STUDIA MATHEMATICA 135 (3) (1999)

What is "local theory of Banach spaces"?

by

ALBRECHT PIETSCH* (Jena)

Abstract. Banach space theory splits into several subtheories. On the one hand, there are an isometric and an isomorphic part; on the other hand, we speak of global and local aspects. While the concepts of *isometry* and *isomorphy* are clear, everybody seems to have its own interpretation of what "local theory" means. In this essay we analyze this situation and propose rigorous definitions, which are based on new concepts of local representability of operators.

Preamble. Of course, the quality of a theorem does not depend on the fact to which theory it belongs. Nevertheless, in order to systematize our knowledge we need criteria that enable us to collect *similar* results in theories.

1. Historical roots

1.1. Let us begin with some quotations.

Pełczyński and Rosenthal [p–r], p. 263: *Localization refers to obtaining quantitative finite-dimensional formulations of infinite-dimensional results.*

Tomczak-Jaegermann [TOM], p. 5: *A property* (*of Banach spaces or of operators acting between them*) *is called local if it can be defined by a quantitative statement or inequality concerning a finite number of vectors or finite-dimensional subspaces.*

Lindenstrauss and Milman [l–m], p. 1151: *The name 'local theory' is applied to two somewhat different topics*:

(a) *The quantitative study of n-dimensional normed spaces as $n \to \infty$.*
(b) *The relation of the structure of an infinite-dimensional space and its finite-dimensional subspaces.*

1991 *Mathematics Subject Classification*: 46B07, 46B08.
*The proof of the Basic Lemma in 5.9 is due to W. B. Johnson.
Research supported by DFG grant PI 322/1-2.

1.2. In 1956, Grothendieck defined the following concept:

L'espace normé E a un type linéaire inférieur à celui d'un espace normé F, si on peut trouver un $M > 0$ fixe tel que tout sous-espace de dimension finie E_1 de E soit isomorphe "à M près" à un sous-espace de dimension finie F_1 de F (Banach–Mazur distance $d(E_1, F_1) \leq 1 + M$); et que E a un type métrique inférieur à celui de F, si la condition précédente est satisfaite pour tout $M > 0$.

In his theory of super-reflexivity, James used to speak of *crudely finite representability* and of *finite representability*, respectively.

1.3. A decisive step was done when Dacunha-Castelle and Krivine introduced the technique of *ultraproducts* in Banach space theory. It soon turned out that X is finitely representable in Y if and only if X is isometric to a subspace of some ultrapower $Y^{\mathcal{U}}$.

Ultraproducts of operators were studied by Pietsch. Next, Beauzamy introduced two (possibly different) concepts of finite representability for operators. A more appropriate approach is due to Heinrich.

1.4. Following Brunel and Sucheston, a property $\mathcal{P}$ is called a *super-property* if it carries over from Y to all X finitely representable in Y. Equivalently, this means that $\mathcal{P}$ is preserved under the formation of ultrapowers and subspaces.

2. Notation

2.1. Let L stand for the class of all (real or complex) Banach spaces. Denote the set of all (bounded linear) operators from X into Y by $\mathfrak{L}(X, Y)$, and write $\mathfrak{L} := \bigcup \mathfrak{L}(X, Y)$, where the union ranges over $X, Y \in \mathsf{L}$.

2.2. An operator $J \in \mathfrak{L}(X, Y)$ is an *injection* if there exists a constant $c > 0$ such that $\|Jx\| \geq c\|x\|$ for all $x \in X$. In the case when $\|Jx\| = \|x\|$, we speak of a *metric injection*. For every (closed linear) subspace M of X, the canonical embedding from M into X is denoted by J_M^X.

2.3. An example of a metric injection is the map $K_X : X \to X^{**}$ that assigns to every $x \in X$ the functional $x^* \mapsto \langle x, x^* \rangle$. For $T \in \mathfrak{L}(X, Y)$, we let $T^{\mathrm{reg}} := K_Y T$, where the superscript $^{\mathrm{reg}}$ stands for *regular*.

2.4. Let $\mathbb{I}$ be an index set. We denote by $l_\infty(\mathbb{I})$ the Banach space of bounded scalar families (ξ_i) with the norm $\|(\xi_i) \,|\, l_\infty(\mathbb{I})\| := \sup_{i \in \mathbb{I}} |\xi_i|$. Note that $l_\infty(\mathbb{I})$ has the *metric extension property*. This means that every operator $T \in \mathfrak{L}(M, l_\infty(\mathbb{I}))$ defined on a subspace M of X admits a norm-preserving extension $T^{\mathrm{ext}} \in \mathfrak{L}(X, l_\infty(\mathbb{I}))$.

If $x \in X$, then $(\langle x, x^* \rangle)$ can be viewed as an element of $l_\infty(B_{X^*})$, where B_{X^*} is the closed unit ball of X^*. In this way, we get the canonical injection

J_X^{inj} from X into $X^{\text{inj}} := l_\infty(B_{X^*})$. For $T \in \mathfrak{L}(X,Y)$, we let $T^{\text{inj}} := J_Y^{\text{inj}}T$, where the superscript $^{\text{inj}}$ stands for *injective.*

2.5. An operator Q from X onto Y is called a *surjection*. In the case when the open unit ball of X is mapped onto the open unit ball of Y, we speak of a *metric surjection*. For every (closed linear) subspace N of Y, the quotient map from Y onto Y/N is denoted by Q_N^Y.

