The Lévy continuity theorem for nuclear groups

by

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Abstract. Let $G$ be an abelian topological group. The Lévy continuity theorem says that if $G$ is an LCA group, then it has the following property (PL): a sequence of Radon probability measures on $G$ is weakly convergent to a Radon probability measure $\mu$ if and only if the corresponding sequence of Fourier transforms is pointwise convergent to the Fourier transform of $\mu$. Bourgain [Do] proved that every locally convex space $G$ has the property (PL). In this paper we prove that the property (PL) is inherited by nuclear groups, a variety of abelian topological groups containing LCA groups and nuclear locally convex spaces, introduced in [B1].

1. Introduction. Let $G$ be an LCA group and $\Gamma$ the dual group. The Bochner theorem may be formulated in the following way:

(a) Every continuous positive-definite function on $G$ is the inverse Fourier transform of a (unique) finite positive Radon measure on $\Gamma$.

This theorem can be extended to inverse limits and countable direct limits of LCA groups. It was also extended to some other classes of abelian topological groups: nuclear locally convex spaces (the Minlos theorem), Hausdorff quotient groups of such spaces (Yang [Y]), locally convex spaces over $p$-adic fields (Myciecki [M]). Trying to give a common generalization of the corresponding results, the author introduced in [B1] the so-called nuclear groups, a variety of abelian topological groups containing LCA groups and nuclear locally convex spaces (the definition and basic properties of nuclear groups are given in Section 5 below). It was proved in [B1, (12.1)] that every nuclear group $G$ satisfies an analogue of (a).

The Lévy continuity theorem may be formulated in the following way:

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A family of Radon probability measures on $\Gamma$ is tight if and only if the corresponding family of inverse Fourier transforms is equicontinuous on $G$.

A sequence $(\mu_n)_{n=1}^\infty$ of Radon probability measures on $G$ is weakly convergent to a Radon probability measure $\mu$ if and only if the corresponding sequence of Fourier transforms is pointwise convergent to the Fourier transform of $\mu$.

An analogue of (β) for nuclear groups was obtained in [B1, (12.5)]. Boulanger [Bo] proved that every nuclear locally convex space $G$ satisfies (γ). The aim of the present paper is to complete the picture by proving that every nuclear group $G$ satisfies (γ) (Theorem 5.3 below). The main idea of the proof is similar to that of [Bo], with vector spaces replaced by their subgroups and quotient groups. Another proof of Theorem 5.3 will be given in [BT].

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2. Notation, terminology and preliminary lemmas. Let $G$ be an abelian topological group (which we abbreviate to a.t. group). Following Hejcean [H], we say that a subset $X$ of $G$ is bounded if, for each neighbourhood $U$ of zero in $G$, one can find a positive integer $n$ and a finite subset $F$ of $G$ such that

$$X \subset F + U + \ldots + U.$$ 

For instance, every precompact subset is bounded. If $G$ is locally compact, then $X$ is bounded if and only if it is precompact. If $G$ is a locally convex space, then $X$ is bounded if and only if it is bounded in the usual sense, i.e. absorbed by every neighbourhood of zero.

By a character of $G$ we mean a homomorphism of $G$ into the multiplicative group of complex numbers with modulus 1. The value of a character $\chi$ at a point $g \in G$ will be denoted by $\chi(g)$ or $(g, \chi)$. The group of all continuous characters of $G$ is denoted by $G^\wedge$. It is usually endowed with the compact-open topology, but we shall also consider other topologies on $G^\wedge$.

Let $\mathcal{S}$ be a family of subsets of $G$ which satisfies the following conditions:

(i) if $X \in \mathcal{S}$, then $-X \in \mathcal{S}$;
(ii) if $X \in \mathcal{S}$ and $Y \subset X$, then $Y \in \mathcal{S}$;
(iii) if $X, Y \in \mathcal{S}$, then $X \cup Y \in \mathcal{S}$;
(iv) if $X, Y \in \mathcal{S}$, then $X + Y \in \mathcal{S}$;
(v) all finite subsets belong to $\mathcal{S}$.

(such a family is sometimes called a boundedness on $G$). Typical examples are the families of finite, compact, precompact or bounded subsets. It is a standard fact that there exists a unique group topology on $G^\wedge$ for which the family of sets of the form

$$\{x \in G^\wedge : |1 - \chi(x)| < \epsilon \text{ for each } x \in G\} \quad (\epsilon > 0, \, x \in \mathcal{S})$$

is a base at zero (only (iii) is needed here). We call it the topology of uniform convergence on elements of $\mathcal{S}$ and denote by $\tau_\mathcal{S}$. Condition (v) implies that $\tau_\mathcal{S}$ is Hausdorff. Condition (iv) implies easily that the family of sets of the form

$$\{x \in G^\wedge : \Re \chi(x) \geq 0 \text{ for each } x \in G\} \quad (X \in \mathcal{S})$$

is also a base at zero for $\tau_\mathcal{S}$. The topology of uniform convergence on finite, compact or bounded subsets of $G$ is called the topology of pointwise convergence, respectively; the corresponding character groups will be denoted by $G_0^\wedge$, $G_1^\wedge$ and $G_2^\wedge$. If $G$ is Hausdorff, then the topology of compact convergence coincides with the compact-open topology on $G^\wedge$. Note that the identity mappings $G_0^\wedge \to G_1^\wedge \to G_2^\wedge$ are continuous.

