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Kadec norms and Borel sets in a Banach space

by

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Abstract. We introduce a property for a couple of topologies that allows us to give simple proofs of some classic results about Borel sets in Banach spaces by Edgar, Schachermayer and Talagrand as well as some new results. We characterize the existence of Kadec type renormings in the spirit of the new results for LUR spaces by Moltó, Orihuela and Troyanski.

1. Introduction. Throughout this paper $(X, \|\cdot\|)$ will denote a Banach space, X^* its dual, w and w^* the weak and weak* topologies respectively. B_X (resp. B_{X^*}) denotes the unit ball of X (resp. X^*). S_X will be the unit sphere of X. We shall also consider topologies on X of convergence on some subsets of the dual space. A subset of B_{X^*} is said to be *norming* (resp. *quasi-norming*) if its w^*-closed convex envelope is B_{X^*} (resp. if the envelope contains an open ball centered at the origin).

A norm $\|\cdot\|$ on X is said to have the *Kadec property* when the weak and norm topologies coincide on the unit sphere. A norm is said to be *locally uniformly rotund* (LUR) if for every sequence (x_n) in the unit sphere and for every point x in the unit sphere such that $\lim_n \|x_n + x\| = 2$ the sequence (x_n) converges to x in norm. LUR norms have the Kadec property. For the proof of this fact and other properties of Banach spaces having an equivalent LUR norm we refer to the book [4]. There exist Banach spaces having a Kadec norm and admitting no equivalent LUR norm [11].

Edgar [5] proved that in a Banach space which admits an equivalent Kadec norm the Borel σ-algebras generated by the weak and norm topologies coincide. He also noted that an analogous result also holds when the Kadec property holds for the weak* topology. Schachermayer [6] proved that a Banach space X that has an equivalent Kadec norm is a Borel set in (X^{**}, w^*). Talagrand [26] showed that the previous two results are not true

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for general Banach spaces, but he proved [25] that for subspaces of weakly compactly generated spaces the Borel sets for the topology of pointwise convergence on a quasi-norming subset of the dual space and the norm Borel sets are the same.

Jayne, Namioka and Rogers [15] introduced the notion of a countable cover by sets of small local d-diameter (SLD) (see Definition 2) for a topological space with respect to some metric d and they noted that if a Banach space X has an equivalent Kadec norm then (X, w) has SLD with respect to the norm, which implies the coincidence of the Borel sets for the norm and weak topologies. In fact, property SLD implies the coincidence of the Borel sets for the original topology and the metric in a wider topological context. Oncina [22] has made a deep study of property SLD showing that a Banach space with SLD for the weak topology with respect to the norm is a Borel set in its bidual. Another approach to the coincidence of the Borel sets and related properties has been given by Hansell in his unpublished preprint [10] using the notion of descriptive topological space. In the context of a Banach space endowed with its weak topology, Hansell's notion of descriptive space is equivalent to property SLD, as pointed out by Moltó, Orihuela, Troyanski and Valdivia [20].

Recently Moltó, Orihuela and Troyanski [19] have characterized the Banach spaces which admit an equivalent LUR norm as those spaces X such that (X, w) satisfies a special case of norm SLD: X has an equivalent LUR norm if and only if (X, w) satisfies Definition 2 below and the weak neighbourhood there is a slice (the intersection with an open half space). See also the comments after Theorem 2.

Our aim in this paper is to show that all the above mentioned positive results on coincidence of Borel σ-algebras and the Borel nature of a Banach space in its bidual stem from a common topological principle which can be used to characterize the existence of Kadec type norms in a Banach space.

In Section 2 we introduce a useful condition (Definition 1) for a couple of topologies that gives a natural approach to the study of Borel sets (Proposition 3). When one of the topologies is given by a metric, our property is equivalent to property SLD (Definition 2, Proposition 2).

In Section 3 we use the framework of topological vector spaces to study the relation between property SLD and the existence of Kadec type equivalent norms. In particular we show that if X is a Banach space such that (X, w) has SLD then the weak and norm topologies coincide on the level sets of some positive homogeneous function (Theorem 1). We also characterize the existence of an equivalent Kadec norm (Theorem 2) in the spirit of the recents results on LUR norms by Moltó, Orihuela and Troyanski [19].

In Section 4 we apply the previous results to WCD Banach spaces taking advantage of the existence of a LUR norm to build Kadec norms for topologies weaker than the weak topology (Theorem 3) and to show the coincidence of Borel sets improving a result by Talagrand. As an application to nonmetric topologies we finish by showing that if K is a Radon–Nikodym compact set then $C(K)$ has an equivalent norm such that the weak and pointwise topologies coincide on the unit sphere (Theorem 4).

Part of the results of this paper have been announced in [23].

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2. Topological results. We begin with the main definition of this paper. Actually the idea is implicit in [25]. We recall that a *network* for some topology is a family of sets not necessarily open such that every open set can be written as a union of sets in the family.

DEFINITION 1. Let X be a set, and τ_1 and τ_2 two topologies on X. A subset $A \subset X$ is said to have *property* $P(\tau_1, \tau_2)$ if there exists a sequence (A_n) of subsets of X such that the family $(A_n \cap U)$ where $n \in \mathbb{N}$ and $U \in \tau_2$ is a network for τ_1, that is, for every $x \in A$ and every $V \in \tau_1$ with $x \in V$ there exist $n \in \mathbb{N}$ and $U \in \tau_2$ such that $x \in A_n \cap U \subset V$.

