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Volume ratios in L_p-spaces

by

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Abstract. There exists an absolute constant c_0 such that for any n-dimensional Banach space E there exists a k-dimensional subspace $F \subset E$ with $k \le n/2$ such that

$$\inf_{\text{ellipsoid } \mathcal{E} \subset B_E} \left(\frac{\operatorname{vol}(B_E)}{\operatorname{vol}(\mathcal{E})} \right)^{1/n} \le c_0 \inf_{\text{zonoid } Z \subset B_F} \left(\frac{\operatorname{vol}(B_F)}{\operatorname{vol}(Z)} \right)^{1/k}.$$

The concept of volume ratio with respect to ℓ_p-spaces is used to prove the following distance estimate for $2 \le q \le p < \infty$:

$$\sup_{F \subset \ell_p,\, \dim F = n} \ \inf_{G \subset L_q,\, \dim G = n} d(F, G) \sim_{c_{pq}} n^{(q/2)(1/q - 1/p)}.$$

Introduction and preliminary notations. The classical volume ratio of an n-dimensional Banach space with respect to the ellipsoid of maximal volume contained in the unit ball is an important geometric quantity. It is intensively studied in the literature, for example in the books of G. Pisier [PSc] and N. Tomczak-Jaegermann [TJ]. More recently, the external volume ratio with respect to linear images of the cube was investigated in [Ba, GE1, GMP, PES, PSc] and the volume ratio with respect to zonoids in [Ba, GMP, GJN, JU2]. As an example, let us note that the volume ratio with respect to zonoids is related to the local unconditional structure of the underlying Banach space.

Our aim in this paper and the closely related previous paper [GJ] is to develop a useful theory of volume ratios with respect to ℓ_p-spaces including those previous concepts. For this purpose, as in [GJ], we define for an n-dimensional (quasi-)normed space E with unit ball B_E and a Banach space Z, the volume ratios

$$\operatorname{vr}(E, Z) := \inf \left\{ \left(\frac{\operatorname{vol}(B_E)}{\operatorname{vol}(T(B_Z))} \right)^{1/n} \,\middle|\, T(B_Z) \subset B_E \right\},$$

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$$\mathrm{vr}(E,S(Z)) := \inf\left\{\left(\frac{\mathrm{vol}(B_E)}{\mathrm{vol}(T(B_F))}\right)^{1/n}\,\middle|\, F\subset Z,\ \dim F=n,\ T(B_F)\subset B_E\right\},$$
$$\mathrm{vr}(E,S_p) := \mathrm{vr}(E,S(L_p)).$$

Note that $\mathrm{vr}(E)=\mathrm{vr}(E,\ell_2)$ is the classical volume ratio with respect to the maximal ellipsoid contained in B_E. Some standard inequalities between the volume ratios follow from the theory of p-stable processes and the type 2, cotype 2 theory, such as:

(a) If $2\le p\le r\le\infty$, then $\mathrm{vr}(E,\ell_r)\le \mathrm{vr}(E,\ell_p)$ because $\ell_{p'}^n$ is $(1+\varepsilon)$-isomorphic to a subspace of $\ell_{r'}^{m(\varepsilon)}$.

(b) If $1<p\le 2$, then the fact that $\ell_{p'}$ has type 2 implies (see [PSc])

$$\frac{1}{c_0\sqrt{p'}}\,\mathrm{vr}(E,\ell_2)\le \mathrm{vr}(E,\ell_p)\le \mathrm{vr}(E,\ell_2).$$

(c) Clearly, for $p=1$,

$$\mathrm{vr}(E,\ell_1)=1.$$

(d) If $1\le p\le 2$, then $c_0\,\mathrm{vr}(E,\ell_2)\le \mathrm{vr}(E,S_p)\le \mathrm{vr}(E,\ell_2)$ since cotype 2 spaces such as ℓ_p, $1\le p\le 2$, have bounded volume ratios with respect to ellipsoids [MiP].

(e) Although for $2<q<p<\infty$ the spaces ℓ_q^n are not uniformly isomorphic to subspaces of ℓ_p, we still have

$$\mathrm{vr}(E,S_p)\le c_0\sqrt{p}\,\mathrm{vr}(E,S_q).$$

We intend to prove this inequality and the following theorem based on an operator ideal approach to volume ratios.

THEOREM 0.1. *Let $1\le p,q<\infty$ and $n\in\mathbb{N}$. Then*

$$\sup_{F\subset\ell_p,\,\dim F=n}\mathrm{vr}(F,S_q)\sim_{c_{pq}}\begin{cases} n^{1/2-1/p} & \text{if } 1\le q\le 2\le p<\infty,\\ n^{(q/2)(1/q-1/p)} & \text{if } 2\le q\le p<\infty,\\ 1 & \text{else,}\end{cases}$$

where c_{pq} is an absolute constant depending only on p,q. In particular, for $2\le q\le p<\infty$,

$$\sup_{F\subset\ell_p,\,\dim F=n}\ \inf_{G\subset L_q,\,\dim G=n} d(F,G)\sim_{c_p} n^{(q/2)(1/q-1/p)}.$$

Here and in the following $a\sim_c b$ means that $c_1^{-1}a\le b\le c_2 a$ with $c_1c_2\le c$. Moreover, $d(E,F)$ is the usual Banach–Mazur distance:

$$d(E,F)=\inf\{\|T:E\to F\|\cdot\|T^{-1}:F\to E\|\},$$

where the infimum is taken over all linear isomorphisms. The last estimate answers a question of Pietsch, which was kindly pointed out to us by A. Hinrichs in the interesting case $2\le q\le p<\infty$. In fact the distance estimate is obtained for random n-dimensional subspaces of ℓ_q^m, where $m\sim n^{q/2}$. Note that for p and q interchanged, the corresponding expression is not known. We deduce further information for the smallest relative projection constant of an n-dimensional subspace of ℓ_q^m.

In the second part of the introduction, we want to concentrate on the more classical volume ratio $\mathrm{vr}(E,\ell_2)$ and the zonoid ratio $\mathrm{vr}(E,\ell_\infty)$. Recall that a *zonotope* is a sum of segments, and more generally, a *zonoid* Z is the linear image $u(B_{\ell_\infty})$ of the unit ball B_{ℓ_∞} of ℓ_∞. Clearly, there are far more zonoids than ellipsoids. One application of the technique we develop here is the following geometric result:

THEOREM 0.2. *Let E be an n-dimensional Banach space. Then there exists a subspace $F\subset E$ of dimension $k\le n/2$ such that*

$$\inf_{\text{ellipsoid } \mathcal{E}\subset B_E}\left(\frac{\mathrm{vol}(B_E)}{\mathrm{vol}(\mathcal{E})}\right)^{1/n}\le c_0\inf_{\text{zonoid } Z\subset B_F}\left(\frac{\mathrm{vol}(B_F)}{\mathrm{vol}(Z)}\right)^{1/k},$$

briefly

$$\mathrm{vr}(E,\ell_2)\le c_0\sup_{F\subset E,\,\dim F\le n/2}\mathrm{vr}(F,\ell_\infty).$$

Here c_0 is an absolute constant.

The theorem has the following geometric interpretation. If the volume ratio with respect to ellipsoids is big, one can find a section, at most half-dimensional, whose volume ratio with respect to the larger class consisting of zonoids is still big. The typical example is ℓ_∞^n, whose unit ball is a zonoid, but (for example by Theorem 0.2) it contains proportional subspaces of distance $\sqrt{n}$ to all zonoids. This observation is initially due to K. Ball [Ba], who also discovered a connection between the theory of 1-summing operators and the maximal volume of inscribed zonoids. Our general result is motivated by corresponding results for ℓ_p^n, $2\le p\le\infty$. There, we can prove the existence of an $n/2$-dimensional subspace $F\subset\ell_p^n$ such that

$$c_0 n^{1/2-1/p}\le \mathrm{vr}(F,\ell_\infty)\le \mathrm{vr}(F,\ell_2^n)\le n^{1/2-1/p}.$$

We do not know whether it suffices to consider proportional subspaces in general.

There is a close relation between volume ratios and other parameters considered in the local theory of Banach spaces, e.g., if $\dim X=n$,

$$c_0\,\mathrm{vr}(X,\ell_\infty)\,\mathrm{vr}(X^*,\ell_\infty)\le \mathrm{gl}_2(X)\le \mathrm{gl}(X)\le\chi_{\mathrm{u}}(X)\le\chi(X)\le d(X,\ell_2^n).$$

Here $\mathrm{gl}(X)$ is the nowadays classical Gordon–Lewis constant and $\mathrm{gl}_2(X)$ the Gordon–Lewis constant for operators on X with range in a Hilbert space (see the next section). $\chi_{\mathrm{u}}(X)$ and $\chi(X)$ are the smallest factorization norms through a Banach lattice and through an n-dimensional Banach space with 1-unconditional basis, respectively. Since these parameters are not further used in this paper, we skip the formal definition (see for example [GMP]). A generalization of this inequality is proved in [GJ, Corollary 3.12].

In the range of the parameters $\mathrm{vr}(E,\ell_p)$ the cases $p=\infty$ and $p=2$ are extremal. However, in view of uniform estimates for all finite-dimensional subspace of an arbitrary Banach space they turn out to be equivalent in the following sense.

THEOREM 0.3. *Let $\alpha>0$ and X be an infinite-dimensional Banach space. Then the following assertions are equivalent.*

(i) *There exists a constant $c_1>0$ such that for all $n\in\mathbb{N}$ and all n-dimensional subspaces $E\subset X$,*
$$\mathrm{vr}(E,\ell_2)\le c_1 n^\alpha.$$

(ii) *There exists a constant $c_2>0$ such that for all $n\in\mathbb{N}$ and all n-dimensional subspaces $E\subset X$,*
$$\mathrm{vr}(E,\ell_\infty)\le c_2 n^\alpha.$$

(iii) *There exists $0<\delta\le 1$ and a constant c_3 such that for all $n\in\mathbb{N}$ and n-dimensional subspaces $E\subset X$, there exists a $[\delta n]$-dimensional subspace $E_\delta\subset E$ such that*
$$\mathrm{vr}(E_\delta,\ell_\infty)\le c_3 n^\alpha.$$

Using these ideas, we get some new insight into the geometry of subspaces of ℓ_p^L for $2\le p<\infty$, which complements the results of Bourgain [Bo], Bourgain and Tzafriri [BT] and Gluskin, Tomczak-Jaegermann and Tzafriri [GTJT]. They were interested in finding $(1+\varepsilon)$-copies of ℓ_p^k, $k=k(\varepsilon,p,L,m)$, $2\le p\le\infty$, in every m-dimensional subspace E of ℓ_p^L. Based on volume ratio estimates, we can prove that there are subspaces $E\subset\ell_p^L$ which do not contain subspaces F of dimension k which are either C-isomorphic to some ℓ_r^k $(1\le r\le\infty)$ or C-complemented in ℓ_p^L.

THEOREM 0.4. *Let $2\le p<\infty$ and $1\le m<L$. Then there exists an m-dimensional subspace $E_m\subset\ell_p^L$ such that for any k-dimensional subspace F of E_m,*
$$k\le C(p,k,m,L)\sqrt{m}\,L^{1/p}\,\mathrm{vr}(F,\ell_\infty),$$
where
$$C(p,k,m,L)\le c_0\min\{\sqrt{p},(L/k)^{1/2-1/p}\}\sqrt{\frac{L}{L-m}\Big(1+\ln\frac{L}{L-m}\Big)}.$$
For $p=\infty$, there exists an m-dimensional subspace $E_m\subset\ell_\infty^L$ such that for all k-dimensional subspaces $F\subset E_m$,
$$k\le c_0\sqrt{m}\sqrt{1+\ln\frac{L}{k}}\sqrt{\frac{L}{L-m}\Big(1+\ln\frac{L}{L-m}\Big)}\,\mathrm{vr}(F,\ell_\infty).$$

Moreover, for all $1\le r\le\infty$, $2\le p\le\infty$ and $F\subset\ell_p^L$,
$$\mathrm{vr}(F,\ell_\infty)\le c_0\min\{d(F,\ell_r^k),\lambda_p(F),\mathrm{gl}_2(F)\}.$$

1. Volume ratios for subspaces and quotients of L_p. In this section, we investigate volume ratios and distance estimates with respect to subspaces and quotient spaces of L_p. An n-dimensional section of the unit ball B_{L_p} in L_p can be represented by a measure μ on $\mathbb{R}^n$ such that
$$S_p(\mu):=\Big\{x\in\mathbb{R}^n\ \Big|\ \|x\|_p:=\Big(\int_{\mathbb{R}^n}|\langle x,y\rangle|^p\,d\mu(y)\Big)^{1/p}\le 1\Big\}.$$
An n-dimensional quotient of the unit ball B_{L_p} is given by
$$Q_p(\mu):=\Big\{\int_{\mathbb{R}^n}f(y)y\,d\mu(y)\ \Big|\ \Big(\int_{\mathbb{R}^n}|f(y)|^p\,d\mu(y)\Big)^{1/p}\le 1\Big\}.$$

Note that $(Q_p(\mu))^\circ=S_{p'}(\mu)$ where $1/p+1/p'=1$. A systematic approach leads to the following four minimax quantities:
$$V(p,q,S,S):=\sup_\mu\inf_\nu\Big(\frac{\mathrm{vol}(S_p(\mu))}{\mathrm{vol}(S_q(\nu))}\Big)^{1/n}=\sup_{F\subset L_p,\ \dim F=n}\mathrm{vr}(F,S_q),$$
$$V(p,q,Q,S):=\sup_\mu\inf_\nu\Big(\frac{\mathrm{vol}(Q_p(\mu))}{\mathrm{vol}(S_q(\nu))}\Big)^{1/n}=\sup_{F\text{ quotient of }L_p,\ \dim F=n}\mathrm{vr}(F,S_q),$$
$$V(p,q,S,Q):=\sup_\mu\inf_\nu\Big(\frac{\mathrm{vol}(S_p(\mu))}{\mathrm{vol}(Q_q(\nu))}\Big)^{1/n}=\sup_{F\subset L_p,\ \dim F=n}\mathrm{vr}(F,\ell_q),$$
$$V(p,q,Q,Q):=\sup_\mu\inf_\nu\Big(\frac{\mathrm{vol}(Q_p(\mu))}{\mathrm{vol}(Q_q(\nu))}\Big)^{1/n}=\sup_{F\text{ quotient of }L_p,\ \dim F=n}\mathrm{vr}(F,\ell_q)$$
subject to the conditions $S_q(\nu)\subset S_p(\mu)$, $S_q(\nu)\subset Q_p(\mu)$, $Q_q(\nu)\subset S_p(\mu)$ and $Q_q(\mu)\subset Q_p(\nu)$, respectively. Santaló's inequality and its inverse due to Bourgain and Milman [BM] imply
$$\mathrm{vol}(S_p(\mu))^{1/n}\,\mathrm{vol}(Q_{p'}(\mu))^{1/n}\sim_{c_0}\frac{1}{n}. \tag{1.1}$$
Therefore, these four expressions can be reformulated in terms of external volume ratios defined in [GJ]. In the next section, we will see that for $q>1$,
$$V(p,q,S,Q)\sim_{c_q}V(p,2,S,Q)\sim_{c_0}\begin{cases}1 & \text{if } 1\le p\le 2,\\ n^{1/2-1/p} & \text{if } 2\le p\le\infty.\end{cases}$$

Using the methods from [GJ] which employ the theory of p-summing and p-nuclear operators, we are able to determine the order of growth of the constants $V(p,q,\cdot,\cdot)$ with respect to the dimension n.

