(b) Finally, let $\{z_n\}$ be an interpolating Blaschke sequence approaching 1, $z_1 = 0$, with m in the w* closure of $\{z_n\}$ and B the corresponding Blaschke product. If $\tau(z) = \frac{1}{2}B(z)$, then it is well known [3] that $(\widehat{\tau} \circ L_m)'(0) = \frac{1}{2}(\widehat{B} \circ L_m)'(0) \neq 0$. This, then, is an example of a compact endomorphism of $H^\infty(D)$ which is not a composition operator but whose spectrum properly contains $\{0, 1\}$.

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The triple-norm extension problem: the nondegenerate complete case

by

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Abstract. We prove that, if A is an associative algebra with two commuting involutions τ and π, if A is a τ-π-tight envelope of the Jordan Triple System $T := H(A,\tau) \cap S(A,\pi)$, and if T is nondegenerate, then every complete norm on T making the triple product continuous is equivalent to the restriction to T of an algebra norm on A.

0. Introduction and preliminaries. The classification of prime Jordan Triple Systems (JTS) is essentially due to E. I. Zel'manov ([Zel2], [Zel3] and [Zel4]). Later Zel'manov's ideas have been clarified and completed in [D'A], [ACCM] and [D'AM]. In the Zel'manov classification of prime nondegenerate JTS's, triples of the following form became crucial. Take an associative algebra A with two commuting involutions τ, π, put

$$H(A,\tau) := \{a \in A : a^\tau = a\} \quad \text{and} \quad S(A,\pi) := \{a \in A : a^\pi = -a\},$$

and consider the JTS $T := H(A,\tau) \cap S(A,\pi)$ under the triple product $\{xyz\} := \frac{1}{2}(xyz + zyx)$.

Let A and T be as above. If $\|\cdot\|$ is an algebra norm on A, then, clearly, the restriction of $\|\cdot\|$ to T is a *triple-norm* on T, i.e., a norm on the vector space T making the triple product of T continuous. The converse question is called the *triple-norm extension problem* [Mor2] (3NEP for short), namely: given a triple-norm $|||\cdot|||$ on T, is there an algebra norm on A whose restriction to T is equivalent to $|||\cdot|||$? To have some possibility of an affirmative answer to the above question, it seems reasonable to assume that A is a *τ-π-tight envelope* of T, which means:

(i) A is generated by T, and

(ii) every nonzero τ-π-invariant ideal of A has nonzero intersection with T.

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Under these assumptions, we showed in [Mor2, Theorem 1.2] that the triple-norm extension problem has an affirmative answer if (and only if) the pentad mapping $\{\ldots\}_5$ is $|||\cdot|||$-continuous, where $\{\ldots\}_5$ is the function from $T\times T\times T\times T\times T$ to T given by $\{t_1t_2t_3t_4t_5\}_5 := \frac{1}{2}(t_1t_2t_3t_4t_5 + t_5t_4t_3t_2t_1)$.

The 3NEP is the natural triple version of the norm extension problem for Jordan algebras (see [RSZ]). Answers to this last problem are involved in several normed versions of the Zel'manov prime theorem for Jordan algebras [Zel1] and related topics (see [FGR], [CR1], [CR2], [Rod, Section F], [RSZ], [CMR1], [CMRZ], [Mor1], [MR1], [MR2], [CMR2] and [CMR3]). In [RSZ], the authors find a general criterion (similar to the one above) for the norm extension problem for Jordan algebras, and prove that, if A is an associative algebra with an involution $*$, if J denotes the Jordan algebra $H(A,*)$, if A is a $*$-tight envelope of J, and if J is nondegenerate, then every complete algebra norm on J is equivalent to the restriction to J of an algebra norm on A. We recall that a JTS is called *nondegenerate* if it has no trivial elements $Q_x=0$, where $Q_x(y)=\{xyx\}$. In this paper we show that, if A is an associative algebra with two commuting involutions τ and π, if A is a τ-π-tight envelope of the Jordan Triple System $T := H(A,\tau)\cap S(A,\pi)$, and if T is nondegenerate, then every complete norm on T making the triple product continuous is equivalent to the restriction to T of an algebra norm on A.

By the way, in the presence of the remaining requirements on A and T as above, the assumption that T is nondegenerate is equivalent to A being semiprime. This is a consequence of the following proposition.

