What is “local theory of Banach spaces”?

by

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Abstract. Banach space theory splits into several subtheories. On the one hand, there are an isometric and an isomorphic part; on the other hand, we speak of global and local aspects. While the concepts of isometry and isomorphy are clear, everybody seems to have its own interpretation of what “local theory” means. In this essay we analyze this situation and propose rigorous definitions, which are based on new concepts of local representability of operators.

Preamble. Of course, the quality of a theorem does not depend on the fact to which theory it belongs. Nevertheless, in order to systematize our knowledge we need criteria that enable us to collect similar results in theories.

1. Historical roots

1.1. Let us begin with some quotations.

Pelczyński and Rosenthal [p–r], p. 263: Localization refers to obtaining quantitative finite-dimensional formulations of infinite-dimensional results.

Tomczak-Jaegermann [TOM], p. 5: A property (of Banach spaces or of operators acting between them) is called local if it can be defined by a quantitative statement or inequality concerning a finite number of vectors or finite-dimensional subspaces.

Lindenstrauss and Milman [l–m], p. 1151: The name ‘local theory’ is applied to two somewhat different topics:

(a) The quantitative study of n-dimensional normed spaces as n → ∞.
(b) The relation of the structure of an infinite-dimensional space and its finite-dimensional subspaces.

1991 Mathematics Subject Classification: 46B07, 46B08.

*The proof of the Basic Lemma in 5.9 is due to W. B. Johnson.
Research supported by DFG grant PI 322/1-2.

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1.2. In 1956, Grothendieck defined the following concept:

*L’espace normé* $E$ *a un type linéaire inférieur à celui d’un espace normé* $F$, *si on peut trouver un* $M > 0$ *faisant que tout sous-espace de dimension finie* $E_1$ *de* $E$ *soit isomorphe “à M près” à un sous-espace de dimension finie* $F_1$ *de* $F$ *(Banach–Mazur distance* $d(E_1, F_1) \leq 1 + M)$; *et que* $E$ *a un type métrique inférieur à celui de* $F$, *si la condition précédente est satisfaite pour tout* $M > 0$.

In his theory of super-reflexivity, James used to speak of *crudely finite representability* and of *finite representability*, respectively.

1.3. A decisive step was done when Dacunha-Castelle and Krivine introduced the technique of *ultraproducts* in Banach space theory. It soon turned out that $X$ is finitely representable in $Y$ if and only if $X$ is isometric to a subspace of some ultrapower $Y^{\text{ul}}$.

Ultraproducts of operators were studied by Pietsch. Next, Beauzamy introduced two (possibly different) concepts of finite representability for operators. A more appropriate approach is due to Heinrich.

1.4. Following Brunel and Sucheston, a property $\mathcal{P}$ is called a *super-property* if it carries over from $Y$ to all $X$ finitely representable in $Y$. Equivalently, this means that $\mathcal{P}$ is preserved under the formation of ultrapowers and subspaces.

2. Notation

2.1. Let $\mathcal{L}$ stand for the class of all (real or complex) Banach spaces. Denote the set of all (bounded linear) operators from $X$ into $Y$ by $\mathcal{L}(X, Y)$, and write $\mathcal{L} := \bigcup \mathcal{L}(X, Y)$, where the union ranges over $X, Y \in \mathcal{L}$.

2.2. An operator $J \in \mathcal{L}(X, Y)$ is an *injection* if there exists a constant $c > 0$ such that $\|Jx\| \geq c\|x\|$ for all $x \in X$. In the case when $\|Jx\| = \|x\|$, we speak of a *metric injection*. For every (closed linear) subspace $M$ of $X$, the canonical embedding from $M$ into $X$ is denoted by $J^X_M$.

2.3. An example of a metric injection is the map $K_X : X \to X^*$ that assigns to every $x \in X$ the functional $x^* \mapsto \langle x, x^* \rangle$. For $T \in \mathcal{L}(X, Y)$, we let $T^{\text{reg}} := K_Y T$, where the superscript $\text{reg}$ stands for regular.

2.4. Let $\mathbb{I}$ be an index set. We denote by $l_\infty(\mathbb{I})$ the Banach space of bounded scalar families $(\xi_i)$ with the norm $\|((\xi_i) | l_\infty(\mathbb{I}))\| := \sup_{i \in \mathbb{I}} |\xi_i|$. Note that $l_\infty(\mathbb{I})$ has the *metric extension property*. This means that every operator $T \in \mathcal{L}(M, l_\infty(\mathbb{I}))$ defined on a subspace $M$ of $X$ admits a norm-preserving extension $T^{\text{ext}} \in \mathcal{L}(X, l_\infty(\mathbb{I}))$.

If $x \in X$, then $(\langle x, x^* \rangle)$ can be viewed as an element of $l_\infty(B_{X^*})$, where $B_{X^*}$ is the closed unit ball of $X^*$. In this way, we get the canonical injection
$J^\text{inj}_X$ from $X$ into $X^\text{inj} := l_\infty(B_X)$. For $T \in \mathfrak{L}(X,Y)$, we let $T^\text{inj} := J^\text{inj}_Y T$, where the superscript $\text{inj}$ stands for injective.

2.5. An operator $Q$ from $X$ onto $Y$ is called a surjection. In the case when the open unit ball of $X$ is mapped onto the open unit ball of $Y$, we speak of a metric surjection. For every (closed linear) subspace $N$ of $Y$, the quotient map from $Y$ onto $Y/N$ is denoted by $Q^N_Y$.

2.6. Given Banach spaces $X_i$ with $i \in \mathbb{I}$, we denote by $[l_\infty(\mathbb{I}), X_i]$ the Banach space of all bounded families $(x_i)$ such that $x_i \in X_i$. Fix an ultrafilter $\mathcal{U}$ on the index set $\mathbb{I}$. The collection of all equivalence classes

$$(x_i)^{\mathcal{U}} := \{(x^\mathcal{U}_i) \in [l_\infty(\mathbb{I}), X_i] : \mathcal{U}\text{-lim}_i \|x^\mathcal{U}_i - x_i\| = 0\}$$

is a Banach space $(X_i)^{\mathcal{U}}$ under the norm $\|(x_i)^{\mathcal{U}}\| := \mathcal{U}\text{-lim}_i \|x_i\|$.

Next, let $T_i \in \mathfrak{L}(X_i, Y_i)$ with $i \in \mathbb{I}$ be a bounded family of operators. Then $(T_i)^{\mathcal{U}} : (x_i)^{\mathcal{U}} \mapsto (T_i x_i)^{\mathcal{U}}$ defines an operator from $(X_i)^{\mathcal{U}}$ into $(Y_i)^{\mathcal{U}}$ such that $\|(T_i)^{\mathcal{U}}\| = \mathcal{U}\text{-lim}_i \|T_i\|$.

These new objects are referred to as ultraproducts.

2.7. If $X_i = X$, $Y_i = Y$ and $T_i = T$, then we speak of ultrapowers, denoted by $X^{\mathcal{U}}$, $Y^{\mathcal{U}}$ and $T^{\mathcal{U}}$, respectively. In this case, there exists a canonical map $J^\mathcal{U}_X$ from $X$ into $X^{\mathcal{U}}$, which sends $x$ to $(x_i)$ with $x_i = x$. Moreover, we define $Q^\mathcal{U}_X : (x_i)^{\mathcal{U}} \mapsto \mathcal{U}\text{-lim}_i K_X x_i$. The right-hand limit is taken with respect to the weak* topology of $X^{**}$. These constructions yield the formula

$T^{\text{reg}} = K_Y T = Q^\mathcal{U}_Y T^{\mathcal{U}} J^\mathcal{U}_X$ for $T \in \mathfrak{L}(X,Y)$. Unfortunately, the canonical embedding $K_Y$ cannot be avoided.

2.8. Throughout this essay, we let

$$T \in \mathfrak{L}(X,Y), \quad T_0 \in \mathfrak{L}(X_0, Y_0), \quad T_1 \in \mathfrak{L}(X_1, Y_1), \quad \text{etc.}$$

3. Subtheories. Imitating Felix Klein, we now describe an Erlanger Programm for operators in Banach spaces.

3.1. Suppose we are given a preordering $\prec$ on $\mathfrak{L}$, the class of all operators. That is, $T_0 \prec T_1$ and $T_1 \prec T_2$ imply $T_0 \prec T_2$, and we have $T \prec T$. A property is said to be $\prec\text{-stable}$ if it is inherited from $T_1$ to every $T_0 \prec T_1$. So every preordering $\prec$ leads to a subtheory that deals with the associated stable properties.

Writing $T_0 \sim T_1$ whenever $T_0 \prec T_1$ and $T_1 \prec T_0$ yields an equivalence relation. Of course, we could also study the concept of $\sim\text{-stability}$.

3.2. Note that a preordering on $\mathfrak{L}$, the class of all Banach spaces, is obtained by letting $X_0 \prec X_1$ if $I_{X_0} \prec I_{X_1}$, where $I_{X_0}$ and $I_{X_1}$ denote the identity maps of $X_0 \in \mathfrak{L}$ and $X_1 \in \mathfrak{L}$, respectively. In this way, every subtheory of operators contains a subtheory of spaces, as a special part.
4. Global representability

4.1. An operator \( T_0 \in \mathcal{L}(X_0, Y_0) \) is *globally representable* in an operator \( T_1 \in \mathcal{L}(X_1, Y_1) \) if there exist \( A_1 \in \mathcal{L}(X_0, X_1) \) and \( B_1 \in \mathcal{L}(Y_1, Y_0) \) such that \( T_0 = B_1T_1A_1 \). That is,

\[
\begin{array}{c}
X_0 \\
A_1
\end{array} \xrightarrow{T_0} \begin{array}{c}
X_1 \\
T_1
\end{array} \xrightarrow{B_1} \begin{array}{c}
Y_1 \\
Y_0
\end{array}
\]

In this case, we write \( T_0 \triangleright^\text{glo} T_1 \). Of course, \( \triangleright^\text{glo} \) is a preordering on \( \mathcal{L} \). The corresponding stable properties are said to be *global*. Obviously, \( X_0 \triangleright^\text{glo} X_1 \) means that \( X_0 \) is isomorphic to a complemented subspace of \( X_1 \).

