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Banach spaces in which all multilinear forms are weakly sequentially continuous

by

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Abstract. We solve several problems in the theory of polynomials in Banach spaces. (i) There exist Banach spaces without the Dunford–Pettis property and without upper p-estimates in which all multilinear forms are weakly sequentially continuous: some Lorentz sequence spaces, their natural preduals and, most notably, the dual of Schreier's space. (ii) There exist Banach spaces X without the Dunford–Pettis property such that all multilinear forms on X and X^* are weakly sequentially continuous; this gives an answer to a question of Dimant and Zalduendo [20]. (iii) The sum of two polynomially null sequences need not be polynomially null; this answers a question of Biström, Jaramillo and Lindström [8] and also of González and Gutiérrez [23]. (iv), (v) The absolutely convex closed hull of a pw-compact set need not be pw-compact; the projective tensor product of two polynomially null sequences need not be a polynomially null sequence. This answers two questions of González and Gutiérrez [23]. (vi) There exists a Banach space without property (P); this answers a question of Aron, Choi and Llavona [5].

1. The setting and the problem. A homogeneous continuous polynomial on a Banach space X is a mapping P of the form $P(x) = A(x, \ldots, x)$ where $A : X \times \ldots \times X \to \mathbb{R}$ is a multilinear continuous map on X. In contrast to the linear setting, continuous polynomials are not usually continuous with respect to the weak topology. Perhaps the simplest example is the map $\|\cdot\|^2$ on ℓ_2 which is not weakly sequentially continuous since the canonical basis is weakly null. It is an interesting open problem to characterize those spaces where all continuous polynomials are weakly sequentially continuous. Of course, if all multilinear forms on a Banach space are weakly sequentially continuous then all polynomials are weakly sequentially continuous. We shall therefore study the weak sequential continuity of multilinear forms.

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The basic examples of spaces where all multilinear forms are weakly sequentially continuous are:

(i) Spaces with the Dunford–Pettis property; this was proved by Ryan [36].

(ii) Tsirelson's original space T^*. This was proved by Alencar *et al.* [1]. Also, it is a consequence of two facts: that the space T^* admits, for all $p < \infty$, upper p-estimates (see [1] and also [16]) and the argument (see below) in [22, 26, 27] that a space has all its polynomials weakly sequentially continuous if it has no lower q-estimates, for $q > 1$.

Both Ryan's proof and the role of upper and lower estimates in polynomial matters are by now rather well understood. Nevertheless, they have remained unrelated so far. An explicit question in this direction, to the best of our knowledge due to M. González, is: what do the spaces with the Dunford–Pettis property and the spaces with upper estimates have in common?

Let us recall some basic facts about upper and lower estimates. Let $p > 1$; the number p^* is defined by $p + p^* = pp^*$. A sequence $\{x_n\}_n$ in a Banach space X is said to be *weakly p-summable* (resp. *weakly p-convergent to x*) if for each $x^* \in X^*$, $\{x^*(x_n)\}_n \in \ell_p$ (resp. $\{x^*(x_n - x)\}_n \in \ell_p$); equivalently, for some constant $C > 0$ and every finite set $a_1, \ldots, a_n \in \mathbb{R}$, $n \in \mathbb{N}$,

$$\Big\|\sum_{i=1}^{n} a_i x_i\Big\| \le C\Big(\sum_{i=1}^{n} |a_i|^{p^*}\Big)^{1/p^*}.$$

For this reason one also says that $\{x_n\}_n$ has an *upper p^*-estimate.* Weakly 1-summable sequences are those admitting upper ∞-estimates. A space is said to have the *W_p-property* ([17]) if bounded sequences admit weakly p-convergent subsequences. Spaces with property W_p are obviously reflexive. A space X is said to have the *weak-W_p-property* (following [17]), or the S_{p^*}-property (following [31]) if weakly null sequences admit weakly p-summable subsequences. In this case we say that the space admits *upper p^*-estimates.*

Analogously, let $q \ge 1$; a sequence $\{x_n\}_n$ in a Banach space X is said to admit a *lower q-estimate* if for some constant $C > 0$ and every finite set $a_1, \ldots, a_n \in \mathbb{R}$, $n \in \mathbb{N}$,

$$\Big\|\sum_{i=1}^{n} a_i x_i\Big\| \ge C\Big(\sum_{i=1}^{n} |a_i|^{q}\Big)^{1/q}.$$

The space X is said to have the *T_q-property* (following [27]) if every normalized weakly null sequence has a subsequence with a lower q-estimate. We shall simply say that X admits *lower q-estimates.*

A space is said to have the *$\mathcal{C}_p$-property* [11] if weakly p-summable sequences are norm convergent, or equivalently, if every operator $\ell_{p^*} \to X$ is compact. It is not hard to see that if X has property W_p then X^* has property $\mathcal{C}_q$ for all $q < p^*$ (see [11]).

We denote by $\mathcal{L}(^N X)$ (resp. $\mathcal{L}_s(^N X)$) the space of continuous N-linear (resp. N-linear symmetric) forms on X. We denote by $\mathcal{P}(X)$ (resp. $\mathcal{P}(^N X)$) the space of all polynomials (resp. N-homogenous polynomials) on X. Recall that there is an identification between the spaces $\mathcal{P}(^N X)$ and $\mathcal{L}_s(^N X)$ given by the polarization formula (see e.g. [34] for details). The intersection of those classes with that of weakly sequentially continuous functions will be denoted with the subscript wsc, as in $\mathcal{P}_{\rm wsc}(^N X)$, $\mathcal{L}_{\rm wsc}(^N X)$, etc.

Concerning operators, $\mathcal{L}(E,F)$ denotes the space of operators between E and F, while $\mathcal{W}(E,F)$ and $\mathcal{K}(E,F)$ are the spaces of weakly compact and compact operators. Following [11], we denote by $\mathcal{C}_\infty(E,F)$ the space of all completely continuous operators from E into F.

A sequence $\{x_n\}_n$ in a Banach space X is said to be *polynomially null* (for short, *$\mathcal{P}$-null*) if $\lim_{n\to\infty} P(x_n) = 0$ for every $P \in \mathcal{P}(X)$. The sequence is called *$\mathcal{P}_N$-null* if $\lim_{n\to\infty} P(x_n) = 0$ for every $P \in \mathcal{P}(^N X)$.

Gonzalo and Jaramillo ([26, 27]; see also [22, 39]) proved that *a basis is $\mathcal{P}_N$-null if it does not admit subsequences with lower N-estimates.* This immediately gives the next result, already proved in [3, 27, 35]:

Proposition 1.1. *If X admits, for all $p < \infty$, upper p-estimates then all polynomials on X are weakly sequentially continuous.*

Moreover, one has

Proposition 1.2. *Let $\{u_i\}_i$ and $\{v_i\}_i$ be equivalent basic sequences in a Banach space X. Then $\{u_i\}_i$ is $\mathcal{P}$-null with respect to polynomials defined on $[u_i]$ if and only if $\{v_i\}_i$ is $\mathcal{P}$-null with respect to polynomials defined on $[v_i]$.*

It is time to introduce the classes we are going to study throughout the paper.

Definition. We will say that a Banach space X is a *$\mathcal{P}$-space* (resp. an *$\mathcal{M}$-space*) if all polynomials on X (resp. all multilinear forms) are weakly sequentially continuous. The space X is a *$\mathcal{P}_N$-space* (resp. an *$\mathcal{M}_N$-space*) if all N-homogeneous polynomials on X (resp. all N-linear forms) are weakly sequentially continuous. The natural identification yields that polynomials are weakly sequentially continuous if and only if symmetric multilinear forms are.

In [29] several classes of reflexive spaces are shown not to be $\mathcal{P}$-spaces. The reader is warned to distrust that the behavior of sequences with respect to polynomials is more or less the same as for functionals: it is not. As a striking example, two equivalent sequences need not be $\mathcal{P}$-null at the same time: if $T : \ell_2 \to C(K)$ is an into isometry, then the sequences $\{(Te_n, 0)\}_n$

and $\{(0, e_n)\}_n$ in $C(K) \times \ell_2$ are equivalent, but the former is $\mathcal{P}$-null while the latter is not.

Let us explicitly state the main problems we have considered and the results obtained:

Q.1. *What do spaces with upper estimates and spaces with the Dunford–Pettis property have in common?*

This will be studied in Section 2. We present a seemingly new approach to study weak sequential continuity of multilinear forms by factorization of some classes (not ideals!) of operators. This exhibits a general framework in which spaces with the Dunford–Pettis property and Tsirelson's space are nothing but extremal cases. Moreover, we describe a *basic strategy* (modelled upon results of Jiménez and Payá [30]) to prove that a space is an $\mathcal{M}$-space (but not to prove that a particular polynomial is weakly sequentially continuous!).