Evidently, if $\tau_1 \subset \tau_2$ then X has $P(\tau_1, \tau_2)$, but this case is not interesting. The relevant case happens when $\tau_2 \subset \tau_1$, for instance, in applications to Banach spaces τ_1 and τ_2 will be the norm and the weak topology respectively. If τ_1 has a countable basis (V_n) then X has $P(\tau_1, \tau_2)$ for any τ_2, because we can take $A_n = V_n$. This happens in particular when (X, τ_1) is metrizable and separable. In fact, we shall use the property introduced in Definition 1 to extend results valid for separable spaces to nonseparable spaces.

Observe that if we take the sequence $(A_n \cap A)$ we can always suppose that $A_n \subset A$. That means that property $P(\tau_1, \tau_2)$ only depends on A equipped with the relative topologies.

To check $P(\tau_1, \tau_2)$ for a given A it is enough to verify the above set inclusion for all the V's belonging to a sub-basis of τ_1, because then A will have $P(\tau_1, \tau_2)$ with the countable family of the finite intersections of sets of the sequence (A_n).

The following proposition contains some other elementary consequences of Definition 1.

PROPOSITION 1. *Let X be a set, τ_1, τ_2 and τ_3 topologies on X, and A a subset of X. Then:*

(i) *If A has $P(\tau_1, \tau_2)$ and $B \subset A$ then B has $P(\tau_1, \tau_2)$.*
(ii) *If A has $P(\tau_1, \tau_2)$ and $P(\tau_2, \tau_3)$ then A has $P(\tau_1, \tau_3)$.*

(iii) *If every point of A has a τ_1-basis of neighbourhoods which is made up of τ_2-closed sets then the sequence (A_n) in Definition 1 can be taken to consist of τ_2-closed sets.*

(iv) *If every set A_n of Definition 1 is τ_2-Borel then for every $V \in \tau_1$ such that $A \subset V$, there is a τ_2-Borel set B satisfying $A \subset B \subset V$. In particular, if A is τ_1-open, or more generally, if A is a G_δ-set for the τ_1-topology, then A is τ_2-Borel.*

Proof. (i) Use the same sequence (A_n).

(ii) If (B_m) is a sequence for $P(\tau_2, \tau_3)$ then it is easy to check that $(A_n \cap B_m)$ satisfies the condition of Definition 1 for $P(\tau_1, \tau_3)$.

(iii) Fix $x \in A$. Take $V \in \tau_1$ with $x \in V$. Take $V_0 \in \tau_1$ such that $x \in V_0$ and $\overline{V_0}^{\tau_2} \subset V$. There exist A_n and $U \in \tau_2$ such that $x \in A_n \cap U \subset V_0$. Thus

$$x \in \overline{A_n}^{\tau_2} \cap U \subset \overline{A_n \cap U}^{\tau_2} \subset \overline{V_0}^{\tau_2} \subset V.$$

(iv) For every $x \in A$ there exist $n_x \in \mathbb{N}$ and $U_x \in \tau_2$ such that $x \in A_{n_x} \cap U_x \subset V$. Now we have

$$A = \bigcup_{x \in A} \{x\} \subset \bigcup_{x \in A} A_{n_x} \cap U_x = \bigcup_{n=1}^{\infty} \Big(A_n \cap \bigcup_{n_x = n} U_x \Big) = B \subset V$$

where B is clearly in $\mathrm{Borel}(X, \tau_2)$.

If $A = \bigcap_{n=1}^{\infty} V_n$ where $V_n \in \tau_1$ we can take τ_2-Borel sets (B_n) such that $A \subset B_n \subset V_n$. Then $A = \bigcap_{n=1}^{\infty} B_n$.

A particularly interesting case occurs when τ_1 is metrizable. In this case the property introduced in Definition 1 agrees with the following one given by Jayne, Namioka and Rogers in [15], which is a special case of their σ-fragmentability.

DEFINITION 2. Let (X, τ) be a topological space and let d be a metric on X. Then X *has a countable cover by sets of small local diameter* (SLD) if for every $\varepsilon > 0$ there exists a decomposition

$$X = \bigcup_{n=1}^{\infty} X_n^{\varepsilon}$$

such that for each $n \in \mathbb{N}$ every point of X_n^{ε} has a relative τ-neighborhood of diameter less than ε.

A Banach space X is said to have *countable Szlenk index* if for every $\varepsilon > 0$, there is a decreasing transfinite countable sequence (C_α) of subsets such that $B_X = \bigcup_\alpha (C_\alpha \setminus C_{\alpha+1})$ and every point of $C_\alpha \setminus C_{\alpha+1}$ has a relative weak neighbourhood in C_α of diameter less than ε. These spaces have been considered by Lancien [18]. Clearly, if X has countable Szlenk index, then (X, w) has $\|\cdot\|$-SLD. However, a separable Banach space X without the Point of Continuity Property does not have countable Szlenk index but (X, w) has $\|\cdot\|$-SLD.

PROPOSITION 2. *Let (X, τ) be a topological space and d a metric on X. Then X has a countable cover by sets of small local diameter if and only if X has $P(d, \tau)$. Moreover, if the closed d-balls are τ-closed then the sets X_n^{ε} in Definition 2 can be taken to be differences of τ-closed sets.*

Proof. If X_n^{ε} are the sets of Definition 2 it is easy to check that the sets (A_n) obtained by arranging $(X_n^{1/m})_{n,m}$ into a sequence by a diagonal process satisfy the condition of Definition 1.

