with respect to Drazin invertibility.*

(b) *Let $\mathfrak{A}$ be a C^*-algebra with identity e, and let $\mathfrak{B}$ be a closed $*$-subalgebra of $\mathfrak{A}$ with $e \in \mathfrak{B}$. Then $\mathfrak{B}$ is inverse closed in $\mathfrak{A}$ with respect to Moore–Penrose invertibility.*

Proof. (a) Let $a \in \mathfrak{B}$ be Drazin invertible of degree k in $\mathfrak{A}$. Then a^k is group invertible in $\mathfrak{A}$ by Lemma 2 and, hence, group invertible in $\mathfrak{A}[a^k]$ by the preceding corollary. Then a^k is also group invertible in $\mathfrak{A}[a]$. Let $b \in \mathfrak{A}[a]$ be the group inverse of a^k. Then, by Lemma 2 again, ba^{k-1} is the Drazin inverse of degree k of a and, evidently, $ba^{k-1} \in \mathfrak{A}[a] \subseteq \mathfrak{B}$.

(b) Let $b \in \mathfrak{A}$ be the Moore–Penrose inverse of $a \in \mathfrak{B}$. Then bb^* is the group inverse of a^*a by Lemma 2, and $bb^* \in \mathfrak{A}[a^*a]$ by Proposition 3. The inverse closedness of C^*-algebras entails that $\mathfrak{A}[a^*a]$ actually coincides with the smallest closed subalgebra of $\mathfrak{A}$ which contains e and a^*a, which implies that $bb^* \in \mathfrak{B}$ (because $a^*a \in \mathfrak{B}$). Since $bb^*a^* = b$, again by Lemma 2, we conclude that $b \in \mathfrak{B}$ as desired. ∎

5. Asymptotic group invertibility and eigenvalue splitting. Beginning with this section, we consider convergent sequences of generalized invertible elements. So let $\mathfrak{A}$ be a Banach algebra with identity again, and let $(a_n) \subseteq \mathfrak{A}$ be a sequence with norm limit $a \in \mathfrak{A}$. The following assertions are well known for the common invertibility:

(A) *If the a_n are invertible, and if $\sup_n \|a_n^{-1}\| < \infty$, then a is invertible, and $a_n^{-1} \to a^{-1}$.*

(B) *If a is invertible, then a_n is invertible for all sufficiently large n, and $a_n^{-1} \to a^{-1}$.*

The trivial example $\mathfrak{A} = \mathbb{C}$ and $a_n = 1$ shows that assertion (B) does not hold for generalized invertibility in general, but assertion (A) remains valid as we will see now.

Define the following sets of sequences of elements of $\mathfrak{A}$:

$$\mathfrak{F} = \{(a_n) : a_n \in \mathfrak{A},\ \sup_n \|a_n\| < \infty\},$$

$$\mathfrak{L} = \{(a_n) : a_n \in \mathfrak{A},\ \lim_{n\to\infty} a_n \text{ exists}\},$$

$$\mathfrak{G} = \{(a_n) : a_n \in \mathfrak{A},\ \lim_{n\to\infty} a_n = 0\}.$$

Provided with elementwise operations and the supremum norm, the sets $\mathfrak{F}$ and $\mathfrak{L}$ become Banach algebras, and $\mathfrak{G}$ becomes a closed two-sided ideal of both $\mathfrak{F}$ and $\mathfrak{L}$. If $\mathfrak{A}$ is an involutive algebra, then we introduce an elementwise involution on $\mathfrak{F}$ which makes $\mathfrak{F}$ as well as its subalgebras $\mathfrak{L}$ and $\mathfrak{G}$ into involutive algebras and, in case $\mathfrak{A}$ is a C^*-algebra, into C^*-algebras. Property (A) of the common invertibility further entails that $\mathfrak{L}$ is an inverse closed subalgebra of $\mathfrak{F}$.

PROPOSITION 4. *Let $\mathfrak{A}$ be a Banach algebra with identity, let $(a_n) \subseteq \mathfrak{A}$ be a convergent sequence with limit a, and suppose all a_n are group invertible with group inverses b_n. Then the following assertions are equivalent*:

(a) $\sup_n \|b_n\| < \infty$.

(b) *a is group invertible, the sequence (b_n) is convergent, and its limit is the group inverse of a.*

Proof. The implication (b)⇒(a) is obvious. So suppose (b_n) is a bounded sequence. Then (b_n) belongs to $\mathfrak{F}$ and is, consequently, the group inverse of (a_n) in this algebra. But the sequence (a_n) belongs to the algebra $\mathfrak{L}$ which is inverse closed in $\mathfrak{F}$. So, Proposition 3 and the inverse closedness of $\mathfrak{L}$ in $\mathfrak{F}$ entail that also $(b_n) \in \mathfrak{L}$, that is, this sequence is convergent. Let b denote its limit. Letting n go to infinity in

$$a_nb_na_n = a_n, \quad b_na_nb_n = b_n, \quad a_nb_n = b_na_n$$

yields

$$aba = a, \quad bab = b, \quad ab = ba.$$

Hence, a is group invertible, and b is its group inverse. ∎

COROLLARY 7. *The assertion of the preceding proposition remains valid for Drazin inverses with uniformly bounded Drazin degree and—in case $\mathfrak{A}$ is a C^*-algebra—also for Moore–Penrose inverses.*

The proof proceeds as that of Proposition 4, only with Corollary 6 in place of Proposition 3. ∎

The problem of norm continuity of the group inversion will be examined in greater detail in the next section. As a preparation, we consider a weaker version of group invertibility of a sequence. A sequence $(a_n) \in \mathfrak{F}$ is called *asymptotically group invertible* if there is a sequence $(b_n) \in \mathfrak{F}$ such that

$$\|a_nb_na_n - a_n\| \to 0, \quad \|b_na_nb_n - b_n\| \to 0, \quad \|a_nb_n - b_na_n\| \to 0$$

as $n \to \infty$. Analogously, one defines *asymptotic Drazin* and *asymptotic Moore–Penrose invertibility.* Thus, the sequence $(a_n) \in \mathfrak{F}$ is asymptotically group (Drazin, Moore–Penrose) invertible if and only if its coset $(a_n) + \mathfrak{G}$ is group (Drazin, Moore–Penrose) invertible in the quotient algebra $\mathfrak{F}/\mathfrak{G}$.

LEMMA 8. *The quotient algebra $\mathfrak{L}/\mathfrak{G}$ is inverse closed in $\mathfrak{F}/\mathfrak{G}$ (with respect to common invertibility).*

Proof. Let (a_n) be a sequence in $\mathfrak{L}$ the coset $(a_n) + \mathfrak{G}$ of which is invertible in $\mathfrak{F}/\mathfrak{G}$. Then there are sequences (b_n) in $\mathfrak{F}$ and (g_n), (h_n) in $\mathfrak{G}$ such that

$$a_nb_n = e + g_n \quad \text{and} \quad b_na_n = e + h_n.$$

If n is sufficiently large ($n \ge n_0$, say), then $\|g_n\| < 1/2$ and $\|h_n\| < 1/2$. Thus, $e + g_n$ and $e + h_n$ are invertible for $n \ge n_0$, and the norms of their inverses are uniformly bounded. So we get

$$a_nb_n(e + g_n)^{-1} = e \quad \text{and} \quad (e + h_n)^{-1}b_na_n = e \quad \text{for all } n \ge n_0,$$

whence the invertibility of the sequence $(a_n)_{n\ge n_0}$ in $\mathfrak{F}$ follows. Its inverse is the sequence $(b_n(e + g_n)^{-1})_{n\ge n_0}$, and since $(a_n)_{n\ge n_0}$ belongs to $\mathfrak{L}$ and $\mathfrak{L}$ is inverse closed in $\mathfrak{F}$, we conclude that $(b_n(e+g_n)^{-1})_{n\ge n_0}$ belongs to $\mathfrak{L}$, i.e. it converges. But then so does $(b_n)_{n\ge n_0} = (b_n(e+g_n)^{-1})_{n\ge n_0}(e+g_n)_{n\ge n_0}$. This implies that $(b_n)_{n\ge n_0}$ and, hence, the sequence (b_n) itself, belong to $\mathfrak{L}$. ∎

Thus, a sequence $(a_n) \in \mathfrak{L}$ is asymptotically group (Drazin, Moore–Penrose) invertible if and only if its coset $(a_n) + \mathfrak{G}$ is group (Drazin, Moore–Penrose) invertible in the quotient algebra $\mathfrak{L}/\mathfrak{G}$. For group invertibility in $\mathfrak{L}/\mathfrak{G}$, we have the following simple criterion.