Q.2. *Do there exist other "types" of spaces where all polynomials are weakly sequentially continuous?*

The answer is affirmative. A first example can be obtained by retouching the main construction in [30]. It is the standard predual of a suitable Lorentz sequence space. The presentation of this example is postponed until Section 5. We first present a more natural example selected from the basic catalogue of "pathological" Banach spaces: the dual of the Schreier space. Although it has the weak Banach–Saks property [24], it can hardly claim to admit upper p-estimates. Moreover, it was proved in [14] that the Schreier space fails the Dunford–Pettis property. This example is discussed in Sections 3 and 6.

And, nonetheless, answers to Q.2 admit gradations, and so they are considered in Sections 4 and 5. Observe that one is trying to do two things at the same time: make N-linear forms weakly sequentially continuous and avoid upper p-estimates. The "property" of not having upper p-estimates can be understood either in a strict sense or one can try to obtain a total negation: there is not a single normalized sequence in the space admitting an upper p-estimate. In Section 4 we consider the total negation for $p = 2 = N$, and in Section 5 we consider multilinear forms without limitation.

The examination of the standard examples shows that when it is the Dunford–Pettis property that is responsible for being a $\mathcal{P}$-space then it appears simultaneously in X and X^* (because, as we observe in passing, there is essentially one known example of Banach space with the Dunford–Pettis property whose dual does not have the Dunford–Pettis property: the space $(\bigoplus_{n=1}^{\infty} \ell_2^n)_{\ell_1}$, see [37]). The situation changes when the reason behind being a $\mathcal{P}$-space is the upper p-estimates: the existence of one single sequence with an upper estimate in a space X, whenever X^* does not contain ℓ_1, produces a non-weakly sequentially continuous polynomial on X^*. Indeed, if there is a normalized sequence with an upper p-estimate then there is a non-compact operator $T : \ell_p \to X$; then the transpose operator $T^* : X^* \to \ell_p^*$ takes a weak Cauchy sequence into the usual basis of ℓ_{p^*} and now the conclusion follows by [29].

We shall also study duals of $\mathcal{P}$-spaces, answering the following question:

Q.3. *Apart from the Dunford–Pettis case, can both X and X^* be $\mathcal{P}$-spaces simultaneously?*

This question was asked by Dimant and Zalduendo [20] for the particular case of reflexive spaces (with the approximation property), where it is equivalent to asking whether $\mathcal{P}(^N X)$ and $\mathcal{P}(^N X^*)$ can be, simultaneously and for all N, reflexive. Another open question formulated in [8] and [23] is:

Q.4. *Is the sum (resp. tensor product) of two polynomially null sequences also polynomially null (resp. weakly null)?*

The first question is relevant to determining the structure of the space of $\mathcal{P}$-null sequences in a Banach space. When X is a $\mathcal{P}$-space or a Λ-*space* (a space where $\mathcal{P}$-null sequences are norm null, see [9]) then the answer is affirmative. In [8] the authors show that the answer to Q.4 is also positive for spaces with a certain polynomial Dunford–Pettis property. Nevertheless, the general answer to the two questions is negative (see Theorem 5.5). While studying polynomial properties of Banach spaces Aron, Choi and Llavona (see [5], and also [28]) introduced property (P) as follows: for any two bounded sequences $\{u_n\}_n$ and $\{v_n\}_n$, if $\lim_{n\to\infty}(P(u_n) - P(v_n)) = 0$ for every polynomial P, then $\lim_{n\to\infty} P(u_n - v_n) = 0$ for all polynomials P. They pose the following question:

Q.5. *Does there exist a space which fails property* (P)*?*

In Section 5 we present the first example of a Banach space failing property (P).

2. When are all multilinear forms weakly sequentially continuous? We begin with bilinear forms and then proceed inductively. Let $B : X \times X \to \mathbb{K}$ be a bilinear form on X. It induces a linear operator $b : X \to X^*$ given by $b(x)(y) = B(x, y)$; in turn, b induces—coordinatewise—an operator

$$\widetilde{b} : c_0^w(X) \to c_0^w(X^*)$$

where $c_0^w(X)$ denotes the space of all weakly null sequences in X. It is well known that $c_0^w(X)$ coincides with the space of all weak*-to-weak continuous operators from X^* into c_0 while $c_0^w(X^*) = \mathcal{W}(X, c_0)$. Thus, one has the diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "b"'] & & c_0 \\ & X^* \arrow[ur, "L"'] & \end{tikzcd}$$

that explains the action of the correspondence $\widetilde{b}$: it sends a weakly null sequence of L (in X) into the weakly null sequence of $L \circ b$ (in X^*).

THEOREM 2.1. *Let X be a Banach space. The following are equivalent*:

(i) $\mathcal{L}(^2X) = \mathcal{L}_{\text{wsc}}(^2X)$.

(ii) *Every operator $T : X \to c_0$ that factorizes through X^* in the form $T = L \circ b$, where L is weak*-to-weak continuous, is completely continuous.*

Proof. Assume that all bilinear mappings on X are weakly sequentially continuous and consider a linear operator $X \to c_0$ that can be factorized through X^* as in the last diagram, where L is weak*-to-weak continuous. If L is represented by the weakly null sequence $\{l_n\}_n$ in X and $x \in X$ then

$$L \circ b(x) = \{\langle bx, l_n\rangle\}_n.$$

Thus, a weakly null sequence $\{x_j\}_j$ in X is transformed into the weakly null sequence $\{\{\langle bx_j, l_n\rangle\}_n\}_j$ in c_0. Observe that if $\{\{\langle bx_j, l_n\rangle\}_n\}_j$ were not norm null in c_0, then for some $\varepsilon > 0$ there would exist subsequences $\{j(k)\}_k$ and $\{n(k)\}_k$ such that

$$\limsup_{k\to\infty} |\langle bx_{j(k)}, l_{n(k)}\rangle| \geq \varepsilon,$$

contrary to the weak sequential continuity of the bilinear form $B(x,y) = \langle bx, y\rangle$.

For the converse, note first that since operators are always weakly (sequentially) continuous, in order to obtain the weak sequential continuity of bilinear forms it is enough to study the behavior of weakly null sequences. So, let $B : X \times X \to \mathbb{K}$ be a bilinear mapping and let $\{(x_n, y_n)\}_n$ be a weakly null sequence. Consider the factorization through the map $L : X^* \to c_0$ given by $L(x) = \{\langle x, x_n\rangle\}_n$. The conclusion is obvious. ■

That bilinear forms are weakly sequentially continuous in spaces with the Dunford–Pettis property is now clear. The condition imposed on the second term of the factorization, to be weak*-to-weak continuous, already implies that the operator has to be weakly compact. Since the Dunford–Pettis property of X means that $\mathcal{W}(X, c_0) \subset \mathcal{C}_\infty(X, c_0)$, condition (ii) is satisfied.

When X has all properties W_p then $\mathcal{L}(X, X^*) \subset \mathcal{C}_\infty(X, X^*)$ or, equivalently, $\mathcal{L}(X, X^*) = \mathcal{K}(X, X^*)$. This equivalence also holds when X has all properties S_p and does not contain ℓ_1: indeed, in that case X^* has all properties T_q ([27]) and then every operator $X \to X^*$ is compact. The particular case of the original Tsirelson space T could also be deduced from Straeuli [38], who proved that all Banach–Saks operators $F \to T$, for any Banach space F, are compact; since T^* has the Banach–Saks property, $\mathcal{L}(T^*, T) = \mathcal{K}(T^*, T)$.

The above results can be generalized to the multilinear case. An N-linear form $A : X^N \to \mathbb{K}$ induces a linear map $\widetilde{A} : X \to \mathcal{L}(^{N-1}X)$ by the formula

$$\widetilde{A}(x) = A(x, \cdot, \ldots, \cdot).$$

Now, a weakly null sequence $\{(x_1(n), \ldots, x_{N-1}(n))\}_n$ in X^{N-1} induces a linear map δ from the space $\mathcal{L}_{\text{wsc}}(^{N-1}X)$ of weakly sequentially continuous $(N-1)$-linear forms on X into c_0 by the formula $\delta(T)(n) = \{T(x_1(n), \ldots \ldots, x_{N-1}(n))\}_n$. One has the diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "\widetilde{A}"'] & & c_0 \\ & \mathcal{L}_{\text{wsc}}(^{N-1}X) \arrow[ur, "\delta"'] & \end{tikzcd}$$

Then one has:

THEOREM 2.2. *Let X be a Banach space and let $N \geq 2$ be an integer. Assume that $\mathcal{L}(^{N-1}X) = \mathcal{L}_{\text{wsc}}(^{N-1}X)$. Then the following are equivalent*:

(i) $\mathcal{L}(^NX) = \mathcal{L}_{\text{wsc}}(^NX)$.