PROPOSITION 0. *Let (A,τ,π) be an associative algebra with two commuting involutions, and put $T := H(A,\tau)\cap S(A,\pi)$. Then T is nondegenerate whenever A is semiprime. Moreover, the converse is true if every τ-π-invariant ideal of A meets T.*

Proof. Assume that A is semiprime. By [Zel3, Lemma 1], it is enough to prove that $H(A,\tau)$ is nondegenerate. Let h be an element of $H(A,\tau)$ such that $hH(A,\tau)h=0$. For $s,s_1\in S(A,\tau)$ and $h_1\in H(A,\tau)$ we have

$$(hsh)h_1(hsh) = h(shh_1hs)h \in hH(A,\tau)h = 0$$

and

$$\begin{aligned}(hsh)s_1(hsh) &= h\{shs_1\}hsh - hs_1hshsh\\ &\in hH(A,\tau)hsh - hs_1hH(A,\tau)h = 0,\end{aligned}$$

so $(hsh)A(hsh)=0$. By the semiprimeness of A, we obtain $hsh=0$ for every $s\in S(A,\tau)$, and consequently $hAh=0$. Again, by the semiprimeness of A, we conclude that $h=0$.

Now, assume that T is nondegenerate and every τ-π-invariant ideal of A meets T. Let I be an ideal of A such that $I^2=0$ (hence $(I^\tau)^2=(I^2)^\tau=0$ and also $(I^\pi)^2=(I^{\tau^\pi})^2=0$). Then $J := I+I^\tau+I^\pi+I^{\tau^\pi}$ is a τ-π-invariant ideal of A. For $x\in J\cap T$ we have

$$\begin{aligned}Q_{Q_{Q_xT}T}T &= \{\{\{xTx\}T\{xTx\}\}T\{\{xTx\}T\{xTx\}\}\}\\ &\in I^2+(I^\tau)^2+(I^\pi)^2+(I^{\tau^\pi})^2=0,\end{aligned}$$

hence, by the nondegeneracy of T we obtain $Q_{Q_xT}T=0$, $Q_xT=0$, $x=0$, i.e., $J\cap T=0$. Therefore $J=0$ (since J is a τ-π-invariant ideal of A not meeting T), and consequently $I=0$. ∎

1. The main result.

The proof of our main result will consist in applying the closed graph theorem to obtain the separate $|||\cdot|||$-continuity of the pentad mapping $\{\ldots\}_5$, so that its joint $|||\cdot|||$-continuity will follow from the principle of uniform boundedness. Finally Theorem 1.2 of [Mor2] will be applied. We begin by stating the following identity (courtesy of E. I. Zel'manov), the verification of which is left to the reader.

LEMMA 1. *Let x,y,z,t,u,c be elements in a special JTS. Then*

$$\begin{aligned}4\{xyztu\}c\{xyztu\} &= 8\{x\{yztuc\}\{xyztu\}\} - 2\{xc(utzyxyztu)\}\\ &\quad - 4x\{yztu\{yztuc\}\}x + xyztucutzyx + utzyxcxyztu.\end{aligned}$$

THEOREM 2. *Let (A,τ,π) be an associative algebra over $\mathbb{K}$ $(=\mathbb{R},\mathbb{C})$ with two commuting involutions. Assume that A is a τ-π-tight envelope of the JTS $T := H(A,\tau)\cap S(A,\pi)$ and that T is nondegenerate. Then for every complete triple-norm $|||\cdot|||$ on T there exists an algebra norm $|\cdot|$ on A making τ and π isometric and having the following properties:*

(1) *The restriction of $|\cdot|$ to T is equivalent to $|||\cdot|||$.*

(2) *If $\widehat{A}$ denotes the completion of $(A,|\cdot|)$, and if τ,π stand for the unique isometric involutions on $\widehat{A}$ that extend τ,π, respectively, then $T = H(\widehat{A},\tau)\cap S(\widehat{A},\pi)$.*

(3) *$(\widehat{A},\tau,\pi)$ is a topological τ-π-tight envelope of T, i.e., $\widehat{A}$ is generated as an associative Banach algebra by T and every nonzero τ-π-invariant ideal of $\widehat{A}$ meets T.*

Proof. Fix y,z,t,u in T, and consider the mapping P^{yztu} from T to T given by $P^{yztu}(x)=\{xyztu\}$ for all x in T. If $\{x_n\}\to 0$ in T and $P^{yztu}(x_n)=\{x_nyztu\}\to l\in T$, then, by Lemma 1, we have $Q_l(c)=lcl=0$ for every $c\in T$. Since T is nondegenerate, $l=0$ and we conclude that P^{yztu} is $|||\cdot|||$-continuous, i.e., the pentad mapping is separately continuous in the first variable. From the equalities

$$\{zyxtu\} = \{\{xyz\}tu\} - \{xyztu\},\qquad \{utzyx\} = \{xyztu\}$$

we deduce that the pentad mapping is also separately continuous in the third and fifth variables. By the uniform boundedness principle, for β,δ