LEMMA 9. *Let $\mathfrak{A}$ be a Banach algebra with identity, and let $(a_n) \in \mathfrak{L}$. The coset $(a_n) + \mathfrak{G}$ is group invertible in $\mathfrak{L}/\mathfrak{G}$ if and only if there is an n_0 and an idempotent p in $\mathfrak{A}$ such that $a_n + p$ is invertible for all $n \ge n_0$, $\sup_{n\ge n_0} \|(a_n + p)^{-1}\| < \infty$, and $\|a_np\| \to 0$ as well as $\|a_np - pa_n\| \to 0$.*

Proof. The algebra $\mathfrak{L}/\mathfrak{G}$ is evidently isometrically isomorphic to $\mathfrak{A}$ with the isomorphism being

(7) $$(a_n)+\mathfrak{G}\mapsto \lim_{n\to\infty} a_n.$$

Hence, $(a_n)+\mathfrak{G}$ is group invertible in $\mathfrak{L}/\mathfrak{G}$ if and only if $a:=\lim a_n$ is group invertible in $\mathfrak{A}$. Let $p\in\mathfrak{A}$ denote the group idempotent of a, i.e.

$$a+p \text{ is invertible},\ ap=0,\ \text{and } ap=pa.$$

Applying the inverse of the isomorphism (7) to these assertions we obtain

$$(a_n+p)+\mathfrak{G} \text{ is invertible in } \mathfrak{L}/\mathfrak{G}, (a_np)\in\mathfrak{G},\ \text{and } (a_np)+\mathfrak{G}=(pa_n)+\mathfrak{G},$$

which is equivalent to the assertions of the lemma. ∎

A remarkable consequence of the asymptotic group invertibility of a convergent sequence (a_n) is the asymptotic splitting of the spectrum of the a_n. Given $\varepsilon>0$, let

$$U_\varepsilon:=\{z\in\mathbb{C}:|z|\le\varepsilon\}\quad\text{and}\quad V_\varepsilon:=\mathbb{C}\setminus U_\varepsilon.$$

PROPOSITION 5. *Let $\mathfrak{A}$ be a Banach algebra with identity, and let $(a_n)\in\mathfrak{L}$ be asymptotically group invertible. Then there are numbers $c_n\ge 0$ with $c_n\to 0$ as $n\to\infty$ and $d>0$ such that*

$$\sigma_\mathfrak{A}(a_n)\subseteq U_{c_n}\cup V_d.$$

Thus, one part of the spectrum tends uniformly to 0, whereas the other part remains bounded away from zero by a positive constant independent of n.

Proof. Let $(a_n)+\mathfrak{G}$ be group invertible in $\mathfrak{L}/\mathfrak{G}$. Then $a:=\lim a_n$ is group invertible in $\mathfrak{A}$ and hence, by Lemma 4, there is a positive number d such that

(8) $$\sigma_\mathfrak{A}(a)\subseteq\{0\}\cup V_d.$$

The upper semicontinuity of the spectrum (see, e.g., [1], p. 26) entails that, given $\varepsilon>0$, there is an n_0 such that $\sigma(a_n)$ lies in the ε-neighbourhood of $\sigma(a)$ for all $n\ge n_0$. Due to (8), this implies that

$$\sigma_\mathfrak{A}(a_n)\subseteq U_\varepsilon\cup V_{d-\varepsilon}\quad\text{for all } n\ge n_0,$$

whence the assertion easily follows. ∎

COROLLARY 8. (a) *Let $\mathfrak{A}$ be a Banach algebra with identity, and let $(a_n)\in\mathfrak{L}$ be asymptotically Drazin invertible. Then there are numbers $c_n\ge 0$ with $c_n\to 0$ as $n\to\infty$ and $d>0$ such that*

$$\sigma_\mathfrak{A}(a_n)\subseteq U_{c_n}\cup V_d.$$

(b) *Let $\mathfrak{A}$ be a C^*-algebra with identity, and let $(a_n)\in\mathfrak{L}$ be asymptotically Moore–Penrose invertible. Then there are numbers $c_n\ge 0$ with $c_n\to 0$ as $n\to\infty$ and $d>0$ such that*

$$s_\mathfrak{A}(a_n)\subseteq[0,c_n]\cup[d,\infty)$$

where $s_\mathfrak{A}(a)$ stands for the set of singular values of $a\in\mathfrak{A}$.

Recall that the singular values are just the non-negative square roots of the (non-negative) numbers in the spectrum of a^*a.

Proof. (a) By Corollary 5, the sequence (a_n) is asymptotically Drazin invertible of degree k if and only if the sequence (a_n^k) is asymptotically group invertible. Thus, we have a splitting of $\sigma_\mathfrak{A}(a_n^k)$ in accordance with the preceding proposition. But $\sigma_\mathfrak{A}(a_n^k)=(\sigma_\mathfrak{A}(a_n))^k$, and so the splitting carries over to the spectra of a_n.

(b) By Lemma 2(b), the sequence (a_n) is asymptotically Moore–Penrose invertible if and only if $(a_n^*a_n)$ is asymptotically group invertible. Thus, the spectra $\sigma_\mathfrak{A}(a_n^*a_n)$ split in accordance with the preceding proposition. Now it is clear from the definition of the singular values that they split in the same manner. Since the spectrum of $a_n^*a_n$ is contained in the non-negative real semiaxis, the neighbourhoods U_{c_n} and V_d can be replaced by real intervals. ∎

It will be important in what follows to be able to replace the (constant) sequence (p) in Lemma 9 by a sequence (q_n) of idempotents which belongs to the inverse closed algebra generated by the sequences (e) and (a_n) and which, in particular, commutes with (a_n) exactly. As a preparation, we need one more lemma.

LEMMA 10. *Let $\mathfrak{A}$ be a Banach algebra with identity, and let (b_n) be a sequence in $\mathfrak{L}$ with limit b. Then*

$$\sigma_\mathfrak{L}((b_n))=\sigma_\mathfrak{A}(b)\cup\bigcup_n\sigma_\mathfrak{A}(b_n).$$

Proof. Let $\lambda\notin\sigma_\mathfrak{L}((b_n))$. Then there is a sequence (c_n) in $\mathfrak{L}$ with

$$(c_n)((b_n)-\lambda(e))=((b_n)-\lambda(e))(c_n)=(e)$$

or, written elementwise,

(9) $$c_n(b_n-\lambda e)=(b_n-\lambda e)c_n=e\quad\text{for all } n.$$

Passing in (9) to the limit, we get (with $c:=\lim c_n$)

(10) $$c(b-\lambda e)=(b-\lambda e)c=e.$$

The identities (9) show that $\lambda\notin\sigma_\mathfrak{A}(b_n)$, and (10) implies that $\lambda\notin\sigma_\mathfrak{A}(b)$. Hence,

$$\sigma_\mathfrak{L}((b_n))\supseteq\sigma_\mathfrak{A}(b)\cup\bigcup_n\sigma_\mathfrak{A}(b_n).$$

For the reverse inclusion, let $\lambda \notin \sigma_{\mathfrak{A}}(b) \cup \bigcup_n \sigma_{\mathfrak{A}}(b_n)$. Set $c := (b - \lambda e)^{-1}$ and $c_n := (b_n - \lambda e)^{-1}$. The invertibility of $b - \lambda e$ and the convergence $b_n - \lambda e \to b - \lambda e$ enforce the uniform boundedness of the elements c_n as well as their convergence to c. Thus, (c_n) is in $\mathfrak{L}$, and it is just the inverse of $(b_n - \lambda e)$ in $\mathfrak{L}$. Consequently, $\lambda \notin \sigma_{\mathfrak{L}}((b_n))$, which proves the reverse inclusion. ∎

The following theorem is a key in our approach to the continuity problem for generalized inverses.