Let us recall the classical definitions of absolutely p-summing and p-nuclear operators. An operator $T : E \to F$ is said to be *p-summing* ($T \in \Pi_p(E,F)$) if there is a constant $C > 0$ such that for all vectors $x_1, \ldots, x_n \in E$,

$$\Big(\sum_{k=1}^{n} \|Tx_k\|_F^p\Big)^{1/p} \le C \sup_{\|x^*\|_{E^*}\le 1} \Big(\sum_{k=1}^{n} |\langle x^*, x_k\rangle|^p\Big)^{1/p}.$$

Then $\pi_p(T) := \inf C$ where the infimum ranges over all constants C satisfying the condition above. The *q-nuclear norm* of a finite rank operator $T : E \to F$ is defined by

$$\nu_q(T) := \inf \|V : \ell_q^m \to F\| \cdot \|D_\tau : \ell_\infty^m \to \ell_q^m\| \cdot \|R : E \to \ell_\infty^m\|,$$

where the infimum is taken over all m, V, R and diagonal operators D_τ such that $T = VD_\tau R$. For $1 \le p \le \infty$ and a linear map $T : X \to Y$, the *p-factorable norm* is defined by $\gamma_p(T) = \inf\{\|R\| \cdot \|S\|\}$ where the infimum is taken over all $S : X \to L_p$ and $R : L_p \to Y^{**}$ such that $\iota_Y T = RS$ and $\iota_Y : Y \to Y^{**}$ is the canonical isometric embedding of Y into its bidual. The gl-*norm* $\mathrm{gl}(T)$ is defined by

$$\mathrm{gl}(T) := \sup \frac{\gamma_1(ST)}{\pi_1(S)}$$

where the supremum is taken over all non-zero $S : Y \to Z$. The *restricted* gl_2-*norm* is defined by restricting to operators with values in ℓ_2,

$$\mathrm{gl}_2(T) := \sup_{S:Y\to\ell_2} \frac{\gamma_1(ST)}{\pi_1(S)}.$$

As usual, we set $\mathrm{gl}(X) = \mathrm{gl}(\mathrm{id}_X)$ and $\mathrm{gl}_2(X) = \mathrm{gl}_2(\mathrm{id}_X)$.

The notion of *volume numbers* will be very convenient for our purpose. Given quasi-normed spaces X, Y and an operator $T : X \to Y$, we define for $n \in \mathbb{N}$ the nth *volume number* by

$$v_n(T) := \sup\Big\{\Big(\frac{\mathrm{vol}(T(B_E))}{\mathrm{vol}(B_F)}\Big)^{1/n} \,\Big|\, E \subset X, T(E) \subset F \subset Y, \dim E = \dim F = n\Big\}.$$

If T is of rank less than n, we set $v_n(T) = 0$. Similar notions were discussed by several authors: Dudley [D], Pisier [PSc], Pajor and Tomczak-Jaegermann [TJ, PT], Mascioni [MAS], Junge [JU1, JU2] and Gordon and Junge [GJ]. Note that for $\dim E = n$ and $T : E \to Y$ of rank n,

$$v_n(T) = \Big(\frac{\mathrm{vol}(T(B_E))}{\mathrm{vol}(B_Y \cap T(E))}\Big)^{1/n}.$$

It is quite useful to compare the volume numbers with Mascioni's [MAS] notion of volume ratio numbers. For an operator $T : X \to Y$ the nth *volume ratio number* is defined by

$$\mathrm{vr}_n(T) := \sup \Big(\frac{\mathrm{vol}(q_S(T(B_X)))}{\mathrm{vol}(B_{Y/S})}\Big)^{1/n}$$

where the supremum is taken over all subspaces $S \subset Y$ such that $\dim(Y/S) = n$, and $q_S : Y \to Y/S$ denotes the corresponding quotient map. We will need several facts. For an operator T on a Hilbert spaces H, the volume numbers can be calculated using the singular values $(a_j(T))_{j\in\mathbb{N}}$ of $\sqrt{T^*T}$ in decreasing order, i.e. the approximation numbers of T (see [GJ]):

$$\mathrm{vr}_k(T) = v_k(T) = \Big(\prod_{j=1}^{k} a_j(T)\Big)^{1/k}. \tag{1.2}$$

If $T : X \to \ell_2$, then the volume ratio numbers are decreasing due to the Aleksandrov–Fenchel inequalities [PSc]. Moreover, using an appropriate orthogonal projection, one has

$$v_n(T) \le \mathrm{vr}_n(T). \tag{1.3}$$

In K-convex Banach spaces there are still "good" projections on finite-dimensional subspaces [PSc, JU1]. A Banach space X is said to be *K-convex* [PSc] if the projection

$$P := \sum_k g_k \otimes g_k : L_2(\Gamma, \mathbb{P}) \to L_2(\Gamma, \mathbb{P})$$

extends to a continuous operator on $L_2(\Gamma, \mathbb{P}; X)$. Here $\{g_k\}$ are orthonormal gaussian variables. The *K-convexity constant* is the norm of this extension:

$$K(X) := \|P \otimes \mathrm{id}_X\|.$$

Clearly, K-convexity is a self-dual property. Pisier [PS1] proved that K-convex spaces are exactly those not containing ℓ_1^n's uniformly. Typical examples are the L_p-spaces which satisfy

$$K(L_p) \le \sqrt{\max\{p, p'\}}. \tag{1.4}$$

Geometrically, we have the following characterization of this notion (see [PSc, JU1]). For any n-dimensional subspace $E \subset X$ there exists a projection $P : X \to E$ such that

$$\Big(\frac{\mathrm{vol}(P(B_X))}{\mathrm{vol}(B_E)}\Big)^{1/n} \le c_0 K(X). \tag{1.5}$$

We will use the following fact probably known to experts. For completeness, we add a short proof.

FACT 1.1. *Let $T : X \to Y$ be an operator and $n \in \mathbb{N}$.*

(i) $v_n(T) \sim_{c_0} \mathrm{vr}_n(T^*)$ *and* $v_n(T^*) \sim_{c_0} \mathrm{vr}_n(T)$.

(ii) *If X is K-convex, then* $\mathrm{vr}_n(T) \le c_0 K(X) v_n(T)$.
(iii) *If Y is K-convex, then* $v_n(T) \le c_0 K(Y)\,\mathrm{vr}_n(T)$.

Proof. (i) The inequalities

$$v_n(T) \le c_0\,\mathrm{vr}_n(T^*) \quad \text{and} \quad \mathrm{vr}_n(T) \le c_0 v_n(T^*)$$

follow from Santaló's inequality and its inverse (see [PSc]), together with the usual duality between subspaces and quotients. Combining these two estimates and the principle of local reflexivity imply

$$v_n(T) \le c_0\,\mathrm{vr}_n(T^*) \le c_0^2 v_n(T^{**}) = c_0^2 v_n(T).$$

The other equivalence in (i) is proved similarly.

(iii) Let $T(E) = F \subset Y$ with $\dim E = \dim F = n$. Since Y is K-convex, there exists a projection $P : Y \to F$ such that $\mathrm{vr}_n(P) \le c_0 K(Y)$. From the submultiplicativity of the volume ratio numbers vr_n [MAS], we obtain

$$\left(\frac{\mathrm{vol}(T(B_E))}{\mathrm{vol}(B_F)}\right)^{1/n} \le \left(\frac{\mathrm{vol}(PT(B_X))}{\mathrm{vol}(B_F)}\right)^{1/n} = \mathrm{vr}_n(PT) \le \mathrm{vr}_n(P)\,\mathrm{vr}_n(T) \le c_0 K(Y)\,\mathrm{vr}_n(T).$$

(ii) follows from (iii) by duality. ■

In the sequel, we will use classical volume estimates for operators on Hilbert spaces. Volume estimates are usually obtained using the *entropy numbers* $e_n(T)$ defined for an operator $T : X \to Y$ between Banach spaces X and Y as follows:

$$e_n(T) := \inf\Big\{\varepsilon \,\Big|\, \exists y_1, \ldots, y_{2^{n-1}} \in Y \text{ such that } T(B_X) \subset \bigcup_{j=1}^{2^{n-1}} y_j + \varepsilon B_Y\Big\}.$$

Note moreover the obvious inequality

$$v_n(T) \le 2 \cdot 2^{k/n} e_k(T). \tag{1.6}$$

One crucial tool for entropy estimates is the rad-norm or the ℓ-norm of a map $u : \ell_2^n \to X$ (see [CP]):

$$\mathrm{rad}(u) := \Big(\int_\Omega \Big\|\sum_{i=1}^n r_i u(e_i)\Big\|_X^2 dP\Big)^{1/2},$$

$$\ell(u) := \Big(\int_\Omega \Big\|\sum_{i=1}^n g_i u(e_i)\Big\|_X^2 dP\Big)^{1/2},$$

where the r_i are independent Rademacher variables with $P(r_k = \pm 1) = 1/2$, and the g_i are independent normalized gaussian variables. A consequence of Carl–Pajor's and Pajor-Tomczak's inequalities (see [PSc]) is that for all $k \in \mathbb{N}$,

$$\sqrt{k}\max\{e_k(u), e_k(u^*)\} \le c_1\sqrt{1+\ln(n/k)}\,\mathrm{rad}(u), \tag{1.7}$$

$$\sqrt{k}\max\{e_k(u), e_k(u^*)\} \le c_1 \ell(u). \tag{1.8}$$

In the sequel, we will also use special cases of this inequality for operators between L_p and ℓ_2. Except for the constant, they can be deduced from classical volume estimates of Dudley and Urysohn. For the constant in the following lemma, we refer to the proof of Theorem 0.2 [GJ, pp. 21–24].

LEMMA 1.2. *Let $1 \le p < \infty$, $f_1, \ldots, f_n \in L_p(\mu)$, $V = \sum_{i=1}^n f_i \otimes e_i : L_{p'} \to \ell_2^n$ be the associated linear operator and $K = V(B_{L_{p'}})$. Then*

$$\sqrt{n}\max\left\{\left(\frac{\mathrm{vol}(K)}{\mathrm{vol}(B_2^n)}\right)^{1/n}, \left(\frac{\mathrm{vol}(B_2^n)}{\mathrm{vol}(K^\circ)}\right)^{1/n}\right\}$$
$$\le \min\{\max(\sqrt{p-1}, 1), \sqrt{n}\}\Big(\int_\Omega \Big(\sum_{i=1}^n |f_i|^2\Big)^{p/2} d\mu\Big)^{1/p}$$
$$\le \min\{\max(\sqrt{p-1}, 1), \sqrt{n}\}\pi_p(V).$$

To simplify notation let us introduce

$$\mathrm{vr}(F, QS_p) := \inf_{Q \text{ quotient of } L_p} \mathrm{vr}(F, S(Q)).$$

We will need the following crucial observation derived from [GJ, Theorem 3.10]:

LEMMA 1.3. *Let $1 \le p \le \infty$, $1/p + 1/p' = 1$, X be a Banach space and $n \in \mathbb{N}$. Then*

$$\frac{1}{c_0\sqrt{p}} \sup_{F \subset X,\, \dim F = n} \mathrm{vr}(F, S_p) \le \sup_{v : X \to \ell_2} \frac{\sqrt{n}\,v_n(v)}{\nu_{p'}(v)} \le c_0 \sup_{F \subset X,\, \dim F = n} \mathrm{vr}(F, S_p)$$

and

$$\sup_{v : X \to \ell_2} \frac{\sqrt{n}\,v_n(v)}{\nu_{p'}(v)} \le c_0\sqrt{p'} \sup_{F \subset X,\, \dim F = n} \mathrm{vr}(F, QS_p).$$

In particular, for any finite-dimensional subspace $F \subset X$,

$$\mathrm{vr}(F, QS_p) \le \mathrm{vr}(F, S_p) \le c_0\sqrt{\max\{p, p'\}}\,\mathrm{vr}(F, QS_p).$$

Proof. Let $F \subset X$ be of dimension n. According to [GJ, Theorem 3.10], there is a map $v : F \to \ell_2^n$ such that $\nu_{p'}(v) < 1$ and

$$\mathrm{vr}(F, S_p) \le c_0\sqrt{p}\sqrt{n}\,v_n(v : F \to \ell_2^n).$$

By the definition of the p'-nuclear norm, there exists an extension $V : X \to \ell_2^n \subset \ell_2$ of v with $\nu_p(V) < 1$. Since $V|_F = v$, we get $v_n(v) \le v_n(V)$ and obtain the left hand inequality. For the right hand inequality, let $v : X \to \ell_2$ and $F \subset X$ be of dimension n. Then $v(F)$ is contained in an n-dimensional subspace H of ℓ_2. Therefore [GJ, Theorem 3.10] implies

$$\sqrt{n}\, v_n(v|_F) \le c_0 \operatorname{vr}(F, S_p)\nu_{p'}(v|_F) \le \operatorname{vr}(F, S_p)\nu_{p'}(v).$$

Taking the supremum over all F implies the right hand inequality.