For the other implication, given $\varepsilon > 0$ just define

$$X_n^{\varepsilon} = \{x \in A_n : \exists U \in \tau,\ x \in U,\ \mathrm{diam}(A_n \cap U) < \varepsilon\}.$$

The "moreover" part is a consequence of Proposition 1(iii).

The following result shows the good Borel behavior of a topological space (X, τ) that has $P(d, \tau)$ for some appropriate metric d. The statement (a) has already been noted by Jayne, Namioka and Rogers in [15] and [17], in terms of property SLD.

PROPOSITION 3. *Let (Y, τ) be a topological space and d a metric on Y stronger than τ and such that closed d-balls are τ-closed. Let X be a subset of Y having $P(d, \tau)$.*

(a) *Considering X with the inherited topologies we have*

$$\mathrm{Borel}(X, \tau) = \mathrm{Borel}(X, d).$$

(b) *If X is d-closed in Y then $X \in \mathrm{Borel}(Y, \tau)$.*

Proof. (a) Evidently every τ-Borel set is a d-Borel set. Conversely, if $V \subset X$ is a d-open set then it has $P(d, \tau)$. As closed d-balls are τ-closed we can apply Proposition 1(iii), (iv) to conclude that V is τ-Borel.

(b) Since X is a G_δ-set in (Y, d), the result follows from Proposition 1(ii), (iv).

The next corollary embraces the applications of property SLD to Banach spaces by Jayne, Namioka and Rogers [15], Oncina [22] and Hansell [10] (this last using the notion of descriptive space) that improve preceding ones by Edgar [5] and Schachermayer [6] on Banach spaces admitting Kadec norms. We shall prove later that Banach spaces having $P(\|\cdot\|, \tau)$ are not very different from Banach spaces that admit an equivalent Kadec norm (Theorem 1).

COROLLARY 1. *Let X be a Banach space and τ a vector topology weaker than the norm topology and such that $\overline{B_X}^{\tau}$ is bounded.*

(a) *If* X *has* $P(\|\cdot\|,\tau)$, *then* $\mathrm{Borel}(X,\|\cdot\|)=\mathrm{Borel}(X,\tau)$.
(b) *If* X *has* $P(\|\cdot\|,w)$, *then* $X\in\mathrm{Borel}(X^{**},w^*)$.

Proof. Note that $\overline{B}_X^{\tau}$ is the unit ball of an equivalent norm on X whose closed balls are τ-closed. Then apply Proposition 3.

Let us remark that $\overline{B}_X^{\tau}$ is bounded, for instance, when τ is the topology of convergence on a norming or a quasi-norming subset of X^*.

We now give an application of Proposition 3 to descriptive topology. Following Fremlin (see [16]), a completely regular topological space X is *Čech-analytic* if for every finite sequence s of positive integers there is a set $A(s)$ open or closed in the Čech–Stone compactification of X such that

$$X=\bigcup_{\sigma\in\mathbb{N}^{\mathbb{N}}}\bigcap_{n=1}^{\infty}A(\sigma|n)$$

where $\sigma|n$ denotes the finite sequence made up from the first n terms of the sequence σ. The notion of Čech-analytic space has some interest in the context of nonseparable and nonmetrizable topological spaces (e.g. a Banach space endowed with its weak topology), where the classical descriptive set theory is not applicable in general. We refer the interested reader to [16] and [10] for more information about Čech-analytic spaces and their applications to Banach spaces.

COROLLARY 2. *Let* (X,τ) *be a topological space. Suppose that there is a set* T *such that* X *can be identified as a subspace of* $\mathbb{R}^T$ *with the pointwise topology which is made up of bounded functions and is complete for the metric* d *on* X *of uniform convergence on* T. *If* X *has* $P(d,\tau)$, *then* X *is a Borel subset of* $\mathbb{R}^T$, *in fact a pointwise* $(F\cap G)_{\sigma\delta}$, *and* (X,τ) *is Čech-analytic.*

Proof. We can assume that d is defined on $\mathbb{R}^T$ and it is stronger than the pointwise topology with pointwise d-closed balls. As X is complete for d, it is d-closed in $\mathbb{R}^T$ and we finish by applying the proofs of Propositions 1 and 3.

According to [16] a sufficient condition for (X,τ) to be Čech-analytic is being homeomorphic to a Borel subset of some compact space. The reasoning above shows that $X\cap[-n,n]^T$ is Borel in $[-n,n]^T$, so it is Borel in $\overline{\mathbb{R}}^T$ where $\overline{\mathbb{R}}$ is the two-point compactification of $\mathbb{R}$. Now, as $X=\bigcup_{n=1}^{\infty}X\cap[-n,n]^T$ it is a Borel set in the compact $\overline{\mathbb{R}}^T$.

Hansell [10] proves that a descriptive topological space is always Čech-analytic, in particular, every Banach space X such that (X,w) has $\|\cdot\|$-SLD is Čech-analytic (see [20]). Corollary 2 contains more information about the structure of X in that particular case.

Under the hypothesis of Corollary 2, it is easy to show that every d-Borel subset of X is pointwise Borel in $\mathbb{R}^T$ and analogously Čech-analytic.