2.6. Given Banach spaces X_i with $i \in \mathbb{I}$, we denote by $[l_\infty(\mathbb{I}), X_i]$ the Banach space of all bounded families (x_i) such that $x_i \in X_i$. Fix an ultrafilter $\mathcal{U}$ on the index set $\mathbb{I}$. The collection of all equivalence classes

$$(x_i)^{\mathcal{U}} := \{(x_i^\circ) \in [l_\infty(\mathbb{I}), X_i] : \mathcal{U}\text{-}\lim_i \|x_i^\circ - x_i\| = 0\}$$

is a Banach space $(X_i)^{\mathcal{U}}$ under the norm $\|(x_i)^{\mathcal{U}}\| := \mathcal{U}\text{-}\lim_i \|x_i\|$.

Next, let $T_i \in \mathfrak{L}(X_i, Y_i)$ with $i \in \mathbb{I}$ be a bounded family of operators. Then $(T_i)^{\mathcal{U}} : (x_i)^{\mathcal{U}} \mapsto (T_i x_i)^{\mathcal{U}}$ defines an operator from $(X_i)^{\mathcal{U}}$ into $(Y_i)^{\mathcal{U}}$ such that $\|(T_i)^{\mathcal{U}}\| = \mathcal{U}\text{-}\lim_i \|T_i\|$.

These new objects are referred to as *ultraproducts.*

2.7. If $X_i = X$, $Y_i = Y$ and $T_i = T$, then we speak of *ultrapowers*, denoted by $X^{\mathcal{U}}$, $Y^{\mathcal{U}}$ and $T^{\mathcal{U}}$, respectively. In this case, there exists a canonical map $J_X^{\mathcal{U}}$ from X into $X^{\mathcal{U}}$, which sends x to (x_i) with $x_i = x$. Moreover, we define $Q_X^{\mathcal{U}} : (x_i)^{\mathcal{U}} \mapsto \mathcal{U}\text{-}\lim_i K_X x_i$. The right-hand limit is taken with respect to the weak* topology of X^{**}. These constructions yield the formula $T^{\text{reg}} = K_Y T = Q_Y^{\mathcal{U}} T^{\mathcal{U}} J_X^{\mathcal{U}}$ for $T \in \mathfrak{L}(X,Y)$. Unfortunately, the canonical embedding K_Y cannot be avoided.

2.8. Throughout this essay, we let

$$T \in \mathfrak{L}(X,Y), \quad T_0 \in \mathfrak{L}(X_0,Y_0), \quad T_1 \in \mathfrak{L}(X_1,Y_1), \quad \text{etc.}$$

3. Subtheories. Imitating Felix Klein, we now describe an *Erlanger Programm* for operators in Banach spaces.

3.1. Suppose we are given a *preordering* $\prec$ on $\mathfrak{L}$, the class of all operators. That is, $T_0 \prec T_1$ and $T_1 \prec T_2$ imply $T_0 \prec T_2$, and we have $T \prec T$. A property is said to be $\prec$*-stable* if it is inherited from T_1 to every $T_0 \prec T_1$. So every preordering $\prec$ leads to a subtheory that deals with the associated stable properties.

Writing $T_0 \sim T_1$ whenever $T_0 \prec T_1$ and $T_1 \prec T_0$ yields an equivalence relation. Of course, we could also study the concept of $\sim$*-stability.*

3.2. Note that a preordering on L, the class of all Banach spaces, is obtained by letting $X_0 \prec X_1$ if $I_{X_0} \prec I_{X_1}$, where I_{X_0} and I_{X_1} denote the identity maps of $X_0 \in \mathsf{L}$ and $X_1 \in \mathsf{L}$, respectively. In this way, every subtheory of operators contains a subtheory of spaces, as a special part.

4. Global representability

4.1. An operator $T_0 \in \mathfrak{L}(X_0, Y_0)$ is *globally representable* in an operator $T_1 \in \mathfrak{L}(X_1, Y_1)$ if there exist $A_1 \in \mathfrak{L}(X_0, X_1)$ and $B_1 \in \mathfrak{L}(Y_1, Y_0)$ such that $T_0 = B_1 T_1 A_1$. That is,

$$\begin{tikzcd} X_1 \arrow[r, "T_1"] & Y_1 \arrow[d, "B_1"] \\ X_0 \arrow[u, "A_1"] \arrow[r, "T_0"'] & Y_0 \end{tikzcd}$$

In this case, we write $T_0 \overset{\text{glo}}{\prec} T_1$. Of course, $\overset{\text{glo}}{\prec}$ is a preordering on $\mathfrak{L}$. The corresponding stable properties are said to be *global*. Obviously, $X_0 \overset{\text{glo}}{\prec} X_1$ means that X_0 is isomorphic to a complemented subspace of X_1.

4.2. A property is called *injective* if it carries over from $T^{\text{inj}} \in \mathfrak{L}(X, Y^{\text{inj}})$ to $T \in \mathfrak{L}(X, Y)$.

4.3. The preordering $T_0 \overset{\text{injglo}}{\prec} T_1$ is defined by $T_0^{\text{inj}} \overset{\text{glo}}{\prec} T_1^{\text{inj}}$. Note that $T_0 \overset{\text{glo}}{\prec} T_1$ implies $T_0 \overset{\text{injglo}}{\prec} T_1$. A property turns out to be $\overset{\text{injglo}}{\prec}$-stable if and only if it is simultaneously *injective* and *global*, which justifies the symbol $\overset{\text{injglo}}{\prec}$. Clearly, $X_0 \overset{\text{injglo}}{\prec} X_1$ means that X_0 is isomorphic to a subspace of X_1.