For the last inequality, let Q be a quotient of L_p, $q : L_p \to Q$ a quotient map, $G \subset Q$ an n-dimensional subspace and $\alpha : G \to F$ be a linear isomorphism which is a contraction. Then $w = P_H v|_F \alpha$ satisfies $\nu_{p'}(w) \le \nu_{p'}(v)$ and admits an extension $W : Q \to H$ such that $W|_G = w$ and $\nu_{p'}(W) \le (1+\varepsilon)\nu_{p'}(v)$. In particular, $B_G \subset B_Q$ implies, by Lemma 1.2,

$$\sqrt{n}\, v_n(w) \le \sqrt{n}\operatorname{vr}_n(W) = \sqrt{n}\operatorname{vr}_n(Wq) \le \max\{\sqrt{p'-1}, 1\}\pi_{p'}(Wq) \le (1+\varepsilon)\sqrt{p'}\,\nu_{p'}(v).$$

Since $\varepsilon > 0$ is arbitrary, we obtain the last assertion. ∎

REMARK 1.4. By the well known inequality $\nu_{p'}(v) \le \nu_{q'}(v)$ for $p \le q$, the lemma immediately implies that for any n-dimensional Banach space F,

$$\operatorname{vr}(F, S_p) \le c_0\sqrt{p}\operatorname{vr}(F, S_q).$$

Moreover, the second assertion implies, for any n-dimensional quotient Q of ℓ_p, the existence of a linear image $T(B_E) \subset B_Q$, $E \subset \ell_p$, such that

$$\left(\frac{\operatorname{vol}(B_Q)}{\operatorname{vol}(T(B_E))}\right)^{1/n} \le c_0\sqrt{p'}.$$

Clearly, this inequality cannot hold with a constant for $p = 1$, which can be used to show that the order $\sqrt{p'}$ is optimal.

For the upper estimate in Theorem 0.1, we use an inequality of Carl.

LEMMA 1.5. *Let $2 \le q, p \le \infty$ and $\alpha = \min\{q/(2p), 1/2\}$. Then any q'-nuclear operator $v : L_p \to \ell_2$ satisfies*

$$\sup_n n^\alpha \operatorname{vr}_n(v) \le \nu_{q'}(v).$$

Proof. It is sufficient to consider $q \le p$. Then the remaining case $p \le q$ follows easily from $\nu_{p'}(v) \le \nu_{q'}(v)$. Now, let H be an n-dimensional subspace of ℓ_2, P_H its orthogonal projection and $(e_i)_{i=1}^n$ an orthogonal basis of H. Using Carl's [Ca] inequality

$$\nu_{p'}(P_H v) \le n^{(q/2)(1/p'-1/q')}\nu_{q'}(P_H v), \tag{1.9}$$

we deduce from Lemma 1.2 that

$$n^{1/2}\operatorname{vr}_n(P_H v) \le \sqrt{n}\left(\frac{\operatorname{vol}(P_H v(B_{L_p}))}{\operatorname{vol}(B_2^n)}\right)^{1/n} \le \pi_{p'}(P_H v) \le \nu_{p'}(P_H v) \le n^{(q/2)(1/p'-1/q')}\nu_{q'}(P_H v) \le n^{1/2-q/(2p)}\nu_{q'}(v). \ ∎$$

REMARK 1.6. A further consequence of Carl's inequality (1.9) is the distance estimate for an n-dimensional subspace $F \subset L_p$ if $2 \le q \le p \le \infty$:

$$\inf_{G \subset \ell_q,\, \dim G = n} d(F, G) \le n^{(q/2)(1/q-1/p)}. \tag{1.10}$$

Here $d(F, G) = \inf\{\|u\| \cdot \|u^{-1}\|\}$ is the Banach–Mazur distance between Banach spaces. This was obtained in collaboration with E. Hinrichs who communicated to us Pietsch's question to find the right order of growth of

$$\sup_{F \subset \ell_p,\, \dim F = n}\ \inf_{G \subset \ell_q,\, \dim G = n} d(F, G).$$

Combined with the trivial estimate

$$d(F, G) \ge \operatorname{vr}(F, S_p),$$

the following proposition will show that the upper estimate (1.10) is essentially best possible. We add a proof of the distance estimate (1.10) for completeness.

Proof of (1.10). Let $F \subset L_p$ be n-dimensional. By Kwapień's factorization theorem [PIE] the minimal distance of F to subspaces of ℓ_q equals the best constant C such that

$$\pi_q(u) \le C\pi_q(u^*)$$

for all operators $u : X \to F$. Moreover, X can be assumed to be n-dimensional since F is n-dimensional. By trace duality applied to Carl's inequality (1.9) combined with Kwapień's estimate for the p-summing norm [PIE] with values in F we obtain

$$\pi_q(u) \le n^{(q/2)(1/q-1/p)}\pi_p(u) \le n^{(q/2)(1/q-1/p)}\pi_p(u^*) \le n^{(q/2)(1/q-1/p)}\pi_q(u^*). \ ∎$$

The proof of Theorem 0.1 relies on the following rather precise estimates.

PROPOSITION 1.7. *Let $2 \le q \le p < \infty$, $n \in \mathbb{N}$ and $m \ge n$. Then a random n-dimensional subspace $F^p \subset \ell_p^m$ satisfies*

$$\mathbb{E}\operatorname{vr}(F^p, QS_q) \ge \frac{c}{\sqrt{p}} m^{1/q-1/p}\frac{\sqrt{n}}{\sqrt{q}\, m^{1/q} + \sqrt{n}}.$$

Moreover, for $p = \infty$,

$$\mathbb{E}\operatorname{vr}(F^\infty, QS_q) \ge c\frac{m^{1/q}}{\sqrt{1+\log(m/n)}} \cdot \frac{\sqrt{n}}{\sqrt{q}\, m^{1/q} + \sqrt{n}}.$$

In particular for $m = n^{q/2}$ for $2 \le q \le p < \infty$, resp. $p = \infty$,

$$\mathbb{E}\operatorname{vr}(F^p, QS_q) \ge \frac{c}{\sqrt{pq}} m^{1/q-1/p} \sim_{c_{pq}} n^{(q/2)(1/q-1/p)},$$

$$\mathbb{E}\,\mathrm{vr}(F^\infty, QS_q) \ge \frac{c}{q}\sqrt{\frac{n}{1+\ln n}},$$

respectively. Here $c>0$ is an absolute constant.

Proof. Let $O(m)$ be the unitary group in $\mathbb{R}^m$ and $e_1,\dots,e_m$ be the unit vectors. We will consider expectations with respect to the normalized Haar measure on $O(m)$. Given $u\in O(m)$ the space

$$F_u = \mathrm{span}\{u(e_1),\dots,u(e_n)\} \subset \ell_p^m$$

is a (random) n-dimensional subspace. With every u, we associate a random partial isometry operator $A_u:\mathbb{R}^n\to\mathbb{R}^m$, $A_u(e_i)=u(e_i)$, $i=1,\dots,n$. Our aim is to estimate

$$\frac{\sqrt{n}\,v_n(A_u^*|_{F_u}:F_u\to\ell_2^n)}{\nu_{q'}(A_u^*:\ell_p^m\to\ell_2^n)}.$$

The crucial point is the following two norm estimates:

$$\mathbb{E}\|A_u^*:\ell_{q'}^m\to\ell_2^n\| \le c_0 m^{-1/2}(\sqrt{q}\,m^{1/q}+\sqrt{n}),$$
$$\mathbb{E}\mathrm{rad}(A_u:\ell_2^n\to\ell_p^m) \le c_0\sqrt{p}\,\sqrt{n}\,m^{1/p-1/2}.$$

In both estimates, we can replace the random unitary u by $m^{-1/2}G$, where $G=(g_{ij})_{i,j=1}^m$ is a random gaussian matrix with independent normalized entries (see the comparison theorem of Marcus–Pisier [MaP]). Hence, we deduce by Chevet's inequality, as in [BG],

$$\begin{aligned}\sqrt{m}\,\mathbb{E}\|A_u^*:\ell_{q'}^m\to\ell_2^n\| &\le c_0\mathbb{E}\Big\|\sum_{i=1,\dots,m,\,j=1,\dots,n} g_{ij}e_i\otimes e_j:\ell_{q'}^m\to\ell_2^n\Big\|\\ &\le c_0\Big(\mathbb{E}\Big\|\sum_{i=1}^m g_ie_i\Big\|_q+\mathbb{E}\Big\|\sum_{j=1}^n g_je_j\Big\|_2\Big)\\ &\le c_0(\sqrt{q}\,m^{1/q}+\sqrt{n}).\end{aligned}$$

As an immediate application, we get

$$\begin{aligned}\mathbb{E}\nu_{q'}(A_u^*:\ell_p^m\to\ell_2^n) &\le \nu_{q'}(\mathrm{id}:\ell_\infty^m\to\ell_{q'}^m)\mathbb{E}\|A_u^*:\ell_{q'}^m\to\ell_2^n\|\\ &\le m^{1/q'-1/2}c_0(\sqrt{q}\,m^{1/q}+\sqrt{n}).\end{aligned}$$

For the second estimate, we invoke again Marcus–Pisier's result and obtain by the symmetry of the gaussian variables and the corresponding Khinchin inequality

$$\begin{aligned}\sqrt{m}\,\mathbb{E}\mathrm{rad}(A_u:\ell_2^n\to\ell_p^m) &\le c_0\mathbb{E}_g\mathbb{E}_r\Big(\sum_{i=1}^m\Big|\sum_{j=1}^n r_jg_{ji}\Big|^p\Big)^{1/p}\\ &= c_0\mathbb{E}_r\mathbb{E}_g\Big(\sum_{i=1}^m\Big|\sum_{j=1}^n r_jg_{ji}\Big|^p\Big)^{1/p} = c_0\mathbb{E}_g\Big(\sum_{i=1}^m\Big|\sum_{j=1}^n g_{ji}\Big|^p\Big)^{1/p}\\ &\le c_0\Big(\sum_{i=1}^m\mathbb{E}_g\Big|\sum_{j=1}^n g_{ji}\Big|^p\Big)^{1/p} \le c_0\sqrt{p}\,\sqrt{n}\,m^{1/p}.\end{aligned}$$

Together with (1.6) and (1.7) this implies

$$\begin{aligned}\mathbb{E}\sqrt{n}\,v_n(A_u:\ell_2^n\to\ell_p^m) \le 4\mathbb{E}\sqrt{n}\,e_n(A_u) &\le 4c_1\mathbb{E}\mathrm{rad}(A_u)\\ &\le 4c_0c_1\sqrt{n}\,\sqrt{p}\,m^{1/p-1/2}.\end{aligned}$$

Since $A_u^*A_u=\mathrm{id}_{\ell_2^n}$ and $A_u(\ell_2^n)=F_u$, we observe that for fixed u, by Lemma 1.3,

$$1=v_n(\mathrm{id})=v_n(A_u)v_n(A_u^*|_{F_u})\le c_0v_n(A_u)\frac{\nu_{q'}(A_u^*)}{\sqrt{n}}\,\mathrm{vr}(F_u,QS_q).$$

By Hölder's inequality this implies

$$\begin{aligned}n=(\mathbb{E}n^{1/3})^3 &\le c_0(\mathbb{E}(\sqrt{n}\,v_n(A_u)\nu_{q'}(A_u^*)\,\mathrm{vr}(F_u,QS_q))^{1/3})^3\\ &\le c_0\mathbb{E}(\sqrt{n}\,v_n(A_u))\mathbb{E}\nu_{q'}(A_u^*)\mathbb{E}\,\mathrm{vr}(F_u,QS_q)\\ &\le 4c_0^3c_1\sqrt{p}\,\sqrt{n}\,m^{1/p-1/2}m^{1/q'-1/2}(\sqrt{q}\,m^{1/q}+\sqrt{n})\mathbb{E}\,\mathrm{vr}(F_u,QS_q)\\ &= 4c_0^3c_1\sqrt{p}\,m^{1/p-1/q}\sqrt{n}\,(\sqrt{q}\,m^{1/q}+\sqrt{n})\mathbb{E}\,\mathrm{vr}(F_u,QS_q).\end{aligned}$$

This proves the assertion. For $\sqrt{n}=m^{1/q}$, we deduce

$$n^{(q/2)(1/q-1/p)} = m^{1/q-1/p} \le 8c_0^3c_1\sqrt{qp}\,\mathbb{E}\,\mathrm{vr}(F_u,QS_q).$$

For $p=\infty$, we use the standard optimization procedure. In fact, let $2\le r<\infty$ to be determined later. Using [MP], we have

$$\begin{aligned}v_n(A_u:\ell_2^n\to\ell_\infty^m) &\le v_n(A_u:\ell_2^n\to\ell_r^m)v_n(\mathrm{id}:\ell_r^m\to\ell_\infty^m)\\ &\le c_0n^{-1/r}v_n(A_u:\ell_2^n\to\ell_r^m).\end{aligned}$$

With this additional factor, we obtain

$$\frac{\sqrt{n}}{\sqrt{q}\,m^{1/q}+\sqrt{n}}m^{1/q} \le 4c_0^4c_1\sqrt{r}\Big(\frac{m}{n}\Big)^{1/r}\mathbb{E}\,\mathrm{vr}(F_u^\infty,QS_q).$$

Here F_u^∞ is considered as a subspace of ℓ_∞^m. Choosing $r=2+\log(m/n)$ yields the assertion. ■

REMARK 1.8. This estimate is essentially sharp in many cases. In fact, we have the obvious upper estimate

$$\mathrm{vr}(F^p,QS_q) \le \inf_{G\subset L_q} d(F^p,G).$$

Here F^p, F^q indicates that F is considered as a subspace of ℓ_p^m, ℓ_q^m, respectively. Since the inclusion $\ell_q^m \subset \ell_p^m$ is a contraction, we clearly have

$$\inf_{G\subset L_q} d(F^p, G) \le \|\mathrm{id} : F^p \to F^q\|.$$

In particular, for $n \ge qm^{q/2}$, we get

$$\frac{m^{1/q-1/p}}{c_0\sqrt{p}} \le \mathbb{E}\,\mathrm{vr}(F^p, QS_q) \le \mathbb{E}\|\mathrm{id} : F^p \to F^q\| \le m^{1/q-1/p}.$$

Hence the estimate is sharp up to the factor $\sqrt{p}$ and the norm of the identity map $\mathrm{id} : \ell_p^m \to \ell_q^m$ is nearly attained on a random n-dimensional subspace of ℓ_p^m. In measure terminology, we see that $V(p,q,S,S)$ is attained for discrete measures $\mu = \nu$ on $\mathbb{R}^n$ where the points are random projections $A_u^*(e_i)$, $i = 1,\dots,n^{q/2}$, and the weights are all equal.