3. Kadec norms. It is convenient for our purposes to give a more general definition of Kadec norms involving topologies different from the weak topology.

DEFINITION 3. Let X be a Banach space and τ a vector topology weaker than the norm topology. An equivalent norm $\|\cdot\|$ is said to be τ-*Kadec* if the norm topology and τ coincide on the unit sphere of $\|\cdot\|$.

The next result appears in [1].

PROPOSITION 4. *A* τ*-Kadec norm* $\|\cdot\|$ *is* τ*-lower semicontinuous, that is, its unit ball is always* τ*-closed.*

Proof. Suppose that $\|\cdot\|$ is not τ-lsc. Then there is a net (x_ω) on the unit sphere S_X and a point x outside the unit ball B_X such that τ-$\lim_\omega x_\omega=x$. Take numbers $t_\omega>1$ such that $\|x+t_\omega(x_\omega-x)\|=\|x\|$. Let $y_\omega=x+t_\omega(x_\omega-x)$. Note that $\{t_\omega\}$ is bounded because $\inf_\omega\|x_\omega-x\|>0$. We deduce that τ-$\lim_\omega y_\omega=x$. Since $\|y_\omega\|=\|x\|$ we should have $\lim_\omega\|y_\omega-x\|=0$, but this is impossible because $\|y_\omega-x\|\ge\|x_\omega-x\|$.

As mentioned in the introduction, LUR norms provide examples of norms with the Kadec property. In fact, it is not difficult to prove that a τ-lower semicontinuous LUR norm is τ-Kadec. At this point it is important to remark that if the unit ball of a Banach space is τ-closed for some vector topology τ, then the new unit ball after a renorming is not necessarily τ-closed. For example, there exists a dual Banach space that admits an equivalent LUR norm but no equivalent dual LUR norm (see the remarks after Theorem 3).

Given two topologies τ_1 and τ_2 on X and a family Σ of subsets of X we shall say that Σ is *good at* $x\in X$ if for every $V\in\tau_1$ with $x\in\tau_1$ there exist $S\in\Sigma$ and $U\in\tau_2$ such that $x\in S\cap U\subset V$. A *good family* means a family good at every point of X. It is easy to see that a family Σ covering X such that on every $S\in\Sigma$ the topologies τ_1 and τ_2 coincide is good and property $P(\tau_1,\tau_2)$ is equivalent to the existence of a countable good family.

The following lemma shows how to make a good family of "thick" sets from a good one made up of "thin" sets.

LEMMA 1. *Let* X *be a vector space,* $\tau_2\subset\tau_1$ *vector topologies on* X *and* Σ *a family good at some* $x\in X$. *Then the family*

$$\{S+W: S\in\Sigma,\ 0\in W\in\tau_1\}$$

is good at x. *Thus, if* Σ *and* Π *are families of subsets of* X *such that for* *every $S \in \Sigma$ and every $W \in \tau_1$ with $0 \in W$ there exists $P \in \Pi$ such that*

$$S \subset P \subset S + W$$

then Π is good if and only if Σ is.

Proof. Given $V \in \tau_1$ with $x \in V$ we shall find $S \in \Sigma$, $0 \in W \in \tau_1$ and $U \in \tau_2$ such that

$$x \in (S + W) \cap U \subset V,$$

As $0 + x \in V$ we can take $W_1, V' \in \tau_1$ with $0 \in W_1$, $x \in V'$ and $W_1 + V' \subset V$. Since Σ is good at x there are $S \in \Sigma$ and $U' \in \tau_2$ such that $x \in S \cap U' \subset V'$. As $0 + x \in U'$ we can find $W_2, U \in \tau_2$ with $0 \in W_2$, $x \in U$ and $W_2 + U \subset U'$. Now take $W = W_1 \cap (-W_2) \in \tau_1$. We show that U and W satisfy the above set inclusion. If $y \in (S + W) \cap U$ then there is $z \in S$ such that $y - z \in W \subset -W_2$ so $z = (z - y) + y \in U'$ thus $z \in S \cap U' \subset V'$. Now as $y - z \in W \subset W_1$ we have $y = (y - z) + z \in V$.

The applications of Kadec type norms to the results developed in Section 2 are contained in the following lemma.

LEMMA 2. *Let $(X, \|\cdot\|)$ be a normed vector space, and $\tau_2 \subset \tau_1$ be vector topologies on X weaker than the norm topology. Suppose that there exists a positive homogeneous function F on X such that:*

(a) *$F(x) \geq c\|x\|$ for some $c > 0$.*
(b) *τ_1 and τ_2 coincide on the set $S = \{x \in X : F(x) = 1\}$.*

Then X has $P(\tau_1, \tau_2)$. In particular, if X is a Banach space that admits an equivalent τ-Kadec norm for some weaker vector topology τ then X has $P(\|\cdot\|, \tau)$.

Proof. Consider the following families of sets: $\Sigma = \{S(t) : t \in [0, \infty)\}$ and the countable one $\Pi = \{A(r, s) : r, s \in \mathbb{Q},\ 0 \leq r \leq s\}$ where

$$S(t) = \{x \in X : F(x) = t\}, \quad A(r, s) = \{x \in X : r \leq F(x) \leq s\}.$$

If $W \in \tau_1$ is a neighbourhood of 0 then it contains some ball $B[0, \delta]$. It is easy to see that for δ small enough

$$S(t) \subset A(t - c\delta, t + c\delta) \subset S(t) + W.$$

The result follows from Lemma 1.