Let $H$ be another a.t. group and $\psi : G \to H$ an algebraic homomorphism. We say that $\psi$ is bounded if the image of every bounded subset of $G$ is a bounded subset of $H$. We say that $\psi$ is bounding if every bounded subset of $H$ is the image of some bounded subset of $G$. If $\psi : G \to H$ is continuous, then it is bounded, which implies that the dual homomorphism $\psi^\wedge : H_0^\wedge \to G_0^\wedge$ given by $\psi^\wedge(x) = x \circ \psi$ is continuous.

Now, let $H$ be a closed subgroup of $G$ and $\psi : G \to G/H$ the canonical homomorphism. It is not hard to see that if $G$ has some bounded neighbourhood of zero, then $\psi$ is bounding. Such a situation occurs if, for instance, $G$ is a normed space or an LCA group.

**Lemma 2.1.** Let $G, H$ be a.t. groups and let $\pi_G : G \times H \to G$ and $\pi_H : G \times H \to H$ be the canonical projections.

(a) A subset $X$ of $G \times H$ is bounded if and only if $\pi_G(X)$ and $\pi_H(X)$ are bounded subsets of $G$ and $H$, respectively.

(b) The group $(G \times H)_0^\wedge$ is canonically topologically isomorphic to $G_0^\wedge \times H_0^\wedge$.

**Proof.** Part (a) is a direct consequence of the definitions. It is a standard fact that the formula

$$\langle (g, h), \sigma(x, \kappa) \rangle = (g, x) \cdot (h, \kappa) \quad (g \in G, \, h \in H, \, x \in G^\wedge, \, \kappa \in H^\wedge)$$

defines an algebraic isomorphism $\sigma : G^\wedge \times H^\wedge \to (G \times H)^\wedge$, and (a) implies that $\sigma$ is a homomorphism between $G_0^\wedge \times H_0^\wedge$ and $(G \times H)_0^\wedge$.

By $\mathcal{M}(G)$ we denote the family of finite positive Radon measures on $G$, and $\mathcal{P}(G) \subset \mathcal{M}(G)$ is the family of Radon probability measures. By the weak topology on $\mathcal{M}(G)$ we mean the topology induced by all functions of
the form

$$\mathcal{M}(G) \ni \mu \mapsto \int_G f(g) \, d\mu(g) \in \mathbb{C}$$

where $f$ is a bounded continuous complex-valued function on $G$. If a net $(\mu_\alpha)$ in $\mathcal{M}(G)$ is weakly convergent to $\mu \in \mathcal{M}(G)$, then we write $\mu_\alpha \rightharpoonup \mu$. A family $\mathcal{S} \subset \mathcal{M}(G)$ is said to be tight if for each $\varepsilon > 0$ there corresponds a compact subset $X$ of $G$ such that $\mu(G \setminus X) \leq \varepsilon$ for each $\mu \in \mathcal{S}$.

By the Fourier transform of a measure $\mu \in \mathcal{M}(G)$ we mean the p.d. (positive-definite) function $\mathcal{F}_\mu : G^\wedge \to \mathbb{C}$ given by

$$\mathcal{F}_\mu(\chi) = \int_G \chi(g) \, d\mu(g) \quad (\chi \in G^\wedge).$$

Since $\mu$ is a Radon measure, $\mathcal{F}_\mu$ is continuous in the compact-open topology on $G^\wedge$.

Let $(\varphi_\alpha)$ be a net of p.d. functions on $G^\wedge$. If it is pointwise convergent to some function $\varphi$, then we write $\varphi_\alpha \rightharpoonup \varphi$. If $(\mu_\alpha)$ is a net in $\mathcal{M}(G)$ with $\mu_\alpha \rightharpoonup \mu \in \mathcal{M}(G)$, then, by definition, $\mathcal{F}_\mu_\alpha \rightharpoonup \mathcal{F}_\mu$. The converse, in general, is not true, even if nets are replaced by usual sequences.

Let $\tau$ be a group topology on $G^\wedge$ such that the functions $\chi \mapsto \chi(g)$, $g \in G$, are continuous, and let $\nu \in \mathcal{M}(G^\tau)$. By the inverse Fourier transform of $\nu$ we mean the p.d. function $\mathcal{F}^{-1} \nu : G \to \mathbb{C}$ given by

$$\mathcal{F}^{-1} \nu(g) = \int_{G^\wedge} \overline{\chi(g)} \, d\nu(\chi) \quad (g \in G).$$

Let $E$ be a topological vector space (all vector spaces occurring are assumed to be real). We may treat $E$ as an (additive) abelian topological group. By $E^*$ we denote the dual space of all continuous linear functionals on $E$, and $E^*_0$ is the space $E^*$ endowed with the topology of uniform convergence on bounded sets. If $E$ is a normed space, then $E^*_0$ is just the dual space with the norm topology.

**Lemma 2.2.** Let $E$ be a topological vector space. Then the formula

$$\langle x, \alpha(f) \rangle = \exp\{2\pi if(x)\} \quad (x \in E, \ f \in E^*)$$

defines a topological isomorphism $\alpha : E^*_0 \to E_0^\circ$.

