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Corrigendum to "On the spectral bound of the generator of a C_0-semigroup"

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by

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Abstract. Some statements of the paper [4] are corrected.

We will use the notation from [4]. For C_0-semigroups $(T(t))_{t\ge 0}$ in a Banach space X satisfying the integrability conditions

$$\int_0^\infty |(T(t)x, x^*)|^p\,dt < \infty, \tag{1}$$

or

$$\{\lambda\in\mathbb{C} : \operatorname{Re}\lambda > 0\}\subset\varrho(A) \quad\text{and}\quad \sup_{s>0}\int_{-\infty}^{\infty} |(R(s+it, A)x, x^*)|^p\,dt < \infty \tag{2}$$

for all $x\in X$, $x^*\in X^*$ and some $p\ge 1$, a significant asymptotic behavior theory was developed recently (see [2, 3], and the references in [4]). In particular, it was shown by G. Weiss (see [5, p. 284, second paragraph]), that if (1) holds for all $x\in X$, $x^*\in X^*$, and some $p\ge 1$, then the spectral bound $s(A)$ of the generator A of $(T(t))_{t\ge 0}$ is negative, and, moreover, there exists δ with $-s(A) > \delta > 0$ such that the resolvent $R(\lambda, A)$ of A is bounded in $\{\lambda\in\mathbb{C} : \operatorname{Re}\lambda > -\delta\}$. If X is a Hilbert space, then the boundedness of $R(\lambda, A)$ in $\{\lambda\in\mathbb{C} : \operatorname{Re}\lambda > -\delta\}$ implies the negativity of the type ω_0 of $(T(t))_{t\ge 0}$, or, in other words, the exponential stability of $(T(t))_{t\ge 0}$ ([5, Theorem 4.2]). For C_0-semigroups in a Hilbert space X satisfying (2) for some $p > 1$ and all $x\in X$, $x^*\in X$, the negativity of ω_0 was obtained by P. Yao and D. Feng [6].

In [4], we introduced generalized integrability conditions by restricting (1) or (2) with $p = 1$ to the subsets $C^\infty(A)$, $C^\infty(A^*)$ of the Banach spaces X, X^*, respectively. The aim of [4] was to obtain, under each such condition,

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the inequality $s(A) < 0$, and then to apply these results to the study of the stability of $(T(t))_{t\geq 0}$.

Assuming $s(A) \geq 0$, the reasoning in the proofs of Theorems 2 and 3 in [4] includes the consecutive consideration of the two a priori possible cases:

1) there exists $\lambda_0 \in \sigma(A)$ with $\operatorname{Re}\lambda_0 \geq 0$;
2) for all $\lambda \in \sigma(A)$ we have $\operatorname{Re}\lambda < 0$, but $s(A) = 0$.

Unfortunately, the arguments given in [4] for the second case contain a gap. In the proof of Theorem 2 ([4, p. 26]), the inequality

$$\sup\{|\operatorname{Re}\lambda|^{-1}|\lambda - \lambda_0|^{-m} : \lambda \in \sigma(A)\} < \infty, \tag{3}$$

where $\lambda_0 \in \varrho(A)$ and $m \in \mathbb{N}$ are fixed, does not, in general, contradict $s(A) = 0$.

For example, if $X = L_2(-\pi, \pi)$, then, for fixed $\lambda_0 \in \varrho(A)$, there exists a C_0-semigroup $(T(t))_{t\geq 0}$ in X such that

$$\sigma(A) = \{\lambda \in \mathbb{C} : |\operatorname{Im}(\lambda - \lambda_0)|(-\operatorname{Re}\lambda)^2 = 1,\ -1 \leq \operatorname{Re}\lambda < 0\}$$

(see [1, Theorem 20.4.2]). Clearly, $s(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} = 0$. On the other hand, for fixed $m \in \mathbb{N}$,

$$\begin{aligned}\sup\{|\operatorname{Re}\lambda|^{-1}|\lambda - \lambda_0|^{-m} &: \lambda \in \sigma(A)\}\\ &\leq \sup\{|\operatorname{Re}\lambda|^{-1}|\operatorname{Im}(\lambda - \lambda_0)|^{-m} : \lambda \in \sigma(A)\}\\ &= \sup\{|\operatorname{Re}\lambda|^{2m-1} : \lambda \in \sigma(A)\} = 1.\end{aligned}$$

Thus, inequality (3) is satisfied.

For this reason, in case 2), we do not obtain a contradiction in the proof of [4, Theorem 3, p. 28], and also in the proof of [4, Proposition 1, p. 31].

The statements of Theorems 2, 3 and Proposition 1 are true under the additional assumption:

"The C_0-semigroup $(T(t))_{t\geq 0}$ is continuous in the uniform operator topology for $t \geq t_0 > 0$".

Then the set $\sigma(A) \cap \{\lambda \in \mathbb{C} : -1 \leq \operatorname{Re}\lambda \leq 0\}$ is compact (see [1, Theorem 16.4.2]). So, the property

$$\sigma(A) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\} \tag{4}$$

is equivalent to $s(A) < 0$, and case 2) is impossible. Moreover, in view of $s(A) = \omega_0$ for such semigroups, $s(A) < 0$ implies $\omega_0 < 0$ (as noted in [4, Remark 1]).

The rest of proofs of the statements indicated above (with the simplifications excluding 2)) remain the same.

In the case of an arbitrary C_0-semigroup $(T(t))_{t\geq 0}$, the conclusions of Theorems 2, 3 and Proposition 1 should be replaced by (4).

Corollary 1 of [4] states that if (1) or (2) with $p = 1$ holds for all $x \in C^\infty(A)$ and $x^* \in C^\infty(A^*)$, then “sufficiently smooth” orbits of $(T(t))_{t\ge 0}$ are uniformly stable:

$$\|T(t)R^m(\lambda_0, A)\| \to 0 \text{ as } t \to \infty, \text{ for some } m \in \mathbb{N}, \text{ with } \lambda_0 \in \varrho(A) \text{ fixed.}$$

The corollary does hold in the previous form as it is based on [4, Theorem 4], and thus, on the property (4). The author does not know whether the statements of Theorems 2, 3 and Proposition 1 remain true in their previous form.

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