Combining Proposition 1, Corollary 1 and the previous lemma we easily obtain the theorems of Edgar and Schachermayer. Note that a more direct proof of Edgar's theorem just needs a special case of Lemma 1 and the idea of point (iv) of Proposition 1. Schachermayer's theorem moreover needs Proposition 1(iii).

COROLLARY 3. *Let X be a Banach space that admits an equivalent Kadec norm. Then* $\mathrm{Borel}(X, \|\cdot\|) = \mathrm{Borel}(X, w)$ *and* $X \in \mathrm{Borel}(X^{**}, w^*)$.

The next theorem is the main result of this section. It provides a converse of Lemma 2 in the metric case. A partial similar result has been proved by Lancien [18].

THEOREM 1. *Let X be a Banach space and τ a vector topology coarser than the norm topology such that $\overline{B_X}^{\tau}$ is bounded. Then the following are equivalent:*

(i) *X has $P(\|\cdot\|, \tau)$ (equivalently, (X, τ) has $\|\cdot\|$-SLD).*
(ii) *There exists a nonnegative symetric homogeneous τ-lower semicontinuous function F on X with $\|\cdot\| \leq F \leq 3\|\cdot\|$ such that the norm topology and τ coincide on the set $S = \{x \in X : F(x) = 1\}$.*

Proof. (ii)⇒(i). This is in fact Lemma 2.

(i)⇒(ii). Assume that X is endowed with a τ-lower semicontinuous equivalent norm $\|\cdot\|$. $B(0, a)$ and $B[0, a]$ are the open and closed balls of center 0 and radius a. As usual $B_X = B[0, 1]$.

Suppose that X has $P(\|\cdot\|, \tau)$ with a sequence (A_n). We can suppose every A_n is star shaped with respect to 0 and norm open. To see that, we are going to modify the sequence in several steps.

STEP 1. Take $A'_n = A_n \cap B_X$.

STEP 2. Take

$$A''_n = \{tx : 0 \leq t \leq 1,\ x \in A'_n\}.$$

We now check that (A''_n) is good for the points of the unit sphere S_X. Let $x \in S_X$ and $\varepsilon > 0$. Applying Lemma 1 we can find $U \in \tau$, $n \in \mathbb{N}$ and $\delta > 0$ such that $x \in A'_n \cap U$ and $\mathrm{diam}((A'_n + B(0, \delta)) \cap U) < \varepsilon$. Now it is clear that

$$A''_n \cap (U \setminus B[0, 1 - \delta]) \subset (A'_n + B(0, \delta)) \cap U.$$

Thus $U' = U \setminus B[0, 1 - \delta] \in \tau$ satisfies $x \in A''_n \cap U'$ and $\mathrm{diam}(A''_n \cap U') < \varepsilon$.

STEP 3. The family

$$\{rA''_n + B(0, \delta) : n \in \mathbb{N},\ r \geq 0,\ \delta > 0,\ r, \delta \in \mathbb{Q}\}$$

is good for X by Lemma 1. Renumbering this family yields the desired (A_n).

Clearly the sets $\overline{A_n}^{\tau}$ are star shaped with respect to 0. Let f_n be the Minkowski functional of $\overline{A_n}^{\tau}$. Since $\overline{A_n}^{\tau} = \{f_n \leq 1\}$ the function f_n is τ-lower semicontinuous. Let $\|f_n\|$ be the supremum of $|f_n(x)|$ with $x \in B_X$. The function F given by the formula

$$F(x) = \|x\| + \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{f_n(x)}{\|f_n\|} + \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{f_n(-x)}{\|f_n\|}$$

is τ-lower semicontinuous and symmetric.

Let $(x_\omega) \subset S$ be a net τ-converging to some $x \in S$. From the τ-lower semicontinuity of $\|\cdot\|$ and f_n we have

$$\|x\| \le \liminf_\omega \|x_\omega\|,$$
$$f_n(x) \le \liminf_\omega f_n(x_\omega),$$
$$f_n(-x) \le \liminf_\omega f_n(-x_\omega).$$

On the other hand, it is not difficult to see that

$$1 \ge \liminf_\omega \|x_\omega\| + \sum_{n=1}^\infty \frac{1}{2^n\|f_n\|} \liminf_\omega f_n(x_\omega) + \sum_{n=1}^\infty \frac{1}{2^n\|f_n\|} \liminf_\omega f_n(-x_\omega).$$

Since $F(x) = 1$, a simple reasoning with lim sup gives the following equalities and the existence of its left members:

$$\lim_\omega \|x_\omega\| = \|x\|,$$
$$\lim_\omega f_n(x_\omega) = f_n(x),$$
$$\lim_\omega f_n(-x_\omega) = f_n(-x),$$

for every $n \in \mathbb{N}$.

Fix $\varepsilon > 0$. By the proof of Proposition 1(iii) there exist $n \in \mathbb{N}$ and $U \in \tau$ such that $x \in A_n \cap U$ and $\operatorname{diam}(\overline{A}_n^\tau \cap U) \le \varepsilon$. In particular, as A_n is norm open then $f_n(x) < 1$ so for ω large enough $f_n(x_\omega) < 1$ and thus $x_\omega \in \overline{A}_n^\tau$. Since for ω large enough we have $x_\omega \in U$ we obtain $\|x_\omega - x\| \le \varepsilon$. This proves that the net (x_ω) converges to x in norm, so the norm topology and τ coincide on S.