in T, the mapping $(\alpha,\gamma,\varepsilon)\mapsto\{\alpha,\beta,\gamma,\delta,\varepsilon\}$ from $T\times T\times T$ to T is jointly $|\!|\!|\cdot|\!|\!|$-continuous. Then a new application of Lemma 1 and the closed graph theorem allow us to derive the separate $|\!|\!|\cdot|\!|\!|$-continuity of the pentad mapping in its second and fourth variables. Now, the pentad is $|\!|\!|\cdot|\!|\!|$-continuous in each of its variables, so, again by the uniform boundedness principle, the pentad mapping is jointly $|\!|\!|\cdot|\!|\!|$-continuous. By [Mor2, Theorem 1.2], there exists an algebra norm $\|\cdot\|$ on A making τ and π isometric and satisfying (1) (with $\|\cdot\|$ instead of $|\cdot|$).

Such a norm also satisfies (2) (again with $\|\cdot\|$ instead of $|\cdot|$). To see this, let B denote the completion of $(A,\|\cdot\|)$, and consider the extension to B (also denoted by τ,π) of the involutions τ,π of A. If x is in $H(B,\tau)\cap S(B,\pi)$, then there exists a sequence $\{x_n\}$ in A such that $x=\|\cdot\|\text{-}\lim\{x_n\}$, so that

$$x=\frac{x+x^{\tau}-x^{\pi}-x^{\tau^{\pi}}}{4}=\|\cdot\|\text{-}\lim\frac{x_n+x_n^{\tau}-x_n^{\pi}-x_n^{\tau^{\pi}}}{4}$$

belongs to T because $(x_n+x_n^{\tau}-x_n^{\pi}-x_n^{\tau^{\pi}})/4$ lies in T for all n and T is closed in B by $\|\cdot\|$-completeness.

Concerning (3), it is obvious that, for B as above, T generates B as a Banach algebra, but we need to "tighten" the envelope. Let I be the largest ideal of B contained in

$$H(B,\tau)\cap H(B,\pi)\oplus S(B,\tau)\cap H(B,\pi)\oplus S(B,\tau)\cap S(B,\pi).$$

I is a closed τ-π-invariant ideal of B, hence we can consider the associative Banach algebra B/I with commuting isometric involutions

$$(a+I)^{\tau}:=a^{\tau}+I,\qquad (a+I)^{\pi}:=a^{\pi}+I.$$

Moreover $I\cap A$ is a τ-π-invariant ideal of A such that $(I\cap A)\cap T=I\cap T=0$, hence $I\cap A=0$ because A is a τ-π-tight envelope of T. Therefore, the natural τ-π-homomorphism $\phi:a\mapsto a+I$ from A into B/I is one-to-one and we can define the definitive algebra norm $|\cdot|$ on A by setting $|a|:=\|a+I\|$ for every a in A. Then for $t\in T$ and $x\in I$ we have

$$\begin{aligned}\|t+x\|&=\left\|\frac{t+x}{2}\right\|+\left\|\left(\frac{t+x}{2}\right)^{\tau}\right\|\ge\left\|t+\frac{x+x^{\tau}}{2}\right\|\\&=\left\|\frac{t+(x+x^{\tau})/2}{2}\right\|+\left\|-\left(\frac{t+(x+x^{\tau})/2}{2}\right)^{\pi}\right\|\\&\ge\left\|t+\frac{x+x^{\tau}-x^{\pi}-x^{\tau^{\pi}}}{4}\right\|=\|t\|\end{aligned}$$

because $x+x^{\tau}-x^{\pi}-x^{\tau^{\pi}}\in I\cap T=0$. It follows that $|t|=\|t\|$ for every t in T, hence property (1) holds. Now, since ϕ is an isometry from $(A,|\cdot|)$ into B/I and $\phi(A)$ is dense in B/I, it extends to a τ-π-isometric isomorphism $\widehat{\phi}$ from $\widehat{A}$ onto B/I, and then

$$H(\widehat{A},\tau)\cap S(\widehat{A},\pi)=\widehat{\phi}^{-1}(H(B/I,\tau)\cap S(B/I,\pi))=\widehat{\phi}^{-1}\phi(T)=T,$$

which proves property (2). Clearly, as T generates A, we see that $\widehat{A}$ is generated as an associative Banach algebra by T. Finally, since, by definition of I, B/I has no nonzero ideals contained in

$$H(B/I,\tau)\cap H(B/I,\pi)\oplus S(B/I,\tau)\cap H(B/I,\pi)\oplus S(B/I,\tau)\cap S(B/I,\pi),$$

every nonzero τ-π-invariant ideal of B/I meets $H(B/I,\tau)\cap S(B/I,\pi)$, which completes the proof. ∎

We conclude the paper by showing that the requirement of completeness for the norm $|\!|\!|\cdot|\!|\!|$ in Theorem 2 cannot be removed. To show the appropriate counter-example, the following result will be useful. As usual, if $(A,*)$ is an algebra with involution, then given n in $\mathbb{N}$, we extend the involution of A to the algebra $M_n(A)$ by setting $(a_{i,j})^*=(a_{j,i}^*)$. In other words, if we identify $M_n(A)\equiv M_n(\mathbb{K})\otimes A$, then $(M_n(A),*)=(M_n(\mathbb{K}),t)\otimes(A,*)$, where t is the familiar transpose involution on $M_n(\mathbb{K})$.