**Proof.** It is a standard fact that $\alpha$ is an algebraic isomorphism (see e.g. [HR, (23.32)]), and it is easy to see that $\alpha$ and $\alpha^{-1}$ are both continuous. $\blacksquare$

Let $X, Y$ be symmetric convex subsets of a vector space $E$. Suppose that $X$ is absorbed by $Y$, i.e. that $X \subset tY$ for some $t > 0$ (we write $X \prec tY$). The Kolmogorov diameters of $X$ with respect to $Y$ are given by

$$d_k(X, Y) = \inf_M \inf \{ t > 0 : X \subset tY + M \} \quad (k = 1, 2, \ldots)$$

where the first infimum is taken over all linear subspaces $M$ of $E$ with $\dim M < k$.

Let $p$ be a seminorm on $E$. We set

$$B_p = \{ x \in E : p(x) \leq 1 \}.$$

We say that $p$ is a pre-Hilbert seminorm if

$$p(x + y)^2 + p(x - y)^2 = 2p(x)^2 + 2p(y)^2 \quad (x, y \in E).$$

Let $E$ be a Hilbert space. By $B_E$ we denote the closed unit ball of $E$. Let $F$ be another Hilbert space and $T : E \to F$ a bounded linear operator. Then the formula $p(x) = \|Tx\|$, $x \in E$, defines a pre-Hilbert seminorm $p$ on $E$. The operator numbers of $T$ are given by

$$d_k(T) = d_k(T(B_E), B_F) = d_k(B_E, T(B_F)) \quad (k = 1, 2, \ldots).$$

We say that $T$ is $\alpha$-approximable, $0 < \alpha < \infty$, if $\sum_{k=1}^{\infty} d_k(T)^\alpha < \infty$.

**Lemma 2.3.** Let $p, q$ be pre-Hilbert seminorms on a vector space $E$, with $B_q \prec B_p$, such that $\sum_{k=1}^{\infty} d_k(B_q, B_p)^\alpha < \infty$. Let $Q$ be an arbitrary subgroup of $E$ and let $\nu$ be a Borel probability measure on $Q^\circ_0$ such that $\Re \mathcal{F}^{-1}(\nu(x)) \geq 1 - \varepsilon$ for each $x \in Q \cap B_p$, where $\varepsilon > 0$. Define

$$Z = \{ x \in Q^\wedge : \Re \chi(x) \geq 0 \text{ for each } x \in Q \cap \frac{1}{2} B_q \}.$$

Then

$$\nu(Q^\wedge \cap Z) \leq 2\varepsilon + \sum_{k=1}^{\infty} d_k(B_q, B_p)^2.$$

This is a direct consequence of [B3, Lemma 3.4]. For details of the proof see [A, Cor. 22.8].

**Lemma 2.4.** Let $p, q$ be pre-Hilbert seminorms on a vector space $E$, with $B_p \prec B_q$, such that $\sum_{k=1}^{\infty} d_k(B_p, B_q)^\alpha < \infty$. Let $Q$ be a subgroup of $E$ and $\chi$ a character of $Q$ such that $\Re \chi(x) \geq 0$ for each $x \in Q \cap B_q$. Then there exists a bounded linear functional $f$ on $E$ such that $\exp\{2\pi if(x)\} = \chi(x)$ for each $x \in Q$, and

$$\|f\| \leq 4 \sum_{k=1}^{\infty} d_k(B_p, B_q).$$

This is a consequence of [B2, Thm. 3.1(i)]. For details of the proof see [A, Lemma 19.13(ii)].

**3. The property (PL).** Let $G$ be an a.t. group. Following [Bo], we say that $G$ has the property (PL) if ($\gamma$) is satisfied, i.e. if $\mathcal{F}_\mu_\alpha \rightharpoonup \mathcal{F}_\mu$ implies that $\mu_\alpha \rightharpoonup \mu$, for any $\mu \in \mathcal{P}(G)$ and any sequence $(\mu_\alpha)_{\alpha=1}^{\infty}$ in $\mathcal{P}(G)$.

**Lemma 3.1.** Let $G$ be an a.t. group with the property (PL) and let $H$ be an arbitrary subgroup of $G$. Then $H$ also has the property (PL).
Proof. Let \( \mu \in \mathcal{P}(H) \) and let \( (\mu_n)_{n=1}^{\infty} \) be a sequence in \( \mathcal{P}(H) \) such that \( F_{\mu_n} \overset{p}{\to} F_{\mu} \). Let \( \iota : H \to G \) be the identity embedding and let \( \mu' = \iota(\mu) \) and \( \mu'_n = \iota(\mu_n) \), \( n = 1, 2, \ldots \). Then \( F_{\mu'_n} = F_{\mu'} \circ \iota \wedge F_{\mu'_n} = F_{\mu_n} \circ \iota \wedge \) for every \( n \), so that \( F_{\mu'_n} \overset{p}{\to} F_{\mu'} \). Since \( G \) has the property (PL), it follows that \( \mu'_n \overset{w}{\to} \mu' \), which means that \( \mu_n \overset{w}{\to} \mu \) (see Lemma 2.1 of [Bo]).