Clearly the constant 3 in statement (ii) of the preceding theorem can be replaced by any constant greater than 1. In fact every function of the form $\|\cdot\| + aF$ with $a > 0$ has the same property. This also shows that the norm can be approximated uniformly by functions with the Kadec property provided at least one such function exists.

Note that S is a norm G_δ-set in $B = \{x \in X : F(x) \le 1\}$, thus (S, τ) is completely metrizable.

A remarkable theorem of Kadec (see [2, p. 177]) shows that every separable Banach space has an equivalent τ-Kadec norm for the topology τ of convergence on a fixed quasi-norming subset of its dual space. The following result characterizes the existence of τ-Kadec norms in general Banach spaces extending Kadec's theorem.

THEOREM 2. *Let X be a Banach space and τ a weaker topology such that $\overline{B}_X^\tau$ is bounded. Then X has an equivalent τ-Kadec norm if and only if X has $P(\|\cdot\|, \tau)$ where the sets (A_n) in Definition 1 are convex, in other words, if there exist convex sets $A_n \subset X$ such that for every $x \in X$ and every $\varepsilon > 0$ there are $n \in \mathbb{N}$ and $U \in \tau$ such that $x \in A_n \cap U$ and* $\operatorname{diam}(A_n \cap U) < \varepsilon$.

Proof. If we begin with (A_n) convex in the proof of Theorem 1 it is easily checked that all the families of sets built there are still convex. Thus F is subadditive and so it is an equivalent τ-Kadec norm.

For the converse assume that the norm of X is τ-Kadec. The proof of Lemma 2 shows that X has $P(\|\cdot\|, \tau)$ with a sequence of differences of closed balls centered at 0. As the closed balls are τ-closed we deduce that the sequence of closed balls with rational radii satisfies what is required.

We do not know if property $P(\|\cdot\|, w)$ implies the existence of an equivalent Kadec norm.

Recently Moltó, Orihuela and Troyanski [19] have given a characterization of the existence of an equivalent LUR norm in a Banach space using a variant of Definition 2. Their result can be reformulated in similar terms to those of Definition 1 as follows: *a Banach space X admits a LUR norm if and only if there exists a sequence of sets $A_n \subset X$ such that for every $x \in X$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ and an open semispace U such that $x \in A_n \cap U$ and* $\operatorname{diam}(A_n \cap U) < \varepsilon$. Note that the topological counterpart of this result is Theorem 1 applied to the weak topology but to deduce that the function F is in fact a Kadec norm we did need a geometric assumption about the sets A_n.

4. Applications. A Banach space X is said to be *weakly countably determined* (WCD) if there exists a sequence (K_n) of w^*-compact subsets of X^{**} such that for every $x \in X$ and every $y \in X^{**} \setminus X$ there is $n \in \mathbb{N}$ with $x \in K_n$ and $y \notin K_n$. WCD Banach spaces generalize in a natural way the weakly compactly generated Banach spaces (WCG), that is, the spaces containing a total weakly compact set. A WCD Banach space admits a LUR norm [28].

The coincidence of Borel families in the following theorem improves one by Talagrand [25] for subspaces of WCG Banach spaces.

THEOREM 3. *Let X be a WCD Banach space and let τ be a Hausdorff vector topology weaker than the weak topology of X. Then X has $P(\|\cdot\|, \tau)$. Moreover, if $\overline{B}_X^\tau$ is bounded then X also admits a τ-Kadec norm topology and*

$$\operatorname{Borel}(X, \|\cdot\|) = \operatorname{Borel}(X, \tau).$$

Proof. We can assume without loss of generality that the sequence (K_n) is closed under finite intersections. We claim that the sequence of w^*-closed convex hulls $\{\overline{\operatorname{co}(K_n)}^{w^*}\}$ also satisfies the above definition. Indeed, fix $x \in X$ and $y \in X^{**} \setminus X$. The set $K = \bigcap_{x \in K_n} K_n$ is a weakly compact set of X containing x. Now, since $\overline{\mathrm{co}(K)}^{w}$ is a weak*-compact convex set not containing y, there is a weak*-open half space H such that $x \in H$ and $y \notin \overline{H}^{w^*}$. By compactness, there is $n \in \mathbb{N}$ such that $x \in K_n \subset H$. As $\overline{\mathrm{co}(K_n)}^{w^*} \subset \overline{H}^{w^*}$ we see that $x \in \overline{\mathrm{co}(K_n)}^{w^*}$ and $y \notin \overline{\mathrm{co}(K_n)}^{w^*}$. This ends the proof of the claim.

First we check that X has $P(w,\tau)$. For every $x \in X$ define

$$S_x = \bigcap_{K_n \ni x} K_n.$$

By definition of WCD it is clear that S_x is a weakly compact subset of X. If we take $\{S_x\}$ as Σ and the traces on X of finite intersections of K_n's as a countable family Π, then the conditions in Lemma 1 are satisfied. Indeed, Σ covers X, and τ and w coincide on every S_x by compactness, so Σ is good for (w,τ). Now let W be a weak neighborhood of 0 and let W' be a weak* neighborhood of 0 in X^{**} such that $W = X \cap W'$. For some increasing sequence (n_j) of integers we have $S_x = \bigcap_j K_{n_j}$. By compactness there are a finite number of K_{n_j}'s whose intersection is contained in $S_x + W'$. So X has convex $P(w,\tau)$.