In the case $n \le cqm^{2/q}$, the estimate for $p = \infty$ is sharp. In fact according to Proposition 1.7 a random n-dimensional subspace $F^\infty \subset \ell_\infty^m$ satisfies

$$\frac{1}{c_0}\sqrt{\frac{n}{1+\ln(m/n)}} \le \mathrm{vr}(F^\infty, QS_q) \le \mathrm{vr}(F^\infty, \ell_2) \le d(F^\infty, \ell_2^n) \le c_0\sqrt{\frac{n}{1+\ln(m/n)}}.$$

For a proof of the last inequality we refer to the recent paper of Guédon [Gue].

Now, we give the proof of Theorem 0.1.

PROPOSITION 1.9. *Let $1 \le p,q < \infty$ and $n \in \mathbb{N}$. Then*

$$\sup_{F\subset \ell_p,\,\dim F=n} \mathrm{vr}(F, S_q) \sim_{c_{pq}} \begin{cases} n^{1/2-1/p} & \text{if } 1 \le q \le 2 \le p < \infty,\\ n^{(q/2)(1/q-1/p)} & \text{if } 2 \le q \le p < \infty,\\ 1 & \text{else},\end{cases}$$

where c_{pq} only depends on p, q.

Proof. Since for $1 \le q \le 2$ subspaces of L_q have uniformly bounded volume ratio (with respect to ℓ_2, see [PSc]), for any finite-dimensional Banach space F we have

$$\mathrm{vr}(F, S_q) \le \mathrm{vr}(F, \ell_2) \le c_0\,\mathrm{vr}(F, S_q),$$

for some absolute constant c_0. Thus, we can assume $2 \le q < \infty$. As recalled earlier the subspaces of L_p also have bounded volume ratio with respect to ellipsoids for $1 \le p \le 2$. Hence for all $F \subset L_p$,

$$1 \le \mathrm{vr}(F, S_q) \le \mathrm{vr}(F, \ell_2) \le c_0.$$

Thus, we can also assume $2 \le p \le \infty$. The case $q = 2$ is well known. In fact, by D. Lewis [L], $\mathrm{vr}(F, \ell_2) \le d(F, \ell_2^n) \le n^{1/2-1/p}$ with equality up to a constant for $F = \ell_p^n$. If $2 \le p \le q < \infty$, we deduce from inequality (1.3) and Lemma 1.5 that for all $v : L_p \to \ell_2$,

$$\sqrt{n}\,v_n(v) \le \nu_{q'}(v).$$

Lemma 1.3 implies

$$\sup_{F\subset L_p} \mathrm{vr}(F, S_q) \le c_0\sqrt{q}.$$

In the case $2 \le q \le p < \infty$, we deduce from Lemma 1.5 and (1.3) for all $v : L_p \to \ell_2$,

$$\sqrt{n}\,v_n(v) \le n^{1/2-q/(2p)}\nu_{q'}(v).$$

By Lemmas 1.3 and 1.5, we deduce that

$$\sup_{F\subset L_p,\,\dim F=n} \mathrm{vr}(F, S_q) \le c_0 n^{(q/2)(1/q-1/p)}.$$

The lower estimate is given by Proposition 1.7 by choosing $m = n^{q/2}$ if $n \le q$. Note that for $n \ge q$ the choice $m = (n/q)^{q/2}$ even yields the slightly better estimate

$$n^{(q/2)(1/p-1/q)} \le c\sqrt{p}\,q^{1/2-q/(2p)} \sup_{F\subset \ell_p} \mathrm{vr}(F, S_q). \ \blacksquare$$

We will use the following refinement of the classical distance estimates.

LEMMA 1.10. *Let $2 \le q < \infty$, and $E \subset \ell_q^m$ an n-dimensional subspace. Then*

$$\max\Big\{1, \frac{\sqrt{n}}{\sqrt{q}\,m^{1/q}}\Big\} \le c_0\,\mathrm{vr}(E, \ell_2). \tag{1.11}$$

For $q = \infty$,

$$\sqrt{\frac{n}{1+\ln(m/n)}} \le C\,\mathrm{vr}(E, \ell_2).$$

Moreover, for all $2 \le q \le \infty$,

$$n^{1/2-1/q} \le c_0\sqrt{m/n}\,\mathrm{vr}(E, \ell_2).$$

Proof. Let $u : \ell_2^n \to E$ be a contraction such that u^{-1} exists. Let $\iota : E \to \ell_q^m$ be the inclusion map and $j : \ell_q^m \to \ell_\infty^m$ the identity map. Then by a well known application of Khinchin's inequality

$$\ell(u) \le \ell(\iota u) \le \sqrt{q}\,\pi_q(\iota u) = \sqrt{q}\,\pi_q(j^{-1}j\iota u) \le \sqrt{q}\,m^{1/q}\|u\| = \sqrt{q}\,m^{1/q}.$$

Hence, we deduce from (1.6) and (1.8) that

$$v_n(u) \le 4e_n(u) \le 4c_1\frac{\ell(u)}{\sqrt{n}} \le 4\frac{\sqrt{q}\,m^{1/q}}{\sqrt{n}}.$$

Since for an isomorphism, we have $1 = v_n(u)v_n(u^{-1})$, this implies

$$\mathrm{vr}(E,\ell_2) = \inf_{\|u\|\le 1} v_n(u^{-1}) \ge \frac{\sqrt{n}}{4c_1\sqrt{q}\,m^{1/q}}.$$

The case $q=\infty$ is an immediate consequence of Gluskin's inequality (for a short proof see [BaP]). Indeed, for any contraction $u:\ell_2^n \to E$ and the canonical embedding $\iota : E \to \ell_\infty^m$, we have

$$\begin{aligned}\sqrt{n}\, v_n(u) = \sqrt{n}\, v_n(\iota u) &\le c_0\sqrt{1+\ln(m/n)}\,\|\iota u\| \\ &\le c_0\sqrt{1+\ln(m/n)}.\end{aligned}$$

For the last estimate, we deduce from $\pi_2(\mathrm{id}:\ell_q^m \to \ell_2^m) \le \sqrt{m}$ and Meyer–Pajor's inequality $v_n(\mathrm{id}:\ell_2^m \to \ell_q^m) \le c_0 n^{1/q-1/2}$ (see [MP]) that for every n-dimensional subspace $E\subset \ell_q^m$,

$$\begin{aligned}1 = v_n(\mathrm{id}) &\le v_n((\mathrm{id}:\ell_2^m\to\ell_q^m)|_E)\,v_n((\mathrm{id}:\ell_q^m\to\ell_2^m)|_E) \\ &\le c_0 n^{1/q-1/2}\,\mathrm{vr}(E,\ell_2)\frac{\pi_2(\mathrm{id})}{\sqrt{n}} \\ &\le c_0 n^{1/q-1/2}\sqrt{m/n}\,\mathrm{vr}(E,\ell_2). \ \blacksquare\end{aligned}$$

It is also well known that the inequality (1.11) is sharp.

LEMMA 1.11. *There exist $\delta>0$ and $C>0$ such that for $2\le p<\infty$ and $\delta p m^{2/p} \le n \le (\frac{\delta}{2}pe^{-p})m$ there exists an n-dimensional subspace $E\subset \ell_p^m$ such that*

$$d(E,\ell_2^n) \le C\frac{\sqrt{n}}{\sqrt{p}\,m^{1/p}}.$$

For $n\le \delta p m^{2/p}$ and $m\ge e^p$, there exists an n-dimensional subspace of ℓ_p^m which is 2-isomorphic to ℓ_2^n.

Proof. In fact the last assertion is well known for m large enough (see for example [FLM]). We want to estimate the size of such m. Using the nowadays standard technique from [GO2], it suffices to choose δ appropriately after proving

$$\mathbb{E}\Big(\sum_{i=1}^{M}|g_i|^p\Big)^{1/p} \ge c\sqrt{p}\,M^{1/p}$$

for $M\ge [e^p]$. Indeed, we have

$$\mathbb{E}\Big(\sum_{i=1}^{[e^p]}|g_i|^p\Big)^{1/p} \ge \mathbb{E}\sup_{i=1,\dots,[e^p]}|g_i| \ge c_1\sqrt{\ln[e^p]} = c_1e^{-2}\sqrt{p}\,([e^p])^{1/p}.$$

For an arbitrary $M\ge[e^p]$, we choose $r\in\mathbb{N}$ such that $re^p\le M\le (r+1)e^p$. Then Minkowski's inequality implies

$$\begin{aligned}\mathbb{E}\Big(\sum_{i=1}^{M}|g_i|^p\Big)^{1/p} &\ge \mathbb{E}\Big(\sum_{j=1}^{r}\Big\{\sum_{i=1}^{[e^p]}|g_{ij}|^p\Big\}^{p/p}\Big)^{1/p} \\ &\ge \Big(\sum_{j=1}^{r}\Big\{\mathbb{E}\Big(\sum_{i=1}^{[e^p]}|g_{ij}|^p\Big)^{1/p}\Big\}^p\Big)^{1/p} \\ &\ge r^{1/p}c_1e^{-2}\sqrt{p}\,([e^p])^{1/p} \ge c_1e^{-3}\sqrt{p}\,M^{1/p}.\end{aligned}$$

Define $M(p)=e^{2+p}$ and for $M\ge M(p)$ consider a subspace $F\subset \ell_p^M$ of dimension $N=\delta p M^{2/p}$ such that $d(F,\ell_2^N)\le 2$. Note that $M\ge N$ implies the additional condition $M^{1-2/p}\ge \delta p$. Then $E=\ell_p^l(F)$ is a subspace of ℓ_p^{Ml} which satisfies

$$d(E,\ell_2^{Nl}) \le 2l^{1/2-1/p}.$$

For given $n\le m$, we choose $M\in\mathbb{N}$ such that

$$\frac{M^{1-2/p}}{p\delta} \le \frac{m}{2n} \le \frac{(M+1)^{1-2/p}}{p\delta}.$$

Let $l\in\mathbb{N}$ be such that $lM\le m\le (l+1)M$. Then

$$\widetilde{n} = l\delta p M^{2/p} \ge lM\frac{2n}{m} \ge n.$$

Any n-dimensional subspace G of $E=\ell_p^l(F)$ satisfies $d(G,\ell_2^n)\le 2l^{1/2-1/p}$. On the other hand, we easily get $\widetilde{n}\le 4n$. Therefore

$$\frac{\sqrt{n}}{\sqrt{p}\,m^{1/p}} \ge \frac{1}{2}\frac{\sqrt{\delta l p}\,M^{1/p}}{\sqrt{2p}\,l^{1/p}M^{1/p}} = \frac{\sqrt{\delta}}{2\sqrt{2}}l^{1/2-1/p} \ge \frac{\sqrt{\delta}}{4\sqrt{2}}d(G,\ell_2^n).$$

This method breaks down if $m/(2n)\le M(p)^{1-2/p}/(p\delta)$, which yields

$$n \ge \frac{\delta p}{2M(p)^{1-2/p}}m \ge \frac{\delta}{2}pe^{-p}m.$$

The other possibility when this method cannot be applied is $M>m$, which implies $2n<p\delta m^{2/p}$. $\blacksquare$

The lemma yields the following local information.

PROPOSITION 1.12. *Let $1\le n\le 4\delta p e^{-p}m$ and $2\le p\le q<\infty$. Then there exists an n-dimensional subspace $E\subset\ell_p^m$ such that*

$$\frac{1}{C\sqrt{q}\,m^{1/q}}\min\{\sqrt{p}\,m^{1/p},\sqrt{n}\} \le \inf \mathrm{vr}(F^q,E) \le \inf d(F^q,E) \le Cm^{1/p-1/q},$$

where the infimum is taken over all n-dimensional subspaces F^q in ℓ_q^m and C is an absolute constant.