THEOREM 1. *Let $\mathfrak{A}$ be a Banach algebra with identity, and let (a_n) be a sequence in $\mathfrak{L}$ with limit a. Then the following assertions are equivalent*:

(a) *a is group invertible in $\mathfrak{A}$.*

(b) *$(a_n) + \mathfrak{G}$ is group invertible in $\mathfrak{L}/\mathfrak{G}$.*

(c) *There exists a sequence $(q_n) \in \mathfrak{L}$ of idempotents such that*:

- ▷ *the sequence $(a_n + q_n)_{n \geq n_0}$ is invertible in $\mathfrak{L}$ for sufficiently large n_0,*
- ▷ *$a_n q_n = q_n a_n$ for all sufficiently large n,*
- ▷ *$a_n q_n \to 0$ as $n \to \infty$.*

If (q_n) has these properties, then it belongs to the algebra $\mathfrak{L}[(a_n)_{n \geq n_0}]$, and the idempotents q_n belong to $\mathfrak{A}[a_n]$ for every $n \geq n_0$.

Proof. The equivalence between (a) and (b) is quite obvious and has already been remarked. The implication (c)⇒(b) is also easy to check: Let (q_n) be as in (c), and set for brevity $A := (a_n) + \mathfrak{G}$ and $Q := (q_n) + \mathfrak{G}$. Then the conditions in (c) imply that Q is an idempotent in $\mathfrak{L}/\mathfrak{G}$ such that $A + Q$ is invertible in $\mathfrak{L}/\mathfrak{G}$ and that $AQ - QA = 0 + \mathfrak{G}$ and $AQ = 0 + \mathfrak{G}$. By Lemma 3, $A = (a_n) + \mathfrak{G}$ is group invertible in $\mathfrak{L}/\mathfrak{G}$.

Now suppose (a) and (b) are satisfied, i.e. let $(a_n) + \mathfrak{G}$ be group invertible in $\mathfrak{L}/\mathfrak{G}$, and let $a = \lim a_n$. By Proposition 5, there are $c_n \geq 0$ with $c_n \to 0$ and $d > 0$ such that

$$\sigma_{\mathfrak{A}}(a_n) \subseteq U_{c_n} \cup V_d \quad \text{for all } n,$$

and by Lemma 4, there is a $d > 0$ such that

$$\sigma_{\mathfrak{A}}(a) \subseteq \{0\} \cup V_d. \tag{11}$$

Without loss of generality, we can assume the two d's occurring in these inclusions to be equal, and the sequence (c_n) to be decreasing. Hence,

$$\sigma_{\mathfrak{A}}(a) \cup \bigcup_{n \geq k} \sigma_{\mathfrak{A}}(a_n) \subseteq U_{c_k} \cup V_d,$$

which, by Lemma 10, is equivalent to

$$\sigma_{\mathfrak{L}}((a_n)_{n \geq k}) \subseteq U_{c_k} \cup V_d \quad \text{for all } k. \tag{12}$$

Let n_0 be a number such that

$$c_{n_0} < \min\{d, 1\}, \tag{13}$$

and let $\mathfrak{B} := \mathfrak{L}[(a_n)_{n \geq n_0}]$ denote the smallest closed and inverse closed subalgebra of $\mathfrak{L}$ which contains the identity sequence (e) and the shortened sequence $(a_n)_{n \geq n_0}$. Then, by (12),

$$\sigma_{\mathfrak{B}}((a_n)_{n \geq n_0}) \subseteq U_{c_{n_0}} \cup V_d. \tag{14}$$

The maximal ideal space of $\mathfrak{B}$ is homeomorphic to $\sigma_{\mathfrak{B}}((a_n)_{n \geq n_0})$ by Lemma 7, and it is the union of its open and disjoint components $\sigma_{\mathfrak{B}}((a_n)_{n \geq n_0}) \cap U_{c_{n_0}}$ and $\sigma_{\mathfrak{B}}((a_n)_{n \geq n_0}) \cap V_d$ by (13) and (14). Of course, one of these components might be empty.

By Shilov's idempotent theorem, there is an idempotent sequence $(q_n)_{n \geq n_0}$ in $\mathfrak{B}$, the Gelfand transform of which is equal to 1 on $\sigma_{\mathfrak{B}}((a_n)_{n \geq n_0}) \cap U_{c_{n_0}}$ and 0 on the other component. If one of the components is empty, then (q_n) is the identity sequence or the zero sequence.

The sequence $(q_n)_{n \geq n_0}$ has the following properties:

(i) Its entries q_n are idempotents.

(ii) $a_n q_n = q_n a_n$ for all $n \geq n_0$.

(iii) The sequence $(a_n + q_n)_{n \geq n_0}$ is invertible in $\mathfrak{B}$ (and, hence, in $\mathfrak{L}$).

(iv) $\varrho_{\mathfrak{B}}((a_n q_n)_{n \geq n_0}) < c_{n_0}$,

where $\varrho_{\mathfrak{B}}(c)$ is the spectral radius of c in $\mathfrak{B}$. Indeed, properties (i) and (ii) are obvious, and (iii) and (iv) follow from Gelfand theory: The Gelfand transforms (with respect to $\mathfrak{B}$) of the sequences $(a_n)_{n \geq n_0}$ and $(a_n + p_n)_{n \geq n_0}$ coincide on $\sigma_{\mathfrak{B}}((a_n)_{n \geq n_0}) \cap V_d$ and, consequently, both have values greater than or equal to d on this component, whereas the Gelfand transform of $(a_n + p_n)_{n \geq n_0}$ is $x + 1$ at $x \in \sigma_{\mathfrak{B}}((a_n)_{n \geq n_0}) \cap U_{c_{n_0}}$. Since $|x| < c_{n_0} < 1$ and $d > 0$, this shows that the Gelfand transform of $(a_n + p_n)_{n \geq n_0}$ is invertible and, hence, the sequence $(a_n + p_n)_{n \geq n_0}$ is invertible in $\mathfrak{B}$. So we obtain property (iii), and (iv) follows similarly: The Gelfand transform of $(a_n q_n)_{n \geq n_0}$ is 0 on $\sigma_{\mathfrak{B}}((a_n)_{n \geq n_0}) \cap V_d$ and x at $x \in \sigma_{\mathfrak{B}}((a_n)_{n \geq n_0}) \cap U_{c_{n_0}}$. This gives (iv) because $|x| < c_{n_0}$.

Now we turn to the limits $a := \lim a_n$ and $q := \lim q_n$ (the latter exists since (q_n) belongs to the subalgebra $\mathfrak{B}$ of $\mathfrak{L}$). We claim that the following assertions hold:

(i′) q is an idempotent element.

(ii′) $aq = qa$.

(iii′) $a + q$ is invertible in $\mathfrak{A}$.

(iv′) $q \in \mathfrak{A}[a]$.

(v′) $\varrho_{\mathfrak{A}}(aq) = 0$.