(ii) *Every operator $T : X \to c_0$ that factorizes through the space $\mathcal{L}(^{N-1}X)$ in the form $T = L \circ b$, where L is weak*-to-weak continuous, is completely continuous.*

Let us remark that the weak* topology we consider on $\mathcal{L}(^NX)$ is as the dual space of the N-fold tensor product $X \otimes_\pi \ldots \otimes_\pi X$. The proof goes as in Theorem 2.1.

This is enough to provide another proof for Ryan's result [36]:

COROLLARY 2.3 (Ryan's result). *Every multilinear form on a Banach space with the Dunford–Pettis property is weakly sequentially continuous.*

Spaces having all S_p-properties and not containing ℓ_1 are also covered by this result since all operators $X \to \mathcal{L}(^{N-1}X)$ are compact. This is a consequence of the fact that the space $\mathcal{L}(^{N-1}X)$ has all $\mathcal{C}_q$-properties (all operators $\ell_q \to \mathcal{L}(^{N-1}X)$ are compact for all $q > 1$; [2], see also [11]).

REMARK. While the Dunford–Pettis property means that all operators $X^* \to c_0$ converge uniformly on weakly null sequences, having weakly sequentially continuous N-linear forms just means that operators $X^* \to c_0$ have to converge uniformly only over those weakly null sequences that are images of weakly null sequences of X^{N-1} under $(N-1)$-linear forms $X^{N-1} \to X^*$. In Section 5 we shall show examples of spaces where $c_0^w(X^*)$ is not covered by the images of weakly null sequences of X^{N-1} under N-linear maps.

An analogous characterization for polynomials can also be obtained. It is enough to recall the identification of polynomials with symmetric linear forms. Moreover, $\mathcal{P}(^N X)$ is also a dual space (dual of the symmetrized projective tensor product). One only has to adapt the proof and determine when an operator "corresponds" to a polynomial. This can be done through the following definition.

Definition. An operator $T : X \to \mathcal{L}_s(^N X)$ is said to be *symmetric* if for every permutation σ of $\{1, \ldots, N\}$ and every set $x_1, \ldots, x_N$ of elements of X one has
$$T(x_1, \ldots, x_N) = T(x_{\sigma(1)}, \ldots, x_{\sigma(N)}).$$

It is now completely straightforward that Theorems 2.1 and 2.2 admit the following variation characterizing $\mathcal{P}$-spaces.

Theorem 2.1.s. *Let X be a Banach space. The following are equivalent*:

(i) $\mathcal{L}_s(^2X) \subset \mathcal{L}_{wsc}(^2X)$.

(ii) *Every operator $T : X \to c_0$ that factorizes through X^* in the form $T = L \circ b$, where L is weak*-to-weak continuous and b is symmetric, is completely continuous.*

Theorem 2.2.s. *Let X be a Banach space and let N be an integer $N \geq 2$. Assume that $\mathcal{L}_s(^{N-1}X) = \mathcal{L}_{wsc}(^{N-1}X)$. Then the following are equivalent*:

(i) $\mathcal{L}_s(^N X) \subset \mathcal{L}_{wsc}(^N X)$.

(ii) *Every operator $T : X \to c_0$ that factorizes through the space $\mathcal{L}_s(^{N-1}X)$ in the form $T = L \circ b$, where L is weak*-to-weak continuous and b is symmetric, is completely continuous.*

Nevertheless, the problem that seems to be behind a unification of Theorems 2.1 and 2.2 with 2.1.s and 2.2.s is:

Problem 1. *Do $\mathcal{P}$-spaces and $\mathcal{M}$-spaces coincide?*

We now pass to describe our strategy to find new examples of $\mathcal{M}$-spaces. Following 2.1 and 2.2, our *basic strategy* to obtain $\mathcal{M}$-spaces will be:

1) to proceed by induction;

2) instead of imposing conditions on all weakly null sequences of the space, we shall allow a controlled number of "types" of weakly null sequences in such a way that each type can be governed in some way.

Let us give a definition that will simplify several statements.

Definition. A sequence $\{x_n\}_n$ in a Banach space X is said to be *multilinearly null* (for short, *$\mathcal{M}$-null*) if for all multilinear mappings $A : X^N \to \mathbb{K}$ and all choices of N subsequences $j_i : \mathbb{N} \to \mathbb{N}$, $i = 1, \ldots, N$,
$$\lim_{n\to\infty} A(x_{j_1(n)}, \ldots, x_{j_N(n)}) = 0.$$

The subsequences j_i have been included in the definition to ease the application of the basic strategy (see the proof of Thm. 2.5); also, to be sure that a space in which every weakly null sequence is $\mathcal{M}$-null is an $\mathcal{M}$-space. The simplest example of an $\mathcal{M}$-null basic sequence is provided by a $\mathcal{P}$-null basic sequence dominated by its subsequences (for instance, although not only, *symmetric or subsymmetric* basic sequences). If a basic sequence is not $\mathcal{M}$-null then for some subsequences $\{j_1(n)\}_n, \ldots, \{j_N(n)\}_n$, the sequence $\{(x_{j_1(n)}, \ldots, x_{j_N(n)})\}_n$ admits a lower estimate ([25], Lemma 1.1). The converse is true for multilinear forms on the product of the closed spans of the subsequence.

The following properties will be useful in the sequel:

Definition. Let A be a class of sequences in a Banach sequence space X. Let $j : X \to \ell_\infty$ be an operator. We say that X has the *$(\infty)_A$-property* with respect to j if bounded sequences $\{x_n\}_n$ in X such that $\lim_{n\to\infty} \|jx_n\|_\infty = 0$ admit subsequences in A.

We shall consider spaces X with basis, so that the operator j will be assumed to be that assigning to each x its coefficients with respect to the basis. Thus, possibly it would have been safer to define a *basis having the $(\infty)_A$-property*; but observe that not all the bases of a space X must have property $(\infty)_A$ simultaneously.

Two specially interesting cases appear when A is chosen to be either the class of sequences equivalent to the canonical basis of c_0 (which we call the *$(\infty)_0$-property*) or the class of sequences equivalent to the canonical basis of ℓ_1 (for short, the *$(\infty)_1$-property*). For instance, the canonical basis of a Lorentz sequence space $d(w; 1)$ has the $(\infty)_1$-property ([4]; also [33], 4.a.3) while the associated functionals span a predual that we denote by $d_*(w)$ in which the basis has the $(\infty)_0$-property (see [10]). The canonical basis of Schreier's space has property $(\infty)_0$ (see e.g. [7]) while that of its dual has the $(\infty)_1$-property.

In [30], Jiménez and Payá proved that all N-linear maps on the predual $d_*(w)$ of a certain Lorentz sequence space are weakly sequentially continuous. Let us make a close inspection of their proof:

Let B be a bilinear mapping that is not weakly sequentially continuous; then there are weakly null sequences $\{x_n\}_n$ and $\{y_n\}_n$ such that $B(x_n, y_n) = 1$ for all n. There is no loss of generality in assuming that both $\{x_n\}_n$ and $\{y_n\}_n$ are normalized blocks of the basis $\{e_n\}_n$. Since $B : X \to X^*$ is continuous, $\{x_n^*\}_n = \{B(x_n,\cdot)\}_n$ is a weakly null sequence in X^*. Two cases may occur:

(i) If $\liminf_{n\to\infty}\|x_n^*\|_\infty = 0$, then a subsequence of $\{x_n^*\}_n$ is equivalent to the basis of ℓ_1, which is impossible.

(ii) If $\|x_n^*\|_\infty \ge \varepsilon > 0$ for some ε and all n, then $B(x_n, e_{j(n)}) \ge \varepsilon$ for some $e_{j(n)}$.

Repeating this procedure with the first coordinate one obtains integer sequences $\{i(n)\}_n$ and $\{j(n)\}_n$ satisfying $B(e_{i(n)}, e_{j(n)}) \ge \varepsilon$ for all n.