Let \( \pi : G \to H \) be a continuous homomorphism of a.t. groups. Consider the following two conditions:

\[ (*) \quad \text{if } \mu \in \mathcal{P}(G) \text{ and if } (\mu_n)_{n=1}^{\infty} \text{ is a sequence in } \mathcal{P}(G) \text{ such that } F_{\mu_n} \overset{p}{\to} F_{\mu}, \text{ then } \pi(\mu_n) \overset{w}{\to} \pi(\mu); \]

\[ (**) \quad \text{if } S \subset \mathcal{P}(G) \text{ is a family of measures such that the family } \{ F_{\mu_n} \}_{n \in S} \text{ is equicontinuous on } G, \text{ then } \{ \pi(\mu_n) \}_{n \in S} \text{ is a tight family of measures on } H. \]

If \( (*) \) is satisfied, then we say that the homomorphism \( \pi \) has the property (PL). If \( (**) \) is satisfied, then we say that \( \pi \) is tightening.

**Lemma 3.2.** Let \( (I, \leq) \) be a directed set and let \( G \) be the limit of an inverse system \( \{ G_i, \pi_{ij}, I \} \) of a.t. groups and continuous homomorphisms. Suppose that to each \( i \in I \) there corresponds some \( j \geq i \) such that the homomorphism \( \pi_{ij} : G_j \to G_i \) has the property (PL). Then the group \( G \) has the property (PL).

**Proof.** Let \( \pi_i : G \to G_i, i \in I \), be the canonical homomorphisms. Let \( \mu \in \mathcal{P}(G) \) and let \( (\mu_n)_{n=1}^{\infty} \) be a sequence in \( \mathcal{P}(G) \) such that \( F_{\mu_n} \overset{p}{\to} F_{\mu} \). Fix an arbitrary \( i \in I \) and choose \( j \geq i \) such that \( \pi_{ij} : G_j \to G_i \) has the property (PL). We have \( F_{\pi_{ij}(\mu)} = F_{\mu} \circ \pi_{ij} \wedge F_{\pi_{ij}(\mu_n)} = F_{\mu_n} \circ \pi_{ij} \wedge \) for every \( n \), which means that \( F_{\pi_{ij}(\mu_n)} \overset{p}{\to} F_{\pi_{ij}(\mu)} \). Therefore \( \pi_{ij}(\pi_{ij}(\mu_n)) \overset{w}{\to} \pi_{ij}(\pi_{ij}(\mu)) \), i.e. \( \pi_i(\mu_n) \overset{w}{\to} \pi_i(\mu) \). Since \( i \in I \) was arbitrary, it follows that \( \mu_n \overset{w}{\to} \mu \) (see Lemma 2.3 of [Bo]).

**Lemma 3.3.** Let \( \varphi \) be a p.d. function on a (not necessarily abelian) group \( G \), with \( \varphi(0) = 1 \). Let \( \varepsilon \in (0, 1) \) and let \( g_1, g_2 \in G \) be such that \( \text{Re } \varphi(g_1) \geq 1 - \varepsilon \), \( i = 1, 2 \). Then \( \text{Re } \varphi(g_1 - g_2) \geq 1 - 4\varepsilon + 2e^2 \geq 1 - 4\varepsilon \).

This next easily from elementary properties of p.d. functions.

The next proposition may be treated as an analogue of the equicontinuity principle for p.d. functions.

**Proposition 3.4.** Let \( G \) be a (not necessarily abelian) Čech-complete group (or even a Baire group) and let \( (\varphi_n)_{n=1}^{\infty} \) be a pointwise convergent sequence of p.d. functions on \( G \) such that the limit function is continuous. Then the sequence \( (\varphi_n) \) is equicontinuous.

**Proof.** Denote the limit function by \( \varphi \). We may assume that \( \varphi(0) = \varphi_n(0) = 1 \). Fix \( \varepsilon \in (0, 1) \) and consider the closed subsets

\[ X_m = \bigcap_{n \geq m} \{ g \in G : \text{Re } \varphi_n(g) \geq 1 - \varepsilon \} \quad (m = 1, 2, \ldots). \]

Since \( \varphi_n \overset{w}{\to} \varphi \), it follows that

\[ V := \{ g \in G : \text{Re } \varphi(g) \geq 1 - \varepsilon/2 \} \subseteq \bigcup_{m=1}^{\infty} X_m. \]

We have \( \text{Int } V \neq \emptyset \) because \( \varphi \) is continuous. Now, a standard category argument shows that there is an index \( m \) such that \( U := \text{Int } X_m \neq \emptyset \). Then \( U \cup U \) is a neighbourhood of zero in \( G \) and, by the previous lemma, we have \( \text{Re } \varphi_n(g) > 1 - 4\varepsilon \) for every \( g \in U \cup U \) and \( n \geq m \).

An a.t. group \( G \) is said to be dually separated if \( G^c \) separates the points of \( G \). If \( K \) is a subgroup of a topological vector space \( E \), then it follows easily from Lemma 2.2 that \( E/K \) is a dually separated group if and only if \( K \) is weakly closed in \( E \) (cf. [B1, (2.3)]).

**Lemma 3.5.** Let \( G \) be a dually separated group and let \( \mu_1, \mu_2 \in \mathcal{P}(G) \). If \( F_{\mu_1} = F_{\mu_2} \), then \( \mu_1 = \mu_2 \).

This is a standard fact. See e.g. Theorem 2.2 of Chapter IV in [VTCh].

**Lemma 3.6.** Let \( G \) be a dually separated group. Let \( \mu \in \mathcal{P}(G) \) and let \( (\mu_n) \) be a net in \( \mathcal{P}(G) \) such that \( F_{\mu_n} \overset{p}{\to} F_{\mu} \). If the family \( \{ \mu_n \} \) is tight, then \( \mu_n \overset{w}{\to} \mu \).