Proof. According to Lemma 1.11 let $E\subset\ell_p^m$ be the n-dimensional space with minimal distance to the Hilbert space. Clearly, for any n-dimensional subspace F^q, by Lemmas 1.10 and 1.11 we get

$$\frac{\sqrt{n}}{\sqrt{q}\,m^{1/q}} \le C\,\mathrm{vr}(F^q,\ell_2) \le C\,\mathrm{vr}(F^q,E)\,\mathrm{vr}(E,\ell_2)$$
$$\le C\,\mathrm{vr}(F^q,E)d(E,\ell_2^n) \le C^2 \max\left\{1, \frac{\sqrt{n}}{\sqrt{p}\,m^{1/p}}\right\} \mathrm{vr}(F^q,E).$$

This yields the assertion. ∎

REMARK 1.13. For given $n \in \mathbb{N}$, the interesting value is $m = n^{p/2}$ or $m = (n/p)^{p/2}$, where we get

$$n^{1/2-p/(2q)} \le \sqrt{q} \inf_{F\subset \ell_q^m} d(E,F) \quad \text{resp.} \quad n^{1/2-p/(2q)} \le \frac{\sqrt{q}}{p^{p/(2q)}} \inf_{F\subset\ell_q^m} d(E,F).$$

Moreover, according to [BLM] for any n-dimensional subspace $E \subset L_p$, there is a 2-isomorphic copy in $\ell_p^{M(n)}$, $M(n) := c_p n^{p/2}(1+\ln n)^3$. Hence

$$\inf_{F\subset\ell_q^M} d(E,F) \le M(n)^{1/p-1/q}.$$

By Proposition 1.12 for $m = M(n)$ this yields

$$\frac{n^{1/2-p/(2q)}}{Cc_p\sqrt{q}\,(\ln n)^{3/q}} \le \sup_{E\subset L_p} \inf_{F\subset\ell_q^{M(n)}} d(E,F) \le (c_p \ln n)^{3(1/p-1/q)} n^{1/2-p/(2q)}.$$

In the sequel, we will consider quotient spaces of ℓ_p. We start with the canonical example.

EXAMPLE 1.14. Let $2 \le q \le p \le \infty$. Then

$$\sqrt{2/(\pi e)}\,n^{1/q-1/p} \le \inf_{X\subset L_q} \mathrm{vr}(\ell_p^n, X) \le \mathrm{vr}(\ell_p^n,\ell_q^n) \le n^{1/q-1/p}.$$

Proof. According to [GJ, Proposition 3.3(ii)],

$$\sqrt{n}\,v_n(v) \le \inf_{X\subset L_q} \mathrm{vr}(\ell_p^n, X)\nu_{q'}(v)$$

for all $v : \ell_p^n \to \ell_2^n$. Since the identity map obviously satisfies $\nu_{q'}(v) \le n^{1/q'}$, the assertion follows by standard estimates of the unit balls: $\mathrm{vol}(B_p^n)^{1/n} \ge 2n^{-1/p}$ and $\sqrt{n}\,\mathrm{vol}(B_2^n)^{1/n} \le \sqrt{2\pi e}$. The upper estimate is obvious. ∎

Now, we determine $V(p,q,Q,Q)$.

PROPOSITION 1.15. *Let $1 < p \le \infty$ and $1 < q < \infty$. Then there exists a constant c_{pq} such that for all $n \in \mathbb{N}$,*

$$\sup_{F \text{ quotient of } \ell_p,\, \dim F = n} \mathrm{vr}(F,\ell_q) \sim_{c_{pq}} \begin{cases} n^{1/2-1/p} & \text{if } 1 < q \le 2 \le p \le \infty, \\ n^{1/q-1/p} & \text{if } 2 \le q \le p \le \infty, \\ 1 & \text{if } 2 \le p \le q < \infty \text{ or } 1 < p \le 2. \end{cases}$$

Proof. For $1 < p \le 2$ quotient spaces of ℓ_p have uniformly bounded volume ratio (with respect to ℓ_2^n) because F^* has type 2 (see [PSc]), hence for all $1 < q < \infty$,

$$1 \le \mathrm{vr}(F,\ell_q) \le \mathrm{vr}(F,\ell_2^n) \le c_0\sqrt{p'}.$$

In particular, for $1 < q \le 2 \le p$ a standard distance estimate [L] implies

$$\sup\{\mathrm{vr}(F,\ell_q) \mid F \text{ quotient of } \ell_p\} \sim_{c_q} \sup\{\mathrm{vr}(F,\ell_2) \mid F \text{ quotient of } \ell_p\}$$
$$\le \sup\{d(F,\ell_2^n) \mid F \text{ quotient of } \ell_p\}$$
$$\le n^{1/2-1/p}.$$

The lower estimate is obtained for $F = \ell_p^n$ (see Example 1.14).

For the following, we consider the case $2 \le p, q$. Let F be a quotient space of ℓ_p with quotient map $q_F : \ell_p \to F$. For an operator $v : F \to \ell_2^n$, we deduce from Lemma 1.2 that

$$\sqrt{n}\,v_n(v) = \sqrt{n}\left(\frac{\mathrm{vol}(vq_F(B_{\ell_p}))}{\mathrm{vol}(B_2^n)}\right)^{1/n} \le \pi_{p'}(vq_F) \le \pi_{p'}(v). \tag{1.12}$$

Let $p \le q$. Then the inequality $\pi_{p'}(v) \le \pi_{q'}(v)$ implies, by [GJ, Theorem 3.7(ii)],

$$\mathrm{vr}(F,\ell_q) \le \sqrt{\pi e/2} \sup_{\pi_{q'}(v)\le 1} \sqrt{n}\,v_n(v) \le \sup_{\pi_{p'}(v)\le 1} \sqrt{n}\,v_n(v)$$
$$\le \sqrt{\pi e/2}\,\mathrm{vr}(F,\ell_p) = \sqrt{\pi e/2}.$$

In the case $2 \le q \le p$, we recall the following inequality from [CAR, Lemma 1]:

$$\pi_{p'}(v) \le (\pi_{p'}(\mathrm{id}_F))^{1-p'/q'}\pi_{q'}(v).$$

Using a well known distance estimate for quotients of ℓ_p (see e.g. [PSc]) and the estimate for the 1-summing norm [GO1], we obtain

$$\pi_{p'}(\mathrm{id}_F) \le d(F,\ell_2^n)\pi_{p'}(\mathrm{id}_{\ell_2^n}) \le n^{1/2-1/p}\sqrt{\pi/2}\,n^{1/2} \le \sqrt{\pi/2}\,n^{1/p'}.$$

Combining these two estimates yields

$$\pi_{p'}(v) \le \sqrt{\pi/2}\,n^{1/q-1/p}\pi_{q'}(v).$$

In view of (1.12) and [GJ, Theorem 3.7(ii)], we obtain

$$\mathrm{vr}(F,\ell_q) \le \sqrt{\pi e/2} \sup_{\pi_{q'}(v)\le 1} \sqrt{n}\,v_n(v) \le \tfrac{1}{2}\pi\sqrt{e}\,n^{1/q-1/p}.$$

The lower estimate is an immediate consequence of Example 1.14. ∎

The next lemma was observed in [GJ, Remark 3.11.1] but it also follows from Lemma 1.3:

LEMMA 1.16. *Let $1 < q < \infty$ and Q be an n-dimensional quotient space of L_q. Then*

$$\mathrm{vr}(Q, S_q) \le c_0 \sqrt{\max(q, q')}.$$

COROLLARY 1.17. *Let $1 < p \le \infty$ and $1 \le q < \infty$. Then there exists a constant c_{pq} such that for all $n \in \mathbb{N}$,*

$$\sup_{F \text{ quotient of } \ell_p,\, \dim F = n} \mathrm{vr}(F, S_q) \sim_{c_{pq}} \begin{cases} n^{1/2-1/p} & \text{if } 1 \le q \le 2 \le p \le \infty, \\ n^{1/q-1/p} & \text{if } 2 \le q \le p \le \infty, \\ 1 & \text{if } 2 \le p \le q < \infty \text{ or } 1 < p \le 2. \end{cases}$$

Proof. As already observed in the introduction, if $1 \le q \le 2$ then L_q has cotype 2 and therefore for any Banach space F,

$$\mathrm{vr}(F, S_q) \le \mathrm{vr}(F, \ell_2) \le c_0\, \mathrm{vr}(F, \ell_2) \le c_0 d(F, \ell_2^n).$$

We also recall that for $1 < p \le 2$ quotient spaces F of ℓ_p have bounded volume ratio, since $\ell_{p'}$ is of type 2, hence

$$1 \le \mathrm{vr}(F, \ell_2) \le c_0 \sqrt{p'}.$$

For $2 \le p \le \infty$, the standard distance estimate $d(F, \ell_2^n) \le n^{1/2-1/p}$ yields the upper estimate. The lower estimate is obtained for $F = \ell_p^n$ (see Example 1.14). Now, let $2 \le q < \infty$. The lower estimate is obtained for $F = \ell_p^n$ using Example 1.14. The upper estimate follows from Proposition 1.15 and Lemma 1.16 upon using

$$\mathrm{vr}(F, S_q) \le \sup_{Q\ n\text{-dimensional quotient of } \ell_q} \mathrm{vr}(Q, S_q)\, \mathrm{vr}(F, \ell_q) \le c_q\, \mathrm{vr}(F, \ell_q). \ ■$$

We finish the investigation of $V(p, q, \cdot, \cdot)$ by proving the following proposition.

PROPOSITION 1.18. *Let $1 < p \le \infty$ and $1 < q < \infty$. Then there exists a constant c_{pq} such that for all $n \in \mathbb{N}$,*

$$\sup\{\mathrm{vr}(F, \ell_q) \mid F \subset \ell_p\} \sim_{c_{pq}} \begin{cases} 1 & \text{if } 1 \le p \le 2, \\ n^{1/2-1/p} & \text{if } 1 < q \le \infty,\ 2 \le p \le \infty. \end{cases}$$

Proof. Let F be an n-dimensional subspace of L_p. Then

$$\mathrm{vr}(F, \ell_\infty) \le \mathrm{vr}(F, \ell_q) \le \mathrm{vr}(F, \ell_2) \le d(F, \ell_2^n).$$

Since for $1 \le p \le 2$ the spaces L_p have cotype 2, the volume ratio is bounded. This yields the upper estimate for $1 \le p \le 2$. The upper estimate for $2 \le p \le \infty$ follows from the standard distance estimate [L]. The lower estimate will be proved in the next section (see Corollary 2.15). ■

To end this section, we apply the results from Bourgain, Lindenstrauss and Milman [BLM] to estimate the volume ratio of quotients of L_p with respect to ℓ_p^n.

PROPOSITION 1.19. *Let $1 < p \le \infty$ and $1/p + 1/p' = 1$. Every n-dimensional quotient space F of L_p satisfies*

$$\mathrm{vr}(F, \ell_p^n) \le c(p)(1 + \ln n)^{3/p'} \begin{cases} 1 & \text{if } 2 \le p \le \infty, \\ n^{1/2-1/p'} & \text{if } 1 < p \le 2, \end{cases}$$

where $c(p)$ only depends on p.

Proof. Considering the dual space $F^* \subset L_{p'}$, we can apply the results of [BLM] for $\varepsilon = 1$. Hence, we can assume that F is a quotient of ℓ_p^m, where

$$m \le c(p) \begin{cases} n^{p'/2}(1 + \ln n)^3 & \text{if } 1 < p \le 2, \\ n(1 + \ln n)^3 & \text{if } 2 \le p \le \infty. \end{cases}$$

Let $V : \ell_p^m \to \mathbb{R}^n$ be the corresponding quotient map and $x_i = V(e_i)$ the image of the ith unit vector. Using the volume formula for quotient spaces of ℓ_p [GJ, Theorem 0.1], we get

$$\begin{aligned} \sqrt{\frac{2}{\pi e^3 \min(p', n)}}\, \mathrm{vol}(B_F)^{1/n} &\le \mathrm{vol}(B_p^n)^{1/n} \Big(\sum_{\mathrm{card}(I) = n} |\det(x_i)_{i \in I}|^{p'} \Big)^{1/(np')} \\ &\le \mathrm{vol}(B_p^n)^{1/n} \binom{m}{n}^{1/(np')} \sup_I |\det(x_i)_{i \in I}|^{1/n} \\ &\le (em/n)^{1/p'}\, \mathrm{vol}(B_p^n)^{1/n} \sup_I |\det(x_i)_{i \in I}|^{1/n}. \end{aligned}$$

But for the subset $I = \{i_1, \ldots, i_n\}$ where the supremum is attained, the corresponding inclusion map $\iota_I : \ell_p^n \to \ell_p^m$, $e_j \mapsto e_{i_j}$, is a contraction, hence $V\iota_I(B_p^n) \subset B_F$ and

$$\mathrm{vol}(V\iota_I(B_p^n)) = |\det(x_i)_{i \in I}|\, \mathrm{vol}(B_p^n).$$

The estimates for m imply the assertion. ■

REMARK 1.20. In the case $p = \infty$, this result was pointed out to the second author by K. Ball. It is an open problem whether the logarithmic factor can be removed.

2. Uniform estimates for volume ratios. In this part, we are interested in volume estimates for the finite-dimensional slices of an operator $T : X \to Y$ between arbitrary Banach spaces. For this, let us introduce the following volume ratio number with respect to a fixed Banach space Z:

$$vr_n(T, Z) := \sup \inf \left(\frac{\mathrm{vol}(T(B_E))}{\mathrm{vol}(u(B_Z))} \right)^{1/n},$$

where the supremum ranges over all n-dimensional subspaces $E \subset X$, $F \subset Y$ such that $T(E) \subset F$ and the infimum ranges over all contractions $u : Z \to F$. By convention, we set $\mathrm{vr}_n(T, Z) = 0$ if no such spaces E and F exist.

REMARK 2.1. If X is n-dimensional, $T : X \to Y$ is of rank n and $T(X) = Y$ then

$$\mathrm{vr}_n(T, Z) = \mathrm{vr}(Y, Z) v_n(T). \tag{2.1}$$

In the case $Z = \ell_p$, $1 \le p \le \infty$, we deduce from [GJ, Theorem 3.7(ii)] that

$$\sqrt{\frac{2}{\pi e}}\, \mathrm{vr}_n(T, \ell_p) \le \sup_{0 \ne v : Y \to \ell_2^n} \frac{\sqrt{n}\, v_n(vT)}{\pi_{p'}(v)} \le \sqrt{\min(p', n)}\, \mathrm{vr}_n(T, \ell_p). \tag{2.2}$$

The next lemma shows that the sequence $(\mathrm{vr}_n(T, \ell_p))_{n \in \mathbb{N}}$ has reasonable monotonicity properties.