Let $A_1 : X \to [e_{i(n)}]$ and $A_2 : X \to [e_{j(n)}]$ be isomorphisms whose existence is guaranteed by the symmetry of the basis. The new bilinear form

$$C(x,y) = B(A_1x, A_2y)$$

satisfies $C(e_n,e_n) \ge \varepsilon$ for all n; thus, the basis is not $\mathcal{P}$-null. So, a formal statement of the argument of Jiménez and Payá is:

THEOREM 2.4. *Let X be a Banach space with a symmetric shrinking $\mathcal{P}$-null basis. If X^* has the $(\infty)_1$-property then X is an $\mathcal{M}$-space.*

Observe that in the proof of the above result the shrinking character of the basis was not used: it just ensures that the $(\infty)_1$-property has a meaning in X^*. Moreover, the only role of the condition "X^* has the $(\infty)_1$-property" is to control those sequences in X^* having $\|\cdot\|_\infty$-norm tending to 0. This suggests that we consider the following class of sequences:

According to [21] a subset $A \subset X^*$ is called an (L)-*set* if for any weakly null sequence $\{x_n\}_n$ in X we have

$$\lim_{n\to\infty}\sup_{x^*\in A}|\langle x_n, x^*\rangle| = 0.$$

We say that a bounded sequence $\{f_n\}_n$ in a dual space X^* is an (L)-*sequence* if the set $\{f_n : n\in\mathbb{N}\}$ is an (L)-set. One has

THEOREM 2.5. *Let X be a Banach space with an $\mathcal{M}$-null basis. If X^* has the $(\infty)_L$-property, then X is an $\mathcal{M}$-space.*

Proof. We proceed inductively. For bilinear forms the result is obvious. Assume now that $\mathcal{L}(^{N-1}X) = \mathcal{L}_{\rm wsc}(^{N-1}X)$. Let A be an N-linear form on X. In order to prove that A is weakly sequentially continuous it is enough to prove that A is weakly sequentially continuous at zero. Assume that there is a normalized weakly null sequence $\{(u_n^1,\ldots,u_n^N)\}_n$ in X^N such that

$$A(u_n^1,\ldots,u_n^N) \ge 1$$

for all n. Consider $u_n^* = A(u_n^1,\ldots,u_n^{(N-1)},\cdot) \in X^*$. Then $\{u_n^*\}_n$ is a seminormalized weakly null sequence in X^* and two cases can occur:

First case: $\liminf_{n\to\infty}\|u_n^*\|_\infty = 0$. In this case $\{u_n^*\}_n$ admits an (L)-subsequence which satisfies $\lim_{n\to\infty} u_n^*(u_n) = 0$, contrary to hypothesis.

Second case: $\|u_n^*\|_\infty \ge \varepsilon$ for some $\varepsilon > 0$ and all n. In this case, there is an integer sequence $\{j_N(n)\}_n$ such that for all n,

$$A(u_n^1,\ldots,u_n^{N-1}, e_{j_N(n)}) \ge \varepsilon.$$

We now consider the sequence $\{A(u_n^1,\ldots,u_n^{N-2},\cdot,e_{j_N(n)})\}_n$ in X^* and proceed in the same way. Thus, in N steps we obtain N subsequences $\{j_i(n)\}_n$ for $i=1,\ldots,N$ such that

$$\liminf_{n\to\infty} A(e_{j_1(n)},\ldots,e_{j_N(n)}) > 0,$$

which contradicts the basis being $\mathcal{M}$-null. ∎

The following proposition provides natural examples of spaces with the $(\infty)_L$-property.

PROPOSITION 2.6. *Let X be a Banach space having a shrinking basis with the $(\infty)_0$-property. Then X^{**} has the $(\infty)_L$-property.*

Proof. Let $\{x_n\}_n$ be a bounded sequence in X^{**} with $\lim_{n\to\infty}\|x_n\|_\infty = 0$ and let $\{f_n\}_n$ be a weakly null sequence in X^*.

In X^*, decompose $f_n = g_n + h_n$ where g_n is finitely supported and $\lim_{n\to\infty}\|h_n\| = 0$. Let $x_n = u_n + v_n$ where the support of u_n coincides with that of g_n. Of course, $\lim_{n\to\infty} h_n(x_n) = 0$. Now, since $\{u_n\}_n$ is a weakly null sequence in X with $\lim_{n\to\infty}\|u_n\|_\infty = 0$ the $(\infty)_0$-property of the basis yields $\lim_{n\to\infty} g_n(u_n) = 0$, and thus $\lim_{n\to\infty} f_n(u_n) = 0$ and the proof is complete. ∎

Using the same argument as in the proof of Theorem 2.4 one obtains

THEOREM 2.7. *Let X be a Banach space with shrinking basis and the $(\infty)_0$-property. If X^* has an $\mathcal{M}$-null basis, then X^* is an $\mathcal{M}$-space.*

3. Polynomials on the dual of Schreier's space. A finite subset A of $\mathbb{N}$ will be called *admissible* if it is either empty or has $\operatorname{card}(A) \le \min(A)$. The *Schreier space* S is obtained as the completion of the space of finite sequences with respect to the norm

$$\|x\| = \sup\Big\{\sum_{n\in A}|x_n| : A \text{ admissible}\Big\}.$$

The Schreier space and the notion of admissible set have been the cornerstones on which most modern counterexamples in Banach space theory involving sequences have been based. It was the first example of a Banach space lacking the *weak Banach–Saks property* (i.e., weakly null sequences admit subsequences having norm convergent arithmetic means). Basic information about S can be found in [7, 15]. For our purposes it is enough to know that the standard unit vectors form a shrinking basis for S with the $(\infty)_0$-property (see [7]). Therefore, as a corollary of Proposition 2.6 one has

COROLLARY 3.1. *The space S^{**} has the $(\infty)_L$-property.*

The space S^* lacks the Dunford–Pettis property since S does (see [13]). It does not have upper p-estimates, since otherwise S would have lower q-estimates, which is impossible since, as a subspace of $C(\omega^\omega)$, it is c_0-saturated (or else: property $(\infty)_0$ implies c_0-saturation, see [32]).

THEOREM 3.2. *The space S^* is an $\mathcal{M}$-space.*

Proof. That the unconditional basis $\{e_n\}_n$ of S^* is $\mathcal{M}$-null follows from the estimate

$$\Big\|\sum_{i=1}^N e_{j_i}\Big\|_{S^*} \le 2\log N$$

which prevents $\{e_n\}_n$ from having lower q-estimates, and the fact that the basis is dominates its subsequences (in S the basis is dominated by its subsequences). Since S has the $(\infty)_0$-property, S^{**} has the $(\infty)_L$-property by Corollary 3.1. The assertion now follows from Theorem 2.7. ∎

REMARK. Although the space S^* has all multilinear forms weakly sequentially continuous, the space S does not. This is due to the estimate in the above proof, which implies (see e.g. [17]) that there is a subsequence of the basis of S^* with an upper estimate, and by duality a subsequence of the basis of S has a lower estimate. Now, since the basis in S dominates its subsequences, the basis has itself a lower estimate. It cannot therefore be $\mathcal{M}$-null.

4. Bilinear forms on Lorentz sequence spaces.

We remind the reader that we would like to do two things at the same time: make all polynomials weakly sequentially continuous and avoid upper estimates. Having proved that they can be done simultaneously, as the case of the dual of the Schreier space shows, we are going to try to do them in a space and its dual at the same time. In this section, we focus on polynomials having degree 2 and bilinear forms, achieving almost optimal results (as will be seen later). Having the *basic strategy* in mind, and since a better quality of a basis increases the quality of its blocks and thus the difficulty of finding bad behaved blocks, we shall consider spaces with symmetric basis.

Let us recall the definition of the Lorentz sequence spaces. Given a sequence w of weights (that is, a decreasing sequence of positive numbers such that $\lim_{n\to\infty} w_n = 0$ and $\sum w_n = \infty$), the Lorentz sequence space $d(w;p)$

is the completion of the space of finite sequences with respect to the norm

$$\|x\| = \sup_\sigma \Big\{\Big(\sum_n |w_{\sigma(n)}x_n|^p\Big)^{1/p}\Big\},$$

where σ runs through all permutations of $\mathbb{N}$. Basic properties of Lorentz sequence spaces can be found in [4] and [33].

THEOREM 4.1. *There exists a Lorentz sequence space $d(w;1)$ such that*

(i) *All bilinear forms on $d(w;1)$ and on its predual $d_*(w)$ are weakly sequentially continuous.*
(ii) *No normalized weakly null sequence exists in $d(w;1)$ admitting an upper or lower 2-estimate.*
(iii) *The predual $d_*(w)$ admits no upper 2-estimates.*
(iv) *The spaces $d_*(w)$ and $d(w;1)$ fail the Dunford–Pettis property.*

Proof. Recall that in [4] it is proved that a Lorentz sequence space $d(w;p)$ in which the partial sums of w (denoted by $s_n = \sum_{i=1}^n w_i$) form a submultiplicative sequence, i.e.

$$\sup_{n,k} \frac{s_{nk}}{s_n s_k} < \infty,$$

admits exactly two non-equivalent symmetric basic sequences [4, Thm. 6]; namely, the unit vector basis of ℓ_p and the usual basis $\{e_n\}_n$ of the space. Moreover, they proved [4, Cor. 4] that every bounded block basic sequence of the basis has a symmetric subsequence. In the particular case of $d(w;1)$ that means that every weakly null normalized sequence has a subsequence equivalent to the basis $\{e_n\}_n$ in $d(w;1)$. Consequently, whenever the above submultiplicative condition is satisfied, in order to have all bilinear forms on $d(w;1)$ weakly sequentially continuous it is enough to ensure that the basis is $\mathcal{M}_2$-null. By the symmetry of the basis it is enough that it is $\mathcal{P}_2$-null, for which we only need to prove that it admits no lower 2-estimates ([26]). Thus, recalling that $\|\sum_{i=1}^n e_i\| = \sum_{i=1}^n w_i$ we require the condition

$$(*)\qquad \liminf_{n\to\infty} \frac{\sum_{i=1}^n w_i}{n^{1/2}} = 0.$$

To make all bilinear forms on its predual $d_*(w)$ weakly sequentially continuous (which, in turn, implies that the basis has no lower 2-estimates) we rely on the symmetry of the basis and on one of the main results of [30], which asserts that it is both necessary and sufficient that w satisfies the condition $w \notin \ell_2$. Hölder's inequality $\sum_{i=1}^n w_i \le (\sum_{i=1}^n w_i^2)^{1/2}|w_1|n^{1/2}$ yields that the condition

$$(**)\qquad \limsup_{n\to\infty} \frac{\sum_{i=1}^n w_i}{n^{1/2}} = \infty$$

is clearly sufficient to imply $w \notin \ell_2$. From (ii), part (iii) follows by duality. Part (iv) follows since the canonical bases of both the space and its dual are weakly null (otherwise, a space with a symmetric non-weakly null basis is ℓ_1).