4.4. So far, we have discussed isomorphic notions. Taking into account isometric aspects, we could define a global preordering by assuming that the operators A_1 and B_1 in $T_0 = B_1 T_1 A_1$ satisfy the condition $\|B_1\| \cdot \|A_1\| \le 1$. We may also require that, for any choice of $\varepsilon > 0$, there exists a factorization such that $\|B_1\| \cdot \|A_1\| \le 1 + \varepsilon$. Another possibility would be to assume that A_1 is an injection and that B_1 is a surjection, metric or not.

5. Operators between finite-dimensional spaces

5.1. The symbol F stands for the collection of all finite-dimensional Banach spaces. Note that, upon identifying isometric copies, F is a set. Throughout, let $E \in \mathsf{F}$ and $F \in \mathsf{F}$. The underlying (real or complex) scalar field (sometimes viewed as a 1-dimensional Banach space) is denoted by $\mathbb{K}$.

5.2. With every operator $T \in \mathfrak{L}(X, Y)$ we associate the *germs*

$$\mathcal{L}(T \mid E, F) := \{BTA \in \mathfrak{L}(E, F) : \|A : E \to X\| \le 1,\ \|B : Y \to F\| \le 1\}.$$

This definition is illustrated by the following diagram:

$$\begin{tikzcd} X \arrow[r, "T"] & Y \arrow[d, "B"] \\ E \arrow[u, "A"] \arrow[r, "BTA"'] & F \end{tikzcd}$$

Roughly speaking, local properties of T can be formulated in terms of the family $\{\mathcal{L}(T \mid E, F) : E, F \in \mathsf{F}\}$. Working with the closed hulls $\overline{\mathcal{L}}(T \mid E, F)$ will turn out to be more elegant.

5.3. The sets $\mathcal{L}(T \mid E, F)$ are bounded and circled, but may look like a hedgehog. For example, $\mathcal{L}(\mathrm{Id} : \mathbb{K} \to \mathbb{K} \mid E, F)$ consists of all $S \in \mathfrak{L}(E, F)$ with $\mathrm{rank}(S) \le 1$ and $\|S\| \le 1$. In the case when T does not attain its norm, $\mathcal{L}(T \mid \mathbb{K}, \mathbb{K})$ is an open disc. On the other hand, $\mathcal{L}(T \mid E, F)$ may also be very nice as shown by $\mathcal{L}(\mathrm{Id} : l_2 \to l_2 \mid E, F)$, which is the closed unit ball of a norm.

5.4. Our first result is obvious.

PROPOSITION. $\mathcal{L}(T \mid E, F) = \mathcal{L}(T^{\mathrm{reg}} \mid E, F)$.

5.5. We now state one of the most important tools of local theory.

PRINCIPLE OF LOCAL REFLEXIVITY. *Let M and N be finite-dimensional subspaces of X^{**} and X^*, respectively. Then for every $\varepsilon > 0$ there exists an isomorphism I_ε from M onto some subspace M_ε of X such that $\|I_\varepsilon\| \le 1+\varepsilon$, $\|I_\varepsilon^{-1}\| \le 1 + \varepsilon$ and*

$$\langle x^{**}, x^* \rangle = \langle I_\varepsilon x^{**}, x^* \rangle \quad \textit{for all } x^{**} \in M \textit{ and } x^* \in N.$$

5.6. The correspondence $S \mapsto S^*$ defines an isometry between $\mathfrak{L}(E, F)$ and $\mathfrak{L}(F^*, E^*)$. The image of a set $\mathcal{L}$ will be denoted by $\mathcal{L}^*$.

PROPOSITION.

$$\mathcal{L}^*(T \mid E, F) \subseteq \mathcal{L}(T^* \mid F^*, E^*) \subseteq (1+\varepsilon)\mathcal{L}^*(T \mid E, F) \quad \textit{for all } \varepsilon > 0.$$

P r o o f. The left-hand inclusion is trivial.

By definition, every operator $S^* \in \mathcal{L}(T^* \mid F^*, E^*)$ can be decomposed in the form $S^* = UT^*V$, where $\|V : F^* \to Y^*\| \le 1$ and $\|U : X^* \to E^*\| \le 1$. Putting $B := K_F^{-1} V^* K_Y$, we get $V = B^*$. The principle of local reflexivity provides us with an isomorphism I_ε from $U^* K_E(E)$ onto some subspace M_ε of X such that $\|I_\varepsilon\| \le 1 + \varepsilon$ and

$$\langle U^* K_E e, T^* V f^* \rangle = \langle I_\varepsilon U^* K_E e, T^* V f^* \rangle \quad \text{for } e \in E \text{ and } f^* \in F^*.$$

Letting $A := I_\varepsilon U^* K_E$, we obtain $\|A : E \to X\| \le 1 + \varepsilon$ and

$$\langle BTAe, f^* \rangle = \langle I_\varepsilon U^* K_E e, T^* V f^* \rangle = \langle U^* K_E e, T^* V f^* \rangle = \langle e, S^* f^* \rangle$$

for $e \in E$ and $f^* \in F^*$. So $S = BTA \in (1+\varepsilon)\mathcal{L}(T \mid E, F)$.

