[12] A. Makagon and H. Salehi, *Structure of periodically distributed stochastic sequences*, in: Stochastic Processes, A Festschrift in Honour of Gopinath Kallianpur, S. Cambanis *et al.* (eds.), Springer, 1993, 245–251.

[13] A. G. Miamee, *Explicit formula for the best linear predictor of periodically correlated sequences*, SIAM J. Math. Anal. 24 (1993), 703–711.

Department of Mathematics
Hampton University
Hampton, Virginia 23668
U.S.A.
E-mail: makagon@fusion.hamptonu.edu

The Hugo Steinhaus Center
for Stochastic Methods
Wrocław Technical University
Wrocław, Poland

Received October 26, 1998 (4194)

STUDIA MATHEMATICA 136 (1) (1999)

Compact endomorphisms of $H^\infty(D)$

by

JOEL F. FEINSTEIN (Nottingham)
and HERBERT KAMOWITZ (Boston, Mass.)

Abstract. Compact composition operators on $H^\infty(G)$, where G is a region in the complex plane, and the spectra of these operators were described by D. Swanton (*Compact composition operators on* $B(D)$, Proc. Amer. Math. Soc. 56 (1976), 152–156). In this short note we characterize all compact endomorphisms, not necessarily those induced by composition operators, on $H^\infty(D)$, where D is the unit disc, and determine their spectra.

Let D be the open unit disc and, as usual, let $H^\infty(D)$ be the algebra of bounded analytic functions on D with $\|f\| = \sup_{z\in D} |f(z)|$. With pointwise addition and multiplication, $H^\infty(D)$ is a well known uniform algebra. In this note we characterize the compact endomorphisms of $H^\infty(D)$ and determine their spectra.

We show that although not every endomorphism T of $H^\infty(D)$ has the form $T(f)(z) = f(\phi(z))$ for some analytic ϕ mapping D into itself, if T is compact, there is an analytic function $\psi : D \to D$ associated with T. In the case where T is compact, the derivative of ψ at its fixed point determines the spectrum of T.

The structure of the maximal ideal space M_{H^∞} is well known. Evaluation at a point $z \in D$ gives rise to an element in M_{H^∞} in the natural way. The remainder of M_{H^∞} consists of singleton Gleason parts and Gleason parts which are analytic discs. An analytic disc, $P(m)$, containing a point $m \in M_{H^\infty}$ is a subset of M_{H^∞} for which there exists a continuous bijection $L_m : D \to P(m)$ such that $L_m(0) = m$ and $\widehat{f}(L_m(z))$ is analytic on D for each $f \in H^\infty(D)$. Moreover, the map L_m has the form

$$L_m(z) = \text{w}^* \lim \frac{z + z_\alpha}{1 + \overline{z}_\alpha z}$$

1991 *Mathematics Subject Classification*: 46J15, 47B48.
This research was supported by EPSRC grant GR/M31132.

for some net $z_\alpha \to m$ in the w* topology, whence

$$\widehat{f}(L_m(z)) = \lim f\left(\frac{z+z_\alpha}{1+\overline{z}_\alpha z}\right)$$

for all $f \in H^\infty(D)$. A fiber M_λ over some $\lambda \in \overline{D} \setminus D$ is the zero set in M_{H^∞} of the function $z - \lambda$. Each part distinct from D is contained in exactly one fiber M_λ. With no loss of generality we let $\lambda = 1$. We also recall that two elements n_1 and n_2 are in the same part if, and only if, $\|n_1 - n_2\| < 2$, where $\|\cdot\|$ is the norm in the dual space $H^\infty(D)^*$.

Now let T be an endomorphism of $H^\infty(D)$, i.e. T is a (necessarily) bounded linear map of $H^\infty(D)$ to itself with $T(fg) = T(f)T(g)$ for all $f, g \in H^\infty(D)$. For a given T, either T has the form $Tf(z) = f(\omega(z))$ for some analytic map $\omega : D \to D$, or $Tf = \widehat{f}(n)1$ for some $n \in M_{H^\infty}$, or there exists an $m \in M_{H^\infty}$, a net $z_\alpha \to m$ in the w* topology and an analytic function $\tau : D \to D$ with $\tau(0) = 0$, for which $Tf(z) = \widehat{f}(L_m(\tau(z)))$ (see [3]). Further, on general principles, if T is an endomorphism of $H^\infty(D)$ there exists a w* continuous map $\phi : M_{H^\infty} \to M_{H^\infty}$ with $\widehat{Tf}(n) = \widehat{f}(\phi(n))$ for all $n \in M_{H^\infty}$. In the last case, $\phi(z) = L_m(\tau(z))$ for $z \in D$.