LEMMA 2.2. *Let $1 \le p \le \infty$, $1 \le k \le n$ and $T : X \to Y$ a linear operator. Then*

$$\mathrm{vr}_n(T, \ell_p) \le C_2 \sqrt{p'}\, (n/k)^{3/2}\, \mathrm{vr}_k(T, \ell_p). \tag{2.3}$$

In particular,

$$\frac{\mathrm{vr}_n(T, \ell_p)}{C_2 \sqrt{p'}\, e^{3/2}} \le \Big(\prod_{j=1}^{n} \mathrm{vr}_j(T, \ell_p)\Big)^{1/n} \le C_2 \sqrt{p'}\, e^{3/2} \Big(\frac{n}{k}\Big)^{3(n-k)/(2n)} \Big(\prod_{j=1}^{k} \mathrm{vr}_j(T, \ell_p)\Big)^{1/k}.$$

Proof. Let $1 \le k \le n$. Without loss of generality we assume $\dim X = \dim Y = \operatorname{rank} T = n$. Let $u : \ell_2^n \to X$ be a Pisier map [PSc, Corollary 7.15] satisfying

$$\sup_k k \max(v_k(u), v_k(u^{-1})) \le C_1 n.$$

According to Remark 2.1 there is an operator $v : Y \to \ell_2^n$ such that $\pi_{p'}(v) = \sqrt{n}$ and

$$\mathrm{vr}_n(T, \ell_p) \le c_0 v_n(vT).$$

If $\{a_j(vTu)\}$ denote the approximation numbers of vTu, we recall from (1.2) that

$$v_k(vTu)^k = \prod_{j=1}^{k} a_j(vTu).$$

In particular, the monotonicity of the singular numbers implies $v_n(vTu) \le v_k(vTu)$. Therefore, from (2.2) we obtain

$$\begin{aligned}\mathrm{vr}_n(T, \ell_p) &\le c_0 v_n(vTuu^{-1}) \le c_0 C_1 v_n(vTu) \le c_0 C_1 v_k(vTu)\\ &\le c_0 C_1^2 \frac{n}{k} \sup_{E_k \subset X,\, \dim E_k = k} v_k(vT|_{E_k})\\ &\le c_0 C_1^2 \frac{n}{k} \sqrt{p'}\, \mathrm{vr}_k(T, \ell_p) \frac{\pi_{p'}(v)}{\sqrt{k}} = c_0 C_1^2 \sqrt{p'} \Big(\frac{n}{k}\Big)^{3/2} \mathrm{vr}_k(T, \ell_p).\end{aligned}$$

So, we proved the first inequality with $C_2 = c_0 C_1^2$.

To prove the second chain of inequalities, it is sufficient to observe that (2.3) implies that for all $1 \le k \le j$,

$$\begin{aligned}\mathrm{vr}_j(T, \ell_p)^k &\le (C_2 \sqrt{p'})^k \prod_{i=1}^{k} \Big(\frac{j}{i}\Big)^{3/2} \mathrm{vr}_i(T, \ell_p)\\ &\le (C_2 \sqrt{p'}\, e^{3/2})^k \frac{j^{3k/2}}{k^{3k/2}} \prod_{i=1}^{k} \mathrm{vr}_i(T, \ell_p).\end{aligned}$$

It is standard to derive from this the formula for the products. ■

The next elementary lemma is crucial for the following.

LEMMA 2.3. *Let $1 \le k \le n$, $T : \ell_2^n \to \ell_2^n$ and $E \subset \ell_2^n$ with $\dim E = k$. Then*

$$v_n(T) \le v_k(T|_E)^{k/n} v_{n-k}(T)^{(n-k)/n}.$$

Proof. As usual, we denote by $a_j(T)$ the singular numbers of T, and set $\tau = (a_j(T))_{j=1}^n$. Since T acts on ℓ_2^n, there are unitaries u, o such that $T = uD_\tau o$, with D_τ the corresponding diagonal operator. Hence, we get

$$v_n(T)^n = |\det(T)| = \prod_{j=1}^{n} a_j(T).$$

Using the orthogonal projection P_E on E, it is easy to derive

$$a_{j+n-k}(T) \le a_j(T|_E).$$

Indeed, for $R : E \to \ell_2^n$ of rank $< j$ the rank of $TP_{E^\perp} + RP_E$ is less than $n - k + j$. Hence

$$a_{n-k+j}(T) \le \inf_R \|T - TP_{E^\perp} - RP_E\| = \inf_R \|TP_E - RP_E\| \le a_j(T|_E).$$

Now, let $H = o^{-1}(\mathrm{span}\{e_1, \ldots, e_{n-k}\}) \subset \ell_2^n$. Then $a_j(T) = a_j(T|_H)$ for all $j = 1, \ldots, n-k$. Therefore, we obtain

$$\begin{aligned}v_n(T)^n &= \prod_{j=1}^{n} a_j(T) = \prod_{j=1}^{n-k} a_j(T|_H) \prod_{j=n-k+1}^{n} a_j(T) \le \prod_{j=1}^{n-k} a_j(T|_H) \prod_{j=1}^{k} a_j(T|_E)\\ &= v_{n-k}(T|_H)^{n-k} v_k(T|_E)^k \le v_{n-k}(T)^{n-k} v_k(T|_E)^k. \ ■\end{aligned}$$

LEMMA 2.4. *Let $1 \le k \le n \in \mathbb{N}$, X and Y be n-dimensional Banach spaces and $T : X \to Y$ be a linear operator. For every k-dimensional subspace $E \subset X$,*

$$v_n(T) \le C_4 v_k(T|_E)^{k/n} v_{n-k}(T)^{(n-k)/n}.$$

Here C_4 denotes an absolute constant.

Proof. Let $u_X : \ell_2^n \to X$ and $u_Y : \ell_2^n \to Y$ be the Pisier maps [PSc] such that for $w \in \{u_X, u_X^{-1}, u_Y, u_Y^{-1}\}$,

$$\sup_k k v_k(w) \le C_1 n.$$

Define $H = u_X^{-1}(E)$. We apply Lemma 2.3 to the operator $u_Y^{-1}Tu_X$ to obtain

$$\begin{aligned} v_n(T) &= v_n(u_Y u_Y^{-1} T u_X u_X^{-1}) = v_n(u_Y) v_n(u_Y^{-1} T u_X) v_n(u_X^{-1}) \\ &\le C_1^2 v_k(u_Y^{-1} T u_X|_H)^{k/n} v_{n-k}(u_Y^{-1} T u_X)^{(n-k)/n} \\ &\le C_1^2 (v_k(u_Y^{-1}) v_k(T|_E) v_k(u_X))^{k/n} (v_{n-k}(u_Y^{-1}) v_{n-k}(T) v_{n-k}(u_X))^{(n-k)/n} \\ &\le C_1^4 \left(\frac{n}{k}\right)^{2k/n} v_k(T|_E)^{k/n} \left(\frac{n}{n-k}\right)^{2(n-k)/n} v_{n-k}(T)^{(n-k)/n} \\ &\le C_1^4 4 v_k(T|_E)^{k/n} v_{n-k}(T)^{(n-k)/n}. \ \blacksquare \end{aligned}$$

REMARK 2.5. Using Santaló's inequality and its inverse together with Fact 1.1, we obtain for every k-dimensional quotient space E of Y with quotient map $q_E : Y \to E$,

$$\mathrm{vr}_n(T) \le C_4 \,\mathrm{vr}_k(q_E T)^{k/n}\, \mathrm{vr}_{n-k}(T)^{(n-k)/n}.$$

COROLLARY 2.6. *Let $1 \le k \le n$ and $T : X \to Y$ be a linear operator. Then*

$$\mathrm{vr}_n(T, \ell_p) \le C_5 \sqrt{p'}\, \mathrm{vr}_{n-k}(T, \ell_p)^{(n-k)/n} \times \sup_{E\subset X,\, \dim E = n}\ \inf_{E_k \subset E,\, \dim E_k = k} \mathrm{vr}_k(T|_{E_k}, \ell_p)^{k/n}.$$

Here $C_5 > 0$ is an absolute constant.

Proof. We can assume $\dim X = \dim Y = n$ and T is of rank n. According to Remark 2.1 (see (2.2)) there exists an operator $v : Y \to \ell_2^n$ with $\pi_{p'}(v) \le \sqrt{n}$ such that

$$\mathrm{vr}_n(T, \ell_p) \le c_0 v_n(vT).$$

Let $E_m \subset E$ be an m-dimensional subspace. Then (2.2) implies

$$v_m(vT\iota_{E_m}) \le \sqrt{p'}\, \mathrm{vr}_m(T\iota_{E_m}, \ell_p) \frac{\pi_{p'}(v)}{\sqrt{m}} \le \sqrt{p'} \sqrt{\frac{n}{m}}\, \mathrm{vr}_m(T\iota_{E_m}, \ell_p).$$

In particular,

$$v_m(vT) \le \sqrt{p'} \sqrt{\frac{n}{m}}\, \mathrm{vr}_m(T, \ell_p).$$

Now, let $E_k \subset E$ be an arbitrary subspace of dimension k. Then from Lemma 2.4 we deduce that

$$\begin{aligned} \mathrm{vr}_n(T, \ell_p) &\le c_0 v_n(vT) \le c_0 C_4 v_k(vT\iota_{E_k})^{k/n} v_{n-k}(vT)^{(n-k)/n} \\ &\le c_0 C_4 \sqrt{p'} \left(\frac{n}{k}\right)^{k/(2n)} \mathrm{vr}_k(T\iota_{E_k}, \ell_p)^{k/n} \left(\frac{n}{n-k}\right)^{(n-k)/(2n)} \\ &\quad \times \mathrm{vr}_{n-k}(T, \ell_p)^{(n-k)/n} \\ &\le c_0 C_4 \sqrt{p'}\, 2\, \mathrm{vr}_k(T\iota_{E_k})^{k/n}\, \mathrm{vr}_{n-k}(T, \ell_p)^{(n-k)/n}. \ \blacksquare \end{aligned}$$

REMARK 2.7. If we replace the p'-summing norm by the p'-nuclear norm, we obtain

$$\mathrm{vr}_n(T, S_p) \le C\sqrt{p'}\, \mathrm{vr}_{n-k}(T, S_p)^{(n-k)/n} \sup_{E\subset X,\, \dim E = n}\ \inf_{E_k \subset E,\, \dim E_k = k} \mathrm{vr}_k(T|_{E_k}, S_p)^{k/n},$$

where C is an absolute constant and $\mathrm{vr}_n(T, S_p) = \inf \mathrm{vr}_n(T, Z)$, with the infimum taken over all n-dimensional subspaces Z of L_p.

We will frequently use the following consequence of the estimates for Gelfand numbers.

LEMMA 2.8. *Let $1 \le j \le L$. Then there exists a $(j-1)$-codimensional subspace $G \subset \ell_\infty^L$ such that*

$$\pi_1(\mathrm{id}|_G : G \to \ell_2^L) \le C_6 L \sqrt{\frac{1 + \ln(L/j)}{j}}.$$

Here C_6 is an absolute constant.

Proof. According to the estimates for the Gelfand numbers of the identity map $I_{12} : \ell_1^L \to \ell_2^L$ (see [SC]), there exists a $(j-1)$-codimensional subspace $G_1 \subset \ell_1^L$ such that

$$\|\mathrm{id}|_{G_1} : G_1 \to \ell_2^L\| \le C_6 \sqrt{\frac{1 + \ln(L/j)}{j}}.$$

Let G be the same space considered as a subspace of ℓ_∞^L. Then

$$\begin{aligned} \pi_1(\mathrm{id}|_G : G \to \ell_2^L) &\le \pi_1(\mathrm{id} : \ell_\infty^L \to \ell_1^L) \|\mathrm{id}|_{G_1} : G_1 \to \ell_2^L\| \\ &\le C_6 L \sqrt{\frac{1 + \ln(L/j)}{j}}. \ \blacksquare \end{aligned}$$

The following lemma is a refinement of a corresponding result of Figiel and Johnson [FJ].

LEMMA 2.9. *Let $1 \le m < n$ and Y be an n-dimensional Banach space. Then there is a subspace $E_m \subset Y$ of dimension m and a contraction $u : \ell_2^m \to E_m$ such that*

$$\pi_1(u^{-1}) \le C_7 \left(\frac{n}{n-m}\right)^{5/2} \sqrt{1 + \ln \frac{n}{n-m}} \sqrt{n},$$

where $C_7 > 0$ is an absolute constant.

Proof. Let $m_1 = [(n-m)/2]$. According to [ST], there exists a subspace $F \subset Y$ with $\dim F = n - m_1$ and operators $a : \ell_2^{n-m_1} \to F$ and $b : F \to \ell_\infty^{n-m_1}$ such that $\|a\| \le 1$ and

$$\|b\| \le C\left(\frac{n}{m_1}\right)^2 \le 4C\left(\frac{n}{n-m}\right)^2$$

and $ba = \mathrm{id} : \ell_2^{n-m_1} \to \ell_\infty^{n-m_1}$. Define $L = n - m_1 \le n$ and $j - 1 = (n-m) - m_1 \ge (n-m)/2$. According to Lemma 2.8, there exists a subspace $G \subset \ell_\infty^{n-m_1}$ of codimension $j-1$ with the corresponding estimate for the 1-summing norm. Let G_2 be the same space considered as a subspace of $\ell_2^{n-m_1}$. Then $E_m = b_2^{-1}(G)$ is of dimension m and the inverse of $a : G_2 \to E_m$ is the restriction $b_2 : E_m \to G_2$ of b. We set $u = a|_{G_2}$ and observe $u^{-1} = b_2$. This map satisfies

$$\pi_1(u^{-1}) \le \pi_1(\mathrm{id}|_G : G \to \ell_2^L)\|b\| \le C_6 n \sqrt{\frac{1+\ln(n/j)}{j}}\, 4C\left(\frac{n}{n-m}\right)^2$$
$$\le 24 C_6 C \sqrt{n}\left(\frac{n}{n-m}\right)^{5/2}\sqrt{1+\ln\frac{n}{n-m}}. \ \blacksquare$$

As an immediate consequence, we see that Bourgain's estimate on ℓ_∞^k copies in large subspaces of ℓ_∞^n cannot essentially be improved. Bourgain [Bo] proved that for each $\tau > 0$, every n-dimensional subspace E of L_∞^N, with $n = N^\delta$, contains a k-dimensional subspace F, $(1+\tau)$-isomorphic to ℓ_∞^k for $k \ge c\tau^5 \frac{\delta^2}{\ln(1/\delta)}\sqrt{n}$.