What remains is to exhibit an example of a suitable sequence w with submultiplicative partial sums and satisfying the two restrictions imposed:

$$0 = \liminf_{n\to\infty} \frac{\sum_{i=1}^n w_i}{n^{1/2}} < \limsup_{n\to\infty} \frac{\sum_{i=1}^n w_i}{n^{1/2}} = \infty.$$

We sketch the construction of w, omitting the full details of this tricky exercise. If we set $s_n = f(n)n^{1/2}$, the function f has to be submultiplicative and satisfy

$$0 = \liminf_{n\to\infty} f(n) < \limsup_{n\to\infty} f(n) = \infty.$$

Moreover, f has to fulfill the condition

$$\frac{f(n+1)}{f(n)} > \sqrt{\frac{n}{n+1}}$$

in order to make (s_n) increasing. We now describe the construction of an f satisfying

$$0 = \liminf_{n\to\infty} f(n) < \limsup_{n\to\infty} \frac{f(n)}{\log n} < \infty.$$

We set

$$f(1+n) = 1 + \ldots + \frac{1}{n}$$

until $f(m_1) > \frac{1}{2}\log m_1$. Then we set

$$f(m_1+n) = \sqrt{\frac{m_1}{m_1+n}}\log(m_1+n)$$

until $f(m_1+\nu_1) < 1$. The process starts again its way up. The only point to be careful about is to make sure that the sequence w is decreasing. Since the last value of w obtained has been $w_{m_1+\nu_1}$, if μ_1 is the smallest integer such that $1/\mu_1 < w_{m_1+\nu_1}$ then we continue the series setting

$$f(m_1+\nu_1+n+1) = f(m_1+\nu_1) + \frac{1}{\mu_1} + \ldots + \frac{1}{\mu_1+n}.$$

In this form, for large m_2 one has $f(m_2) > \frac{1}{2}\log m_2$; find now ν_2 so that $f(m_2+\nu_2) < 1/2$ and continue. At the points m_N the value of f is proportional to log while at the points $m_N+\nu_N$ it tends to 0. ∎

One might guess that it should not be difficult to reproduce the preceding arguments for N-linear forms. Nevertheless, observe that condition $(*)$ becomes

$$\liminf_{n\to\infty} \frac{\sum_{i=1}^n w_i}{n^{1/N}} = 0,$$

while $(**)$ becomes

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^n w_i}{n^{1/N^*}} = \infty.$$

It can be proved that an increasing submultiplicative sequence cannot oscillate between two different powers. Worse yet, the tantalizing possibility of extending the construction and making it valid for all N simultaneously cannot be realized: a decreasing sequence with submultiplicative partial sums necessarily belongs to some ℓ_p, which yields the existence of some non-weakly sequentially continuous polynomial on $d_*(w)$.

5. Multilinear forms on Lorentz sequence spaces. Preduals of Lorentz sequence spaces can be managed through Theorem 2.4. The Lorentz spaces $d(w;1)$ themselves can be treated in that way upon realizing that the predual space has property $(\infty)_0$ and then appealing to Theorem 2.7. However, in this section we try something different: to prove that a space is an $\mathcal{M}$-space assuming no knowledge about the dual; i.e., involving only properties of the space. The cost of this process is that the symmetry of the basis cannot be relaxed. We have:

THEOREM 5.1. *Let X be a Banach space with a symmetric boundedly complete basis which is polynomially null (equivalently, $\mathcal{M}$-null). If X has the $(\infty)_1$-property, then X is an $\mathcal{M}$-space.*

We will need several lemmas. Recall that if $0 \neq \alpha = \sum_{i=1}^\infty \alpha_i e_i \in X$ and $\{p_n\}$ is an increasing sequence of integers then the sequence

$$y_n^{(\alpha)} = \sum_{i=p_n+1}^{p_{n+1}} \alpha_{i-p_n} e_i$$

is called a *block basis generated by the vector* α. We first show the following lemma on decomposition of block bases that may be of independent interest:

LEMMA 5.2. *Let X be a Banach space with a symmetric boundedly complete basis. Let $u_n = \sum_{i=p_n+1}^{p_{n+1}} a_i e_i$ be a block basis of $\{e_n\}_n$ and assume that $|a_{p_n+1}| \geq \ldots \geq |a_{p_{n+1}}|$. Then there is a subsequence $\{u_{n_k}\}_k$ of $\{u_n\}_n$ such that $\{u_{n_k}\}_k$ and $\{y_k^{(\alpha)} + z_k\}_k$ are equivalent, where:*

(i) *$\{y_n^{(\alpha)}\}_n$ is a block basis generated by a vector $\sum_{i=1}^\infty \alpha_i e_i \in X$.*
(ii) $\lim_{k\to\infty} \|z_k\|_\infty = 0$.

Moreover,

$$\sum_{k=1}^{\infty} \|u_{n_k} - (y_k^{(\alpha)} + z_k)\| \le \frac{1}{2C}$$

where C is the unconditional constant of the block basis $\{u_n\}_n$.

Proof. Of course, if $\liminf_{n\to\infty} \|u_n\|_\infty = 0$ then the result is obvious by passing to a subsequence. Assume then that $\|u_n\|_\infty \ge \varepsilon > 0$ for all n.

Let $q_n = p_{n+1} - p_n$. If $\sup_n q_n < \infty$, then by passing to a subsequence we may assume that $q_n = r$ for all n. In particular,

$$u_n = \sum_{i=1}^{r} a_{p_n+i} e_{p_n+i}.$$

Then, since each $\{a_{p_n+i}\}_n$, for $i = 1, \ldots, r$, is a bounded sequence, there is an infinite set of integers H such that $\lim_{n\in H,\, n\to\infty} a_{p_n+i} = \alpha_i$ for $i = 1, \ldots, r$. We now construct an increasing sequence $\{n_k\}_k$ of integers such that

$$\sum_{i=1}^{r} |a_{p_{n_k}+i} - \alpha_i| \le \frac{1}{2^{k+1}C}$$

for $i = 1, \ldots, k$, where C is the unconditional basis constant of $\{e_n\}_n$. Then, if $\alpha = \sum_{i=1}^{r} \alpha_i e_i$ we consider the sequence $\{y_k^{(\alpha)}\}_k$ defined by

$$y_k^{(\alpha)} = \sum_{i=1}^{r} \alpha_i e_{p_{n_k}+i}.$$

Since

$$\sum_{k=1}^{\infty} \|u_{n_k} - y_k^{(\alpha)}\| \le \frac{1}{2C},$$

it follows (see e.g. [33]) that $\{u_{n_k}\}_k$ and $\{y_k^{(\alpha)}\}_k$ are equivalent as required.