**Proof.** Suppose the contrary, i.e. that \( \mu_n \overset{w}{\to} \mu \). Then there is a finer net \( (\mu'_n) \) for which \( \mu \) is not a weak cluster point. Being tight, the family \( \{ \mu_n \} \) is weakly relatively compact in \( \mathcal{P}(G) \) (see e.g. Theorem 3.6 of Chapter I in [VTCh]). So, there is a net \( (\mu'_n) \) finer than \( (\mu'_n) \) which converges to some \( \mu'' \in \mathcal{P}(G) \). We have \( \mu'' \neq \mu, \text{ otherwise } \mu \) would be a cluster point of \( (\mu'_n) \). Then the net \( (F_{\mu'_n}) \) is pointwise convergent to \( F_{\mu} \) and \( F_{\mu''} \); hence \( F_{\mu} = F_{\mu''} \). By Lemma 3.5, we obtain \( \mu = \mu'' \), which is a contradiction.

**Lemma 3.7.** Let \( \pi : G \to H \) be a continuous homomorphism of a.t. groups. Suppose that \( G \) is dually separated and \( G^c \) is Čech-complete. If \( \pi \) is tightening, then it has the property (PL).

**Proof.** Let \( \mu \in \mathcal{P}(G) \) and let \( (\mu_n)_{n=1}^{\infty} \) be a sequence in \( \mathcal{P}(G) \) with \( F_{\mu_n} \overset{p}{\to} F_{\mu} \). By Lemma 3.4, \( (F_{\mu_n})_{n=1}^{\infty} \) is an equicontinuous family of functions on \( G \) (the function \( F_{\mu} \) is continuous on \( G \) and hence on \( G^c \)). If \( \pi \) is tightening, then \( (\pi(\mu_n))_{n=1}^{\infty} \) is a tight family of measures on \( H \). We have \( F_{\pi(\mu)} = F_{\mu} \circ \pi^\wedge \wedge F_{\pi(\mu_n)} = F_{\mu_n} \circ \pi^\wedge \) for every \( n \), which implies that \( F_{\pi(\mu_n)} \overset{w}{\to} F_{\pi(\mu)} \). Hence, by Lemma 3.6, \( \pi(\mu_n) \overset{w}{\to} \pi(\mu) \).
Lemma 3.8. Let $G, D, H$ be a.t. groups with $D$ discrete. Identify $G$ with the open subgroup $G \times \{0\}$ of $G \times D$. Let $\pi : G \times D \to H$ be a continuous homomorphism such that the restriction $\pi|G : G \to H$ is tightening. Then $\pi$ is also tightening.

Proof. Let $S \subset \mathcal{P}(G \times D)$ be a family of measures such that $\{F_\mu\}_{\mu \in S}$ is an equicontinuous family of functions on $(G \times D)_0^1$. For each $d \in D$, let $G_d = G \times \{d\}$ be the corresponding coset modulo $G$. For $\mu \in S$ and $d \in D$, let $\mu_d \in \mathcal{M}(G)$ be the measure given by $\mu_d(A) = \mu(A \cap G_d)$ for Borel subsets $A \subset G$. (I.e., $\mu_d$ is the restriction of $\mu$ to $G_d$.) Then we may write $\mu = \sum_{d \in D} \mu_d$ for $\mu \in S$. To prove that the family $\{\pi(\mu_d)\}_{\mu \in S}$ is tight, it is enough to show the following two assertions:

(I) To each $\varepsilon > 0$ there corresponds a finite subset $I \subset D$ such that $\mu(G \times I) \geq 1 - \varepsilon$ for each $\mu \in S$.

(II) For each $d \in D$, the family $\{\pi(\mu_d)\}_{\mu \in S}$ is tight.

Let $\psi_G : G \times D \to G$ and $\psi_D : G \times D \to D$ be the canonical projections. Consider the dual homomorphisms $\psi_G^* : G_0^1 \to (G \times D)_0^1$ and $\psi_D^* : D_0^1 \to (G \times D)_0^1$. We have $F_{\psi_G(\mu)} = F_\mu \circ \psi_G^*$ and $F_{\psi_D(\mu)} = F_\mu \circ \psi_D^*$ for $\mu \in S$. Therefore $\{F_{\psi_G(\mu)}\}_{\mu \in S}$ and $\{F_{\psi_D(\mu)}\}_{\mu \in S}$ are equicontinuous families of functions on $G_0^1$ and $D_0^1$, respectively. The Lebesgue theorem for discrete groups implies that $\{\psi_D(\mu)\}_{\mu \in S}$ is a tight family of measures on $D_0^1$.

Let $\sigma : G \to H$ be the restriction of $\pi$ to $G$. Since $\{F_{\sigma(\mu)}\}_{\mu \in S}$ is equicontinuous and $\sigma$ is tightening, it follows that

(III) the family $\{\sigma(\mu_d)\}_{\mu \in S}$ is tight.

To prove (II), fix $d \in D$ and let $\tau : H \to H$ be the shift $h \mapsto h + \pi(d)$. A direct verification shows that $\pi(\mu_d) = \tau \sigma(\mu_d)$ for $\mu \in S$. Therefore it is enough to show that the family $\{\sigma(\mu_d)\}_{\mu \in S}$ is tight. This, however, follows immediately from (III), because $\mu_d \leq \mu$ and thus $\sigma(\mu_d) \leq \sigma(\mu)$ for $\mu \in S$.