COROLLARY 2.10. *For every $0 < \delta < 1$ and n-dimensional Banach space X, there exists a $[\delta n]$-dimensional subspace E such that for every k-dimensional subspace $F \subset E$,*

$$k \le C(\delta)\sqrt{n}\, d(F, \ell_\infty^k).$$

Here

$$C(\delta) \le C_7 (1-\delta)^{-5/2}\sqrt{1-\ln(1-\delta)}.$$

Proof. Indeed, let $m = [\delta n]$, $E = E_m$ and $F \subset E$. If $T : \ell_\infty^k \to F$ is an isomorphism, we get

$$k \le \pi_1(\mathrm{id}_{\ell_\infty^k}) = \|T\|\cdot\|T^{-1}\|\pi_1(\mathrm{id}_F) \le \|T\|\cdot\|T^{-1}\|\pi_1(u^{-1})\|u\|$$
$$\le \|T\|\cdot\|T^{-1}\|\left(\frac{n}{n-m}\right)^{5/2}\sqrt{1+\ln\frac{n}{n-m}}\sqrt{n}.$$

Taking the infimum over all isomorphisms, we deduce the assertion together with the required estimate of $C(\delta)$. $\blacksquare$

COROLLARY 2.11. *Let Y be an n-dimensional Banach space. For $1 \le m < n$ there exists an m-dimensional subspace E_m of Y such that for all k-dimensional subspaces $F \subset E_m$ with $1 \le k \le m$,*

$$\mathrm{vr}(F, \ell_2) \le C_7\left(\frac{n}{n-m}\right)^{5/2}\sqrt{\frac{n}{k}\left(1+\ln\frac{n}{n-m}\right)}\,\mathrm{vr}(F, \ell_\infty).$$

Here $C_7 > 0$ is the constant of Lemma 2.9.

Proof. Let $E_m \subset Y$ be an m-dimensional space obtained from Lemma 2.9. Let $F \subset E_m$ be of dimension k and $H = u^{-1}(F)$. Since $u(B_H) \subset B_F$, we obtain from [Ba, GJ, Theorem 3.7(ii)], i.e. (2.2) for $p = \infty$, and Lemma 2.9,

$$\mathrm{vr}(F, \ell_2) \le v_k(u^{-1}) \le \mathrm{vr}(F, \ell_\infty)\frac{\pi_1(u^{-1})}{\sqrt{k}}$$
$$\le C_7\left(\frac{n}{n-m}\right)^{5/2}\sqrt{1+\ln\frac{n}{n-m}}\sqrt{\frac{n}{k}}\,\mathrm{vr}(F,\ell_\infty). \ \blacksquare$$

In the case of ℓ_p-spaces, $2 \le p \le \infty$, this technique implies the following strengthening of Theorem 0.4.

PROPOSITION 2.12. *There exists an absolute constant $C_8 > 0$ such that for all $2 \le p \le \infty$:*

(a) *Let $1 \le k \le m < n \le L$ and E be an n-dimensional subspace of ℓ_p^L. There exists a subspace $E_m \subset E$ of dimension m such that for every k-dimensional subspace $F \subset E_m$,*

$$\max\left\{1, k^{1-1/p}L^{-1/2}, \frac{\sqrt{k}}{\sqrt{p}\,L^{1/p}}\right\}$$
$$\le C_8\sqrt{\frac{n}{k}}\left(\frac{n}{n-m}\right)^{5/2}\sqrt{1+\ln\frac{n}{n-m}}\,\mathrm{vr}(F,\ell_\infty).$$

For $p = \infty$,

$$\sqrt{\frac{k}{1+\ln(L/k)}} \le C_8\sqrt{\frac{n}{k}}\left(\frac{n}{n-m}\right)^{5/2}\sqrt{1+\ln\frac{n}{n-m}}\,\mathrm{vr}(F,\ell_\infty).$$

(b) *There exists a* $(j-1)$*-codimensional subspace* $G_j \subset \ell_p^L$ *such that for every* k*-dimensional subspace* $F \subset G_j$,

$$\max\left\{1, k^{1-1/p}L^{-1/2}, \frac{\sqrt{k}}{\sqrt{p}\,L^{1/p}}\right\} \le C_8 \frac{L}{\sqrt{k}\sqrt{j}}\sqrt{1+\ln(L/j)}\,\mathrm{vr}(F,\ell_\infty).$$

For $p=\infty$,

$$\sqrt{\frac{k}{1+\ln(L/k)}} \le C_8 \frac{L}{\sqrt{k}\sqrt{j}}\sqrt{1+\ln(L/j)}\,\mathrm{vr}(F,\ell_\infty).$$

REMARK 2.13. Observe that for any k-dimensional subspace $F \subset \ell_p^L$,

$$\mathrm{vr}(F,\ell_\infty) \le \gamma_p(\mathrm{id}_F) \le \lambda_p(F) \le \min\{\sqrt{p}\,d(F,\ell_2^k), k^{1/2-1/p}\}.$$

Here $\gamma_p(\mathrm{id}_F)$ is the p-factorable norm of id_F, and $\lambda_p(F) = \inf\{\|P\|\}$, with the infimum taken over all projections of ℓ_p^n onto F, is the relative projection constant of F. Indeed, the first estimate follows from $\mathrm{vr}(\ell_p,\ell_\infty) \le 1$. Clearly, the γ_p-norm can be estimated by the relative projection constant. Using Maurey's factorization theorem [Mau] together with the fact that the type 2 constant of ℓ_p is less than $\sqrt{p}$, we see that the relative projection constant can be estimated by the distance to the Hilbert space of F. The estimate of the other term in the minimum was proved by D. Lewis [L].

Proof (of Proposition 2.12). (a) Indeed, let $E_m \subset E$ be the space obtained in Lemma 2.9. Then according to Lemma 1.10 for all $F \subset E_m$ of dimension k,

$$\begin{aligned}\max\left\{1, k^{1-1/p}L^{-1/2}, \frac{\sqrt{k}}{\sqrt{p}\,L^{1/p}}\right\} &\le c_0\,\mathrm{vr}(F,\ell_2)\\ &\le c_0 C_7\left(\frac{n}{n-m}\right)^{5/2}\sqrt{\frac{n}{k}\left(1+\ln\frac{n}{n-m}\right)}\\ &\quad\times \mathrm{vr}(F,\ell_\infty),\end{aligned}$$

where the last inequality is obtained in Corollary 2.11. For the case $p=\infty$, we use the corresponding lower bound for the classical volume from Lemma 1.10.

In the proof of (b) we apply Lemma 2.8 directly. There exists a $(j-1)$-codimensional subspace $G \subset \ell_\infty^L$ such that

$$\pi_1(\mathrm{id}|_G : G \to \ell_2^L) \le C_6 L\sqrt{\frac{1+\ln(L/j)}{j}}.$$

For any k-dimensional subspace $F \subset G_j$, we deduce from Lemma 1.10 as in

the proof of Corollary 2.11 that

$$\begin{aligned}\max\left\{1, k^{1-1/p}L^{-1/2}, \frac{\sqrt{k}}{\sqrt{p}\,L^{1/p}}\right\} &\le c_0\,\mathrm{vr}(F,\ell_2) \le c_0 v_k(\mathrm{id}_F : F \to \ell_2^L)\\ &\le c_0^2\,\mathrm{vr}(F,\ell_\infty)\frac{\pi_1(\mathrm{id}_F : F\to \ell_2^L)}{\sqrt{k}}\\ &\le c_0^3\,\mathrm{vr}(F,\ell_\infty)\frac{L}{\sqrt{k}\sqrt{j}}\sqrt{1+\ln(L/j)}.\end{aligned}$$

The proof in the case $p=\infty$ is again the same using the appropriate lower bound from Lemma 1.10 for the classical volume ratio. ∎

REMARK 2.14. To illustrate this proposition, we consider the case $p=\infty$ and $j = L - L/\ln L$, $m = L/\ln L$. Then there exists an m-dimensional subspace $E \subset \ell_\infty^L$ (and in fact a subspace chosen at random has this property) such that for all k-dimensional subspaces $F \subset E$,

$$k \le c_0\sqrt{L(1+\ln L)}\,\mathrm{vr}(F,\ell_\infty).$$

Hence a k-dimensional subspace $F \subset E$ can be 2-complemented in ℓ_∞^L or 2-isomorphic to the Hilbert space only for $k \le \sqrt{L(1+\ln L)}$. This should be compared with Bourgain's [Bo] result on the existence of 2-isomorphic copies of ℓ_∞^k in large subspaces of ℓ_∞^L mentioned before the proof of Corollary 2.10.

Proof of Theorem 0.4. Let $1 \le k \le m < L$. If $m \le L/2$ we consider an $n = 2m$-dimensional subspace in which we find according to Proposition 2.12(a) an m-dimensional subspace such that for all k-dimensional subspaces F,

$$\max\left\{1, k^{1-1/p}L^{-1/2}, \frac{\sqrt{k}}{\sqrt{p}\,L^{1/p}}\right\} \le 8C_8\sqrt{2m/k}\,\mathrm{vr}(F,\ell_\infty).$$

Using $k = (k^{1-1/p}L^{-1/2})(L^{1/2}k^{1/p}) \le 8C_8\,\mathrm{vr}(F,\ell_\infty)\sqrt{m}\,L^{1/p}(L/k)^{1/2-1/p}$, we deduce the assertion. For $m \ge 2n$, we choose a $j = n-m$-codimensional subspace according to (b) and deduce the assertion. The proof for $p=\infty$ is similar. ∎

COROLLARY 2.15. *For* $2 \le p \le \infty$ *and* $0 < \delta < 1$ *there exists a constant* $C(\delta)$ *such that for all* $L \in \mathbb{N}$ *there exists a* $k = [\delta L]$*-dimensional subspace* E *of* ℓ_p^L *such that*

$$k^{1/2-1/p} \le C(\delta)\,\mathrm{vr}(E,\ell_\infty) \le C(\delta)k^{1/2-1/p}.$$

Here

$$C(\delta) \le C_9\left(\delta^{-1/2} + \sqrt{\frac{1}{1-\delta}\left(1+\ln\frac{1}{1-\delta}\right)}\right)$$

and $C_9 > 0$ *is an absolute constant.*

Proof. The upper estimate is clear, since by D. Lewis [L] any k-dimensional subspace $E \subset L_p$ satisfies

$$\mathrm{vr}(E,\ell_\infty) \le \mathrm{vr}(E,\ell_2) \le d(E,\ell_2^L) \le k^{1/2-1/p}.$$

Let $k = m = [\delta L]$. If $\delta \le 1/2$, we set $n = 2m$ and apply Proposition 2.12(a) to an arbitrary $2m$-dimensional subspace E'. Then we get a k-dimensional subspace $E \subset E'$ such that

$$k^{1/2-1/p} \le \sqrt{L/k}\, C_8 2^3\, \mathrm{vr}(E,\ell_\infty) = 8C_8\delta^{-1/2}\, \mathrm{vr}(F,\ell_\infty).$$

For $\delta \ge 1/2$, we set $j-1 = L-m = L-[\delta L]$. Then the space $E = G_j$ from Proposition 2.12(b) has dimension $k = [\delta L] \ge L/2$ and satisfies

$$\begin{aligned} k^{1/2-1/p} &\le \sqrt{L/k}\, C_8 \frac{L}{\sqrt{k}\sqrt{j}} \sqrt{1+\ln(L/j)}\, \mathrm{vr}(E,\ell_\infty) \\ &\le 2C_8\sqrt{(1-\delta)^{-1}(1-\ln(1-\delta))}\, \mathrm{vr}(E,\ell_\infty). \ \blacksquare \end{aligned}$$

REMARK 2.16. The corollary yields a generalization of Sobczyk's estimate of the relative projection constant for subspaces of ℓ_p^L. In fact Sobczyk [So] proved that the eigenspace $E_l \subset \ell_p^{2^l}$ of the normalized Walsh $2^l \times 2^l$-matrix satisfies $\lambda_p(E_l) \ge c(2^l)^{1/2-1/p}$. This shows that D. Lewis' upper estimate $\lambda_p(F) \le k^{1/2-1/p}$ for arbitrary k-dimensional subspaces $F \subset \ell_p$ cannot be improved. Using

$$\frac{1}{C(\delta)} k^{1/2-1/p} \le \mathrm{vr}(F,\ell_\infty) \le \mathrm{vr}(F,\ell_p) \le \lambda_p(F) \le k^{1/2-1/p}$$

our argument yields in addition examples of maximal relative projection inside ℓ_p^L of dimension $k = [\delta L]$ for any $0 < \delta < 1$.

Now, we are able to prove the main formula in this section

PROPOSITION 2.17. *Let $1 \le k < n$ and $T : X \to Y$ be a linear operator. Then*

$$\mathrm{vr}_n(T,\ell_2) \le C_{10}\left(\frac{n}{n-k}\right)^{3k/n} \mathrm{vr}_k(T,\ell_\infty)^{k/n}\, \mathrm{vr}_{n-k}(T,\ell_2)^{(n-k)/n}.$$

Here $C_{10} > 0$ denotes an absolute constant.