4.2. A property is called *injective* if it carries over from \( T_{\text{inj}} \in \mathcal{L}(X, Y_{\text{inj}}) \) to \( T \in \mathcal{L}(X, Y) \).

4.3. The preordering \( T_0 \triangleright^\text{inj glo} T_1 \) is defined by \( T_0 \triangleright^\text{inj} T_1 \triangleright^\text{glo} T_1 \). Note that \( T_0 \triangleright T_1 \) implies \( T_0 \triangleright^\text{inj glo} T_1 \). A property turns out to be *\( \triangleright \)-stable* if and only if it is simultaneously *injective* and *global*, which justifies the symbol \( \triangleright^\text{inj glo} \). Clearly, \( X_0 \triangleright^\text{inj glo} X_1 \) means that \( X_0 \) is isomorphic to a subspace of \( X_1 \).

4.4. So far, we have discussed isomorphic notions. Taking into account isometric aspects, we could define a global preordering by assuming that the operators \( A_1 \) and \( B_1 \) in \( T_0 = B_1T_1A_1 \) satisfy the condition \( \|B_1\| \cdot \|A_1\| \leq 1 \). We may also require that, for any choice of \( \varepsilon > 0 \), there exists a factorization such that \( \|B_1\| \cdot \|A_1\| \leq 1 + \varepsilon \). Another possibility would be to assume that \( A_1 \) is an injection and that \( B_1 \) is a surjection, metric or not.

5. Operators between finite-dimensional spaces

5.1. The symbol \( F \) stands for the collection of all finite-dimensional Banach spaces. Note that, upon identifying isometric copies, \( F \) is a set. Throughout, let \( E \in F \) and \( F \in F \). The underlying (real or complex) scalar field (sometimes viewed as a 1-dimensional Banach space) is denoted by \( \mathbb{K} \).

5.2. With every operator \( T \in \mathcal{L}(X, Y) \) we associate the *germs*

\[ \mathcal{L}(T \mid E, F) := \{ BTA \in \mathcal{L}(E, F) : \|A : E \to X\| \leq 1, \|B : Y \to F\| \leq 1 \} \]

This definition is illustrated by the following diagram:
Roughly speaking, local properties of $T$ can be formulated in terms of the family $\{\mathcal{L}(T : E, F) : E, F \in \mathcal{F}\}$. Working with the closed hulls $\overline{\mathcal{L}}(T : E, F)$ will turn out to be more elegant.

5.3. The sets $\mathcal{L}(T | E, F)$ are bounded and circled, but may look like a hedgehog. For example, $\mathcal{L}(\text{Id} : \mathbb{K} \to \mathbb{K} | E, F)$ consists of all $S \in \mathcal{L}(E, F)$ with rank$(S) \leq 1$ and $\|S\| \leq 1$. In the case when $T$ does not attain its norm, $\mathcal{L}(T | \mathbb{K}, \mathbb{K})$ is an open disc. On the other hand, $\mathcal{L}(T | E, F)$ may also be very nice as shown by $\mathcal{L}(\text{Id} : l_2 \to l_2 | E, F)$, which is the closed unit ball of a norm.

5.4. Our first result is obvious.

**Proposition.** $\mathcal{L}(T | E, F) = \mathcal{L}(T^{\text{reg}} | E, F)$.

5.5. We now state one of the most important tools of local theory.

**Principle of Local Reflexivity.** Let $M$ and $N$ be finite-dimensional subspaces of $X^{**}$ and $X^*$, respectively. Then for every $\varepsilon > 0$ there exists an isomorphism $I_\varepsilon$ from $M$ onto some subspace $M_\varepsilon$ of $X$ such that $\|I_\varepsilon\| \leq 1 + \varepsilon$, $\|I_\varepsilon^{-1}\| \leq 1 + \varepsilon$ and

$$\langle x^{**}, x^* \rangle = \langle I_\varepsilon x^{**}, x^* \rangle \quad \text{for all} \quad x^{**} \in M \quad \text{and} \quad x^* \in N.$$ 

5.6. The correspondence $S \mapsto S^*$ defines an isometry between $\mathcal{L}(E, F)$ and $\mathcal{L}(F^*, E^*)$. The image of a set $\mathcal{L}$ will be denoted by $\mathcal{L}^*$.

**Proposition.**

$$\mathcal{L}^*(T | E, F) \subseteq \mathcal{L}(T^* | F^*, E^*) \subseteq (1 + \varepsilon)\mathcal{L}^*(T | E, F) \quad \text{for all} \quad \varepsilon > 0.$$

**Proof.** The left-hand inclusion is trivial.

By definition, every operator $S^* \in \mathcal{L}(T^* | F^*, E^*)$ can be decomposed in the form $S^* = UT^*V$, where $\|V : F^* \to Y^*\| \leq 1$ and $\|U : X^* \to E^*\| \leq 1$. Putting $B := K_{F^*}^{-1}V^*K_Y$, we get $V = B^*$. The principle of local reflexivity provides us with an isomorphism $I_\varepsilon$ from $U^*K_E(E)$ onto some subspace $M_\varepsilon$ of $X$ such that $\|I_\varepsilon\| \leq 1 + \varepsilon$ and

$$\langle U^*K_E e, T^*V f^* \rangle = \langle I_\varepsilon U^*K_E e, T^*V f^* \rangle \quad \text{for} \quad e \in E \quad \text{and} \quad f^* \in F^*.$$ 

Letting $A := I_\varepsilon U^*K_E$, we obtain $\|A : E \to X\| \leq 1 + \varepsilon$ and

$$\langle BTA e, f^* \rangle = \langle I_\varepsilon U^*K_E e, T^*V f^* \rangle = \langle U^*K_E e, T^*V f^* \rangle = \langle e, S^* f^* \rangle \quad \text{for} \quad e \in E \quad \text{and} \quad f^* \in F^*.$$ 

So $S = BTA \in (1 + \varepsilon)\mathcal{L}(T | E, F)$. 

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X \begin{array}{c}
T \end{array} Y