Since a WCD Banach space admits a Kadec norm, it has convex $P(\|\cdot\|, w)$. Now X has $P(\|\cdot\|,\tau)$ by Proposition 1(ii) with convex sets. The existence of a τ-Kadec equivalent norm follows from Theorem 2, and the coincidence of Borel sets follows from Corollary 1.

Using the general definition of a countably determined topological space (X,τ_1) in terms of usc maps one can prove that X has $P(\tau_1,\tau_2)$ for every weaker Hausdorff topology τ_2, but it is not clear if that implies the coincidence of Borel sets. For example, in the preceding theorem, if we want to prove the coincidence of Borel sets for τ and the weak topology directly from the fact that X has $P(w,\tau)$ we have to check that $X \cap K_n$ is τ-Borel, which is not evident except in the case of a WCG space. Roughly speaking that was the argument of Talagrand [25], but WCD spaces were introduced some years later.

In the particular case of a dual WCD space, when τ is the weak* topology it is known that the space admits an equivalent dual LUR norm [8]. Without the hypothesis of WCD the result may not be true: the space $J(\omega_1)$ is a dual with the Radon–Nikodym property, so it admits an equivalent LUR norm [9], but $\mathrm{Borel}(J(\omega_1), w^*)$ is a proper subset of $\mathrm{Borel}(J(\omega_1), w) = \mathrm{Borel}(J(\omega_1), \|\cdot\|)$ (see [7]). A natural generalization of dual WCD are the dual spaces X^* such that $(B_{X^{**}}, w^*)$ is a Corson compact set but in this case there may be no dual LUR norm [12].

The next corollary is inspired by a result of [5] for WCG spaces.

COROLLARY 4. *Let Y be a Banach space and τ a vector topology weaker than the weak topology of Y such that the unit ball $\overline{B}_Y^{\tau}$ is bounded. If X is a WCD norm closed subspace of Y then X is a τ-Borel set in Y.*

Proof. Note that τ is Hausdorff. We deduce from Theorem 3 that X has $P(\|\cdot\|,\tau)$. Now apply Proposition 3(b).

It is not difficult to see that under the conditions of Corollary 3 if X is $K_{\sigma\delta}$ in (X^{**}, w^*) (for example if X is WCG) then it is an $F_{\sigma\delta}$ in (Y,τ) while the proof of Corollary 4 shows that X is an $(F \cap G)_{\sigma\delta}$. It is not known if a WCD Banach space is always a $K_{\sigma\delta}$ in (X^{**}, w^*) (see [4, Problem VI.3]).

It is known that K-analytic topological spaces are Čech-analytic for every Hausdorff weaker topology. The same result is not true in general for WCD topological spaces. The next corollary gives a positive answer in the particular case of Banach spaces and "reasonable" topologies.

COROLLARY 5. *Let X be a WCD Banach space and τ the topology of convergence on a quasi-norming subset of X^*. Then (X,τ) is Čech-analytic.*

Proof. Using an equivalent norm we can suppose that τ is given by a norming subset. Then apply Corollary 2.

Let us mention here that it is a consequence of Proposition 2 and Theorem 3 that under the hypothesis of Corollary 5, (X,τ) is σ-fragmentable and, in particular, the τ-compact subsets of X are fragmentable (see [3] for the definitions and some consequences).

A typical situation is the case of $C(K)$ spaces with the pointwise topology. There is a huge family of compact spaces K called Valdivia compact sets such that $C(K)$ admits a LUR norm which makes the unit ball pointwise closed [27]. So the results above are applicable, in particular the Borel sets for the norm and pointwise topologies coincide. Recently Haydon, Jayne, Namioka and Rogers [13] have shown that if K is a totally ordered set that is compact in its order topology then $C(K)$ admits a norm with the Kadec property for the pointwise topology so the same coincidence of Borel sets holds.

A different class of compact spaces where we can check directly the coincidence of Borel sets in $C(K)$ for the weak and pointwise topologies is the class of Radon–Nikodym compact spaces. Originally, a compact space is called Radon–Nikodym when it is homeomorphic to a w^*-compact subset of a dual with the Radon–Nikodym property. Equivalently, *a compact set K is Radon–Nikodym if and only if there exists a stronger lower semicontinuous metric d on K such that every Radon measure on K is the restriction of a Radon measure on (K,d)* ([21] and [14]).

THEOREM 4. *Let K be a Radon–Nikodym compact space. Then $C(K)$ has an equivalent pointwise lower semicontinuous norm such that on its unit* *sphere the weak and pointwise topologies coincide,* $C(K)$ *has* $P(w, t_p(K))$ *and*

$$\mathrm{Borel}(C(K), w) = \mathrm{Borel}(C(K), t_p(K)).$$

Proof. A continuous function on K is d-uniformly continuous. Indeed, suppose not. Then we can take sequences (x_n) and (y_n) in K such that $\lim_n d(x_n, y_n) = 0$ while $|f(x_n) - f(y_n)| \geq \delta$ for some $\delta > 0$. By taking an ultrafilter we make the sequences converge to the limits x and y respectively. But by the lower semicontinuity of d we have $d(x, y) = 0$ so $x = y$ and this contradicts the continuity of f.