The assertions (i′) and (ii′) follow immediately from (i) and (ii). Assertion (iii′) is a consequence of (iii) and of property (A) of the common invertibility (see the beginning of the present section).

For the proof of (iv′), recall that $(q_n)_{n\ge n_0} \in \mathfrak{B} = \mathfrak{L}[(a_n)_{n\ge n_0}]$. By Lemma 7, there are rational functions r_k with poles outside $\sigma_{\mathfrak{B}}((a_n)_{n\ge n_0})$ such that $r_k((a_n)_{n\ge n_0})$ converges in $\mathfrak{B}$ to $(q_n)_{n\ge n_0}$ as $k\to\infty$. By Lemma 10, the poles of the r_k then also lie outside $\sigma(a)$ and outside $\sigma(a_n)$ for all $n\ge n_0$. So it makes sense to form the elements $r_k(a)$ as well as $r_k(a_n)$, and it is easy to see that the limits $\lim_{n\to\infty} r_k(a_n)$ exist and that

$$\lim_{n\to\infty} r_k(a_n) = r_k(a) \quad \text{for all } k.$$

From

$$\|q - r_k(a)\| = \lim_{n\to\infty} \|q_n - r_k(a_n)\|$$
$$\le \sup_{n\ge n_0} \|q_n - r_k(a_n)\| = \|(q_n)_{n\ge n_0} - r_k((a_n))_{n\ge n_0}\|_{\mathfrak{L}}$$

we infer that $r_k(a)\to q$ as $k\to\infty$, whence it follows that q belongs to $\mathfrak{A}$.

For (v′), recall that

$$\sigma_{\mathfrak{A}}(aq) \subseteq \sigma_{\mathfrak{L}}((a_nq_n)_{n\ge n_0}) = \sigma_{\mathfrak{B}}((a_nq_n)_{n\ge n_0}),$$

which together with (iv) yields $\sigma_{\mathfrak{A}}(aq) \subseteq U_{c_{n_0}}$. Now we consider a and q as elements of $\mathfrak{A}[a]$. The maximal ideal space of $\mathfrak{A}[a]$ is (by Lemma 7) homeomorphic to $\sigma_{\mathfrak{A}}(a)$, and $\sigma_{\mathfrak{A}}(a) \subseteq \{0\}\cup V_d$ by (11). Assume the Gelfand transform of q is 1 at some $x\in\sigma(a)\cap V_d$. Since the Gelfand transform of a is x at every $x\in\sigma(a)$, this would imply $x\in\sigma(aq)$, which is impossible because $|x|\ge d > c_{n_0}$ and $\varrho(aq)\le c_{n_0}$. Thus, the Gelfand transform of q is 0 (of course, 0 and 1 are the only possible values of the Gelfand transform of an idempotent) on $\sigma(a)\cap V_d$, whence via (11) it follows that $\sigma(aq)=\{0\}$. This proves (v).

Let p denote the group idempotent of a. We will show that $p=q$. We have seen in the proof of Proposition 3 that $p\in\mathfrak{A}[a]$. Let $\widehat{a}, \widehat{p}$ and $\widehat{q}$ be the Gelfand transforms of a, p and q. The invertibility of $a+p$ implies that $(\widehat{a}+\widehat{p})(x)\ne 0$ for every maximal ideal x and, in particular, $\widehat{p}(x)=1$ for $x\in\sigma(a)\cap\{0\}$. Similarly, we conclude from $\varrho(ap)=0$ that $(\widehat{a}\widehat{p})(x)=0$ for every maximal ideal x and, in particular, $\widehat{p}(x)=0$ for $x\in\sigma(a)\cap V_d$. These considerations remain valid for q in place of p as well. Thus, $\widehat{p}=\widehat{q}$ or, equivalently, $q=p+r$ with some r in the radical of $\mathfrak{A}[a]$. Now we have

$$p+r = q = q^2 = (p+r)^2 = p+2pr+r^2$$

or $r(e-2p-r)=0$. The element $e-2p$ is invertible (its spectrum is contained in $\{-1,1\}$), r is in the radical, and so $e-2p-r$ is invertible. This implies $r=0$ and, hence, $p=q$.

Now we can finish the proof of assertion (c) of the theorem. We have already shown that the a_n+q_n are invertible for large n, that the norms of their inverses are uniformly bounded, and that $a_nq_n = q_na_n$ for all large n. In order to see that $a_nq_n\to 0$, write

$$a_nq_n = a_np + a_n(q_n-p)$$

and take into account that $\|a_np\|\to 0$ by Lemma 9 and $\|q_n-p\| = \|q_n-q\| \to 0$ as we have just seen.

Further, we have chosen the sequence (q_n) so that it belongs to $\mathfrak{L}[(a_n)_{n\ge n_0}]$, and it remains to show that the idempotents q_n belong to $\mathfrak{A}[a_n]$ for every $n\ge n_0$. If r_k are the rational functions introduced above, then

$$\|q_n - r_k(a_n)\| \le \sup_{n\ge n_0} \|q_n - r_k(a_n)\| = \|(q_n)_{n\ge n_0} - r_k((a_n)_{n\ge n_0}\|_{\mathfrak{L}} \to 0,$$

hence, $r_k(a_n)\to q_n$ as $k\to\infty$, and $q_n\in\mathfrak{A}[a_n]$. ∎

6. Continuity of generalized inversion. Let us start with some evident criteria for continuity.

LEMMA 11. (a) *Let $\mathfrak{A}$ be a Banach algebra with identity, and let $(a_n)\subseteq\mathfrak{A}$ be a convergent sequence with limit a such that every a_n as well as a are group invertible. Then the group inverses of a_n converge to the group inverse of a if and only if the group idempotents of a_n converge to the group idempotent of a.*

(b) *Let $\mathfrak{A}$ be a Banach algebra with identity, and let $(a_n)\subseteq\mathfrak{A}$ be a convergent sequence with limit a such that every a_n as well as a are Drazin invertible of degree k. Then the Drazin inverses of a_n converge to the Drazin inverse of a if and only if the group inverses of a_n^k converge to the group inverse of a^k.*

(c) *Let $\mathfrak{A}$ be a C^*-algebra with identity, and let $(a_n)\subseteq\mathfrak{A}$ be a convergent sequence with limit a such that every a_n as well as a are Moore–Penrose invertible. Then the Moore–Penrose inverses of a_n converge to the Moore–Penrose inverse of a if and only if the group inverses of $a_n^*a_n$ converge to the group inverse of a^*a.*

Proof. The assertion (a) follows from the relations between the group inverse and the group idempotent of a given element mentioned in Lemma 3, and to prove (b) (resp. (c)), one has to invoke the relations between the Drazin inverse (resp. the Moore–Penrose inverse) of a_n and the group inverse of a_n^k (resp. of $a_n^*a_n$) as explained in the proof of Lemma 2. ∎

For a more interesting criterion, we introduce the notion of similarity of idempotents. Let $\mathfrak{A}$ be an algebra with identity. We call two idempotents p and q in $\mathfrak{A}$ *similar* (and write $p\sim q$ in this case) if there is an invertible element c in $\mathfrak{A}$ such that $q = c^{-1}pc$. Clearly, similarity is an equivalence relation in the set of all idempotents of $\mathfrak{A}$. We need two lemmata.

LEMMA 12. *Let $\mathfrak{A}$ be a Banach algebra with identity e, and let p and q be idempotents in $\mathfrak{A}$ with $\|p-q\| < \|2p-e\|^{-1}$. Then $p \sim q$.*

Proof. Let $c := p + q - e$. Then, in any case, $cq = pc$. Further, we have $c = (2p - e) + (q - p)$ where the element $2p - e$ is invertible and is its own inverse. Then, by Neumann series, c is invertible if only

$$\|q-p\| < \|(2p-e)^{-1}\|^{-1} = \|2p-e\|^{-1}.$$

Thus, if this inequality holds, then $p \sim q$. ∎

Observe that the element c as well as its inverse belong to the smallest closed subalgebra of $\mathfrak{A}$ which contains p, q and e.