A \begin{array}{c}
E \end{array} B

E \begin{array}{c}
BTA \end{array} F
```
We now formulate a corollary of the preceding result.

**Proposition.**
\[ L(T \mid E \mid F) \subseteq (1 + \varepsilon) L(T \mid E \mid F) \quad \text{for all } \varepsilon > 0. \]

The next statement even holds for compact operators. However, we will only need the finite-dimensional case.

**Proposition.** If \( T \) has finite rank, then \( L(T^* \mid F^* \mid E^*) \) is closed.

**Proof.** The closed unit balls
\[ \{ U : \| U : X^* \to E^* \| \leq 1 \} \quad \text{and} \quad \{ V : \| V : F^* \to Y^* \| \leq 1 \} \]
are compact in the weak* topologies induced by the seminorms
\[ p_{e,x^*}(U) := |\langle e, U x^* \rangle| \quad \text{and} \quad p_{y,f^*}(V) := |\langle y, V f^* \rangle|. \]
Moreover, the norm topology of \( \mathcal{L}(F^*, E^*) \) coincides with the weak* topology obtained from \( p_{e,f^*}(S^*) := |\langle e, S^* f^* \rangle| \). Since \( T \) can be written in the form
\[ T = \sum_{i=1}^{N} x_i^* \otimes y_i, \]
we get
\[ \langle e, UT^* V f^* \rangle = \sum_{i=1}^{N} \langle e, U x_i^* \rangle \langle y_i, V f^* \rangle. \]
So the bilinear map \( (U, V) \mapsto UT^* V \) is continuous, which in turn implies the compactness of \( L(T^* \mid F^* \mid E^*) \).

I am very indebted to W. B. Johnson for a proof of the following result, which is the crucial device in our understanding of local theory.

**Basic Lemma.** \( \overline{L}(T \mid E, F) \subseteq (1 + \varepsilon) L(T \mid E, F) \quad \text{for all } \varepsilon > 0. \)

**Proof.** If \( T \) has finite rank, then it follows from the previous proposition that \( L(T^* \mid E, F) \) is closed. Hence, by 5.7,
\[ \overline{L}(T \mid E, F) \subseteq L(T^* \mid E, F) \subseteq (1 + \varepsilon) L(T \mid E, F). \]

To treat the case \( \text{rank}(T) = \infty \), we let \( m := \dim(E) \) and \( n := \dim(F) \). If \( d := 2m + n \), then there exist \( A_0 \in B(l_2^d, X) \) and \( B_0 \in B(Y, l_2^d) \) such that
\[ B_0 TA_0 \] is the identity map of \( l_2^d \). Given \( S \in \overline{L}(T \mid E, F) \) and \( \delta > 0 \), we find a decomposition \( S = S_1 + S_2 \) such that \( S_1 \in L(T \mid E, F) \) and \( \| S_2 \| \leq \delta \). Write \( S_1 = B_1 TA_1 \) with \( \| A_1 \| \leq 1 \) and \( \| B_1 \| \leq 1 \). Let \( M \) be the range of \( B_0 TA_1 \), and let \( N \) be the null space of \( B_1 TA_0 \). Then \( \text{cod}(M^\perp) = \dim(M) \leq m \) and \( \text{cod}(N) \leq n \). Hence \( \text{cod}(M^\perp \cap N) \leq m + n, \) which is equivalent to \( \dim(M^\perp \cap N) \geq m \). This implies that \( M^\perp \cap N \) contains an \( m \)-dimensional subspace \( H \). Let \( J \) denote the embedding from \( H \) into \( l_2^d \), while \( Q \) stands for the orthogonal projection from \( l_2^d \) onto \( H \). So \( QB_0 TA_0 J \) is the identity.
map of $H$. John’s theorem provides us with an isomorphism $U \in \mathfrak{L}(E, H)$ such that $\|U\| \cdot \|U^{-1}\| \leq \sqrt{m}$.

We obtain $S_2 = S_2U^{-1}QB_0TA_0JU = B_2TA_2$, where $B_2 := S_2U^{-1}QB_0$ and $A_2 := A_0JU$. The main purpose of the construction above was to get the formulas $B_1TA_0J = O$ and $QB_0TA_1 = O$, which in turn yield

$$B_1TA_2 = B_1TA_0JU = O \quad \text{and} \quad B_2TA_1 = S_2U^{-1}QB_0TA_1 = O.$$ 

Hence

$$S = S_1 + S_2 = B_1TA_1 + B_2TA_2 = (B_1 + B_2)T(A_1 + A_2).$$ 

Moreover,

$$\|B_2\| \cdot \|A_2\| \leq \|S_2U^{-1}QB_0\| \cdot \|A_0JU\| \leq \delta \sqrt{m} \|B_0\| \cdot \|A_0\|.$$ 

Clearly, $U$ may be chosen such that $\|A_2\| = \|B_2\|$. Then

$$\|A_1 + A_2\| \cdot \|B_1 + B_2\| \leq (1 + \sqrt{\delta \sqrt{m} \|B_0\| \cdot \|A_0\|})^2 \leq 1 + \varepsilon$$

whenever $\delta > 0$ is sufficiently small. This proves that $S \in (1 + \varepsilon)\mathcal{L}(T \mid E, F)$.

5.10. The preceding result can be stated in the following form.

**Theorem.** $\mathcal{L}(T \mid E, F) = \bigcap_{\varepsilon > 0} (1 + \varepsilon)\mathcal{L}(T \mid E, F)$.

5.11. In the theory of ultraproducts, we have a striking counterpart of the principle of local reflexivity; see [kue], [ste] and [hei 1], p. 8.

**Kürsten–Stern Lemma.** Let $M$ and $N$ be finite-dimensional subspaces of $(X_i)^{\omega}$ and $((X_i)^{\omega})^*$, respectively. Then for every $\varepsilon > 0$ there exists an isomorphism $I_\varepsilon$ from $N$ onto some subspace $N_\varepsilon$ of $(X_i)^{\omega}$ such that $\|I_\varepsilon\| \leq 1 + \varepsilon$, $\|I_\varepsilon^{-1}\| \leq 1 + \varepsilon$ and

$$\langle x, x^* \rangle = \langle x, I_\varepsilon x^* \rangle \quad \text{for all} \ x \in M \ \text{and} \ x^* \in N.$$
5.12. For completeness, we provide another standard result of the theory of ultraproducts.

Lemma. Assume that \( x_1 = (x_{1i})^U, \ldots, x_m = (x_{mi})^U \in (X_i)^U \) are linearly independent. Then, given \( \varepsilon > 0 \), there exists \( U \in \mathcal{U} \) such that
\[
\left\| \sum_{h=1}^{m} \xi_h x_{hi} \right\| \leq (1 + \varepsilon) \left\| \sum_{h=1}^{m} \xi_h x_h \right\| \quad \text{for } \xi_1, \ldots, \xi_m \in \mathbb{K} \text{ and } i \in U.
\]

Proof. First of all, fix \( \delta > 0 \) and choose a finite \( \delta \)-net \( \mathcal{N} \) in the unit sphere \( S_1^m \) of \( l_1^m \). Next, pick \( U \in \mathcal{U} \) such that
\[
\left\| \sum_{h=1}^{m} \xi_h x_{hi} \right\| - \left\| \sum_{h=1}^{m} \xi_h x_h \right\| \leq \delta
\]
for \( (\xi_h) \in \mathcal{N} \) and \( i \in U \). Clearly, we may assume that \( \mathcal{N} \) contains the unit vectors of \( l_1^m \). Then it follows that \( \left\| x_{hi} \right\| - \left\| x_h \right\| \leq \delta \) for \( h = 1, \ldots, m \) and \( i \in U \). Putting \( c := \max \{ \left\| x_1 \right\|, \ldots, \left\| x_m \right\| \} \), we get
\[
\left\| \sum_{h=1}^{m} \xi_h x_{hi} \right\| - \left\| \sum_{h=1}^{m} \xi_h x_h \right\| \leq (1 + 2c + \delta) \delta
\]
for \( (\xi_h) \in S_1^m \) and \( i \in U \). Hence, by homogeneity,
\[
\left\| \sum_{h=1}^{m} \xi_h x_{hi} \right\| \leq \left\| \sum_{h=1}^{m} \xi_h x_h \right\| + (1 + 2c + \delta) \delta \sum_{h=1}^{m} |\xi_h|
\]
for \( \xi_1, \ldots, \xi_m \in \mathbb{K} \) and \( i \in U \). Since \( x_1, \ldots, x_m \) are linearly independent, we find a constant \( b > 0 \) such that
\[
\sum_{h=1}^{m} |\xi_h| \leq b \left\| \sum_{h=1}^{m} \xi_h x_h \right\| \quad \text{for } \xi_1, \ldots, \xi_m \in \mathbb{K},
\]
which implies
\[
\left\| \sum_{h=1}^{m} \xi_h x_{hi} \right\| \leq (1 + (1 + 2c + \delta)b\delta) \left\| \sum_{h=1}^{m} \xi_h x_h \right\|
\]
for \( \xi_1, \ldots, \xi_m \in \mathbb{K} \) and \( i \in U \). Choosing \( \delta > 0 \) sufficiently small completes the proof.

5.13. We are now in a position to establish an analogue of 5.7.

Proposition.
\[
\mathcal{L}(T \mid E, F) \subseteq \mathcal{L}(T^U \mid E, F) \subseteq (1 + \varepsilon)\mathcal{L}(T \mid E, F) \quad \text{for all } \varepsilon > 0.
\]

Proof. The left-hand inclusion follows from \( T^\text{reg} = Q^U_Y T^U X^U \); see 2.7.
Decompose the operator \( S \in \mathcal{L}(T^{\mathcal{U}}|E,F) \) in the form \( S = BT^{\mathcal{U}}A \) with \( \|A : E \to X^{\mathcal{U}}\| \leq 1 \) and \( \|B : Y^{\mathcal{U}} \to F\| \leq 1 \). Write

\[
A = \sum_{h=1}^{m} e_h^* \otimes x_h \quad \text{and} \quad B = \sum_{k=1}^{n} y_k^* \otimes f_k,
\]

where \( x_1, \ldots, x_m \in X^{\mathcal{U}} \) and \( y_1^*, \ldots, y_n^* \in (Y^{\mathcal{U}})^* \) are linearly independent.

By 5.11, we find an isomorphism subspace \( F \) of \( \mathcal{L}_x \) from \( F := \text{span}\{y_1^*, \ldots, y_n^*\} \) onto some subspace \( F \) of \( (Y^*)^{\mathcal{U}} \) such that \( \|F\| \leq 1 + \varepsilon \) and

\[