4. Subgroups and quotients of Hilbert spaces. Let $E$ be a (real) Hilbert space. The scalar product of vectors $x, y \in E$ is denoted by $(x, y)$ or just by $xy$. It follows from Lemma 2.2 that the formula

$$
\langle y, \xi(x) \rangle = \exp\{2\pi i xy\} \quad (x, y \in E)
$$

defines a topological isomorphism $\zeta : E \to E_0^1$. Next, let $K$ be a closed additive subgroup of $E$. Define

$$
Q = \{x \in E : (x, y) \in \mathbb{Z} \text{ for each } y \in K\}.
$$

It is clear that $Q$ is a weakly closed subgroup of $E$. Let $\psi : E \to E/K$ be the canonical mapping. If $x \in Q$, then $\xi(x)$ is a continuous character of $E$.

trivial on $K$; it induces a continuous character $\xi(x)$ of $E/K$ by the formula

$$
\langle y, \xi(x) \rangle = \exp\{2\pi i xy\} \quad (x \in Q, y \in E).
$$

It is clear that the mapping $\xi : Q \to (E/K)_0^1$ thus defined is an algebraic isomorphism. In fact, $\xi : Q \to (E/K)_0^1$ is a topological isomorphism ($\xi$ is continuous because $\psi$ is bounded; $\xi^{-1}$ is continuous because $\psi$ is bounded).

Let $\iota : Q \to E$ be the identity embedding. The composition $E \xrightarrow{\iota} E_0^1 \xrightarrow{\psi} Q_0^1$ is a continuous homomorphism trivial on $K$, therefore it induces a continuous homomorphism $\eta : E/K \to Q_0^1$ given by

$$
\langle y, \eta(\xi(x)) \rangle = \exp\{2\pi i xy\} \quad (x \in E, y \in Q).
$$

Observe that $\eta$ is injective if and only if $K$ is weakly closed in $E$. If $\mu \in \mathcal{P}(E/K)$, then $\nu = \eta(\mu) \in \mathcal{P}(Q_0^1)$ and $F^{-1}{\nu}(x) = F_{\mu}(\xi(x))$ for each $x \in Q$, which can be verified directly. In what follows, by the canonical homomorphisms $Q \to (E/K)_0^1$ and $E/K \to Q$ we mean the homomorphisms $\xi$ and $\eta$ defined above.

Now, suppose we are given two Hilbert spaces $E_1, E_2$ with weakly closed subgroups $K_1, K_2$, respectively, and a bounded linear operator $T : E_1 \to E_2$ with $T(K_1) \subset K_2$. Let $\psi_i : E_i \to E_i/K_i$ for $i = 1, 2$, be the canonical mappings. Then the formula $\tau \psi_1 = \psi_2 T$ defines a continuous homomorphism $\tau : E_1/K_1 \to E_2/K_2$, as shown in the following diagram:

$$
\begin{array}{ccc}
E_1 & \xrightarrow{T} & E_2 \\
\downarrow{\psi_1} & & \downarrow{\psi_2} \\
E_1/K_1 & \xrightarrow{T \psi_1} & E_2/K_2
\end{array}
$$

We say that the homomorphism $\tau$ is induced by $T$.

Under these assumptions, the following is true:

Lemma 4.1. (a) Let $\mu_1 \in \mathcal{P}(E_1/K_1)$ and let $\mu_2 = \pi(\mu_1) \in \mathcal{P}(E_2/K_2)$. Let $\varepsilon$ and $r$ be some fixed positive numbers and let $A = 16r^{-1}e^{-1/2}B_{E_2}$.

Suppose that

$$
\sum_{k=1}^{\infty} d_k(T)^{2/3} \leq 1,
$$

Re $F_{\mu_1}(\xi_1(x)) \geq 1 - \varepsilon$ for each $x \in Q_1 \cap r B_{E_1}$.

Then $\mu_2(\xi_2(A)) \geq 1 - 3\varepsilon$.

(b) If the operator $T$ is $2/3$-approximable, then $\pi$ is tightening.

Remark. Condition (3) may be replaced by $\sum_{k=1}^{\infty} d_k(T) \leq c$ where $c$ is some universal constant. Similarly, $2/3$-approximable operators in (b) may be replaced by $1$-approximable. The proofs of these assertions need certain additional preparations and will be given elsewhere.
Proof. (a) For $i = 1, 2$, define 
$$Q_i = \{ x \in E_i : (x, y) \in Z \text{ for each } y \in K_i \}$$
and let $\xi_i : Q_i \to (E_i/K_i)^w$ and $\eta_i : E_i/K_i \to Q_i^\circ$ be the corresponding canonical homomorphisms. Let $T^* : E_2 \to E_1$ be the adjoint operator given by

$$(x, T^*y) = (Tx, y) \quad (x \in E_1, y \in E_2).$$

Then $T^*(Q_2) \subset Q_1$ and we obtain the following commutative diagrams of continuous homomorphisms:

$$\begin{array}{ccc}
Q_2 & \xrightarrow{T^*} & Q_1 \\
\downarrow{\xi_2} & & \downarrow{\xi_1} \\
(E_2/K_2)^w & \xrightarrow{\pi^w} & (E_1/K_1)^w \\
\downarrow{\eta_2} & & \downarrow{\eta_1} \\
(Q_2^\circ) & \xrightarrow{(T^*)^\circ} & (Q_1^\circ) \\
\end{array}$$