For a given endomorphism T, if the induced map ϕ maps D to itself, then T is commonly called a *composition operator*. Compact composition operators on H^∞ were completely characterized in [4]. However, in general, $L_m(\tau(z))$ need not be in D, and so not every endomorphism of $H^\infty(D)$ is a composition operator. It is these endomorphisms that we discuss here. Trivially, for any $n \in M_{H^\infty} \setminus D$, the map T defined by $Tf(z) = \widehat{f}(n)1$ is a compact endomorphism of $H^\infty(D)$ which is not a composition operator.

Now let $P(m)$ be an analytic part and let T be an endomorphism defined by $Tf(z) = \widehat{f}(L_m(\tau(z)))$ as discussed above. Also suppose that $\phi : M_{H^\infty} \to M_{H^\infty}$ is such that $\widehat{Tf} = \widehat{f} \circ \phi$. Assume that T is compact. We claim that $\tau(D)$ is a compact subset of D in the Euclidean topology. Indeed, if we regard the endomorphism T as an operator from $H^\infty(D)$ into $C(M_{H^\infty})$, then T is compact if, and only if, ϕ is w* to norm continuous on M_{H^∞} (see [2]). Since M_{H^∞} is itself compact and connected (in the w* topology), $\phi(M_{H^\infty})$ must be compact and connected in the norm topology on M_{H^∞}, and so ϕ maps M_{H^∞} into a norm compact connected subset of $P(m)$. Therefore the range, $\phi(D) = L_m(\tau(D))$, is contained in a norm compact subset of $P(m)$, and further, since L_m^{-1} is an isometry in the Gleason norms on $P(m)$ and D (see [1]), $\tau(D) = L_m^{-1}(\phi(D))$ is contained in a compact subset of D in the norm topology on D. Since the norm, Euclidean and w* topologies on D coincide, $\overline{\tau(D)}$ is a compact subset of D in these three topologies. As a consequence, $\widehat{\tau}(M_{H^\infty}) \subset D$.

Next consider two maps of $H^\infty(D)$ into itself. The first, C_{L_m}, is defined by $C_{L_m}(f)(z) = \widehat{f}(L_m(z))$, and the second, C_τ, by $C_\tau(f)(z) = f(\tau(z))$. Then $(C_{L_m} \circ C_\tau)(f)(z) = C_{L_m}(f \circ \tau)(z) = \widehat{f \circ \tau}(L_m(z))$ and $(C_\tau \circ C_{L_m})(f)(z) = \widehat{f}(L_m(\tau(z))) = Tf(z)$. But if B is a Banach space and S_1 and S_2 are any two bounded linear maps from B to B, the spectra satisfy $\sigma(S_1S_2) \setminus \{0\} = \sigma(S_2S_1) \setminus \{0\}$. Thus we see that $\sigma(T) \setminus \{0\} = \sigma(C_{L_m} \circ C_\tau) \setminus \{0\}$.

Since f is analytic on a neighborhood of the range of $\widehat{\tau}$ which is a subset of D, a standard functional calculus argument gives $\widehat{f \circ \tau}(L_m(z)) = f(\widehat{\tau}(L_m(z)))$. If we let $\psi(z) = \widehat{\tau}(L_m(z))$ we see that $C_{L_m} \circ C_\tau$ is a compact composition operator in the usual sense, and so if $z_0 \in D$ is the unique fixed point of ψ, and $\mathbb{N}$ is the set of positive integers, then $\sigma(T) = \{(\psi'(z_0))^n : n \in \mathbb{N}\} \cup \{0, 1\}$.

To summarize, we have shown the following.

THEOREM. *If T is a compact endomorphism of $H^\infty(D)$, then either T has one-dimensional range, i.e. $Tf = \widehat{f}(n)1$ for some $n \in M_{H^\infty}$, or T is a composition operator in the usual sense, or T has the form $Tf(z) = \widehat{f}(L_m(\tau(z)))$ where τ is described above. In the last case, there is a compact composition operator C_ψ such that $\sigma(T) = \sigma(C_\psi) = \{(\psi'(z_0))^n : n \in \mathbb{N}\} \cup \{0, 1\}$ where $z_0 \in D$ is the unique fixed point of ψ.*

We conclude with two examples showing differences between composition operators and general endomorphisms.

(a) With the same terminology and symbols, suppose $\widehat{\tau}$ is constant on $P(m)$, i.e. $\widehat{\tau}(P(m)) = \{\widehat{\tau}(m)\}$. Since T is compact, $\widehat{\tau}(m) \in D$. Then using C_τ and C_{L_m} as before, we show that $T^2f = \widehat{f}(n)1$ for some $n \in P(m)$. Indeed, $(C_{L_m} \circ C_\tau)f = f(t_0)1$ where $t_0 = \widehat{\tau}(m) \in D$. Then we see that

$$\begin{aligned} T^2f &= [(C_\tau \circ C_{L_m}) \circ (C_\tau \circ C_{L_m})]f = [C_\tau \circ (C_{L_m} \circ C_\tau) \circ C_{L_m}]f \\ &= [C_\tau \circ (C_{L_m} \circ C_\tau)](\widehat{f} \circ L_m) = C_\tau(\widehat{f}(L_m(t_0))1) = \widehat{f}(L_m(t_0))1. \end{aligned}$$

Letting $n = L_m(t_0)$ gives the result.