\langle T^{\mathcal{U}}x, y_k^* \rangle = \langle T^{\mathcal{U}}x, I \cdot y_k^* \rangle \quad \text{for } h = 1, \ldots, m \text{ and } k = 1, \ldots, n.
\]

Fix representations \( x_h = (x_h)^{\mathcal{U}} \) and \( I \cdot y_k^* = (y_k^* )^{\mathcal{U}} \) with \( x_h \in X \) and \( y_k^* \in Y^* \). By the preceding lemma, there exists \( U \in \mathcal{U} \) such that

\[
\left\| \sum_{h=1}^{m} \xi_h x_h \right\| \leq (1 + \varepsilon) \left\| \sum_{h=1}^{m} \xi_h x_h \right\| \quad \text{and} \quad \left\| \sum_{k=1}^{n} \eta_k y_k^* \right\| \leq (1 + \varepsilon) \left\| \sum_{k=1}^{n} \eta_k y_k^* \right\|
\]

for \( \xi_1, \ldots, \xi_m \in \mathbb{K} \), \( \eta_1, \ldots, \eta_n \in \mathbb{K} \) and \( i \in U \). Letting

\[
A_i := \sum_{h=1}^{m} e_h^* \otimes x_{hi} \quad \text{and} \quad B_i := \sum_{k=1}^{n} y_k^* \otimes f_k
\]

yields operators with \( \|A_i : E \to X\| \leq 1 + \varepsilon \) and \( \|B_i : Y \to F\| \leq 1 + \varepsilon \). Moreover, in view of

\[
\langle T^{\mathcal{U}}x, y_k^* \rangle = \langle T^{\mathcal{U}}x, I \cdot y_k^* \rangle = \mathcal{U}\lim_{i} \langle Tx_{hi}, y_k^* \rangle,
\]

it may be achieved that

\[
\|BT^{\mathcal{U}}A - B_iTA_i\| = \left\| \sum_{h=1}^{m} \sum_{k=1}^{n} (\langle T^{\mathcal{U}}x_h, y_k^* \rangle - \langle Tx_{hi}, y_k^* \rangle) e_h^* \otimes f_k \right\|
\]

becomes as small as we please. So

\[
S = BT^{\mathcal{U}}A \in (1 + \varepsilon)^2 \mathcal{L}(T|E,F) \subseteq (1 + \varepsilon)^3 \mathcal{L}(T|E,F).
\]

5.14. Finally, we summarize the most important results of this section.

**THEOREM.** The closed germs \( \mathcal{L}(-|E,F) \) coincide for the operators \( T \), \( T_\text{reg} \), \( T^{**} \) and \( T^{\mathcal{U}} \).

That is, from the local point of view we cannot distinguish between \( T \), \( T_\text{reg} \), \( T^{**} \) and \( T^{\mathcal{U}} \).

6. Local representability

6.1. Let \( c \geq 0 \). We say that an operator \( T_0 \in \mathcal{L}(X_0, Y_0) \) is locally \( c \)-representable in an operator \( T_1 \in \mathcal{L}(X_1, Y_1) \) if for \( \varepsilon > 0 \), for any choice of
finite-dimensional spaces \( E \) and \( F \), for \( A_0 \in \mathfrak{L}(E, X_0) \) and \( B_0 \in \mathfrak{L}(Y_0, F) \) there exist \( A_1 \in \mathfrak{L}(E, X_1) \) and \( B_1 \in \mathfrak{L}(Y_1, F) \) such that
\[
B_1 T_1 A_1 = B_0 T_0 A_0 \quad \text{and} \quad \|B_1\| \cdot \|A_1\| \leq (c + \varepsilon)\|B_0\| \cdot \|A_0\|.
\]
In other words, we assume that
\[
\mathcal{L}(T_0 | E, F) \subseteq (c + \varepsilon)\mathcal{L}(T_1 | E, F)
\]
whenever \( \varepsilon > 0 \) or, by 5.10, that
\[
\mathcal{L} \left( T_0 | E, F \right) \subseteq c\mathcal{L} \left( T_1 | E, F \right).
\]

Roughly speaking, it is required that the finite-dimensional structure of \( T_1 \) is at least as rich as that of \( T_0 \).

6.2. An operator \( T_0 \in \mathfrak{L}(X_0, Y_0) \) is said to be \textit{locally representable} in an operator \( T_1 \in \mathfrak{L}(X_1, Y_1) \) if the above conditions hold for some \( c \geq 0 \). In this case, we write \( T_0 \precloc T_1 \). Of course, \( \precloc \) is a preordering on \( \mathfrak{L} \).

6.3. We now restate Theorem 5.14.

**Theorem.** The operators \( T, T^{reg}, T^{**} \) and \( T^{d} \) are locally equivalent.

6.4. We see from 5.6 that the concept of local representability is stable under duality.

**Proposition.** Let \( c \geq 0 \). Then the following are equivalent:

(1) \( T_0 \) is locally \( c \)-representable in \( T_1 \).

(2) \( T_0^* \) is locally \( c \)-representable in \( T_1^* \).

6.5. The local nature of the above notion can also be seen from the next criterion.

**Proposition.** Let \( c \geq 0 \). Then the following are equivalent:

(1) \( T_0 \) is locally \( c \)-representable in \( T_1 \).

(2) For \( \varepsilon > 0 \), for \( x_{01}, \ldots, x_{0m} \in X_0 \) and for \( y_{01}^*, \ldots, y_{0n}^* \in Y_0^* \) we can find \( x_{11}, \ldots, x_{1m} \in X_1 \) and \( y_{11}^*, \ldots, y_{1n}^* \in Y_1^* \) such that
\[
\langle T_1 x_{1k}, y_{1k}^* \rangle = \langle T_0 x_{0h}, y_{0k}^* \rangle \quad \text{for} \ h = 1, \ldots, m \text{ and } k = 1, \ldots, n,
\]
\[
\left\| \sum_{k=1}^{m} \xi_k x_{1k} \right\| \leq \left\| \sum_{k=1}^{m} \xi_k x_{0h} \right\| \quad \text{and} \quad \left\| \sum_{k=1}^{n} \eta_k y_{1k}^* \right\| \leq (c + \varepsilon)\left\| \sum_{k=1}^{n} \eta_k y_{0k}^* \right\|
\]
whenever \( \xi_1, \ldots, \xi_m \in \mathbb{K} \) and \( \eta_1, \ldots, \eta_n \in \mathbb{K} \).

**Proof.** We consider the operators \( A_0 := J_{M}^{X_0} \) and \( B_0 := Q_{N}^{Y_0} \), where \( M := \text{span}\{x_{01}, \ldots, x_{0m}\} \) and \( N := \{y \in Y_0 : \langle y, y_{01}^* \rangle = \ldots = \langle y, y_{0n}^* \rangle = 0 \} \).
Note that \((Y_0/N)^*\) can be identified with \(\text{span}\{y_{01}^*, \ldots, y_{0n}^*\}\). Choose \(A_1\) and \(B_1\) such that
\[
B_1T_1A_1 = B_0T_0A_0, \quad \|A_1\| \leq \|A_0\| \quad \text{and} \quad \|B_1\| \leq (c + \varepsilon)\|B_0\|.
\]
Then we may put \(x_{1h} := A_1x_{0h}\) and \(y_{1k}^* := B_1^*x_{0k}^*\). This proves that
\((1) \Rightarrow (2)\).

In order to verify the reverse implication, we represent \(A_0 \in \mathfrak{L}(E, X_0)\) and \(B_0 \in \mathfrak{L}(Y_0, F)\) in the form
\[
A_0 = \sum_{h=1}^m e_h^* \otimes x_{0h} \quad \text{and} \quad B_0 = \sum_{k=1}^n y_{0k}^* \otimes f_k.
\]
Pick \(x_{11}, \ldots, x_{1m}\) and \(y_{11}^*, \ldots, y_{1n}^*\) as described in \((2)\). Then the operators
\[
A_1 := \sum_{h=1}^m e_h^* \otimes x_{1h} \quad \text{and} \quad B_1 := \sum_{k=1}^n y_{1k}^* \otimes f_k
\]
satisfy the conditions
\[
B_1T_1A_1 = B_0T_0A_0, \quad \|A_1\| \leq \|A_0\| \quad \text{and} \quad \|B_1^*f^*\| \leq (c + \varepsilon)\|B_0^*f^*\|.
\]

**6.6.** Next, we connect the concepts of local and global representability with the help of ultrapowers. This result should be compared with Theorem 1.2 in [hei 1].

**Theorem.** Let \(c \geq 0\). Then the following are equivalent:

1. \(T_0\) is locally \(c\)-representable in \(T_1\).
2. There exist operators \(A \in \mathfrak{L}(X_0, X_1^U)\) and \(B \in \mathfrak{L}(Y_1^U, Y_0^{**})\), where \(U\) is an ultrafilter on a suitable index set \(\mathcal{I}\), such that
\[
T_0^\text{reg} = BT_1^U A \quad \text{and} \quad \|B\| \cdot \|A\| \leq c.
\]

**Proof.** In view of 5.13, we conclude from \(T_0^\text{reg} = BT_1^U A\) that
\[
\mathcal{L}(T_0 | E, F) = \mathcal{L}(T_0^\text{reg} | E, F) \subseteq c\mathcal{L}(T_1^U | E, F) \subseteq (1 + \varepsilon)c\mathcal{L}(T_1 | E, F).
\]
This proves that \((2) \Rightarrow (1)\).

We consider the index set \(\mathcal{I}\) formed by all triples \(i = (M, N, \varepsilon)\). Here \(M\) is any finite-dimensional subspace of \(X_0\), and \(N\) is any finite-codimensional subspace of \(Y_0\). As usual, \(\varepsilon > 0\). For \(i = (M, N, \varepsilon)\) and \(i_0 = (M_0, N_0, \varepsilon_0)\), we write \(i \geq i_0\) if \(M \supseteq M_0, N \subseteq N_0\) and \(0 < \varepsilon \leq \varepsilon_0\). Furthermore, fix some ultrafilter \(U\) on \(\mathcal{I}\) that contains all sections \(\{i \in \mathcal{I} : i \geq i_0\}\) with \(i_0 \in \mathcal{I}\). Choose \(A_i \in \mathfrak{L}(M, X_1)\) and \(B_i \in \mathfrak{L}(Y_1, Y_0/N)\) such that \(\|A_i\| \leq 1, \|B_i\| \leq c + \varepsilon\) and \(B_iT_1A_i = Q_i^N T_0 J_{X_0}^{X_N} M\). Let
\[
x_i := \begin{cases} A_i x & \text{if } x \in M, \\ 0 & \text{if } x \not\in M. \end{cases}
\]