Assume now, by passing to a subsequence if necessary, that $\{q_n\}_n$ is an increasing sequence. Then we choose an infinite set $H_1 \subset \mathbb{N}$ such that there is α_1 with $\lim_{n\in H_1,\, n\to\infty} a_{p_n+1} = \alpha_1$ and for $n \in H_1$,

$$|a_{p_n+1} - \alpha_1| \le \frac{1}{2^3 C}.$$

Now, consider the sequence $\{a_{p_n+2}\}_{n\in H_1}$. Again, we may choose α_2 and an infinite set $H_2 \subset H_1$ of integers such that $\lim_{n\in H_2,\, n\to\infty} a_{p_n+2} = \alpha_2$ and for $i = 1, 2$ and $n \in H_2$,

$$|a_{p_n+i} - \alpha_i| \le \frac{1}{2^{3+i}C}.$$

By repeating this procedure, we construct a sequence $\{H_k\}_k$ of sets of integers where $H_{k+1} \subset H_k$ and a sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ where $|\alpha_1| \ge |\alpha_2| \ge \ldots$ such that for $i = 1, \ldots, k$ and $n \in H_k$,

$$|a_{p_n+i} - \alpha_i| \le \frac{1}{2^{k+i+1}C}.$$

Consider now the diagonal set $H = \{n_k\}_k$ and define

$$y_k^{(\alpha)} = \sum_{i=1}^{k} \alpha_i e_{p_{n_k}+i} \quad \text{and} \quad z_k = \sum_{i=k+1}^{q_{n_k}} a_{p_{n_k}+i} e_{p_{n_k}+i}.$$

Then $\{u_{n_k}\}_k$ and $\{y_k^{(\alpha)} + z_k\}_k$ are equivalent, which follows from

$$\sum_{k=1}^{\infty} \|u_{n_k} - (y_k^{(\alpha)} + z_k)\| \le \sum_{k=1}^{\infty} \Big(\sum_{i=1}^{k} |a_{p_n+i} - \alpha_i|\Big) \le \frac{1}{2C}$$

(see e.g. [33]).

It is clear that

$$\sup_n \Big\| \sum_{i=1}^{n} \alpha_i e_i \Big\| < \infty,$$

and using the fact that the basis is boundedly complete we deduce that $\alpha = \sum_{i=1}^{\infty} \alpha_i e_i \in X$ and $\{y_k^{(\alpha)}\}$ is a block basis generated by the vector α.

In particular, $\lim_{k\to\infty} \alpha_k = 0$. Then $\|z_k\|_\infty = |a_{p_{n_k}+k+1}| \le |a_{p_{n_k}+k}|$ and since $\lim_{k\to\infty} |a_{p_{n_k}+k} - \alpha_k| = 0$ it follows that $\lim_{k\to\infty} \|z_k\|_\infty = 0$. ■

LEMMA 5.3. *Let X be a Banach space with symmetric basis $\{e_n\}_n$ and let $\{y_n^{(\alpha)}\}_n$ be a block basis generated by a vector $\alpha \in X$. If $\{e_n\}_n$ is $\mathcal{P}$-null (equivalently, $\mathcal{M}$-null), then $\{y_n^{(\alpha)}\}_n$ is also $\mathcal{M}$-null.*

Proof. Let $\alpha = \sum_{n=1}^{\infty} \alpha_n e_n \in X$ be such that $y_n^{(\alpha)} = \sum_{i=p_n+1}^{p_{n+1}} \alpha_{i-p_n} e_i$ where $\{p_n\}_n$ is an increasing sequence of integers. Without loss of generality we may assume that $\|\alpha\| = 1$. Note that it is enough to find a subsequence of $\{y_n^\alpha\}_n$ which is $\mathcal{P}$-null.

If $\sup_n (p_{n+1} - p_n) < \infty$ then by [4] there is a subsequence equivalent to the basis. The result now follows from the fact that if a sequence $\{y_n\}_n$ is equivalent to the basis and $\{e_n\}_n$ is $\mathcal{P}$-null then so is $\{y_n\}_n$. Indeed, if $\{y_n\}_n$ were not $\mathcal{P}$-null then it would have a lower estimate. Then also $\{e_n\}_n$ would have one, which is impossible.

Otherwise, if $\sup_n (p_{n+1} - p_n) = \infty$, by taking a subsequence we may assume that $\lim_{n\to\infty} (p_{n+1} - p_n) = \infty$. Let P be a polynomial; since P is uniformly continuous on bounded sets, for $C > 0$ (the unconditional constant) and $\varepsilon > 0$ there is $\delta > 0$ such that

$$|P(x+h) - P(x)| \le \varepsilon$$

whenever $\|x\| \leq C$ and $\|h\| \leq \delta$. Now, we may choose an integer N such that $\|\sum_{i=N+1}^{\infty} \alpha_i e_i\| \leq \delta$. Then $p_{n+1} - p_n > N$ for n large enough and

$$|P(y_n^{(\alpha)})| \leq \Big|P\Big(\sum_{i=p_n+1}^{p_{n+1}} \alpha_{i-p_n} e_i\Big) - P\Big(\sum_{i=p_n+1}^{p_n+N} \alpha_{i-p_n} e_i\Big)\Big| + \Big|P\Big(\sum_{i=p_n+1}^{p_n+N} \alpha_{i-p_n} e_i\Big)\Big| \leq \varepsilon + \Big|P\Big(\sum_{i=p_n+1}^{p_n+N} \alpha_{i-p_n} e_i\Big)\Big|.$$

The conclusion follows since the sequence $\{\sum_{i=p_n+1}^{p_n+N} \alpha_{i-p_n} e_i\}_n$ is formed by blocks of bounded length with respect to the basis $\{e_n\}_n$ and consequently is $\mathcal{M}$-null. ∎

We can now prove Theorem 5.1:

Proof of Theorem 5.1. It is enough to prove that any block basis $\{u_k\}_k$ is a $\mathcal{P}$-null sequence. Indeed, if $\{v_n\}_n$ is a weakly null normalized sequence in X then by a perturbation argument passing to a subsequence there is a block basis $\{u_n\}_n$ such that $\lim_{k\to\infty} \|v_{n_k} - u_k\| = 0$. Since P is uniformly continuous on bounded sets, we have

$$\lim_{k\to\infty} |P(u_k) - P(v_{n_k})| = 0.$$

Let $\{p_n\}_n$ be an increasing sequence and let $u_n = \sum_{i=p_n+1}^{p_{n+1}} a_i e_i$ be a block basis.

In order to prove that $\{u_n\}_n$ is $\mathcal{P}$-null we may assume without loss of generality that $|a_{p_n+1}| \geq \ldots \geq |a_{p_{n+1}}|$ (note that there is an isomorphism $X \to X$ which takes u_n to $\sum_{i=p_n+1}^{p_{n+1}} a_i^* e_i$ where (a_i^*) is the non-decreasing rearrangement of $(|a_i|)_{i=p_n+1}^{p_{n+1}}$).

By using Lemma 5.2 we may extract a subsequence $\{u_{n_k}\}_k$ of $\{u_n\}_n$ such that

$$\lim_{k\to\infty} \|u_{n_k} - (y_k^{(\alpha)} + z_k)\| = 0,$$

where $\{y_k^{(\alpha)}\}_k$ is a block sequence generated by a vector α in X and the sequence $\{z_k\}_k$ satisfies $\lim_{k\to\infty} \|z_k\|_\infty = 0$.

Since X has the $(\infty)_1$-property and $\{z_k\}_k$ is weakly null it follows that $\lim_{k\to\infty} \|z_k\| = 0$. So, it is possible to extract again a subsequence of $\{u_{n_k}\}_k$, still denoted by $\{u_{n_k}\}_k$, which satisfies $\lim_{k\to\infty} \|u_{n_k} - y_k^{(\alpha)}\| = 0$. By Lemma 5.3 we conclude that $\{u_{n_k}\}_k$ is $\mathcal{M}$-null. ∎

We can now present a nice example:

THEOREM 5.4. *There is a Lorentz sequence space $d(w;1)$ such that:*

(i) *All multilinear forms on $d_*(w)$ and $d(w;1)$ are weakly sequentially continuous.*

(ii) *No normalized weakly null sequence exists in $d(w;1)$ having an upper p-estimate for $p > 1$.*

(iii) *The predual $d_*(w)$ admits no upper p-estimates for any $p > 1$.*

(iv) *The spaces $d_*(w)$ and $d(w;1)$ fail the Dunford–Pettis property.*

Proof. It is enough to choose a sequence w of weights such that for all $p > 1$,

$$0 = \liminf_{N\to\infty} \frac{\sum_{i=1}^N w_i}{N^{1/p}} \leq \limsup_{N\to\infty} \frac{\sum_{i=1}^N w_i}{N^{1/p}} = \infty.$$

Note that the condition means that for all $p > 1$,

$$0 = \liminf_{N\to\infty} \frac{\|\sum_{i=1}^N e_i\|}{N^{1/p}} \leq \liminf_{N\to\infty} \frac{\|\sum_{i=1}^N e_i\|}{N^{1/p}} = \infty,$$

where $\{e_n\}_n$ is the usual basis in $d(w;1)$. This means that such a basis does not have lower or upper estimates (nor subsequences with lower or upper estimates, by symmetry). By duality, the same holds for the basis of $d_*(w)$. In particular both bases are polynomially null. It is also clear that there is no difficulty in choosing a sequence of weights satisfying that condition.

The space $d(w;1)$ is an $\mathcal{M}$-space by Theorem 5.1. All multilinear forms on $d_*(w)$ are weakly sequentially continuous by Theorem 2.4 since the basis is $\mathcal{M}$-null. Another way to see it is using [30] since $w \notin \ell_p$ for all $p < \infty$ (this holds since w is decreasing).