LEMMA 13. *Let $\mathfrak{A}$ be a Banach algebra with identity e, let p and q be idempotents in $\mathfrak{A}$ with $p \sim q$, and suppose that the algebra $p\mathfrak{A}p := \{pap : a \in \mathfrak{A}\}$ is a finite-dimensional linear space. Then the algebra $q\mathfrak{A}q$ is also finite-dimensional, and*

$$\dim p\mathfrak{A}p = \dim q\mathfrak{A}q.$$

Proof. Let c be an invertible element in $\mathfrak{A}$ such that $q = c^{-1}pc$, set $\dim p\mathfrak{A}p =: l$, and let $qa_1q, \ldots, qa_{l+1}q$ be arbitrary elements in $q\mathfrak{A}q$. The $l+1$ elements $pca_1c^{-1}p, \ldots, pca_{l+1}c^{-1}p$ are linearly dependent by assumption; hence, there are numbers $\alpha_1, \ldots, \alpha_{l+1}$, not all zero, such that

$$\alpha_1 pca_1c^{-1}p + \ldots + \alpha_{l+1}pca_{l+1}c^{-1}p = 0.$$

Multiplying this equality by c^{-1} from the left and by c from the right hand side yields

$$\alpha_1 qa_1q + \ldots + \alpha_{l+1}qa_{l+1}q = 0,$$

i.e. any $l+1$ elements of $q\mathfrak{A}q$ are linearly dependent. Thus, $\dim p\mathfrak{A}p \geq \dim q\mathfrak{A}q$, and the reverse inequality follows analogously. ∎

Now we can formulate and prove our criterion for the continuity of group inversion in Banach algebras.

THEOREM 2. *Let $\mathfrak{A}$ be a Banach algebra with identity, let $(a_n) \subseteq \mathfrak{A}$ be a convergent sequence with limit $a \in \mathfrak{A}$, and suppose that a and every a_n are group invertible with corresponding group idempotents p resp. p_n. Further assume that $p\mathfrak{A}p$ is a finite-dimensional algebra. Then the following assertions are equivalent*:

(a) *The group inverses b_n of a_n converge to the group inverse b of a.*
(b) *The group idempotents p_n of a_n are similar to the group idempotent p of a.*

Proof. (a)⇒(b). If $a_n \to a$ and $b_n \to b$ then, by Lemma 11(a), we have $p_n \to p$. From Lemma 12 we conclude that $p_n \sim p$ for all sufficiently large n.

(b)⇒(a). By Theorem 1 and its proof, the group invertibility of a entails the existence of a sequence (q_n) of idempotents such that

$$a_n + q_n \text{ is invertible and } q_n \in \mathfrak{A}[a_n] \tag{15}$$

for all sufficiently large n, and

$$\|q_n - p\| \to 0. \tag{16}$$

We shall verify that $p_n = q_n$ for all sufficiently large n. By (16), this implies that $p_n \to p$, which gives the assertion via Lemma 11(a).

From (16) and Lemma 12 we infer that $q_n \sim p$ for all sufficiently large n, and by hypothesis we have $p_n \sim p$ for all sufficiently large n. Thus, by Lemma 13,

$$\dim p_n\mathfrak{A}p_n = \dim p\mathfrak{A}p = \dim q_n\mathfrak{A}q_n. \tag{17}$$

Further we have $(a_n + q_n)p_n = p_nq_n$ (recall that $a_np_n = 0$ by Lemma 3), and hence

$$p_n = (a_n + q_n)^{-1}p_nq_n$$

for all sufficiently large n. Multiplying this equality by q_n and taking into account that all the occurring factors commute with each other (since $p_n \in \mathfrak{A}[a_n]$ by Proposition 3 and its proof, and since $q_n \in \mathfrak{A}[a_n]$ for all sufficiently large n by (15)), we obtain

$$p_n = p_nq_n = q_np_n = q_np_nq_n$$

for all large n. Thus, for these n, the idempotent p_n belongs to the algebra $q_n\mathfrak{A}q_n$, and $p_n\mathfrak{A}p_n$ is a subalgebra of $q_n\mathfrak{A}q_n$. Since both algebras are finite-dimensional and have the same dimension by (17), we conclude that even

$$p_n\mathfrak{A}p_n = q_n\mathfrak{A}q_n,$$

i.e. there are $d_n \in \mathfrak{A}$ such that $q_n = p_nd_np_n$ for all sufficiently large n. But then

$$a_nq_n = a_np_nd_np_n = 0,$$

which together with (15) and Lemma 3 yields that q_n is the group idempotent of a_n. Consequently, $q_n = p_n$, and we are done. ∎

This theorem is no longer valid if the finiteness condition is violated. To have an example, let $B(L^2)$ denote the algebra of all bounded linear operators on the Lebesgue space L^2 over the real line, and let, for every interval M, χ_M stand for the operator of multiplication by the characteristic function of M. The operators $A_n := (1/n)\chi_{[0,1]} + \chi_{[1,\infty)}$ converge in the norm of $B(L^2)$ to the operator $A := \chi_{[1,\infty)}$. Both the operator A and the operators A_n are group invertible; their group inverses are $\chi_{[1,\infty)}$ and $n\chi_{[0,1]} + \chi_{[1,\infty)}$, and their group idempotents are $\chi_{(-\infty,1]}$ and $\chi_{(-\infty,0]}$, respectively. These idempotents are similar; for, if $U : L^2 \to L^2$ is the shift operator $(Uf)(t) = f(t-1)$, then

$$\chi_{(-\infty,0]} = U^{-1}\chi_{(-\infty,1]}U,$$

but the group inverses of the A_n do not converge.

7. The operator case. Now we are going to specialize the results of the preceding sections to the case of linear operators. Let us start with recalling some results concerning the existence of generalized inverses of operators. We denote by $\operatorname{Im} A$ and $\operatorname{Ker} A$ the range and the kernel of the linear operator A. Further, let I stand for the identity operator.

LEMMA 14. *Let E be a linear space and $L(E)$ denote the algebra of all linear operators on E.*

(a) *Every operator $A \in L(E)$ is generalized invertible.*

(b) *$A \in L(E)$ is group invertible if and only if $\operatorname{Im} A \cap \operatorname{Ker} A = \{0\}$ and $\operatorname{Im} A + \operatorname{Ker} A = E$.*

Proof. (a) Every linear subspace of E has an algebraic complement. In particular, there are subspaces E_K and E_I of E such that

$$E = \operatorname{Ker} A \dotplus E_K \quad \text{and} \quad E = E_I \dotplus \operatorname{Im} A.$$

The restriction of A to E_K is a bijection between E_K and $\operatorname{Im} A$. So one can define a linear operator B on E as being 0 on E_I and $(A|_{E_K})^{-1}$ on $\operatorname{Im} A$, and this operator satisfies the identity $ABA = A$. (Actually, it is even a symmetric inverse.)

(b) Let $A \in L(E)$ be group invertible with group inverse B and group idempotent P, and set $Q = I - P$. Then

$$Q = BA, \quad Q = AB, \quad A = (B+P)^{-1}Q, \quad A = Q(B+P)^{-1},$$

and these identities imply that

$$\text{(18)} \quad \operatorname{Ker} A \subseteq \operatorname{Ker} Q, \quad \operatorname{Im} Q \subseteq \operatorname{Im} A, \quad \operatorname{Ker} Q \subseteq \operatorname{Ker} A, \quad \operatorname{Im} A \subseteq \operatorname{Im} Q.$$

Hence, $\operatorname{Ker} A = \operatorname{Ker} Q$ and $\operatorname{Im} A = \operatorname{Im} Q$, and the assertion follows from $\operatorname{Im} Q \cap \operatorname{Ker} Q = \{0\}$ and $\operatorname{Im} Q + \operatorname{Ker} Q = E$.