Clearly, \( A : x \mapsto (x_i)^{il} \) defines an operator \( A \in \mathcal{L}(X_0, X_1^{il}) \) with \( \| A \| \leq 1 \). Next, given \((y_i)^{il} \in Y_1^{il}\), we pick \((y_i^0)^{il} \in Y_0^{il}\) such that \( Q_{N_i}^{0} y_i^0 = B_i y_i \) and \( \| y_i^0 \| \leq (1 + c) \| B_i y_i \| \). Put \( B : (y_i)^{il} \mapsto \mathcal{U}\)-lim, \( K_{Y_0} y_i^0 \), where the limit is taken with respect to the weak* topology of \( Y_0^{**} \). Then \( \| B \| \leq c \) and \( T_0^{reg} = B T_1^{il} A \).

6.7. A property is said to be
- **local** if it is \( \text{loc} \)-stable,
- **global** if it is \( \text{glo} \)-stable,
- **regular** if it carries over from \( T_{\text{reg}} \in \mathcal{L}(X,Y^{**}) \) to \( T \in \mathcal{L}(X,Y) \),
- **injective** if it carries over from \( T_{\text{inj}} \in \mathcal{L}(X,Y^{\text{inj}}) \) to \( T \in \mathcal{L}(X,Y) \),
- **ultrapower-stable** if, for all ultrafilters \( U \), it is inherited from \( T \in \mathcal{L}(X,Y) \) to \( T^{il} \in \mathcal{L}(X^{il},Y^{il}) \).

6.8. Thanks to 6.6, we are able to exhibit the typical ingredients of local representability. Theorem. A property is local if and only if it is regular, ultrapower-stable and global.

7. Local operator schemes

7.1. We now define a preordering on the set of all operators acting between finite-dimensional spaces. If \( S_0 \in \mathcal{L}(E_0,F_0) \) and \( S_1 \in \mathcal{L}(E_1,F_1) \), then \( S_0 \leq S_1 \) means that \( S_0 \in \mathcal{L}(S_1 | E_0, F_0) \). In other words, there exist \( A_1 \) and \( B_1 \) such that

\[
S_0 = B_1 S_1 A_1, \quad \| A_1 : E_0 \to E_1 \| \leq 1 \quad \text{and} \quad \| B_1 : F_1 \to F_0 \| \leq 1.
\]

7.2. A **local operator scheme** is a rule that assigns to every pair \((E,F)\) of finite-dimensional Banach spaces a compact subset \( G(E,F) \) of \( \mathcal{L}(E,F) \) such that the following conditions are satisfied:

1. If \( S_0 \in \mathcal{L}(E_0,F_0) \) and \( S_1 \in G(E_1,F_1) \), then \( S_0 \leq S_1 \) implies that \( S_0 \in G(E_0,F_0) \).
2. For \( S_1 \in G(E_1,F_1) \) and \( S_2 \in G(E_2,F_2) \) there exists \( S \in G(E,F) \) such that \( S_1 \leq S \) and \( S_2 \leq S \).
3. The sets \( G(E,F) \) are uniformly bounded. That is, \( \| S \| \leq c \) for all \( S \in G(E,F) \), where the constant \( c > 0 \) does not depend on \( E \) and \( F \).

7.3. The closed germs \( G(T | E,F) := \mathcal{L}(T | E,F) \) of any operator \( T \) constitute a local operator scheme. Indeed, in order to verify \( (2) \) we let
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7.4. From the philosophical point of view, the following observation is the most important result of this paper. Operators at hand can be reconstructed from their germs, and finite-dimensional pieces can be glued together to form new operators. Our considerations are based on an isometric concept of local equivalence: $T_0 \sim_1 T_1$ if and only if $\mathcal{L}(T_0 | E, F) = \mathcal{L}(T_1 | E, F)$ for all finite-dimensional test spaces $E$ and $F$.

**Theorem.** There is a one-to-one correspondence between $\sim_1$-equivalence classes of operators and local operator schemes.

**Proof.** We only need to show that every local operator scheme can be obtained from an operator $T$, which will be produced by ultraproduct techniques. The underlying index set $I$ consists of all triples $i = (S : E \to F)$ with $S \in \mathcal{G}(E, F)$. Condition (2) implies that $I$ is upwards directed. So we can find an ultrafilter $U$ that contains all sections $U(i_0) := \{ i \in I : i \geq i_0 \}$ with $i_0 \in I$. Construct the ultraproducts $X := (X_i)^U$, $Y := (Y_i)^U$ and $T := (T_i)^U$, where $X_i := E$, $Y_i := F$ and $T_i := S_i$.

Fix $i_0 = (S_0 : E_0 \to F_0) \in I$. If $i \in U(i_0)$, then we find operators $A_i$ and $B_i$ such that $S_i = B_i T_i A_i$, $\|A_i : E_0 \to X_i\| \leq 1$ and $\|B_i : Y_i \to F_0\| \leq 1$. Put $A_i := O$ and $B_i := O$ whenever $i \notin U(i_0)$. Define $A : e \mapsto (A_i e)^U$ and $B : (y_i)^U \mapsto \lim_i B_i y_i$.

Then $\|A : E_0 \to X\| \leq 1$ and $\|B : Y \to F_0\| \leq 1$. Finally,

$$BT Ae = \lim_i B_i T_i A_i e = S_0 e$$

for all $e \in E_0$.

This means that $\mathcal{G}(E_0, F_0) \subseteq \mathcal{L}(T | E_0, F_0)$.
In order to prove the reverse inclusion, let \( \| A : E_0 \to X \| \leq 1 \) and \( \| B : Y \to F_0 \| \leq 1 \). Write

\[
A = \sum_{h=1}^{m} e^*_h \otimes x_{hi} \quad \text{and} \quad B = \sum_{k=1}^{n} y^*_k \otimes f_k,
\]

where \( x_1, \ldots, x_m \in X \) and \( y^*_1, \ldots, y^*_n \in Y^* \) are linearly independent. By 5.11, we find an isomorphism \( I_x \) from \( F := \text{span}\{y^*_1, \ldots, y^*_n\} \) onto some subspace \( F_e \) of \( (Y^*_e)^2 \) such that \( \| I_x \| \leq 1 + \varepsilon \) and

\[
\langle Tx_h, y^*_k \rangle = \langle Tx_h, I_x y^*_k \rangle \quad \text{for} \quad h = 1, \ldots, m \quad \text{and} \quad k = 1, \ldots, n.
\]

Fix representations \( x_h = (x_{hi})^{2l} \) and \( I_x y^*_k = (y^*_k)_{2l} \) with \( x_{hi} \in X_i \) and \( y^*_k \in Y^*_i \). In view of Lemma 5.12, there exists \( U \in \mathcal{U} \) such that

\[
\left\| \sum_{h=1}^{m} \xi_h x_{hi} \right\| \leq (1 + \varepsilon) \left\| \sum_{h=1}^{m} \xi_h x_{hi} \right\| \quad \text{and} \quad \left\| \sum_{k=1}^{n} \eta_k y^*_k \right\| \leq (1 + \varepsilon) \left\| \sum_{k=1}^{n} \eta_k y^*_k \right\|
\]

for \( \xi_1, \ldots, \xi_m \in \mathbb{K}, \eta_1, \ldots, \eta_n \in \mathbb{K} \) and \( i \in U \). Letting

\[
A_i := \sum_{h=1}^{m} e^*_h \otimes x_{hi} \quad \text{and} \quad B_i := \sum_{k=1}^{n} y^*_k \otimes f_k
\]

yields operators with \( \| A_i : E_0 \to X_i \| \leq 1 + \varepsilon \) and \( \| B_i : Y \to F_0 \| \leq 1 + \varepsilon \). Moreover, in view of \( \langle Tx_h, y^*_k \rangle = \langle Tx_h, I_x y^*_k \rangle = \mathcal{U}\text{-lim}_i \langle T_i x_{hi}, y^*_{ki} \rangle \), it may be achieved that

\[
\| BTA - B_i T_i A_i \| = \left\| \sum_{h=1}^{m} \sum_{k=1}^{n} \langle (Tx_h, y^*_k) - \langle T_i x_{hi}, y^*_{ki} \rangle \rangle e^*_h \otimes f_k \right\|
\]

becomes as small as we please. By (1), we have \( B_i T_i A_i \in (1 + \varepsilon)^2 \mathcal{G}(E_0, F_0) \). So \( BTA \in \mathcal{G}(E_0, F_0) \) as \( \varepsilon \to 0 \), which gives \( \mathcal{L}(T | E_0, F_0) \subseteq \mathcal{G}(E_0, F_0) \).

7.5. Assigning to \((E, F)\) the closed unit ball of \( \mathcal{L}(E, F) \), we get a local operator scheme. Any generating operator \( T_0 \) is maximal with respect to the preordering \( \prec \); that is, \( T \prec T_0 \) for all operators \( T \). There is a reflexive and separable Banach space whose identity map is universal in this sense. Indeed, choosing a dense sequence \( E_{m1}, E_{m2}, \ldots \) in the Minkowski compactum of all \( m \)-dimensional Banach spaces with \( m = 1, 2, \ldots \), we may take the \( l_2 \)-sum of the double sequences \((E_{mn})\) so obtained.

8. Injective-local representability

8.1. Let \( c \geq 0 \). We say that an operator \( T_0 \in \mathcal{L}(X_0, Y_0) \) is \textit{injective-locally} \( c \)-\textit{representable} in an operator \( T_1 \in \mathcal{L}(X_1, Y_1) \) if for \( \varepsilon > 0 \), for any choice of a finite-dimensional space \( E \) and \( N = 1, 2, \ldots \), for \( A_0 \in \mathcal{L}(E, X_0) \)
and $B_0 \in \mathfrak{L}(Y_0, l_\infty^N)$ there exist $A_1 \in \mathfrak{L}(E, X_1)$ and $B_1 \in \mathfrak{L}(Y_1, l_\infty^N)$ such that

$$B_1T_1A_1 = B_0T_0A_0 \quad \text{and} \quad \|B_1\| \cdot \|A_1\| \leq (c + \varepsilon)\|B_0\| \cdot \|A_0\|.$$  

In other words, we assume that

$$\mathcal{L}(T_0 | E, l_\infty^N) \subseteq (c + \varepsilon)\mathcal{L}(T_1 | E, l_\infty^N) \quad \text{whenever} \quad \varepsilon > 0$$

or, by 5.10, that

$$\|B_1\| \leq c\|B_0\|.$$ 

This means that, compared with Definition 6.1, we only use a special type of test spaces $F$. The main point is that $l_\infty^N$ has the metric extension property.