To prove (ii), let $\{u_n\}_n$ be a weakly null sequence of normalized blocks of $\{e_n\}_n$. If $\|u_n\|_\infty \geq \varepsilon$ for some $\varepsilon > 0$, then some coefficient λ_j in u_j is greater than or equal to ε. Thus,

$$\Big\|\sum_{j=1}^N u_j\Big\| \geq \Big\|\sum_{i=1}^N \varepsilon e_{j_i}\Big\| \geq \varepsilon \Big\|\sum_{i=1}^N e_i\Big\|.$$

Hence, if $\{u_j\}_j$ has an upper p-estimate then so does $\{e_j\}_j$, which is impossible.

Otherwise, $\lim_{n\to\infty} \|u_n\|_\infty = 0$, and some subsequence is equivalent to the canonical basis of ℓ_1 and thus it cannot be weakly null and normalized simultaneously.

Finally, the predual space $d_*(w)$ does not admit upper p-estimates, for $p > 1$, since obviously the basis does not admit any upper estimate. ∎

Other solved problems and further open problems. The above example provides a solution to Question 4 in the introduction (see [8] and [23], Problem 4.8(2)).

THEOREM 5.5. *The sum of two $\mathcal{P}$-null sequences need not be $\mathcal{P}$-null.*

Proof. Consider the sequences $\{(e_n^*, 0)\}_n$ and $\{(0, e_n)\}_n$ in $d_*(w) \times d(w;1)$ where $\{e_n^*\}_n$ and $\{e_n\}_n$ are the canonical bases in $d_*(w)$ and $d(w;1)$ respectively. Their sum $\{(e_n^*, e_n)\}_n$ is not $\mathcal{P}$-null; indeed, the 2-homogeneous polynomial $P : d_*(w) \times d(w;1) \to \mathbb{K}$ given by $P(x,y) = \langle x, y \rangle$ is not weakly sequentially continuous since $P(e_n^*, e_n) = 1$ for all $n \in \mathbb{N}$. ∎

Minor modifications provide negative answers to Problems 4.8(1) and (3) of [23].

Problem 4.8(3) is: *If A is a pw-compact set, is the absolutely convex closed convex hull of A pw-compact?* Recall that a net in a Banach space is said to be *pw-convergent* to x if, for every continuous polynomial, the images of the elements of the net converge to the image of x. The *pw*-topology is the topology that corresponds to the *pw*-convergence.

Although $\{(e_n^*, 0)\}_n \cup \{(0, e_n)\}_n \cup \{(0,0)\}$ is a *pw*-compact set in $d_*(w) \times d(w;1)$, its convex hull is not since the sequence $\{\frac{1}{2}(e_n^*, e_n)\}_n$ admits no *pw*-convergent subsequences.

Analogously, Problem 4.8(1) asking if the projective tensor product of $\mathcal{P}$-null sequences must be at least weakly null has a negative answer. It is clear that a space X fails the Dunford–Pettis property if and only if there exist weakly null sequences in X and X^* whose tensor product is not weakly null in $X \otimes_\pi X^*$ (see [19] for further information). Our example provides a space $X = d_*(w)$ in which there exist $\mathcal{P}$-null sequences in X and X^* whose tensor product is not even weakly null.

Property (P) was introduced by Aron, Choi and Llavona in [5] as follows: for any two bounded sequences $\{u_n\}_n$ and $\{v_n\}_n$ if $\lim_{n\to\infty}(P(u_n) - P(v_n)) = 0$ for every polynomial P, then $\lim_{n\to\infty} P(u_n - v_n) = 0$ for all polynomials P. Spaces with the Dunford–Pettis property have property (P) ([5]), and more generally $\mathcal{P}$-spaces. Choi and Kim [18] have shown that spaces with non-trivial type have property (P); in particular, superreflexive spaces. The existence of a space without property (P) has been so far unknown.

THEOREM 5.6. *The space $d_*(w) \times d(w;1)$ fails property* (P).

Proof. Indeed, since $\{(e_n^*, 0)\}_n$ and $\{(0, e_n)\}_n$ are $\mathcal{P}$-null in $d_*(w) \times d(w;1)$, we have $\lim_{n\to\infty}(P(e_n^*, 0) - P(0, e_n)) = 0$ for every polynomial P. However, the difference sequence $\{(e_n^*, -e_n)\}_n$ is not $\mathcal{P}$-null, as we have already shown. ∎

It has also been proved that the product of two spaces with property (P) may fail property (P).

In [18], while trying to give a clue for the existence of a counterexample, Choi and Kim give a sufficient condition for a space to fail property (P): *If X is a Λ-space and the space of n-homogeneous polynomials on X is separable for all n, then X does not have property* (P). Certainly, the space $d_*(w) \times d(w;1)$ fails to be a Λ-space (recall that this means that polynomially null sequences are norm null); moreover, since $d(w;1)$ contains ℓ_1 the dual space is already non-separable. This shows that their conditions are far from necessary.

We now state some questions that might be interesting.

PROBLEM 1. Do $\mathcal{P}$-spaces and $\mathcal{M}$-spaces coincide?

We have avoided to face that problem throughout the paper, although it already appeared in Section 2. Another seemingly difficult question with rather deep connections has already been asked in [20].

PROBLEM 2. Does there exist a reflexive space X such that both X and X^* are $\mathcal{M}$-spaces?

Maybe not too far, although not too close, is to ask:

PROBLEM 3. Does there exist a reflexive $\mathcal{M}$-space without upper estimates?

Perhaps a cleaner statement would be:

PROBLEM 4. Does there exist a superreflexive $\mathcal{M}$-space?

6. Further examples. The purpose of this section is to obtain some variations of the spaces presented. In the case of Lorentz spaces we replace symmetric by subsymmetric bases; in the Schreier case we replace the family of admissible sets by other suitable families.

Lorentz-like spaces. Recall that the definition of Lorentz sequence space involves the group of all permutations of $\mathbb{N}$, and this is the reason that makes the basis symmetric. Replace the group of all permutations of $\mathbb{N}$ by a subgroup G and add displacements to the right (i.e., increasing mappings $\mathbb{N} \to \mathbb{N}$). With $G+D$ one can construct Lorentz-like sequence spaces $d_G(w;1)$ whose canonical basis is no longer symmetric: it only admits permutations in G. The basis of $d_G(w;1)$ is subsymmetric.

The proof that $d_G(w;1)$ has property $(\infty)_1$ and that its natural predual has the $(\infty)_0$-property can be done in a standard way. Thus, one has

THEOREM 6.1. *There exists a Lorentz-like space $d_G(w;1)$ such that:*

(i) *All multilinear forms and polynomials on $d_G(w;1)$ and its predual $d_{*G}(w)$ are weakly sequentially continuous.*

(ii) *No normalized weakly null sequence in $d_G(w;1)$ admits an upper p-estimate.*

(iii) *The predual $d_{*G}(w)$ admits no upper p-estimate for any $p > 1$.*

It is worth mentioning that the simplest example of such spaces, which corresponds to the the choice $G = \{\mathrm{id}\}$ and $D = \{\text{displacements}\}$, can be easily represented as an isometric (uncomplemented) subspace of the space of continuous functions on $[0,1]$. The uncomplementability is a consequence of the failure of the Dunford–Pettis property.

Schreier-like spaces. There is a wide class of spaces, quite often used as counterexamples. They have been called in [15] Schreier-like spaces. We give a brief description.

In what follows $P_\infty(\mathbb{N})$ and $P_f(\mathbb{N})$ denote the sets of all infinite (resp. finite) subsets of $\mathbb{N}$. We consider the following construction: Let $\mathcal{F}$ be a family of finite sets of $\mathbb{N}$ such that

(i) if $G \subset F \in \mathcal{F}$ then $G \in \mathcal{F}$,
(ii) $\{n\} \in \mathcal{F}$ for all $n \in \mathbb{N}$,
(iii) for all $Z \in P_\infty(\mathbb{N})$ there exists $B \subset Z$ such that $B \notin \mathcal{F}$.

Then $\mathcal{F}$ is a countable compact metric space under the topology induced by identifying $\mathcal{F}$ with $\{1_F : F \in \mathcal{F}\}$ equipped with the topology of pointwise convergence. The sequence $\{x_n\}_n$ defined by $x_n(F) = 1_F(n)$ is pointwise convergent to 0 in $\mathcal{F}$, and each x_n is continuous on $\mathcal{F}$.

Given an *adequate* family $\mathcal{F}$ of subsets of $\mathbb{N}$, the Banach space $S_\mathcal{F}$ is constructed as the completion of the space of finite sequences with respect to the norm

$$\|x\| = \sup\Big\{\sum_{n\in E} |x_n| : E \in \mathcal{F}\Big\},$$

for which the standard basis $\{e_n\}_n$ is a weakly null unconditional basis.