Let, conversely, $\operatorname{Im} A \cap \operatorname{Ker} A = \{0\}$ and $\operatorname{Im} A + \operatorname{Ker} A = E$, and let P stand for the projection operator from E onto $\operatorname{Ker} A$ parallel to $\operatorname{Im} A$. Then $AP = PA = 0$ and $A + P$ is invertible; thus, A is group invertible with group idempotent P. ∎

PROPOSITION 6. *Let E be a Banach space and $B(E)$ denote the Banach algebra of all bounded linear operators on E.*

(a) *$A \in B(E)$ is generalized invertible if and only if $\operatorname{Im} A$ is closed and if there are closed subspaces E_K and E_I of E such that*

$$E = \operatorname{Ker} A \dotplus E_K \quad \text{and} \quad E = E_I \dotplus \operatorname{Im} A.$$

(b) *$A \in B(E)$ is group invertible if and only if $\operatorname{Im} A$ is closed, $\operatorname{Im} A \cap \operatorname{Ker} A = \{0\}$, and $\operatorname{Im} A + \operatorname{Ker} A = E$.*

(c) *$A \in B(E)$ is Drazin invertible of degree k if and only if $\operatorname{Im} A^k$ is closed, $\operatorname{Im} A^k \cap \operatorname{Ker} A^k = \{0\}$, and $\operatorname{Im} A^k + \operatorname{Ker} A^k = E$.*

(d) *If H is a Hilbert space, then $A \in B(H)$ is Moore–Penrose invertible if and only if $\operatorname{Im} A$ is closed.*

Proof. (a) Let A be generalized invertible, let B be one of its generalized inverses, and set $P = AB$ and $Q = BA$. Then P and Q are projection operators, and from

$$\operatorname{Im} A = \operatorname{Im} ABA \subseteq \operatorname{Im} AB \subseteq \operatorname{Im} A,$$
$$\operatorname{Ker} A \subseteq \operatorname{Ker} BA \subseteq \operatorname{Ker} ABA = \operatorname{Ker} A$$

we conclude that

$$\operatorname{Im} A = \operatorname{Im} P \quad \text{and} \quad \operatorname{Ker} A = \operatorname{Ker} Q.$$

Now the necessity of the conditions follows since the kernel and the range of a bounded projection operator are closed subspaces each of which is the direct complement of the other. The sufficiency can be checked as in part (a) of the preceding lemma (with the boundedness of $(A|_{E_K})^{-1}$ being a consequence of Banach's theorem on the inverse operator).

(b) The necessity part is a consequence of Lemma 14 and of part (a) of the present proposition. The sufficiency can be seen as in Lemma 14(b) (with the boundedness of P being a consequence of the closed graph theorem).

(c) This is an immediate consequence of Corollary 5.

(d) If A is Moore–Penrose invertible, then the range of A is closed by (a). The reverse conclusion can be proved in the same way as Lemma 14(a); one only has to choose E_K and E_I as the orthogonal complements of $\operatorname{Ker} A$ and $\operatorname{Im} A$, respectively. This choice makes AB and BA self-adjoint projections. ∎

LEMMA 15. *Let E be a Banach space, let $P, Q \in B(E)$ be projection operators, and suppose that $\dim \operatorname{Im} P < \infty$. Then the following assertions are equivalent:*

(a) *$P \sim Q$.*

(b) *$\dim \operatorname{Im} P = \dim \operatorname{Im} Q$.*

Proof. (a)⇒(b). We have $P = C^{-1}QC$ and $Q = CPC^{-1}$ with some invertible operator C, and the assertion follows from the inequality

$$\dim \operatorname{Im} AB \le \min\{\dim \operatorname{Im} A, \dim \operatorname{Im} B\}$$

holding for arbitrary linear operators A, B.

(b)⇒(a). We have $E = \operatorname{Im} P \dotplus \operatorname{Im}(I-P)$ and $E = \operatorname{Im} Q \dotplus \operatorname{Im}(I-Q)$. Consider the restriction of $I-Q$ to the subspace $\operatorname{Im}(I-P)$ of E. The operator $(I-Q)(I-P)$ is a Fredholm operator of index zero, the space $\operatorname{Im} P$ lies in the kernel of that operator, and the intersection of $\operatorname{Im} Q$ with the range of that operator is $\{0\}$. Hence, there are subspaces E_1, E_2 and E_3, E_4 of E such that

$$E = \operatorname{Ker}(I-Q)(I-P) \dotplus E_1 \quad \text{and} \quad \operatorname{Ker}(I-Q)(I-P) = \operatorname{Im} P \dotplus E_2$$

as well as

$$E = E_3 \dotplus \operatorname{Im}(I-Q)(I-P) \quad \text{and} \quad E_3 = \operatorname{Im} Q \dotplus E_4.$$

The zero index of $(I-Q)(I-P)$ ensures that

$$\dim(\operatorname{Im} P \dotplus E_2) = \dim(\operatorname{Im} Q \dotplus E_4),$$

and since $\dim \operatorname{Im} P = \dim \operatorname{Im} Q$ by hypothesis, we also have $\dim E_2 = \dim E_4$. Further, all these dimensions are finite, and the restriction of $I-Q$ to E_1 acts bijectively between this space and $\operatorname{Im}(I-Q)(I-P)$. Choose linear bijections C_1 from $\operatorname{Im} P$ onto $\operatorname{Im} Q$ and C_2 from E_2 onto E_4, and define a linear operator C from E into E as follows: let C act on $\operatorname{Im} P$ as C_1, on E_2 as C_2, and on E_1 as $I-Q$. This operator is bounded (the operators C_1 and C_2 act between finite-dimensional spaces and are, thus, bounded, and the restriction of the bounded operator $I-Q$ is bounded again), it is invertible (the inverse acts on $\operatorname{Im} Q$ as C_1^{-1}, on E_4 as C_2^{-1}, and on $\operatorname{Im}(I-Q)(I-P)$ as $((I-Q)|_{E_1})^{-1}$), and its inverse is bounded by Banach's theorem on the inverse operator. It is easy to check that $C^{-1}QC = P$, i.e. P and Q are similar. ∎

Now we can specialize Theorem 2 to the operator case.

THEOREM 3. *Let E be a Banach space, and let $(A_n) \subseteq B(E)$ be a norm convergent sequence of operators with limit $A \in B(E)$. Suppose that the operator A and all the operators A_n are group invertible, and that the kernel of A has a finite dimension. Then the following assertions are equivalent*:

(a) *The group inverses of A_n converge in the norm of $B(E)$ to the group inverse of A.*

(b) *The kernels of A_n are finite-dimensional, and satisfy* $\dim \operatorname{Ker} A_n = \dim \operatorname{Ker} A$ *for all sufficiently large n.*

Proof. Let P and P_n denote the group idempotents of A and A_n, respectively. The kernel dimension of A resp. A_n coincides with the range dimension of P resp. P_n by (18), thus, the finiteness condition of Theorem 2 is satisfied. The equivalence of the conditions (b) in Theorem 2 and in the present theorem is a consequence of (18) and of Lemma 15. ∎

COROLLARY 9. *Let E be a Banach space, and let $(A_n) \subseteq B(E)$ be a norm convergent sequence of operators with limit $A \in B(E)$. Suppose that the operator A and all the operators A_n are Drazin invertible of degree k, and that the kernel of A^k has a finite dimension. Then the following assertions are equivalent*:

(a) *The Drazin inverses of A_n converge in the norm of $B(E)$ to the Drazin inverse of A.*