Proposition 3. *Let (A,τ,π) be an algebra with two commuting involutions, satisfying the following two conditions:*

(1) $A=A^2$.
(2) $T:=H(A,\tau)\cap S(A,\pi)$ *generates* A.

Then, for n in $\mathbb{N}$, $T_n:=H(M_n(A),\tau)\cap S(M_n(A),\pi)$ generates $M_n(A)$.

Proof. Let $n\in\mathbb{N}$, and let S_n denote the subalgebra of $M_n(A)$ generated by T_n. If $\{e_{i,j}\}_{i,j=1,\dots,n}$ denotes the usual system of matrix units for $M_n(\mathbb{K})$, then it is enough to show that, for all $i,j\in\{1,\dots,n\}$, $e_{i,j}\otimes A$ is contained in S_n. Since $e_{i,i}\otimes T\subset S_n$, certainly the above happens if $i=j$. Now, fix $i,j\in\{1,\dots,n\}$ with $i\ne j$, and define $W:=\{a\in A: e_{i,j}\otimes a\in S_n\}$. Since, for $a\in A$, $w\in W$, and $t\in T$, we have

$$e_{i,j}\otimes wa=(e_{i,j}\otimes w)(e_{j,j}\otimes a),\qquad e_{i,j}\otimes ta=[(e_{i,j}+e_{j,i})\otimes t](e_{j,j}\otimes a),$$

and $(e_{i,j}+e_{j,i})\otimes T\subset S_n$, it follows that $WA\subseteq W$ and $TA\subseteq W$. In this way, $P:=\{x\in A: xA\subseteq W\}$ is a right ideal (hence a subalgebra) of A containing T. Therefore $P=A$, i.e. $A^2\subseteq W$. Since $A=A^2$, we conclude that $W=A$. ∎

Let $\mathbb{K}=\mathbb{R},\mathbb{C}$. For $n\in\mathbb{N}$ and $\varepsilon=\pm1$, we consider the involutions on $M_{2n}(\mathbb{K})$ given by $a\mapsto s^{-1}a^ts$, where a^t denotes the transpose of a and $s:=\operatorname{diag}\{\overbrace{q,\dots,q}^{n}\}$ with $q:=\left(\begin{smallmatrix}0&1\\\varepsilon&0\end{smallmatrix}\right)$. Depending on ε, these involutions are called the *symmetric* (if $\varepsilon=1$) and the *symplectic* (if $\varepsilon=-1$) involutions on $M_{2n}(\mathbb{K})$, and are denoted by τ and π, respectively. These two involutions pass to the algebra $M_\infty(\mathbb{K})$ (of all countably infinite matrices over

$\mathbb{K}$ with only a finite number of nonzero entries) by regarding $M_\infty(\mathbb{K})$ as $\bigcup_{n\in\mathbb{N}} M_{2n}(\mathbb{K})$ in the most natural way. Then, by [Mor2, Corollary 2.6], there exists a triple-norm $|\!|\!|\cdot|\!|\!|$ on $T := H(M_\infty(\mathbb{K}),\tau) \cap S(M_\infty(\mathbb{K}),\pi)$ such that there is no algebra norm on $M_\infty(\mathbb{K})$ whose restriction to T is equivalent to $|\!|\!|\cdot|\!|\!|$. Notice that, since $M_\infty(\mathbb{K})$ is a simple associative algebra, T is nondegenerate by Proposition 0, and in order to show that $M_\infty(\mathbb{K})$ is a τ-π-tight envelope of T it only remains to check that $M_\infty(\mathbb{K})$ is generated by T. Clearly $T_2 := \{\left(\begin{smallmatrix} 0 & \lambda \\ \mu & 0 \end{smallmatrix}\right) : \lambda, \mu \in \mathbb{K}\}$ generates $M_2(\mathbb{K})$, and, by Proposition 3, $M_{2n}(\mathbb{K})$ is generated by $T_{2n} := H(M_{2n}(\mathbb{K}),\tau) \cap S(M_{2n}(\mathbb{K}),\pi)$, which actually establishes the desired conclusion.

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