5.7. We now formulate a corollary of the preceding result.

Proposition.

$$\mathcal{L}(T \mid E,F) \subseteq \mathcal{L}(T^{**} \mid E,F) \subseteq (1+\varepsilon)\mathcal{L}(T \mid E,F) \quad \text{for all } \varepsilon > 0.$$

5.8. The next statement even holds for compact operators. However, we will only need the finite-dimensional case.

Proposition. *If T has finite rank, then $\mathcal{L}(T^* \mid F^*, E^*)$ is closed.*

Proof. The closed unit balls

$$\{U : \|U : X^* \to E^*\| \le 1\} \quad \text{and} \quad \{V : \|V : F^* \to Y^*\| \le 1\}$$

are compact in the weak* topologies induced by the seminorms

$$p_{e,x^*}(U) := |\langle e, Ux^*\rangle| \quad \text{and} \quad p_{y,f^*}(V) := |\langle y, Vf^*\rangle|.$$

Moreover, the norm topology of $\mathfrak{L}(F^*, E^*)$ coincides with the weak* topology obtained from $p_{e,f^*}(S^*) := |\langle e, S^* f^*\rangle|$. Since T can be written in the form

$$T = \sum_{i=1}^{N} x_i^* \otimes y_i,$$

we get

$$\langle e, UT^*Vf^*\rangle = \sum_{i=1}^{N} \langle e, Ux_i^*\rangle\langle y_i, Vf^*\rangle.$$

So the bilinear map $(U,V) \mapsto UT^*V$ is continuous, which in turn implies the compactness of $\mathcal{L}(T^* \mid F^*, E^*)$.

5.9. I am very indebted to W. B. Johnson for a proof of the following result, which is the crucial device in our understanding of local theory.

Basic Lemma. $\overline{\mathcal{L}}(T \mid E,F) \subseteq (1+\varepsilon)\mathcal{L}(T \mid E,F)$ *for all* $\varepsilon > 0$.

Proof. If T has finite rank, then it follows from the previous proposition that $\mathcal{L}(T^{**} \mid E,F)$ is closed. Hence, by 5.7,

$$\overline{\mathcal{L}}(T \mid E,F) \subseteq \mathcal{L}(T^{**} \mid E,F) \subseteq (1+\varepsilon)\mathcal{L}(T \mid E,F).$$

To treat the case $\operatorname{rank}(T) = \infty$, we let $m := \dim(E)$ and $n := \dim(F)$. If $d := 2m+n$, then there exist $A_0 \in \mathfrak{L}(l_2^d, X)$ and $B_0 \in \mathfrak{L}(Y, l_2^d)$ such that B_0TA_0 is the identity map of l_2^d. Given $S \in \overline{\mathcal{L}}(T \mid E,F)$ and $\delta > 0$, we find a decomposition $S = S_1 + S_2$ such that $S_1 \in \mathcal{L}(T \mid E,F)$ and $\|S_2\| \le \delta$. Write $S_1 = B_1TA_1$ with $\|A_1\| \le 1$ and $\|B_1\| \le 1$. Let M be the range of B_0TA_1, and let N be the null space of B_1TA_0. Then $\operatorname{cod}(M^\perp) = \dim(M) \le m$ and $\operatorname{cod}(N) \le n$. Hence $\operatorname{cod}(M^\perp \cap N) \le m+n$, which is equivalent to $\dim(M^\perp \cap N) \ge m$. This implies that $M^\perp \cap N$ contains an m-dimensional subspace H. Let J denote the embedding from H into l_2^d, while Q stands for the orthogonal projection from l_2^d onto H. So QB_0TA_0J is the identity map of H. John's theorem provides us with an isomorphism $U \in \mathfrak{L}(E, H)$ such that $\|U\| \cdot \|U^{-1}\| \leq \sqrt{m}$.

$$\begin{tikzcd}
E \arrow[r, "S_1"] \arrow[d, "A_1"'] & F \\
X \arrow[r, "T"] & Y \arrow[u, "B_1"'] \arrow[d, "B_0"] \\
N \subset l_2^d \arrow[r, "I_d"] \arrow[u, "A_0"] & l_2^d \supset M \arrow[d, "Q"] \\
H \arrow[r, "I_H"] \arrow[u, "J"] & H \arrow[d, "U^{-1}"] \\
E \arrow[r, "I_E"] \arrow[u, "U"] & E
\end{tikzcd}$$

We obtain $S_2 = S_2 U^{-1} Q B_0 T A_0 J U = B_2 T A_2$, where $B_2 := S_2 U^{-1} Q B_0$ and $A_2 := A_0 J U$. The main purpose of the construction above was to get the formulas $B_1 T A_0 J = O$ and $Q B_0 T A_1 = O$, which in turn yield

$$B_1 T A_2 = B_1 T A_0 J U = O \quad \text{and} \quad B_2 T A_1 = S_2 U^{-1} Q B_0 T A_1 = O.$$

Hence

$$S = S_1 + S_2 = B_1 T A_1 + B_2 T A_2 = (B_1 + B_2) T (A_1 + A_2).$$

Moreover,

$$\|B_2\| \cdot \|A_2\| \leq \|S_2 U^{-1} Q B_0\| \cdot \|A_0 J U\| \leq \delta \sqrt{m}\, \|B_0\| \cdot \|A_0\|.$$

Clearly, U may be chosen such that $\|A_2\| = \|B_2\|$. Then

$$\|A_1 + A_2\| \cdot \|B_1 + B_2\| \leq \left(1 + \sqrt{\delta \sqrt{m}\, \|B_0\| \cdot \|A_0\|}\,\right)^2 \leq 1 + \varepsilon$$

whenever $\delta > 0$ is sufficiently small. This proves that $S \in (1+\varepsilon)\mathcal{L}(T \,|\, E, F)$.