Consider the measure $\nu_2 = \eta_2(\mu_2) \in \mathcal{P}(Q_2^\circ)$. We have

$$\mathcal{F}^{-1}\nu_2 = \mathcal{F}\mu_2 \circ \xi_2 = \mathcal{F}\mu_1 \circ \pi^w \circ \xi_2 = \mathcal{F}\mu_1 \circ \xi_1 \circ T^*\nu_2.$$ 

Let $p$ be the continuous pre-Hilbert seminorm on $E_2$ given by

$$p(x) = r^{-1}\|T^*x\| \quad (x \in E_2).$$

Then

$$d_k(B_{E_2}, B_p) = r^{-1}d_k(T^*) \quad (k = 1, 2, \ldots).$$

A standard argument based on the polar decomposition of $T^*$ shows that there exists another continuous pre-Hilbert seminorm $q$ on $E_2$ with $B_{E_2} \prec B_1 \prec B_q$ and such that

$$d_k(B_q, B_p) = \epsilon^{1/2}d_k(T^*)^{1/3},$$

$$d_k(B_{E_2}, B_q) = \epsilon^{-1/2}d_k(T^*)^{2/3} \quad (k = 1, 2, \ldots).$$

Since $d_k(T^*) = d_k(T)$, from (3) we obtain

(6) \quad $\sum_{k=1}^{\infty} d_k(B_q, B_p)^2 \leq \epsilon,$

(7) \quad $\sum_{k=1}^{\infty} d_k(B_{E_2}, B_q) \leq \epsilon^{-1/2}.$

If $x \in Q_2 \cap B_p$, then $T^*x \in Q_1 \cap rB_{E_1}$. Hence, by (5) and (4), we have

(8) \quad $\text{Re}\mathcal{F}^{-1}\nu_2(x) \geq 1 - \epsilon$ \quad for each $x \in Q_2 \cap B_p$.

Set

$$Z = \{ \chi \in Q_2^\circ : \text{Re} \chi(x) \geq 0 \quad \text{for each } x \in Q_2 \cap B_q \}.$$

Applying Lemma 2.3, (8) and (6), we obtain

(9) \quad $\nu_2(Q_2^\circ \setminus Z) \leq 3\epsilon.$

Now, take an arbitrary $\chi \in Z$. By Lemma 2.4, there exists a bounded linear functional $f$ on $E_2$ such that

(10) \quad $\exp(2\pi if(x)) = \chi(x)$ \quad for each $x \in Q_2,$

(11) \quad $\|f\| \leq 4 \sum_{k=1}^{\infty} d_k(B_{E_2}, B_q) \leq 16r^{-1}\epsilon^{-1/2}.$

Let $y \in E_2$ be given by $f(y) = (x, y)$ for $x \in E_2$. By (11) and (7), we have

$$\|y\| = \|f\| \leq 10 \sum_{k=1}^{\infty} d_k(B_{E_2}, B_q) \leq 16r^{-1}\epsilon^{-1/2}.$$  

Thus $y \in A$. Condition (10) means that $\eta_2\psi_2(y) = \chi$. Since $\chi \in Z$ was arbitrary, it follows that $Z \subset \eta_2\psi_2(A)$. Hence

$$\nu_2(Z) = \eta_2(\mu_2)(Z) = \mu_2(\eta_2^{-1}(Z)) \subset \mu_2(\eta_2^{-1}(\eta_2\psi_2(A))) = \mu_2(\psi_2(A))$$

because $\eta_2$ is injective ($K_2$ was assumed to be weakly closed in $E_2$). In view of (9), this completes the proof of (a).

(b) Let $S \subset \mathcal{P}(E_1/K_1)$ be a family of measures such that $\{\mathcal{F}\mu : \mu \in S\}$ is an equicontinuous family of functions on $(E_1/K_1)^w$. Suppose that $\sum_{k=1}^{\infty} d_k(T)^{2/3} < \infty$. Using the polar decomposition of $T$ etc., we can find a Hilbert space $E_2$ and bounded linear operators $T' : E_1 \to E_2'$ and

$$T'' : E_2' \to E_2$$

with $T = T''T'$ such that $\sum_{k=1}^{\infty} d_k(T')^{2/3} \leq 1$ and $T''$ is compact. Let $K_1$ be the weak closure of $T'(K_1)$ in $E_2$. It is not hard to see that $T''(K_2') \subset K_2$. We obtain the canonical commutative diagram

$$\begin{array}{ccc}
E_1 & \xrightarrow{T} & E_2' \\
\downarrow{\psi_1} & & \downarrow{\psi_2} \\
E_1/K_1 & \xrightarrow{\pi'} & E_2/K_2' \\
\end{array}$$

where $\pi''\pi' = \pi$.