One way to have $\widehat{\tau}$ constant on $P(m)$ is for τ to be continuous at 1 in the usual sense.

A more interesting example, perhaps, is to define τ by

$$\tau(z) = \tfrac{1}{2} z e^{(z+1)/(z-1)},$$

and $m \in M_{H^\infty}$ as the w* limit of a real net x_α approaching 1. Then

$$\widehat{\tau}(L_m(z)) = \lim_\alpha \tau\left(\frac{z+x_\alpha}{1+\overline{x}_\alpha z}\right) = 0,$$

and so $T^2f = \widehat{f}(m)1$ for all $f \in H^\infty(D)$. In both cases, $\sigma(T) = \{0, 1\}$.

(b) Finally, let $\{z_n\}$ be an interpolating Blaschke sequence approaching 1, $z_1 = 0$, with m in the w* closure of $\{z_n\}$ and B the corresponding Blaschke product. If $\tau(z) = \frac{1}{2}B(z)$, then it is well known [3] that $(\widehat{\tau} \circ L_m)'(0) = \frac{1}{2}(\widehat{B} \circ L_m)'(0) \neq 0$. This, then, is an example of a compact endomorphism of $H^\infty(D)$ which is not a composition operator but whose spectrum properly contains $\{0, 1\}$.

References

[1] S. Dineen, J. F. Feinstein, A. G. O'Farrell and R. M. Timoney, *A fixed-point theorem for holomorphic maps*, Proc. Roy. Irish Acad. Sect. A 94 (1994), 77–84.

[2] N. Dunford and J. Schwartz, *Linear Operators: Part I*, Interscience, New York, 1958.

[3] J. Garnett, *Bounded Analytic Functions*, Academic Press, London, 1981.

[4] D. Swanton, *Compact composition operators on B(D)*, Proc. Amer. Math. Soc. 56 (1976), 152–156.

School of Mathematical Sciences
University of Nottingham
Nottingham NG7 2RD, England
E-mail: Joel.Feinstein@nottingham.ac.uk

Department of Mathematics
University of Massachusetts at Boston
100 Morrissey Boulevard
Boston, MA 02125-3393
U.S.A.
E-mail: hkamo@cs.umb.edu

Received November 10, 1998 (4200)

STUDIA MATHEMATICA 136 (1) (1999)

The triple-norm extension problem: the nondegenerate complete case

by

A. MORENO GALINDO (Granada)

Abstract. We prove that, if A is an associative algebra with two commuting involutions τ and π, if A is a τ-π-tight envelope of the Jordan Triple System $T := H(A,\tau) \cap S(A,\pi)$, and if T is nondegenerate, then every complete norm on T making the triple product continuous is equivalent to the restriction to T of an algebra norm on A.

0. Introduction and preliminaries. The classification of prime Jordan Triple Systems (JTS) is essentially due to E. I. Zel'manov ([Zel2], [Zel3] and [Zel4]). Later Zel'manov's ideas have been clarified and completed in [D'A], [ACCM] and [D'AM]. In the Zel'manov classification of prime nondegenerate JTS's, triples of the following form became crucial. Take an associative algebra A with two commuting involutions τ, π, put

$$H(A,\tau) := \{a \in A : a^\tau = a\} \quad \text{and} \quad S(A,\pi) := \{a \in A : a^\pi = -a\},$$

and consider the JTS $T := H(A,\tau) \cap S(A,\pi)$ under the triple product $\{xyz\} := \frac{1}{2}(xyz + zyx)$.

Let A and T be as above. If $\|\cdot\|$ is an algebra norm on A, then, clearly, the restriction of $\|\cdot\|$ to T is a *triple-norm* on T, i.e., a norm on the vector space T making the triple product of T continuous. The converse question is called the *triple-norm extension problem* [Mor2] (3NEP for short), namely: given a triple-norm $|||\cdot|||$ on T, is there an algebra norm on A whose restriction to T is equivalent to $|||\cdot|||$? To have some possibility of an affirmative answer to the above question, it seems reasonable to assume that A is a τ-π-*tight envelope* of T, which means:

(i) A is generated by T, and

(ii) every nonzero τ-π-invariant ideal of A has nonzero intersection with T.

1991 *Mathematics Subject Classification*: Primary 17C65, 46K70.

Key words and phrases: Jordan triple systems, JB^*-triples, norm extension problem.

Partially supported by DGICYT Grant PB95-1146 and Junta de Andalucía grant FQM 0199.