5.10. The preceding result can be stated in the following form.

THEOREM. $\overline{\mathcal{L}}(T \,|\, E, F) = \bigcap_{\varepsilon > 0} (1 + \varepsilon) \mathcal{L}(T \,|\, E, F)$.

5.11. In the theory of ultraproducts, we have a striking counterpart of the principle of local reflexivity; see [kue], [ste] and [hei 1], p. 8.

KÜRSTEN–STERN LEMMA. *Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be finite-dimensional subspaces of $(X_i)^{\mathcal{U}}$ and $((X_i)^{\mathcal{U}})^*$, respectively. Then for every $\varepsilon > 0$ there exists an isomorphism $\boldsymbol{I}_\varepsilon$ from $\boldsymbol{N}$ onto some subspace $\boldsymbol{N}_\varepsilon$ of $(X_i^*)^{\mathcal{U}}$ such that $\|\boldsymbol{I}_\varepsilon\| \leq 1 + \varepsilon$, $\|\boldsymbol{I}_\varepsilon^{-1}\| \leq 1 + \varepsilon$ and*

$$\langle \boldsymbol{x}, \boldsymbol{x}^* \rangle = \langle \boldsymbol{x}, \boldsymbol{I}_\varepsilon \boldsymbol{x}^* \rangle \quad \textit{for all } \boldsymbol{x} \in \boldsymbol{M} \textit{ and } \boldsymbol{x}^* \in \boldsymbol{N}.$$

5.12. For completeness, we provide another standard result of the theory of ultraproducts.

LEMMA. *Assume that* $\boldsymbol{x}_1 = (x_{1i})^{\mathcal{U}}, \ldots, \boldsymbol{x}_m = (x_{mi})^{\mathcal{U}} \in (X_i)^{\mathcal{U}}$ *are linearly independent. Then, given* $\varepsilon > 0$, *there exists* $U \in \mathcal{U}$ *such that*

$$\Big\| \sum_{h=1}^{m} \xi_h x_{hi} \Big\| \le (1+\varepsilon) \Big\| \sum_{h=1}^{m} \xi_h \boldsymbol{x}_h \Big\| \quad \textit{for } \xi_1, \ldots, \xi_m \in \mathbb{K} \textit{ and } i \in U.$$

Proof. First of all, fix $\delta > 0$ and choose a finite δ-net $\mathcal{N}$ in the unit sphere $\mathbb{S}_1^m$ of l_1^m. Next, pick $U \in \mathcal{U}$ such that

$$\Big| \Big\| \sum_{h=1}^{m} \xi_h x_{hi} \Big\| - \Big\| \sum_{h=1}^{m} \xi_h \boldsymbol{x}_h \Big\| \Big| \le \delta$$

for $(\xi_h) \in \mathcal{N}$ and $i \in U$. Clearly, we may assume that $\mathcal{N}$ contains the unit vectors of l_1^m. Then it follows that $|\|x_{hi}\| - \|\boldsymbol{x}_h\|| \le \delta$ for $h = 1, \ldots, m$ and $i \in U$. Putting $c := \max\{\|\boldsymbol{x}_1\|, \ldots, \|\boldsymbol{x}_m\|\}$, we get

$$\Big| \Big\| \sum_{h=1}^{m} \xi_h x_{hi} \Big\| - \Big\| \sum_{h=1}^{m} \xi_h \boldsymbol{x}_h \Big\| \Big| \le (1 + 2c + \delta)\delta$$

for $(\xi_h) \in \mathbb{S}_1^m$ and $i \in U$. Hence, by homogeneity,

$$\Big\| \sum_{h=1}^{m} \xi_h x_{hi} \Big\| \le \Big\| \sum_{h=1}^{m} \xi_h \boldsymbol{x}_h \Big\| + (1 + 2c + \delta)\delta \sum_{h=1}^{m} |\xi_h|$$

for $\xi_1, \ldots, \xi_m \in \mathbb{K}$ and $i \in U$. Since $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ are linearly independent, we find a constant $b > 0$ such that

$$\sum_{h=1}^{m} |\xi_h| \le b \Big\| \sum_{h=1}^{m} \xi_h \boldsymbol{x}_h \Big\| \quad \text{for } \xi_1, \ldots, \xi_m \in \mathbb{K},$$

which implies

$$\Big\| \sum_{h=1}^{m} \xi_h x_{hi} \Big\| \le (1 + (1 + 2c + \delta) b\delta) \Big\| \sum_{h=1}^{m} \xi_h \boldsymbol{x}_h \Big\|$$

for $\xi_1, \ldots, \xi_m \in \mathbb{K}$ and $i \in U$. Choosing $\delta > 0$ sufficiently small completes the proof.

5.13. We are now in a position to establish an analogue of 5.7.

PROPOSITION.

$$\mathcal{L}(T \,|\, E, F) \subseteq \mathcal{L}(T^{\mathcal{U}} \,|\, E, F) \subseteq (1+\varepsilon)\mathcal{L}(T \,|\, E, F) \quad \textit{for all } \varepsilon > 0.$$

Proof. The left-hand inclusion follows from $T^{\text{reg}} = Q_Y^{\mathcal{U}} T^{\mathcal{U}} J_X^{\mathcal{U}}$; see 2.7.