Fix a d-dense set $(x_\alpha)_{\alpha \in \Gamma}$. Now we define the seminorms O_n as follows:

$$O_n(f) = \sup_\alpha \sup\{|f(x) - f(x_\alpha)| : d(x, x_\alpha) \leq 1/n\}.$$

Clearly O_n is pointwise lower semicontinuous and since every $f \in C(K)$ is d-uniformly continuous, for every $\delta > 0$ there exists $n \in \mathbb{N}$ such that $O_n(f) < \delta$.

Define a new norm by the formula

$$|||f||| = \|f\| + \sum_{n=1}^{\infty} \frac{1}{2^n} O_n(f).$$

Evidently $\|\cdot\| \leq |||\cdot||| \leq 3\|\cdot\|$. Thus $|||\cdot|||$ is an equivalent norm in $C(K)$. It is also not hard to check that the unit ball of $|||\cdot|||$ is pointwise closed.

We now check that the weak and pointwise topologies coincide on $S = \{f \in C(K) : |||f||| = 1\}$. Let (f_ω) be a net in S pointwise converging to $f \in S$. Take a Radon measure μ with $\|\mu\| \leq 1$ that we suppose already defined on $\mathrm{Borel}(K, d)$ and take $\varepsilon > 0$.

From the pointwise lower semicontinuity of $\|\cdot\|$ and O_n, reasoning as in Theorem 1 we deduce that $\lim_\omega O_n(f_\omega) = O_n(f)$ for every $n \in \mathbb{N}$.

Now fix $n \in \mathbb{N}$ such that $O_n(f) \leq \varepsilon/8$. Then for ω large enough $O_n(f_\omega) \leq \varepsilon/6$. Since μ has a d-separable d-support we can fix $F \subset \Gamma$ finite such that

$$|\mu|\Big(\bigcup_{\alpha \in F} B[x_\alpha, 1/n]\Big) > |\mu|(K) - \frac{\varepsilon}{4}.$$

If ω is large enough then $|f_\omega(x_\alpha) - f(x_\alpha)| \leq \varepsilon/6$ for $\alpha \in F$. So $|f_\omega(x) - f(x)| \leq \varepsilon/2$ for every $x \in \bigcup_{\alpha \in F} B(x_\alpha, 1/n)$.

If we have in mind that $\|f\|$ and $\|f_\omega\|$ are bounded by 1, an easy calculus gives

$$|\mu(f_\omega - f)| \leq \int |f_\omega - f|\, d|\mu| \leq \varepsilon,$$

which implies that (f_ω) converges weakly to f.

Now apply Lemma 2 to deduce that $C(K)$ has $P(w, t_p(K))$. Since the unit ball is pointwise closed the weak and pointwise topologies have the

same Borel sets by Proposition 1(iv); moreover, every weakly open set is a countable union of differences of pointwise closed sets.

Clearly Theorem 3 is still true for a continuous image of a Radon–Nikodym compactum. We know no example of a compact space with different Borel sets for the weak and pointwise topologies.

Note that if K is Radon–Nikodym compact and $(C(K), w)$ has $\|\cdot\|$-SLD, then $(C(K), t_p(K))$ has $\|\cdot\|$-SLD. In particular, K has the Namioka property (see [15]).

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Interpolation of real method spaces via some ideals of operators

by

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Abstract. Certain operator ideals are used to study interpolation of operators between spaces generated by the real method. Using orbital equivalence a new reiteration formula is proved for certain real interpolation spaces generated by ordered pairs of Banach lattices of the form $(X, L_\infty(w))$. As an application we extend Ovchinnikov's interpolation theorem from the context of classical Lions–Peetre spaces to a larger class of real interpolation spaces. A description of certain abstract $\mathcal{J}$-method spaces is also presented.

0. Introduction. The Riesz–Thorin–Marcinkiewicz interpolation theorems are important tools in classical and modern analysis. Recall that the Riesz–Thorin theorem states that if a linear operator T is bounded from L_{p_j} into L_{q_j} for $j = 0, 1$ then T is bounded from L_p into L_q, where $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_0$ and $0 < \theta < 1$. It is natural to ask if under the same assumptions we can improve the conclusion: for example we ask if it is possible to find a smaller range space Y such that T is bounded from L_p into Y. It was known for a long time that if $q_0 < p_0$ or $q_1 < p_1$ the result is not sharp. Finally in [13] Ovchinnikov obtained a sharp version of the Riesz–Thorin–Marcinkiewicz theorem: under the same assumptions of the classical Riesz–Thorin–Marcinkiewicz theorem we can conclude that T maps continuously L_p into the Lorentz space $L_{q,r}$ with $1/r = (1-\theta)\max\{1/q_0, 1/p_0\} + \theta\max\{1/q_1, 1/p_1\}$. The proof of this remarkable result is based on the application of a factorization theorem of Bennett [1], which states that the inclusion map $\ell_p \hookrightarrow \ell_\infty$ is a $(p,1)$-summing operator, to prove a new interpolation theorem for operators acting on weighted sequence ℓ_p-spaces modelled on the set $\mathbb{Z}$ of integers. A simple application of the reiteration theorem allows Ovchinnikov to prove his general interpolation theorem for Lions–Peetre scales.

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