Proof. There is no loss of generality to assume $\dim X = \dim Y = n$ and T of rank n. Let $m = k$ and $F = E_m \subset Y$ be a k-dimensional space satisfying the assertion of Corollary 2.11. Furthermore let $E = T^{-1}(F)$ and $T_{EF} : E \to F$ the astriction of T. Then we get

$$\begin{aligned} \mathrm{vr}_k(T|_E,\ell_2) &= \mathrm{vr}(F,\ell_2) v_k(T_{EF}) \\ &= C_7\left(\frac{n}{n-k}\right)^{5/2} \sqrt{\frac{n}{k}\left(1+\ln\frac{n}{n-k}\right)}\, \mathrm{vr}(F,\ell_\infty) v_k(T_{EF}) \\ &\le C_7\left(\frac{n}{n-k}\right)^{3} \sqrt{\frac{n}{k}}\, \mathrm{vr}_k(T,\ell_\infty). \end{aligned}$$

Now apply Corollary 2.6 for $p=2$ together with the estimate $(n/k)^{k/n} \le 1.5$ to get

$$\begin{aligned} \mathrm{vr}_n(T,\ell_2) &\le C_5\sqrt{2}\, \mathrm{vr}_{n-k}(T,\ell_2)^{(n-k)/k} \sup_E \inf_{E_k \subset E} \mathrm{vr}_k(T|_{E_k},\ell_2)^{k/n} \\ &\le C_5\sqrt{2}\, \mathrm{vr}_{n-k}(T,\ell_2)^{(n-k)/k} C_7\left(\frac{n}{n-k}\right)^{3k/n} \sqrt{2}\, \mathrm{vr}_k(T,\ell_\infty)^{k/n}. \ \blacksquare \end{aligned}$$

REMARK 2.18. For many Banach spaces (see below), we have good control of the least constant $c(E,\delta)$ satisfying

$$\mathrm{vr}_{[(1-\delta)n]}(\mathrm{id}_E,\ell_2) \le c(E,\delta)\, \mathrm{vr}(E).$$

In this case, we get

$$\mathrm{vr}(E) \le c(E,\delta)^{\delta^{-1}} \sup_{F \subset E,\, \dim F = [\delta n]} \mathrm{vr}(F,\ell_\infty).$$

Indeed, a good control for $c(E,\delta)$ is available for $E = \ell_p^n$. Moreover, if we assume that John's [J] map $u : \ell_2^n \to E$ satisfies

$$v_{[(1-\delta)n]}(u^{-1}) \le c(E,\delta) v_n(u^{-1}) = c(E,\delta)\, \mathrm{vr}(E),$$

then for all $[(1-\delta)n]$-dimensional subspaces $F \subset E$, we have

$$\mathrm{vr}(F) \le v_{[(1-\delta)n]}(u^{-1}) \le c(E,\delta)\, \mathrm{vr}(E).$$

Estimates for $c(E,\delta)$ are known in the following cases:

1. If E has enough symmetries, then the ellipsoid of maximal volume is a multiple of the ℓ-ellipsoid (see [PSc]) and therefore

$$\begin{aligned} \sqrt{k}\, v_k(u^{-1}) &\le \ell((u^{-1})^*) \le K(E)\ell^*(u^{-1}) = K(E)\frac{n}{\ell(u)} \\ &\le K(E)\sqrt{n}\, v_n(u^{-1}). \end{aligned}$$

Here $\ell(u)^2 = \mathbb{E}\|\sum_{i=1}^n g_i u(e_i)\|^2$ for independent normalized gaussian variables is the well known ℓ-norm. Hence $c(E,\delta) \le K(E)(1-\delta)^{-1/2}$.

2. A similar argument applies for spaces with 1-symmetric basis $e_1,\ldots\ldots,e_n$. Then

$$\Big\|\sum_{i=1}^n e_i\Big\|_E \Big\|\sum_{i=1}^n e_i^*\Big\|_{E^*} = n.$$

Since this space has enough symmetries, the John map is a multiple of the identity. The volume estimate of the ℓ-norm is replaced by a corresponding one for the Rademacher average [CP] (see (1.6)). This yields the estimate $c(E,\delta) \le c_0(1-\delta)^{-1/2}(1-\log(1-\delta))^{1/2}$.

Theorem 0.2 will be deduced from the following

COROLLARY 2.19. *Let $\alpha \in \mathbb{R}$. Then there exists a constant $C(\alpha)$ such that for all $n \in \mathbb{N}$ and linear operators $T : X \to Y$,*

$$n^{\alpha} \operatorname{vr}_n(T, \ell_2) \le C(\alpha) \sup_{k \le n/2} k^{\alpha} \operatorname{vr}_k(T, \ell_\infty).$$

Proof. Using Lemma 2.2 (see (2.3)), we get

$$\begin{aligned} S := \sup_{k \le n} k^{\alpha} \operatorname{vr}_k(T, \ell_2) &\le c_0 \sup_{k \le n/2} \max((2k)^{\alpha}, (2k+1)^{\alpha}) \operatorname{vr}_{2k}(T, \ell_2) \\ &\le c_0 c_1 \sup_{k \le n/2} \max((2k)^{\alpha}, (2k+1)^{\alpha}) \operatorname{vr}_k(T, \ell_2)^{1/2} \operatorname{vr}_k(T, \ell_\infty)^{1/2} \\ &\le c_0 c_1 \sup_{1 \le k \le n/2} \frac{\max((2k)^{\alpha}, (2k+1)^{\alpha})}{k^{\alpha}} S^{1/2} \big(\sup_{k \le n/2} k^{\alpha} \operatorname{vr}_k(T, \ell_\infty) \big)^{1/2}. \end{aligned}$$

If $\alpha \ge 0$, we can put $C(\alpha) = c_0^2 c_1^2 9^{\alpha}$. For $\alpha \le 0$, we can choose $C(\alpha) = c_0^2 c_1^2$. ∎

The case $T = \mathrm{id}_E$ and $= 0$ implies Theorem 0.2:

THEOREM 2.20. *There exists an absolute constant $C > 0$ such that for all n-dimensional Banach spaces E,*

$$\operatorname{vr}(E, \ell_2) \le C \sup_{F \subset E,\, \dim F \le n/2} \operatorname{vr}(F, \ell_\infty).$$

COROLLARY 2.21. *Let $n \in \mathbb{N}$, $m = [n/4]$ and $T : X \to Y$ be a linear operator. Then*

$$\operatorname{vr}_n(T, \ell_2) \le C_0 \Big(\prod_{j=1}^{m} \operatorname{vr}_j(T, \ell_\infty) \Big)^{1/m}.$$

Proof. Let $2^k \le n \le 2^{k+1}$. Using Lemma 2.2, we can assume $n = 2^k$. Inductively, we obtain

$$\begin{aligned} \operatorname{vr}_n(T, \ell_2) &\le C \operatorname{vr}_{n/2}(T, \ell_2)^{1/2} \operatorname{vr}_{n/2}(T, \ell_\infty)^{1/2} \\ &\le C C^{1/2} \operatorname{vr}_{n/4}(T, \ell_2)^{1/4} \operatorname{vr}_{n/4}(T, \ell_\infty)^{1/4} \operatorname{vr}_{n/2}(T, \ell_\infty)^{1/2} \le \ldots \\ &\le C^2 \prod_{j=1}^{k} \operatorname{vr}_{n/2^j}(T, \ell_\infty)^{1/2^j} \operatorname{vr}_1(T, \ell_\infty)^{1/n}. \end{aligned}$$

For $j < k$ and $n/2^{j+1} < l \le n/2^j$, we deduce from Lemma 2.2 that

$$\operatorname{vr}_{n/2^j}(T, \ell_\infty) \le c_0 \operatorname{vr}_l(T, \ell_\infty).$$

On the other hand, there are $n/2^{j+1}$ such numbers l, hence

$$\operatorname{vr}_{n/2^j}(T, \ell_\infty) \le c_0 \Big(\prod_{n/2^{j+1} < l \le n/2^j} \operatorname{vr}_l(T, \ell_\infty) \Big)^{2 \cdot 2^j / n}.$$

Recollecting this, we obtain

$$\operatorname{vr}_n(T, \ell_2) \le C^2 c_0 \Big(\prod_{1 \le l \le n/2} \operatorname{vr}_l(T, \ell_\infty) \Big)^{2/n}. \ ∎$$

Now, we give the proof of Theorem 0.3.

PROPOSITION 2.22. *Let $\alpha \in \mathbb{R}$ and $T : X \to Y$ be a linear operator. Then the following assertions are equivalent.*

(i) *There exists a constant $c_1 > 0$ such that for all $n \in \mathbb{N}$,*

$$\operatorname{vr}_n(T, \ell_2) \le c_1 n^{\alpha}.$$

(ii) *There exists a constant $c_2 > 0$ such that for all $n \in \mathbb{N}$,*

$$\operatorname{vr}_n(T, \ell_\infty) \le c_2 n^{\alpha}.$$

(iii) *There exists $0 < \delta \le 1$ and a constant c_3 such that for all $n \in \mathbb{N}$ and n-dimensional subspaces $E \subset X$, there exists a subspace E_δ of dimension $k \ge \delta n$ with*

$$\operatorname{vr}_k(T|_{E_\delta}, \ell_2) \le c_3 n^{\alpha}.$$

(iv) *There exists $0 < \delta \le 1$ and a constant c_4 such that for all $n \in \mathbb{N}$ and n-dimensional subspaces $E \subset X$, there exists a subspace E_δ of dimension $k \ge \delta n$ with*

$$\operatorname{vr}_k(T|_{E_\delta}, \ell_\infty) \le c_4 n^{\alpha}.$$

Proof. The implications (i)⇒(ii), (i)⇒(iii), (iii)⇒(iv) and (ii)⇒(iv) are obvious.

Let us show (iii)⇒(i). We can assume that T is of finite rank by considering all the restrictions to large finite-dimensional subspaces. Then

$$S := \sup_k k^{-\alpha} \operatorname{vr}_k(T, \ell_2)$$

is a finite number and there exists $n \in \mathbb{N}$ and an n-dimensional subspace $E \subset X$ such that

$$S \le 2n^{-\alpha} \operatorname{vr}_n(T|_E, \ell_2).$$

By (iii) there exists a $k \ge \delta n$-dimensional subspace $E' \subset E$ such that

$$\operatorname{vr}_k(T|_{E'}, \ell_2) \le c_2 n^{\alpha}.$$

Corollary 2.6 yields

$$\begin{aligned} S &\le 2n^{-\alpha} \operatorname{vr}_n(T|_E, \ell_2) \le C n^{-\alpha} \operatorname{vr}_{n-k}(T|_E, \ell_2)^{(n-k)/n} \operatorname{vr}_k(T|_{E'}, \ell_2)^{k/n} \\ &\le C n^{-\alpha} (n-k)^{\alpha((n-k)/n)} S^{(n-k)/n} n^{\alpha n/k} c_2^{k/n}. \end{aligned}$$

Hence, we deduce from $n/k \le \delta^{-1}$ that

$$S \le C^{\delta^{-1}} c_2 \Big(\frac{n-k}{n} \Big)^{\alpha(n-k)/k}.$$

In the case $\alpha \geq 0$, we have
$$((n-k)/n)^{\alpha(n-k)/k} \leq 1.$$
In the case $\alpha < 0$, we deduce
$$\left(\frac{n-k}{n}\right)^{\alpha(n-k)/k} = \left(\frac{n}{n-k}\right)^{\frac{n-k}{n}\cdot\frac{-\alpha n}{k}} \leq e^{-\alpha n/k} \leq e^{-\alpha/\delta}.$$

For (iv)⇒(iii), assume $E \subset X$ is of dimension $n > 1$ and T of rank n. According to Corollary 2.11, there is a subspace $F_1 \subset T(E)$ of dimension $\geq n/2$ such that for all $F \subset F_1$ of dimension k,

(2.4) $$\mathrm{vr}(F, \ell_2) \leq c_0\sqrt{n/k}\,\mathrm{vr}(F, \ell_\infty).$$

Let $E_\delta \subset T^{-1}(F_1)$ be a subspace of dimension $k = \dim E_\delta \geq \delta[n/2] \geq \delta n/4$ such that
$$\mathrm{vr}_k(T|_{E_\delta}, \ell_\infty) \leq c_4 n^\alpha.$$
Then we obtain from (2.4), for $F_\delta = T(E_\delta) \subset F_1$,
$$\mathrm{vr}_k(T|_{E_\delta}, \ell_2) = \mathrm{vr}_k(F_\delta, \ell_2)v_k(T_{E_\delta F_\delta}) \leq c_0\sqrt{n/k}\,\mathrm{vr}_k(F_\delta, \ell_\infty)v_k(T_{E_\delta F_\delta})$$
$$= c_0\sqrt{n/k}\,\mathrm{vr}_k(T|_{E_\delta}, \ell_\infty) \leq c_0 2\delta^{-1/2} c_4 n^\alpha. \ \blacksquare$$

Condition (ii) will be used to construct examples in the forthcoming paper [GJ2].

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The Lévy continuity theorem for nuclear groups

by

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Abstract. Let G be an abelian topological group. The Lévy continuity theorem says that if G is an LCA group, then it has the following property (PL): a sequence of Radon probability measures on G is weakly convergent to a Radon probability measure μ if and only if the corresponding sequence of Fourier transforms is pointwise convergent to the Fourier transform of μ. Boulicaut [Bo] proved that every nuclear locally convex space G has the property (PL). In this paper we prove that the property (PL) is inherited by nuclear groups, a variety of abelian topological groups containing LCA groups and nuclear locally convex spaces, introduced in [B1].

1. Introduction. Let G be an LCA group and Γ the dual group. The Bochner theorem may be formulated in the following way:

(α) *Every continuous positive-definite function on G is the inverse Fourier transform of a (unique) finite positive Radon measure on Γ.*

This theorem can be extended to inverse limits and countable direct limits of LCA groups. It was also extended to some other classes of abelian topological groups: nuclear locally convex spaces (the Minlos theorem), Hausdorff quotient groups of such spaces (Yang [Y]), locally convex spaces over p-adic fields (Mądrecki [M]). Trying to give a common generalization of the corresponding results, the author introduced in [B1] the so-called nuclear groups, a variety of abelian topological groups containing LCA groups and nuclear locally convex spaces (the definition and basic properties of nuclear groups are given in Section 5 below). It was proved in [B1, (12.1)] that every nuclear group G satisfies an analogue of (α).

The Lévy continuity theorem may be formulated in the following way:

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