Decompose the operator $S \in \mathcal{L}(T^{\mathcal{U}} \,|\, E, F)$ in the form $S = \boldsymbol{B} T^{\mathcal{U}} \boldsymbol{A}$ with $\|\boldsymbol{A} : E \to X^{\mathcal{U}}\| \le 1$ and $\|\boldsymbol{B} : Y^{\mathcal{U}} \to F\| \le 1$. Write

$$\boldsymbol{A} = \sum_{h=1}^{m} e_h^* \otimes \boldsymbol{x}_h \quad \text{and} \quad \boldsymbol{B} = \sum_{k=1}^{n} \boldsymbol{y}_k^* \otimes f_k,$$

where $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m \in X^{\mathcal{U}}$ and $\boldsymbol{y}_1^*, \ldots, \boldsymbol{y}_n^* \in (Y^{\mathcal{U}})^*$ are linearly independent. By 5.11, we find an isomorphism $\boldsymbol{I}_\varepsilon$ from $\boldsymbol{F} := \operatorname{span}\{\boldsymbol{y}_1^*, \ldots, \boldsymbol{y}_n^*\}$ onto some subspace $\boldsymbol{F}_\varepsilon$ of $(Y^*)^{\mathcal{U}}$ such that $\|\boldsymbol{I}_\varepsilon\| \le 1 + \varepsilon$ and

$$\langle T^{\mathcal{U}} \boldsymbol{x}_h, \boldsymbol{y}_k^* \rangle = \langle T^{\mathcal{U}} \boldsymbol{x}_h, \boldsymbol{I}_\varepsilon \boldsymbol{y}_k^* \rangle \qquad \text{for } h = 1, \ldots, m \text{ and } k = 1, \ldots, n.$$

Fix representations $\boldsymbol{x}_h = (x_{hi})^{\mathcal{U}}$ and $\boldsymbol{I}_\varepsilon \boldsymbol{y}_k^* = (y_{ki}^*)^{\mathcal{U}}$ with $x_{hi} \in X$ and $y_{ki}^* \in Y^*$. By the preceding lemma, there exists $U \in \mathcal{U}$ such that

$$\Big\| \sum_{h=1}^{m} \xi_h x_{hi} \Big\| \le (1+\varepsilon) \Big\| \sum_{h=1}^{m} \xi_h \boldsymbol{x}_h \Big\| \quad \text{and} \quad \Big\| \sum_{k=1}^{n} \eta_k y_{ki}^* \Big\| \le (1+\varepsilon) \Big\| \sum_{k=1}^{n} \eta_k \boldsymbol{y}_k^* \Big\|$$

for $\xi_1, \ldots, \xi_m \in \mathbb{K}$, $\eta_1, \ldots, \eta_n \in \mathbb{K}$ and $i \in U$. Letting

$$A_i := \sum_{h=1}^{m} e_h^* \otimes x_{hi} \quad \text{and} \quad B_i := \sum_{k=1}^{n} y_{ki}^* \otimes f_k$$

yields operators with $\|A_i : E \to X\| \le 1 + \varepsilon$ and $\|B_i : Y \to F\| \le 1 + \varepsilon$. Moreover, in view of

$$\langle T^{\mathcal{U}} \boldsymbol{x}_h, \boldsymbol{y}_k^* \rangle = \langle T^{\mathcal{U}} \boldsymbol{x}_h, \boldsymbol{I}_\varepsilon \boldsymbol{y}_k^* \rangle = \mathcal{U}\text{-}\lim_i \langle T x_{hi}, y_{ki}^* \rangle,$$

it may be achieved that

$$\|\boldsymbol{B} T^{\mathcal{U}} \boldsymbol{A} - B_i T A_i\| = \Big\| \sum_{h=1}^{m} \sum_{k=1}^{n} (\langle T^{\mathcal{U}} \boldsymbol{x}_h, \boldsymbol{y}_k^* \rangle - \langle T x_{hi}, y_{ki}^* \rangle) e_h^* \otimes f_k \Big\|$$

becomes as small as we please. So

$$S = \boldsymbol{B} T^{\mathcal{U}} \boldsymbol{A} \in (1+\varepsilon)^2 \overline{\mathcal{L}}(T \,|\, E, F) \subseteq (1+\varepsilon)^3 \mathcal{L}(T \,|\, E, F).$$

5.14. Finally, we summarize the most important results of this section.

THEOREM. *The closed germs $\overline{\mathcal{L}}(\cdot \,|\, E, F)$ coincide for the operators T, T^{reg}, T^{**} and $T^{\mathcal{U}}$.*

That is, from the local point of view we cannot distinguish between T, T^{reg}, T^{**} and $T^{\mathcal{U}}$.

6. Local representability

6.1. Let $c \ge 0$. We say that an operator $T_0 \in \mathfrak{L}(X_0, Y_0)$ is *locally c-representable* in an operator $T_1 \in \mathfrak{L}(X_1, Y_1)$ if for $\varepsilon > 0$, for any choice of finite-dimensional spaces E and F, for $A_0 \in \mathfrak{L}(E, X_0)$ and $B_0 \in \mathfrak{L}(Y_0, F)$ there exist $A_1 \in \mathfrak{L}(E, X_1)$ and $B_1 \in \mathfrak{L}(Y_1, F)$ such that

$$B_1 T_1 A_1 = B_0 T_0 A_0 \quad \text{and} \quad \|B_1\| \cdot \|A_1\| \le (c+\varepsilon)\|B_0\| \cdot \|A_0\|.$$

In other words, we assume that

$$\mathcal{L}(T_0 \,|\, E, F) \subseteq (c+\varepsilon)\mathcal{L}(T_1 \,|\, E, F) \qquad \text{whenever } \varepsilon > 0$$

or, by 5.10, that

$$\overline{\mathcal{L}}(T_0 \,|\, E, F) \subseteq c\overline{\mathcal{L}}(T_1 \,|\, E, F).$$

Roughly speaking, it is required that the finite-dimensional structure of T_1 is at least as rich as that of T_0.