8.2. An operator $T_0 \in \mathfrak{L}(X_0, Y_0)$ is said to be injective-locally representative in an operator $T_1 \in \mathfrak{L}(X_1, Y_1)$ if the above conditions hold for some $c \geq 0$. In this case, we write $T_0 \stackrel{\text{injloc}}{\prec} T_1$. Of course, $\stackrel{\text{injloc}}{\prec}$ is a preordering on $\mathfrak{L}$. Note that $T_0 \stackrel{\text{loc}}{\prec} T_1$ implies $T_0 \stackrel{\text{injloc}}{\prec} T_1$.

8.3. In the following, we need a folklore result.

**Lemma.** Let $\varepsilon > 0$, $A \in \mathfrak{L}(E, X)$ and $B \in \mathfrak{L}(X, l_\infty(\mathbb{I}))$. Then there exist $B_0 \in \mathfrak{L}(X, l_\infty^N)$ and $V \in \mathfrak{L}(l_\infty^N, l_\infty(\mathbb{I}))$ such that $\|B_0\| \leq \|B\|$, $\|V\| \leq 1 + \varepsilon$ and

$$E \xrightarrow{A} X \xrightarrow{B} l_\infty(\mathbb{I})$$

$$\begin{array}{c}
E \\
\downarrow^{B_0} \\
\downarrow^{l_\infty^N} \\
\downarrow^{V}
\end{array}
\begin{array}{c}
X \\
\downarrow^{B} \\
\downarrow^{\mathbb{I}}
\end{array}
$$

In addition, we may arrange that $N \leq (3 + 2/\varepsilon)^n$, with $n := \dim(E)$ replaced by $2n$ in the complex case.

**Proof.** Choose functionals $x_1^*, \ldots, x_N^* \in X^*$ of norm 1 whose restrictions to $M := A(E)$ constitute a minimal finite $\delta$-net in the unit sphere of $M^*$, where $\delta := \varepsilon/(1 + \varepsilon)$. Since $(1 - \delta)\|x\| \leq \max_i |(x, x_i^*)|$ for all $x \in M$, the map $B_0 : x \mapsto \|B||(x, x_i^*)|$ is invertible on $B_0(M)$. Finally, $V$ can be obtained as a norm-preserving extension of $BB_0^{-1}$. The estimate of $N$ follows from entropy theory.

8.4. The symbol $\stackrel{\text{injloc}}{\prec}$ is justified by the next criterion.

**Proposition.** Let $c \geq 0$. Then the following are equivalent:

(1) $T_0$ is injective-locally $c$-representable in $T_1$.

(2) $T_0^{\text{inj}}$ is locally $c$-representable in $T_1^{\text{inj}}$. 
Proof. (1)⇒(2). Look at the commutative diagram

which is obtained as follows: Let $A_0$ and $C_0$ be given. Applying Lemma 8.3 to $T_0 A_0$ and $J^{inj}_{Y_0}$, we find $B_0$ and $V_0$ such that $\|B_0\| \leq 1$ and $\|V_0\| \leq 1 + \varepsilon$. Choose $A_1$ and $B_1$ with $\|B_1\| \cdot \|A_1\| \leq (c + \varepsilon)\|B_0\| \cdot \|A_0\|$. Finally, let $V_1$ be any norm-preserving extension of $B_1$ and put $C_1 := C_0 V_0 V_1$. Then

$$\|C_1\| \cdot \|A_1\| \leq \|C_0\| \cdot \|V_0\| \cdot \|V_1\| \cdot \|A_1\| \leq (1 + \varepsilon)\|C_0\| \cdot \|B_1\| \cdot \|A_1\| \leq (1 + \varepsilon)(c + \varepsilon)\|C_0\| \cdot \|A_0\|.$$ 

(2)⇒(1). Now we use the commutative diagram

Let $A_0$ and $B_0$ be given. Fix any norm-preserving extension $C_0$ of $B_0$. Choose $A_1$ and $C_1$. Finally, let $B_1 := C_1 J^{inj}_{Y_1}$.

8.5. Next, we state a counterpart of 6.8.

**Theorem.** A property is injective-local if and only if it is injective, ultrapower-stable and global.

8.6. In the case of injective-local representability the list of locally equivalent operators, given in 6.3, can be extended by $T^{inj}$.

**Theorem.** The operators $T$, $T^{reg}$, $T^{inj}$, $T^{**}$ and $T^{\text{ul}}$ are injective-locally equivalent.

**Proof.** Note that $\mathcal{L}(T^{inj} \mid E, l^N_{\infty}) = \mathcal{L}(T \mid E, l^N_{\infty})$. 
8.7. We now establish an analogue of 6.5.

Proposition. Let \( c \geq 0 \). Then the following are equivalent:

1. \( T_0 \) is injective-locally \( c \)-representable in \( T_1 \).
2. For \( \varepsilon > 0 \) and for \( x_{01}, \ldots, x_{0m} \in X_0 \) we can find \( x_{11}, \ldots, x_{1m} \in X_1 \) such that
   \[
   \left\| \sum_{h=1}^{m} \xi_h x_{1h} \right\| \leq \left\| \sum_{h=1}^{m} \xi_h x_{0h} \right\| \quad \text{and} \quad \left\| \sum_{h=1}^{m} \xi_h T_0 x_{0h} \right\| \leq (c + \varepsilon) \left\| \sum_{h=1}^{m} \xi_h T_1 x_{1h} \right\|
   \]
   whenever \( \xi_1, \ldots, \xi_m \in \mathbb{K} \).
3. For \( \varepsilon > 0 \), for any finite-dimensional space \( E \), and for \( A_0 \in \mathcal{L}(E, X_0) \) there exists \( A_1 \in \mathcal{L}(E, X_1) \) such that
   \[
   \| A_1 e \| \leq \| A_0 e \| \quad \text{and} \quad \| T_0 A_0 e \| \leq (c + \varepsilon) \| T_1 A_1 e \| \quad \text{whenever} \quad e \in E.
   \]

Proof. (1) \( \Rightarrow \) (2). Let \( A_0 := J_{X_0}^M \) with \( M := \text{span}\{x_{01}, \ldots, x_{0m}\} \). Applying 8.3 to \( T_0 A_0 \) and \( J_{X_0}^{\text{inj}} \), we find \( B_0 \) and \( V \) with \( J_{X_0}^{\text{inj}} T_0 A_0 = V B_0 T_0 A_0 \), \( \| B_0 \| \leq 1 \) and \( \| V \| \leq 1 + \varepsilon \). Choose \( A_1 \in \mathcal{L}(M, X_1) \) and \( B_1 \in \mathcal{L}(Y_1, l_\infty^N) \) such that
   \[
   B_1 T_1 A_1 = B_0 T_0 A_0, \quad \| A_1 \| \leq \| A_0 \| = 1 \quad \text{and} \quad \| B_1 \| \leq (c + \varepsilon) \| B_0 \| \leq c + \varepsilon.
   \]

Putting \( x_{1h} := A_1 x_{0h} \), we finally arrive at
   \[
   \left\| \sum_{h=1}^{m} \xi_h x_{1h} \right\| \leq \left\| \sum_{h=1}^{m} \xi_h x_{0h} \right\|
   \]

and
   \[
   \left\| \sum_{h=1}^{m} \xi_h T_0 x_{0h} \right\| = \left\| \sum_{h=1}^{m} \xi_h J_{X_0}^{\text{inj}} T_0 A_0 x_{0h} \right\| \leq \| V \| \cdot \left\| \sum_{h=1}^{m} \xi_h B_0 T_0 A_0 x_{0h} \right\|
   \]

   \[
   \leq (1 + \varepsilon) \left\| \sum_{h=1}^{m} \xi_h B_1 T_1 A_1 x_{0h} \right\|
   \]

   \[
   \leq (1 + \varepsilon) (c + \varepsilon) \left\| \sum_{h=1}^{m} \xi_h T_1 x_{1h} \right\|.
   \]

(2) \( \Rightarrow \) (3). Write \( A_0 \in \mathcal{L}(E, X_0) \) in the form \( A_0 = \sum_{h=1}^{m} e_h^* \otimes x_{0h} \), and let \( A_1 := \sum_{h=1}^{m} e_h^* \otimes x_{1h} \).

(3) \( \Rightarrow \) (1). Given \( A_0 \in \mathcal{L}(E, X_0) \) and \( B_0 \in \mathcal{L}(Y_0, l_\infty^N) \), we choose an operator \( A_1 \in \mathcal{L}(E, X_1) \) such that \( \| A_1 e \| \leq \| A_0 e \| \) and \( \| T_0 A_0 e \| \leq (c + \varepsilon) \| T_1 A_1 e \| \) whenever \( e \in E \). Of course, \( \| A_1 \| \leq \| A_0 \| \). It follows from
   \[
   \| B_0 T_0 A_0 e \| \leq \| B_0 \| \cdot \| T_0 A_0 e \| \leq (c + \varepsilon) \| B_0 \| \cdot \| T_1 A_1 e \|
   \]

that \( T_1 A_1 e \mapsto B_0 T_0 A_0 e \) yields a well-defined operator from \( T_1 A_1(E) \) into \( l_\infty^N \). If \( B_1 \) is any norm-preserving extension, then \( \| B_1 \| \leq (c + \varepsilon) \| B_0 \| \).
Remark. In the case when $T_0$ and $T_1$ are identity maps, the inequalities from (2) pass into
\[ \left\| \sum_{h=1}^{m} \xi_h x_{1h} \right\| \leq \left\| \sum_{h=1}^{m} \xi_h x_{0h} \right\| \leq (c + \varepsilon) \left\| \sum_{h=1}^{m} \xi_h x_{1h} \right\|. \]
This yields an upper estimate of the Banach–Mazur distance between $\text{span}\{x_{01}, \ldots, x_{0m}\}$ and $\text{span}\{x_{11}, \ldots, x_{1m}\}$. More precisely, it turns out that the relation $\text{span}^{\text{injloc}}$ extends the concept of crudely finite representability to the setting of operators.

8.8. The next criterion is an analogue of 6.6, from which it could be derived via 8.4. We prefer, however, to give a direct proof.

**Proposition.** Let $c \geq 0$. Then the following are equivalent:

1. $T_0$ is injective-locally $c$-representable in $T_1$.
2. There exist operators $A \in \mathcal{L}(X_0, X_1^U)$ and $B \in \mathcal{L}(Y_1^U, Y_0^{\text{inj}})$, where $\mathcal{U}$ is an ultrafilter on a suitable index set $I$, such that

\[ T_0^{\text{inj}} = BT_1^U A \quad \text{and} \quad \|B\| \cdot \|A\| \leq c. \]

**Proof.** If $T_0^{\text{inj}} = BT_1^U A$, then

\[ \mathcal{L}(T_0 \mid E, l^N_\infty) = \mathcal{L}(T_0^{\text{inj}} \mid E, l^N_\infty) \subseteq \|B\| \cdot \|A\| \mathcal{L}(T_1^U \mid E, l^N_\infty) \]