(b) $\dim \operatorname{Ker} A_n^k = \dim \operatorname{Ker} A^k$ *for all sufficiently large n.*

Proof. Immediate from the preceding theorem and Lemma 11(b). ∎

COROLLARY 10. *Let H be a Hilbert space, and let $(A_n) \subseteq B(H)$ be a norm convergent sequence of operators with limit $A \in B(H)$. Suppose that the operator A and all the operators A_n are Moore–Penrose invertible, and that the kernel of A has a finite dimension. Then the following assertions are equivalent*:

(a) *The Moore–Penrose inverses of A_n converge in the norm of $B(H)$ to the Moore–Penrose inverse of A.*

(b) $\dim \operatorname{Ker} A_n = \dim \operatorname{Ker} A$ *for all sufficiently large n.*

Proof. Recalling that $\operatorname{Ker} B^*B = \operatorname{Ker} B$ for arbitrary operators $B \in B(H)$, one gets this result immediately from Theorem 3 combined with Lemma 11(c). ∎

COROLLARY 11. *Let H be a Hilbert space, r a positive integer, and let N_r denote the set of all normally solvable operators $A \in B(H)$ with kernel dimension k. Then the mapping assigning to every $A \in N_r$ its Moore–Penrose inverse is continuous on N_r.*

Proof. An operator is normally solvable if and only if its range is closed. So the assertion is just a reformulation of the preceding corollary. ∎

Let us remark that the continuity of this mapping on the subset $\Phi_{r,s}$ of N_r of all Fredholm operators with kernel dimension r and range codimension s is well known (see, e.g., [4], Vol. I, Ch. 4, Cor. 13.1).

8. Regularization, and asymptotic splitting of approximation numbers. In this concluding section, we mention some further consequences of Theorem 1 which have their roots in numerical analysis. The first one concerns the possibility of regularizing a (bad) asymptotically group invertible sequence by adding a sequence tending to zero in norm in order to get a (good) sequence whose elements have uniformly bounded group inverses (which then form, by Lemma 11(a), a norm convergent sequence).

THEOREM 4. *Let $\mathfrak{A}$ be a Banach algebra with identity, let $(a_n) \subseteq \mathfrak{A}$ be a convergent sequence with limit $a \in \mathfrak{A}$, and suppose this sequence is* *asymptotically group invertible. Then there is a sequence* $(g_n) \in \mathfrak{G}$ *such that*

(a) $a_n g_n = g_n a_n$ *for all* n,
(b) $a_n - g_n$ *is group invertible for all sufficiently large* n, *and*
(c) *the group inverses of* $a_n - g_n$ *converge to the group inverse of* a.

Proof. Let (q_n) be the sequence of idempotents introduced in the proof of Theorem 1, and set $g_n := a_n q_n$. Since $a_n q_n = q_n a_n$, one has $a_n g_n = g_n a_n$. Further,

$$(a_n - g_n) q_n = (a_n - a_n q_n) q_n = 0,$$

and the elements $a_n - g_n + q_n$ are invertible for all sufficiently large n due to $\|g_n\| = \|a_n q_n\| \to 0$ and to condition (c) in Theorem 1. Thus, the $a_n - g_n$ are group invertible, and the q_n are the associated group idempotents for all sufficiently large n. Finally, the uniform boundedness of the inverses of $a_n + q_n$ (which is a consequence of the invertibility of $(a_n + q_n) + \mathfrak{G}$ in $\mathfrak{L}/\mathfrak{G}$) implies the uniform boundedness of the inverses of $a_n - g_n + q_n$ and, hence, of the group inverses of $a_n - g_n$. Proposition 4 yields the convergence of the group inverses of $a_n - g_n$ to the group inverse of a. ∎

The second consequence concerns the asymptotic splitting of the approximation numbers of the elements of an asymptotically group invertible sequence. Let E be a Banach space. We define the kth *approximation number* $s_k^E(A)$ of an operator $A \in B(E)$ by

$$s_k^E(A) = \inf\{\|A - F\| : F \in L(E),\ \operatorname{codim} \operatorname{Im} F \geq k\},$$

but we have to mention that this name is also used for other (related) numbers in the literature.

We prepare the proof of the announced splitting result by a few lemmata.

LEMMA 16. *Let* E *be an infinite-dimensional Banach space,* $A \in B(E)$, *and let* $P \in B(E)$ *be a projection operator with norm* 1 *and finite kernel dimension* r *such that* $A = PA = AP$. *Then*

$$s_k^E(A) = \begin{cases} 0 & \text{if } k \leq r, \\ s_{k-r}^{PE}(PAP) & \text{if } k > r. \end{cases}$$

Proof. We conclude from $\operatorname{Im} PB \subseteq \operatorname{Im} P$ that

$$\operatorname{codim} \operatorname{Im} PA \geq \operatorname{codim} \operatorname{Im} P = \dim \operatorname{Ker} P = r.$$

Hence,

$$s_r^E(A) = \inf\{\|A - F\| : \operatorname{codim} \operatorname{Im} F \geq r\} \leq \|A - PA\| = 0,$$

whence $s_0^E(A) = \ldots = s_r^E(A) = 0$.

Let now $k > r$. Since $\|A - PFP\| = \|P(A - F)P\| \leq \|P\|^2 \|A - F\|$ for all $F \in B(E)$, one has

(19) $\inf\{\|A - PFP\| : \operatorname{codim} \operatorname{Im} F \geq k\} \leq \inf\{\|A - F\| : \operatorname{codim} \operatorname{Im} F \geq k\}.$

If F runs through the set of all operators in $B(E)$ with $\operatorname{codim}_E \operatorname{Im} F \geq k$, then PFP runs through the operators in $B(PE)$ with $\operatorname{codim}_{PE} \operatorname{Im} PFP \geq k - r$. Thus, (19) gives the estimate

$$s_{k-r}^{PE}(A) \leq s_k^E(A) \quad \text{for all } k > r.$$

Conversely, if $\operatorname{codim}_{PE} \operatorname{Im} PFP \geq k - r$ for an operator F, then we have $\operatorname{codim}_E \operatorname{Im} PFP \geq k$ and, consequently,

$$\inf\{\|A - F\| : \operatorname{codim}_E \operatorname{Im} F \geq k\} \leq \inf\{\|A - PFP\| : \operatorname{codim}_{PE} \operatorname{Im} PFP \geq k - r\},$$

i.e. $s_k^E(A) \leq s_{k-r}^{PE}(A)$ for all $k > r$. ∎

COROLLARY 12. *In case* $\|P\| > 1$, *the approximation numbers* $s_0^E(A), \ldots, s_r^E(A)$ *are all zero, whereas* $s_k^E(A)$ *and* $s_{k-r}^{PE}(PAP)$ *for* $k > r$ *are equivalent in the sense that*

$$s_k^E(A) \leq s_{k-r}^{PE}(PAP) \leq \|P\|^2 s_k^E(A).$$

COROLLARY 13. *Let* E *be a Banach space, and let* $A \in B(E)$ *be a group invertible operator with group inverse* B *and group idempotent* P. *If* $\dim \operatorname{Ker} A = r < \infty$, *then* $s_0^E(A) = \ldots = s_r^E(A) = 0$, *whereas*

$$s_{r+1}^E(A) \leq \|B\|^{-1} \leq \|I - P\|^2 s_{r+1}^E(A).$$

Proof. If F is a Banach space and $C \in B(F)$ is an invertible operator, then

$$s_1^F(C) = \|C^{-1}\|^{-1}$$

(see [2] for the proof). Applying this to the operator $C = (I - P)A(I - P)$ thought of as acting on $F = (I - P)E$, and taking into account Lemma 16 and its corollary, one obtains the assertion. ∎

Thus, the $(r+1)$st approximation number is equivalent to the norm of the group inverse. We conjecture that these numbers even coincide.