6.2. An operator $T_0 \in \mathfrak{L}(X_0, Y_0)$ is said to be *locally representable* in an operator $T_1 \in \mathfrak{L}(X_1, Y_1)$ if the above conditions hold for some $c \ge 0$. In this case, we write $T_0 \overset{\text{loc}}{\prec} T_1$. Of course, $\overset{\text{loc}}{\prec}$ is a preordering on $\mathfrak{L}$.

6.3. We now restate Theorem 5.14.

Theorem. *The operators T, T^{reg}, T^{**} and $T^{\mathcal{U}}$ are locally equivalent.*

6.4. We see from 5.6 that the concept of local representability is stable under duality.

Proposition. *Let $c \ge 0$. Then the following are equivalent*:

(1) *T_0 is locally c-representable in T_1.*
(2) *T_0^* is locally c-representable in T_1^*.*

6.5. The local nature of the above notion can also be seen from the next criterion.

Proposition. *Let $c \ge 0$. Then the following are equivalent*:

(1) *T_0 is locally c-representable in T_1.*
(2) *For $\varepsilon > 0$, for $x_{01}, \dots, x_{0m} \in X_0$ and for $y_{01}^*, \dots, y_{0n}^* \in Y_0^*$ we can find $x_{11}, \dots, x_{1m} \in X_1$ and $y_{11}^*, \dots, y_{1n}^* \in Y_1^*$ such that*

$$\langle T_1 x_{1h}, y_{1k}^* \rangle = \langle T_0 x_{0h}, y_{0k}^* \rangle \quad \textit{for } h = 1, \dots, m \textit{ and } k = 1, \dots, n,$$

$$\Big\| \sum_{h=1}^m \xi_h x_{1h} \Big\| \le \Big\| \sum_{h=1}^m \xi_h x_{0h} \Big\| \quad \textit{and} \quad \Big\| \sum_{k=1}^n \eta_k y_{1k}^* \Big\| \le (c+\varepsilon) \Big\| \sum_{k=1}^n \eta_k y_{0k}^* \Big\|$$

whenever $\xi_1, \dots, \xi_m \in \mathbb{K}$ and $\eta_1, \dots, \eta_n \in \mathbb{K}$.

Proof. We consider the operators $A_0 := J_M^{X_0}$ and $B_0 := Q_N^{Y_0}$, where $M := \operatorname{span}\{x_{01}, \dots, x_{0m}\}$ and $N := \{y \in Y_0 : \langle y, y_{01}^* \rangle = \ldots = \langle y, y_{0n}^* \rangle = 0\}$.

Note that $(Y_0/N)^*$ can be identified with $\operatorname{span}\{y_{01}^*, \dots, y_{0n}^*\}$. Choose A_1 and B_1 such that

$$B_1 T_1 A_1 = B_0 T_0 A_0, \quad \|A_1\| \le \|A_0\| \quad \text{and} \quad \|B_1\| \le (c+\varepsilon)\|B_0\|.$$

Then we may put $x_{1h} := A_1 x_{0h}$ and $y_{1k}^* := B_1^* x_{0k}^*$. This proves that (1)⇒(2).

In order to verify the reverse implication, we represent $A_0 \in \mathfrak{L}(E, X_0)$ and $B_0 \in \mathfrak{L}(Y_0, F)$ in the form

$$A_0 = \sum_{h=1}^{m} e_h^* \otimes x_{0h} \quad \text{and} \quad B_0 = \sum_{k=1}^{n} y_{0k}^* \otimes f_k.$$

Pick $x_{11}, \dots, x_{1m}$ and $y_{11}^*, \dots, y_{1n}^*$ as described in (2). Then the operators

$$A_1 := \sum_{h=1}^{m} e_h^* \otimes x_{1h} \quad \text{and} \quad B_1 := \sum_{k=1}^{n} y_{1k}^* \otimes f_k$$

satisfy the conditions

$$B_1 T_1 A_1 = B_0 T_0 A_0, \quad \|A_1 e\| \le \|A_0 e\| \quad \text{and} \quad \|B_1^* f^*\| \le (c+\varepsilon)\|B_0^* f^*\|.$$

6.6. Next, we connect the concepts of local and global representability with the help of ultrapowers. This result should be compared with Theorem 1.2 in [hei 1].

THEOREM. *Let $c \ge 0$. Then the following are equivalent*:

(1) *T_0 is locally c-representable in T_1.*

(2) *There exist operators $\boldsymbol{A} \in \mathfrak{L}(X_0, X_1^{\mathcal{U}})$ and $\boldsymbol{B} \in \mathfrak{L}(Y_1^{\mathcal{U}}, Y_0^{**})$, where $\mathcal{U}$ is an ultrafilter on a suitable index set $\mathbb{I}$, such that*

$$T_0^{\text{reg}} = \boldsymbol{B} T_1^{\mathcal{U}} \boldsymbol{A} \quad \textit{and} \quad \|\boldsymbol{B}\| \cdot \|\boldsymbol{A}\| \le c.$$

Proof. In view of 5.13, we conclude from $T_0^{\text{reg}} = \boldsymbol{B} T_1^{\mathcal{U}} \boldsymbol{A}$ that

$$\mathcal{L}(T_0 \,|\, E, F) = \mathcal{L}(T_0^{\text{reg}} \,|\, E, F) \subseteq c\mathcal{L}(T_1^{\mathcal{U}} \,|\, E, F) \subseteq (1+\varepsilon)c\mathcal{L}(T_1 \,|\, E, F).$$

This proves that (2)⇒(1).