Some examples worth mentioning are: c_0 (taking as $\mathcal{F}$ the family of one-point sets); the Schreier space, obtained by taking as $\mathcal{F}$ the family of *admissible* sets; and the space of Schachermayer (see [17]), obtained by choosing as $\mathcal{F}$ the family of *totally admissible* sets. See [14, 15] for further examples of Schreier-like spaces.

In a Schreier-like space it is possible to translate (some) topological properties of the space into algebraic properties of the family $\mathcal{F}$ (see [15]). Two useful functions will be needed. If B is a (maybe infinite) subset of $\mathbb{N}$, then $|B|$ denotes the cardinality of B and we define $\phi(B) = \max\{|A\cap B| : A \in \mathcal{F}\}$.

The function g_n is defined by

$$g_n(B_1,\ldots,B_n,B) = \max\{|A\cap B| : A \in \mathcal{F},\ A\cap B_i \neq \emptyset,\ i = 1,\ldots,n\}.$$

The family $\mathcal{F}$ is said to satisfy the *n-condition* if every sequence $\{B_n\}$ of finite sets such that $\phi(B_n) \to \infty$ contains a subsequence such that for some function $h : P_f(\mathbb{N})^n \to \mathbb{N}$,

$$g_n(B_i, B_{i+1},\ldots,B_{i+n}) \le h(B_i, B_{i+1},\ldots,B_{i+n-1}).$$

For instance the family of admissible sets satisfies the 1-condition. The family of lunatic sets [15] satisfies the 2-condition, but not the 1-condition.

The reason for introducing the n-condition is the following lemma:

LEMMA 6.2. *If for some n the family $\mathcal{F}$ satisfies the n-condition, then $S_\mathcal{F}$ has the $(\infty)_0$-property.*

Moreover, if there is a family $\{A_n\}_n$ of disjoint sets satisfying the condition $\liminf_{n\to\infty} \phi(A_n)/|A_n| > 0$, then the canonical basis of $S_\mathcal{F}$ admits no upper estimates, hence $\{e_n^*\}_n$ admits no lower estimates and consequently is $\mathcal{P}$-null. Under these conditions, whenever A belongs to $\mathcal{F}$ then DA belongs to $\mathcal{F}$ where D is a displacement to the right, then the basis is dominated by its subsequences. Thus "$\mathcal{P}$-null" implies "$\mathcal{M}$-null". This yields:

PROPOSITION 6.3. *Let $\mathcal{F}$ be an adequate family satisfying the n-condition. If $\{e_n\}_n$ is dominated by its subsequences then $(S_\mathcal{F})^*$ is an $\mathcal{M}$-space.*

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References

[1] R. Alencar, R. Aron and S. Dineen, *A reflexive space of holomorphic functions in infinitely many variables*, Proc. Amer. Math. Soc. 90 (1984), 407–411.
[2] R. Alencar, R. Aron and G. Fricke, *Tensor products of Tsirelson's space*, Illinois J. Math. 31 (1987), 17–23.
[3] R. Alencar and K. Floret, *Weak-strong continuity of multilinear mappings and the Pełczyński–Pitt theorem*, J. Math Anal. Appl. 206 (1997), 532–546.
[4] Z. Altshuler, P. G. Casazza and B. L. Lin, *On symmetric basic sequences in Lorentz sequence spaces*, Israel J. Math. 15 (1993), 140–155.
[5] R. M. Aron, Y. S. Choi and J. G. Llavona, *Estimates by polynomials*, Bull. Austral. Math. Soc. 52 (1995), 475–486.
[6] R. M. Aron and S. Dineen, *Q-reflexive Banach spaces*, Rocky Mountain J. Math. 27 (1997), 1009–1025.
[7] B. Beauzamy et J. T. Lapresté, *Modèles étalés des espaces de Banach*, Hermann, Paris, 1984.
[8] P. Biström, J. A. Jaramillo and M. Lindström, *Polynomial compactness in Banach spaces*, Rocky Mountain J. Math., to appear.

[9] T. Carne, B. Cole and T. Gamelin, *A uniform algebra of analytic functions on a Banach space*, Trans. Amer. Math. Soc. 314 (1989), 639–659.
[10] P. G. Casazza and B. L. Lin, *On symmetric basis sequences in Lorentz sequence spaces II*, Israel J. Math. 17 (1974), 191–218.
[11] J. M. F. Castillo, *On Banach spaces X such that $L(L_p, X) = K(L_p, X)$*, Extracta Math. 10 (1995), 27–36.
[12] J. M. F. Castillo, R. García and R. Gonzalo, *Stability properties of the class of Banach spaces in which all multilinear forms are weakly sequentially continuous*, preprint, 1999.
[13] J. M. F. Castillo and M. González, *The Dunford–Pettis property is not a three-space property*, Israel J. Math. 81 (1993), 297–299.
[14] J. M. F. Castillo and M. González, *New results on the Dunford–Pettis property*, Bull. London Math. Soc. 27 (1995), 599–605.
[15] J. M. F. Castillo, M. González and F. Sánchez, *M-ideals of Schreier type and the Dunford–Pettis property*, in: Non-Associative Algebra and its Applications, S. González (ed.), Kluwer, 1994, 81–85.
[16] J. M. F. Castillo and F. Sánchez, *Remarks on some basic properties of Tsirelson's space*, Note Mat. 13 (1993), 117–122.
[17] —, —, *Weakly p-compact, p-Banach-Saks, and super-reflexive Banach spaces*, J. Math. Anal. Appl. 185 (1994), 256–261.
[18] Y. S. Choi and S. G. Kim, *Polynomial properties of Banach spaces*, J. Math. Anal. Appl. 190 (1995), 203–210.
[19] J. C. Díaz, *Non-containment of ℓ_1 in projective tensor products of Banach spaces*, Rev. Mat. Univ. Complut. Madrid 3 (1990), 121–124.
[20] V. Dimant and I. Zalduendo, *Bases in spaces of multilinear forms over Banach spaces*, J. Math. Anal. Appl. 200 (1996), 548–566.
[21] G. Emmanuele, *A dual characterization of spaces not containing ℓ_1*, Bull. Polish Acad. Sci. 34 (1986), 155–160.
[22] J. D. Farmer, *Polynomial reflexivity in Banach spaces*, Israel J. Math. 87 (1994), 257–273.
[23] M. González and J. M. Gutiérrez, *Gantmacher type theorems for holomorphic mappings*, Math. Nachr. 186 (1997), 131–145.
[24] —, —, *Polynomials on Schreier's space*, preprint, 1997.
[25] R. Gonzalo, *Multilinear forms, subsymmetric polynomials and spreading models*, J. Math. Anal. Appl. 202 (1996), 379–397.
[26] R. Gonzalo and J. A. Jaramillo, *Smoothness and estimates of sequences in Banach spaces*, Israel J. Math. 89 (1995), 321–341.
[27] —, —, *Compact polynomials between Banach spaces*, Proc. Roy. Irish Acad. Sect. A 95 (1995), 213–226.
[28] J. M. Gutiérrez, J. A. Jaramillo and J. G. Llavona, *Polynomials and geometry of Banach spaces*, Extracta Math. 10 (1995), 79–114.
[29] J. A. Jaramillo and A. Prieto, *Weak-polynomial convergence on a Banach space*, Proc. Amer. Math. Soc. 118 (1993), 463–468.
[30] M. Jiménez Sevilla and R. Payá, *Norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces*, Studia Math. 137 (1998), 99–112.
[31] H. Knaust and E. Odell, *Weakly null sequences with upper ℓ_p-estimates*, in: Lecture Notes in Math. 1470, Springer, 1990, 85–107.
[32] D. Leung, *On c_0-saturated Banach spaces*, Illinois J. Math. 39 (1995), 15–29.
[33] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer, Berlin, 1977.

[34] J. Mujica, *Complex Analysis in Banach Spaces*, North-Holland Math. Stud. 120, North-Holland, Amsterdam, 1996.
[35] A. Pełczyński, *A property of multilinear operations*, Studia Math. 16 (1957), 173–182.
[36] R. A. Ryan, *Dunford–Pettis properties*, Bull. Acad. Polon. Sci. 27 (1979), 373–379.
[37] C. Stegall, *Duals of certain spaces with the Dunford–Pettis property*, Notices Amer. Math. Soc. 19 (1972), A-799.
[38] E. Straeuli, *On Hahn–Banach extensions for certain operator ideals*, Arch. Math. (Basel) 47 (1986), 49–54.
[39] M. Valdivia, *Complemented subspaces and interpolation properties in spaces of polynomials*, J. Math. Anal. Appl. 208 (1997), 1–30.

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