Take an arbitrary $\epsilon > 0$. Since $\xi_1 : Q_1 \to (E_1/K_1)^w$ is a topological isomorphism, $\{\mathcal{F}\mu : \mu \in S\}$ is an equicontinuous family of functions on $Q_1$. So, there is some $\delta > 0$ such that $\text{Re}\mathcal{F}\mu(\xi_1(x)) \geq 1 - \epsilon$ for every $x \in Q_1 \cap rB_{E_1}$ and $\mu \in S$. Let $A = 16r^{-1}\epsilon^{-1/2}B_{E_2}$ and let $X = \psi_2(T''(A))$. Then $X$ is a compact subset of $E_2/K_2$. Now, take any $\mu \in S$. By (a), we have $\pi'(\mu)(\psi_2(A)) \geq 1 - 3\epsilon$. Hence

$$\pi''(\mu)(X) \supset \pi''(\mu)(\psi_2(T''(A))) = \pi''(\pi'(\mu)) = 1 - \epsilon.$$
By an EKD-group we mean a group of the form $(E/K) \times D$ where $D$ is a discrete abelian group and $K$ is a weakly closed subgroup of a Hilbert space $E$. We shall identify $E/K$ with the corresponding subgroup of $(E/K) \times D$.

**Lemma 4.2.** Let $G = (E/K) \times D$ be an EKD-group. Then the group $G'_{\infty}$ is Čech-complete.

**Proof.** By Lemma 2.1(b), the group $G'_{\infty}$ is topologically isomorphic to $(E/K)'_{\infty} \times D'_{\infty}$. Let $Q$ be defined as in (1). Since the canonical mapping $\xi : Q \to (E/K)'_{\infty}$ is a topological isomorphism, the group $(E/K)'_{\infty}$ is Čech-complete. Hence $G'_{\infty}$ is Čech-complete because $D'_{\infty}$ is compact. ■

Let $G_1 = (E_1/K_1) \times D_1$ and $G_2 = (E_2/K_2) \times D_2$ be EKD-groups and let $\pi : G_1 \to G_2$ be a continuous homomorphism with $\pi(E_1/K_1) \subset E_2/K_2$. We say that $\pi$ is $\alpha$-approximable, $0 < \alpha < \infty$, if the restriction $\pi_{E_1/K_1} : E_1/K_1 \to E_2/K_2$ is induced by an $\alpha$-approximable operator $T : E_1 \to E_2$ (see diagram (2)).

**Lemma 4.3.** Every 2/3-approximable homomorphism of EKD-groups has the property (PL).

**Proof.** Let $\pi : (E_1/K_1) \times D_1 \to (E_2/K_2) \times D_2$ be a 2/3-approximable homomorphism of EKD-groups. Then the restriction $\sigma : E_2/K_2 \to E_2/K_2$ of $\pi$ to $E_2/K_2$ is induced by a 2/3-approximable operator $T : E_1 \to E_2$. Lemma 4.1(b) says that $\sigma$ is tightening. Hence $\pi$ is tightening according to Lemma 3.8. It is now enough to apply Lemmas 3.7 and 4.2. ■

5. Nuclear groups. Nuclear groups were defined in [B1, (7.1)] (an equivalent definition is given by Lemma 5.1 below). They form a class of a.t. groups with the following properties:

(1) every LCA group is nuclear;

(2) a topological vector space $G$ is nuclear if and only if $G$ is a nuclear locally convex space;

(3) every subgroup of a nuclear group is nuclear;

(4) every Hausdorff quotient group of a nuclear group is nuclear;

(5) the product of an arbitrary family of nuclear groups is nuclear;

(6) the direct sum of a countable family of nuclear groups is nuclear.

The proofs of these assertions are given in [B1, Sect. 7]. Moreover, if $G$ is a Čech-complete nuclear group, then the group $G'_{\infty}$ is nuclear [A1, (20.36)].

Let $F$ be a vector space and $\tau$ a topology on $F$ such that $F$ is an additive topological group. We say that $F$ is a locally convex vector group if it is separated and has a base at zero consisting of symmetric convex sets. A locally convex vector group $F$ is called a nuclear vector group if to each symmetric convex neighbourhood $U$ of zero in $F$ there corresponds another symmetric convex neighbourhood $V$ with $d_0(V, U) \leq k^{-1}$ for every $k$.

**Lemma 5.1.** An a.t. group $G$ is nuclear if and only if it is topologically isomorphic to a group of the form $H/K$, where $H$ is a subgroup of a nuclear vector group $F$ and $K$ is a closed subgroup of $H$.

This follows from [B1, (9.4) and (9.6)].

**Lemma 5.2.** Let $K$ be a closed subgroup of a nuclear vector group $F$. Then the quotient group $F/K$ is topologically isomorphic to a dense subgroup of the limit of an inverse system $(G_i, \pi_{ij}, I)$ of EKD-groups with the following property: to each $i \in I$ there corresponds some $j \geq i$ such that the homomorphism $\pi_{ij} : G_j \to G_i$ is 1/2-approximable.

This is a reformulation of Theorem 3.4 of Galindo [G]. The number 1/2 may be replaced here by an arbitrary $\alpha \in (0, \infty)$.

**Theorem 5.3.** Every nuclear group has the property (PL).

**Proof.** Let $G$ be a nuclear group. By Lemma 5.1, there exist a nuclear vector group $F$ and a closed subgroup $K$ of $F$ such that $G$ is topologically isomorphic to a subgroup of $F/K$. By Lemma 3.1, we may assume that $G = F/K$. That $F/K$ has the property (PL) follows from Lemmas 5.2, 3.1, 3.2 and 4.3. ■

References


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