\[ \subseteq (1 + \varepsilon)c \mathcal{L}(T_1 \mid E, l^N_\infty). \]

This proves that \((2) \Rightarrow (1)\).

We consider the index set $I$ formed by all pairs $i = (M, \varepsilon)$. Here $M$ is any finite-dimensional subspace of $X_0$ and $\varepsilon > 0$. For $i = (M, \varepsilon)$ and $i_0 = (M_0, \varepsilon_0)$, we write $i \geq i_0$ if $M \supseteq M_0$ and $0 < \varepsilon \leq \varepsilon_0$. Furthermore, fix some ultrafilter $\mathcal{U}$ on $I$ that contains all sections $\{i \in I : i \geq i_0\}$ with $i_0 \in I$.

Choose $A_i \in \mathcal{L}(M, X_1)$ such that

\[ \|A_i x\| \leq \|x\| \quad \text{and} \quad \|T_0 x\| \leq (c + \varepsilon) \|T_1 A_i x\| \quad \text{for} \ x \in M. \]

Let

\[ x_i := \begin{cases} A_i x & \text{if} \ x \in M, \\ o & \text{if} \ x \not\in M, \end{cases} \quad \text{and} \quad y_i := \begin{cases} T_1 A_i x & \text{if} \ x \in M, \\ o & \text{if} \ x \not\in M. \end{cases} \]

Clearly, $A : x \mapsto (x_i)^U$ defines an operator $A \in \mathcal{L}(X_0, X_1^U)$ with $\|A\| \leq 1$.

Next, it follows from

\[ \|T_0^{\text{inj}} x\| = \|T_0 x\| \leq \mathcal{U}\text{-lim}_{i} (c + \varepsilon) \|T_1 A_i x\| = c \|(y_i)^U\| \]

that $B_0 : (y_i)^U \mapsto T_0^{\text{inj}} x$ yields a map from the range of $T_1^U A$ into $Y_0^{\text{inj}}$. Choosing any norm-preserving extension, we obtain the required operator $B \in \mathcal{L}(Y_1^U, Y_0^{\text{inj}})$ with $\|B\| \leq c$. Hence \((1) \Rightarrow (2)\).
9. Local and injective-local distances. In the definition of local representability the dimensions of the test spaces $E$ and $F$ are arbitrary. Now we introduce a gradation.

9.1. For $n = 1, 2, \ldots$, the $n$th local distance between $T_0$ and $T_1$ is defined by

$$l_n(T_0, T_1) := \inf \left\{ c \geq 0 : \mathcal{L}(T_0 \mid E, F) \subseteq c \mathcal{L}(T_1 \mid E, F), \ \dim(E) \leq n, \ \dim(F) \leq n \right\}.$$ 

If such a constant $c \geq 0$ does not exist, we let $l_n(T_0, T_1) := \infty$. This happens if and only if $\text{rank}(T_0) \geq n > \text{rank}(T_1)$. In view of 5.10, we have

$$l_n(T_0, T_1) := \min \left\{ c \geq 0 : \overline{\mathcal{L}}(T_0 \mid E, F) \subseteq c \overline{\mathcal{L}}(T_1 \mid E, F), \ \dim(E) \leq n, \ \dim(F) \leq n \right\}.$$ 

9.2. Note that

$$\|T_0\| \cdot \|T_1\|^{-1} = l_1(T_0, T_1) \leq l_2(T_0, T_1) \leq \ldots \leq l_n(T_0, T_1) \leq \ldots$$

The growth of this sequence measures the deviation of $T_1$ from $T_0$.

For operators $T_0$, $T_1$ and $T_2$ between arbitrary couples of Banach spaces, we have a multiplicative triangle inequality:

$$l_n(T_0, T_2) \leq l_n(T_0, T_1) l_n(T_1, T_2).$$

Moreover, 5.6 yields

$$l_n(T_0^*, T_1^*) = l_n(T_0, T_1).$$

9.3. For $n = 1, 2, \ldots$, the $n$th injective-local distance between $T_0$ and $T_1$ is defined by

$$i_n(T_0, T_1) := \inf \left\{ c \geq 0 : \mathcal{L}(T_0 \mid E, l_N^\infty) \subseteq c \mathcal{L}(T_1 \mid E, l_N^\infty), \ \dim(E) \leq n, \ N = 1, 2, \ldots \right\}.$$ 

If such a constant $c \geq 0$ does not exist, we let $i_n(T_0, T_1) := \infty$. This happens if and only if $\text{rank}(T_0) \geq n > \text{rank}(T_1)$. In view of 5.10, we have

$$i_n(T_0, T_1) := \min \left\{ c \geq 0 : \overline{\mathcal{L}}(T_0 \mid E, l_N^\infty) \subseteq c \overline{\mathcal{L}}(T_1 \mid E, l_N^\infty), \ \dim(E) \leq n, \ N = 1, 2, \ldots \right\}.$$ 

9.4. In analogy with 9.2,

$$\|T_0\| \cdot \|T_1\|^{-1} = i_1(T_0, T_1) \leq i_2(T_0, T_1) \leq \ldots \leq i_n(T_0, T_1) \leq \ldots$$

and

$$i_n(T_0, T_2) \leq i_n(T_0, T_1) i_n(T_1, T_2).$$

However, $i_n(T_0^*, T_1^*)$ and $i_n(T_0, T_1)$ may behave quite differently.
We now compare the local distances with the injective-local distances.

**Proposition.** For every \( \varepsilon > 0 \) there exists a natural number \( a > 1 \) such that
\[
i_n(T_0, T_1) \leq (1 + \varepsilon) i_{a^n}(T_0, T_1) \quad \text{for } n = 1, 2, \ldots
\]

**Proof.** Put \( a := 4 + \lceil 2/\varepsilon \rceil \) (real case) and \( a := (4 + \lceil 2/\varepsilon \rceil)^2 \) (complex case). Note that
\[
c_n := \inf \left\{ c \geq 0 : \mathcal{L}(T_0 | E, l_{N_0}^\infty) \leq c \mathcal{L}(T_1 | E, l_{N_0}^\infty), \quad \dim(E) \leq n \right\}
\]
\[
\leq i_{a^n}(T_0, T_1).
\]

We now construct a commutative diagram:

\[
\begin{array}{ccc}
X_0 & \xrightarrow{T_0} & Y_0 \\
| \downarrow A_0 & & \downarrow C_0 \\
E & \xrightarrow{V} & l_{N_0}^\infty \\
| \downarrow A_1 & & \downarrow C_1 \\
X_1 & \xrightarrow{T_1} & Y_1
\end{array}
\]

Let \( \delta > 0, A_0 \) and \( B_0 \) be given. Applying Lemma 8.3 to \( T_0 A_0 \) and \( B_0 \), we find \( C_0 \) and \( V \) such that \( \|C_0\| \leq \|B_0\| \) and \( \|V\| \leq 1 + \varepsilon \). Choose \( A_1 \) and \( C_1 \) with \( \|C_1\| : \|A_1\| \leq (c_n + \delta)\|C_0\| : \|A_0\| \). Finally, putting \( B_1 := VC_1 \), we arrive at \( \|B_1\| : \|A_1\| \leq (1 + \varepsilon)(c_n + \delta)\|B_0\| : \|A_0\| \), which proves that
\[
i_n(T_0, T_1) \leq (1 + \varepsilon)(c_n + \delta) \leq (1 + \varepsilon) i_{a^n}(T_0, T_1) + \delta.
\]

**Remark.** An inequality of the form \( i_n(T_0, T_1) \leq c i_{\varphi(n)}(T_0, T_1) \) can only hold if \( \varphi : \mathbb{N} \to \mathbb{N} \) grows exponentially: \( \varphi(n) \gg a^n \) with \( a > 1 \); see 11.6.

9.6. Here is another consequence of Theorem 5.14.

**Theorem.** The \( l_n \)-distances between \( T, T^{\text{reg}}, T^{**} \) and \( T^{d*} \) equal 1. In the case of \( i_n \)-distances the same is true for \( T, T^{\text{reg}}, T^{\text{inj}}, T^{**} \) and \( T^{d*} \).

9.7. Distances between spaces \( X_0 \) and \( X_1 \) are obtained as the distances between the associated identity maps \( I_{X_0} \) and \( I_{X_1} \). In the infinite-dimensional case, we have \( i_n(X_0, X_1) \leq n \) and \( i_n(X_0, X_1) \leq \sqrt{n} \). It turns out that \( i_n(X_0, X_1) \) is the infimum of all constants \( c \geq 1 \) with the following property: For every \( n \)-dimensional subspace \( M_0 \) of \( X_0 \) there exists an \( n \)-dimensional subspace \( M_1 \) of \( X_1 \) such that the Banach–Mazur distance \( d(M_0, M_1) \) is less than or equal to \( c \).
9.8. For \( n = 1, 2, \ldots \) and \( T \in \mathfrak{L}(X, Y) \), the \( n \)th Bernstein number \( b_n(T) \) is the supremum of all constants \( b \geq 0 \) such that \( \|Tx\| \geq b\|x\| \) for every \( x \) in some \( n \)-dimensional subspace of \( X \). In the case when \( \dim(X) < n \), we let \( b_n(T) := 0 \). Obviously,

\[
\|T\| = b_1(T) \geq b_2(T) \geq \ldots \geq b_n(T) \geq \ldots \geq 0.
\]

9.9. Roughly speaking, the following estimate shows that the decay of the sequence \((b_n(T))\) depends continuously on \( T \).

Proposition. \( b_n(T_0) \leq i_n(T_0, T_1)b_n(T_1) \).

Proof. If \( b_n(T_0) > b > 0 \), then there exists an \( n \)-dimensional subspace \( E_0 \) of \( X_0 \) such that

\[
\|T_0x\| \geq b\|x\| \quad \text{for all } x \in E_0.
\]

Let \( A_0 \in \mathfrak{L}(E_0, X_0) \) denote the embedding map from \( E_0 \) into \( X_0 \). Choose \( A_1 \in \mathfrak{L}(E_0, X_1) \) such that

\[
\|A_1x\| \leq \|A_0x\| \quad \text{and} \quad \|T_0A_0x\| \leq (i_n(T_0, T_1) + \varepsilon)\|T_1A_1x\|
\]

for all \( x \in E_0 \). Note that \( A_1x = 0 \) implies \( x = 0 \). So \( E_1 := A_1(E_0) \) is an \( n \)-dimensional subspace of \( X_1 \). Hence it follows from

\[
(i_n(T_0, T_1) + \varepsilon)\|T_1A_1x\| \geq \|T_0A_0x\| \geq b\|A_0x\| \geq b\|A_1x\|
\]

that

\[
\frac{b}{i_n(T_0, T_1) + \varepsilon} \leq b_n(T_1).
\]

Finally, letting \( b \to b_n(T_0) \) and \( \varepsilon \to 0 \) completes the proof.

10. Operators in Hilbert spaces