Now we can verify the announced splitting property of the approximation numbers.

THEOREM 5. *Let* E *be an infinite-dimensional Banach space, and let* $(A_n) \subseteq B(E)$ *be a norm convergent sequence with limit* $A \in B(E)$. *Suppose that* A *is group invertible, and that* $\dim \operatorname{Ker} A =: r < \infty$. *Then*

$$s_r^E(A_n) \to 0 \quad \text{as } n \to \infty,$$

and there is a constant $C > 0$ *such that*

$$s_{r+1}^E(A_n) \geq C \quad \text{for all sufficiently large } n.$$

Proof. Let the operators $Q_n, G_n \in B(E)$ be defined as the elements q_n, g_n in the proof of Theorem 4. The operators Q_n converge to the group idempotent P of A, hence, by Lemmata 12 and 15, $\dim \operatorname{Im} Q_n = r$ for all sufficiently large n. Thus, $\operatorname{codim} \operatorname{Im} A_n(I-Q_n) \geq \operatorname{codim} \operatorname{Im}(I-Q_n) = r$, whence

$$s_r^E(A_n) \leq \|A_n - A_n(I-Q_n)\| = \|A_n Q_n\| \to 0.$$

For the proof of the second assertion recall from Theorem 4 and its proof that the operators $A_n - G_n$ are group invertible with group idempotents Q_n. Hence, $\dim \operatorname{Ker}(A_n - G_n) = \dim \operatorname{Im} Q_n = r$ for all large n, say $n \geq n_0$. Let B_n denote the group inverse of $A_n - G_n$. We infer from Corollary 13 that

$$s_{r+1}^E(A_n) \leq \|B_n\|^{-1} \leq \|I - Q_n\|^2 s_{r+1}^E(A_n)$$

and, thus,

$$s_{r+1}^E(A_n) \geq \inf_{n \geq n_0} \frac{\|B_n\|^{-1}}{\|I-Q_n\|^2} \tag{20}$$

for all $n \geq n_0$. The norms of the operators B_n are uniformly bounded above by Theorems 3 and 4, and the norms of the $I - Q_n$ are uniformly bounded above because of the norm convergence $Q_n \to P$ and below because $P \neq I$. Thus, the infimum in (20) is finite and positive. ∎

We conclude with a result which can be considered as an analogue of the preceding theorem for Moore–Penrose invertible operators.

THEOREM 6. *Let H be an infinite-dimensional Hilbert space, and let $(A_n) \subseteq B(H)$ be a norm convergent sequence with limit $A \in B(H)$. Suppose that A is Moore–Penrose invertible, and that $\dim \operatorname{Ker} A =: r < \infty$.*

(a) *A_n is Moore–Penrose invertible, and $\dim \operatorname{Ker} A_n \leq r$ for all sufficiently large n.*

(b) *$A_n^* A_n$ is group invertible, and $\dim \operatorname{Ker} A_n^* A_n \leq r$ for all sufficiently large n.*

(c) *There exist numbers $c_n \geq 0$ with $c_n \to 0$ as $n \to \infty$ and $d > 0$ such that $\sigma(A_n^* A_n) \subseteq [0, c_n] \cup [d, \infty)$ for all n.*

(d) *$\sigma(A_n^* A_n) \cap [0, c_n]$ consists of a finite number of pairwise distinct eigenvalues $\lambda_1, \ldots, \lambda_s$, the corresponding eigenspaces $H_1, \ldots, H_s$ have finite dimension, and*

$$\sum_{i=1}^{s} \dim H_i = r.$$

Proof. (a) Every Moore–Penrose invertible operator with finite-dimensional kernel is a Φ^+-operator by Proposition 6(d). It is well known that the class of all Φ^+-operators is open in $B(H)$ and that the function $B \mapsto \dim \operatorname{Ker} B$ is upper semicontinuous on this class (see [11], Chapter I, Theorem 3.9). Thus, if A is Moore–Penrose invertible and $\dim \operatorname{Ker} A =: r$, then the A_n are Φ^+-operators with $\dim \operatorname{Ker} A_n \leq r$ for all sufficiently large n. Together with Proposition 6(d), this proves the assertion.

(b) This is an immediate consequence of (a), Lemma 2 and the fact that $\operatorname{Ker} A_n^* A_n = \operatorname{Ker} A_n$ for every $A_n \in B(H)$.

(c) Combine (b) and Proposition 5.

(d) If A is Moore–Penrose invertible, then A^*A is group invertible by Lemma 2. So we can apply Theorems 1 and 4 with A^*A and $(A_n^* A_n)$ in place of a and (a_n). What results is the existence of a sequence (Q_n) of idempotents and of operators $G_n := A_n^* A_n Q_n$ with $\|G_n\| \to 0$ such that the operators $A_n^* A_n - G_n$ are group invertible with group idempotents Q_n and that their group inverses converge to the group inverse of A^*A. Further we infer from the last assertion of Theorem 1 that the Q_n belong to the commutative C^*-algebra generated by $A_n^* A_n$ and the identity operator and are, hence, self-adjoint idempotents. Consequently, Q_n is the orthogonal projection from H onto the kernel of $A_n^* A_n - G_n$. Further we know from Theorem 3 that $\dim \operatorname{Ker}(A_n^* A_n - G_n) = \dim \operatorname{Ker} A^* A = r$ for all sufficiently large n, hence, $\dim \operatorname{Im} Q_n = r$.

The orthogonality of the idempotents Q_n implies that H is the orthogonal sum of its subspaces $\operatorname{Im} Q_n$ and $\operatorname{Im}(I - Q_n)$, and from

$$A_n^* A_n = Q_n A_n^* A_n Q_n + (I - Q_n) A_n^* A_n (I - Q_n)$$

we obtain

$$\sigma(A_n^* A_n) = \sigma(A_n^* A_n|_{\operatorname{Im} Q_n}) \cup \sigma(A_n^* A_n|_{\operatorname{Im}(I-Q_n)}).$$

Let c_n and d be as in (c). Since $A_n^* A_n|_{\operatorname{Im} Q_n} = G_n|_{\operatorname{Im} Q_n}$ and $\|G_n\| \to 0$, we see that

$$\sigma(A_n^* A_n|_{\operatorname{Im} Q_n}) \cap [d, \infty) = \emptyset$$

for all sufficiently large n. Similarly we have

$$A_n^* A_n|_{\operatorname{Im}(I-Q_n)} = (A_n^* A_n - G_n)|_{\operatorname{Im}(I-Q_n)},$$

and the convergence of the group inverses of the $A_n^* A_n - G_n$ entails that the norms and, thus, the spectra of these group inverses are bounded above by a positive constant. Hence, the spectra of $A_n^* A_n|_{\operatorname{Im}(I-Q_n)}$ are bounded below by a positive constant, which implies that

$$\sigma(A_n^* A_n|_{\operatorname{Im}(I-Q_n)}) \cap [0, c_n] = \emptyset$$

for all sufficiently large n. Consequently,

$$\sigma(A_n^* A_n|_{\operatorname{Im} Q_n}) = \sigma(A_n^* A_n) \cap [0, c_n]$$

for all large n. Now one has to take into account that $A_n^* A_n|_{\operatorname{Im} Q_n}$ is a self-adjoint operator acting on an r-dimensional space in order to get the desired result. ∎

The upper semicontinuity of the kernel dimension implies that, if A has a finite-dimensional kernel and if ε is small enough, then

$$\dim \operatorname{Ker} B \leq \dim \operatorname{Ker} A \quad \text{for all } B \text{ with } \|B - A\| < \varepsilon.$$

Further, there are trivial examples showing that the kernel dimension of B can be strictly less than the kernel dimension of A. Nevertheless, as the previous theorem shows, the information about the kernel dimension of A is stored in B in some sense if only ε is small enough.

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