10.1. Throughout this section, we let \( T \in \mathfrak{L}(H, K) \), \( T_0 \in \mathfrak{L}(H_0, K_0) \) and \( T_1 \in \mathfrak{L}(H_1, K_1) \), where \( H, H_0, H_1 \) and \( K, K_0, K_1 \) denote Hilbert spaces. Since in this setting all s-numbers coincide, we may work with the Bernstein numbers whose local nature is evident.

10.2. Theorem. All non-compact operators between Hilbert spaces are locally equivalent.

Proof. If \( \|A : E \to H\| \leq 1 \) and \( \|B : K \to F\| \leq 1 \), then there exist factorizations \( A : E \xrightarrow{A_0} l_2 \xrightarrow{U} H \) and \( B : K \xrightarrow{V} l_2 \xrightarrow{B_0} F \) such that \( \|A_0\| = \|A\| \) and \( \|U\| = 1 \), \( \|B_0\| = \|B\| \) and \( \|V\| = 1 \). Putting \( T_0 := VTU \), we get the diagram
That is, \( BTA = B_0T_0A_0 = (B_0)I(VTU A_0) \), where \( I \) denotes the identity map of \( l_2 \). This proves that every operator \( T \) is locally \( \|T\| \)-representable in \( I \).

Conversely, \( I \) is even globally representable in all non-compact operators, since these operators fail to be strictly singular.

**Remark.** In separable Hilbert spaces, all non-compact operators are globally equivalent.

10.3. As just observed, in the setting of Hilbert spaces there is almost no difference between local and global properties. This assertion will be confirmed by the next result.

**Theorem.** Let \( c \geq 0 \). Then, for compact operators \( T_0 \) and \( T_1 \), the following are equivalent:

1. \( T_0 \) is globally representable in \( T_1 \).
2. \( T_0 \) is locally representable in \( T_1 \).
3. \( T_0 \) is injective-locally representable in \( T_1 \).
4. There exists \( c \geq 0 \) such that \( b_n(T_0) \leq cb_n(T_1) \) for \( n = 1, 2, \ldots \)

**Proof.** The implications (1) \( \Rightarrow \) (2) and (2) \( \Rightarrow \) (3) are trivial, and (3) \( \Rightarrow \) (4) follows from 9.9.

In order to verify (4) \( \Rightarrow \) (1), we assume that the underlying spaces are infinite-dimensional. Consider Schmidt representations

\[
T_0 = \sum_{n=1}^{\infty} b_n(T_0)x_{0n}^* \otimes y_{0n} \quad \text{and} \quad T_1 = \sum_{n=1}^{\infty} b_n(T_1)x_{1n}^* \otimes y_{1n},
\]

where \( (x_{0n}), (y_{0n}), (x_{1n}) \) and \( (y_{1n}) \) are orthonormal sequences. Moreover, define \( x_{0n}^* : x \mapsto (x|x_{0n}) \) and \( x_{1n}^* : x \mapsto (x|x_{1n}) \). Since \( b_n(T_0) \leq cb_n(T_1) \), we find a sequence \( (\beta_n) \) such that \( b_n(T_0) = \beta_n b_n(T_1) \) and \( 0 \leq \beta_n \leq c \). Finally, letting

\[
A_1 := \sum_{n=1}^{\infty} x_{0n}^* \otimes x_{1n} \quad \text{and} \quad B_1 := \sum_{n=1}^{\infty} \beta_n y_{1n}^* \otimes y_{0n}
\]

yields the factorization \( T_0 = B_1T_1A_1 \).

10.4. The following formulas can be proved by standard techniques:

\[
\mathcal{L}(T \mid l_2^n, l_2^n) = \{ S \in \mathcal{L}(l_2^n, l_2^n) : b_1(S) \leq b_1(T), \ldots , b_n(S) \leq b_n(T) \}
\]

and

\[
l_n(T_0, T_1) = \iota_n(T_0, T_1) = \max \{ b_1(T_0)b_1(T_1)^{-1}, \ldots , b_n(T_0)b_n(T_1)^{-1} \}.
\]
11. Examples

11.1. Obviously, belonging to an operator ideal is a global property. Every maximal quasi-Banach ideal \( \mathfrak{A} \) can be obtained as the collection of all operators \( T \in \mathfrak{A} \) for which \( \sup \{ \alpha(S) : S \in \mathcal{L}(T | E, F), E, F \in \mathfrak{F} \} \) is finite. So \( \mathfrak{A} \) is uniquely determined if we know the underlying quasi-ideal norm \( \alpha \) only for operators acting between finite-dimensional spaces. This observation implies that belonging to a maximal quasi-Banach operator ideal is a local property.

11.2. Global, but non-local classes:
- completely continuous operators,
- weakly compact operators,
- operators with the Radon–Nikodým property,
- strictly singular operators,
- operators with separable range.

11.3. Global and ultrapower-stable, but non-regular classes:
- nuclear operators.

11.4. Local, but non-injective classes:
- approximable operators,
- operators with \( p \)-summable approximation numbers,
- \( p \)-integral operators, \( p \neq 2 \),
- \( L_p \)-factorable operators, \( p \neq 2 \).

11.5. Injective-local classes:
- compact operators,
- super weakly compact operators,
- 2-integral operators,
- \( L_2 \)-factorable operators,
- \( p \)-summing operators,
- operators of Rademacher type \( p \),
- operators of Rademacher cotype \( q \),
- UMD-operators.

11.6. For positive sequences \( (\alpha_n) \) and \( (\beta_n) \), the asymptotic relation \( \alpha_n \asymp \beta_n \) means that \( \alpha_n = O(\beta_n) \) and \( \beta_n = O(\alpha_n) \).

We now describe the asymptotic behaviour of the distances \( l_n(L_p, L_q) \) and \( i_n(L_p, L_q) \) as \( n \to \infty \). Since infinite-dimensional \( L_p \) spaces are locally equivalent, it is enough to think of \( L_p[0, 1] \) or \( l_p \). In all cases that are settled until now, we have \( l_n(L_p, L_q) \asymp n^\lambda \) and \( i_n(L_p, L_q) \asymp n^\mu \).
The following diagrams show the values of the exponents $\lambda$ and $\mu$ if $1 < p, q < \infty$. On the border-lines $p = 1$, $q = 1$, $p = \infty$ and $q = \infty$, discontinuities may occur. We see that the sequences $(l_n(L_p, L_q))$ and $(i_n(L_p, L_q))$ behave differently for $2 < q < p < 1$.

The $\lambda$-diagram is due to A. Hinrichs (unpublished), while the $\mu$-diagram was established by A. Hinrichs and T. Kaufhold and by Y. Gordon and M. Junge. In particular, we have

$$\sqrt{n/q} \leq i_n(L_{\infty}, L_q)$$ and $$l_n(L_{\infty}, L_q) \leq n^{1/q}$$ if $2 \leq q < \infty$,

which implies the remark in 9.5 by letting $q := n/(4c^2)$.

12. Final remarks. By 6.5 and 8.7 we have the following

Criteria. An operator $T_0$ is locally $c$-representable in $T_1$ if and only if for $\varepsilon > 0$, for $x_0, \ldots, x_m \in X_0$ and for $y_0, \ldots, y_n \in Y_0^*$ we can find $x_1, \ldots, x_m \in X_1$ and $y_1, \ldots, y_n \in Y_1^*$ such that

$$\langle T_1x_{1h}, y_{1k}^* \rangle = \langle T_0x_{0h}, y_{0k}^* \rangle$$ for $h = 1, \ldots, m$ and $k = 1, \ldots, n,$

$$\left\| \sum_{h=1}^m \xi_hx_{1h} \right\| \leq \left\| \sum_{h=1}^m \xi_hx_{0h} \right\| \quad \text{and} \quad \left\| \sum_{k=1}^n \eta_ky_{1k}^* \right\| \leq (c + \varepsilon) \left\| \sum_{k=1}^n \eta_ky_{0k}^* \right\|$$

whenever $\xi_1, \ldots, \xi_m \in \mathbb{K}$ and $\eta_1, \ldots, \eta_n \in \mathbb{K}$.

An operator $T_0$ is injective-locally $c$-representable in $T_1$ if and only if for $\varepsilon > 0$ and for $x_0, \ldots, x_m \in X_0$ we can find $x_1, \ldots, x_m \in X_1$ such that

$$\left\| \sum_{h=1}^m \xi_hx_{1h} \right\| \leq \left\| \sum_{h=1}^m \xi_hx_{0h} \right\| \quad \text{and} \quad \left\| \sum_{h=1}^m \xi_hT_0x_{0h} \right\| \leq (c + \varepsilon) \left\| \sum_{h=1}^m \xi_hT_1x_{1h} \right\|$$

whenever $\xi_1, \ldots, \xi_m \in \mathbb{K}$. 


So the concept of local representability is based on elements and functionals, while injective-local representability is formulated in terms of elements only. This means that the latter concept works on a lower level. Clearly, there is a third concept: surjective-local representability. But this can be obtained just by duality, and nothing really new happens.

Since the local theory reduces infinite-dimensional statements to finite-dimensional ones, it requires a deep knowledge about \( n \)-dimensional normed linear spaces and about operators acting between them, where the dimension \( n \) is understood to be large. Consequently, the finite-dimensional theory is often viewed as a part, or even the main part, of local theory.

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*Received February 22, 1999*