We consider the index set $\mathbb{I}$ formed by all triples $i = (M, N, \varepsilon)$. Here M is any finite-dimensional subspace of X_0, and N is any finite-codimensional subspace of Y_0. As usual, $\varepsilon > 0$. For $i = (M, N, \varepsilon)$ and $i_0 = (M_0, N_0, \varepsilon_0)$, we write $i \ge i_0$ if $M \supseteq M_0$, $N \subseteq N_0$ and $0 < \varepsilon \le \varepsilon_0$. Furthermore, fix some ultrafilter $\mathcal{U}$ on $\mathbb{I}$ that contains all sections $\{i \in \mathbb{I} : i \ge i_0\}$ with $i_0 \in \mathbb{I}$. Choose $A_i \in \mathfrak{L}(M, X_1)$ and $B_i \in \mathfrak{L}(Y_1, Y_0/N)$ such that $\|A_i\| \le 1$, $\|B_i\| \le c + \varepsilon$ and $B_i T_1 A_i = Q_N^{Y_0} T_0 J_M^{X_0}$. Let

$$x_i := \begin{cases} A_i x & \text{if } x \in M, \\ o & \text{if } x \notin M. \end{cases}$$

Clearly, $\boldsymbol{A} : x \mapsto (x_i)^{\mathcal{U}}$ defines an operator $\boldsymbol{A} \in \mathfrak{L}(X_0, X_1^{\mathcal{U}})$ with $\|\boldsymbol{A}\| \le 1$. Next, given $(y_i)^{\mathcal{U}} \in Y_1^{\mathcal{U}}$, we pick $(y_i^\circ)^{\mathcal{U}} \in Y_0^{\mathcal{U}}$ such that $Q_N^{Y_0} y_i^\circ = B_i y_i$ and $\|y_i^\circ\| \le (1+\varepsilon)\|B_i y_i\|$. Put $\boldsymbol{B} : (y_i)^{\mathcal{U}} \mapsto \mathcal{U}\text{-}\lim_i K_{Y_0} y_i^\circ$, where the limit is taken with respect to the weak* topology of Y_0^{**}. Then $\|\boldsymbol{B}\| \le c$ and $T_0^{\mathrm{reg}} = \boldsymbol{B} T_1^{\mathcal{U}} \boldsymbol{A}$.

6.7. A property is said to be

- *local* if it is $\overset{\mathrm{loc}}{\prec}$-stable,
- *global* if it is $\overset{\mathrm{glo}}{\prec}$-stable,
- *regular* if it carries over from $T^{\mathrm{reg}} \in \mathfrak{L}(X, Y^{**})$ to $T \in \mathfrak{L}(X, Y)$,
- *injective* if it carries over from $T^{\mathrm{inj}} \in \mathfrak{L}(X, Y^{\mathrm{inj}})$ to $T \in \mathfrak{L}(X, Y)$,
- *ultrapower-stable* if, for all ultrafilters $\mathcal{U}$, it is inherited from $T \in \mathfrak{L}(X, Y)$ to $T^{\mathcal{U}} \in \mathfrak{L}(X^{\mathcal{U}}, Y^{\mathcal{U}})$.

6.8. Thanks to 6.6, we are able to exhibit the typical ingredients of local representability.

THEOREM. *A property is local if and only if it is regular*, *ultrapower-stable and global.*

7. Local operator schemes

7.1. We now define a preordering on the set of all operators acting between finite-dimensional spaces. If $S_0 \in \mathfrak{L}(E_0, F_0)$ and $S_1 \in \mathfrak{L}(E_1, F_1)$, then $S_0 \le S_1$ means that $S_0 \in \mathcal{L}(S_1 \mid E_0, F_0)$. In other words, there exist A_1 and B_1 such that

$$S_0 = B_1 S_1 A_1, \quad \|A_1 : E_0 \to E_1\| \le 1 \quad \text{and} \quad \|B_1 : F_1 \to F_0\| \le 1.$$

7.2. A *local operator scheme* is a rule that assigns to every pair (E, F) of finite-dimensional Banach spaces a compact subset $\mathcal{G}(E, F)$ of $\mathfrak{L}(E, F)$ such that the following conditions are satisfied:

(1) If $S_0 \in \mathfrak{L}(E_0, F_0)$ and $S_1 \in \mathcal{G}(E_1, F_1)$, then $S_0 \le S_1$ implies that $S_0 \in \mathcal{G}(E_0, F_0)$.

(2) For $S_1 \in \mathcal{G}(E_1, F_1)$ and $S_2 \in \mathcal{G}(E_2, F_2)$ there exists $S \in \mathcal{G}(E, F)$ such that $S_1 \le S$ and $S_2 \le S$.

(3) The sets $\mathcal{G}(E, F)$ are uniformly bounded. That is, $\|S\| \le c$ for all $S \in \mathcal{G}(E, F)$, where the constant $c > 0$ does not depend on E and F.

7.3. The closed germs $\mathcal{G}(T \mid E, F) := \overline{\mathcal{L}}(T \mid E, F)$ of any operator T constitute a local operator scheme. Indeed, in order to verify (2) we let

$S_i \in \overline{\mathcal{L}}(T \,|\, E_i, F_i)$ for $i = 1, 2$. Given $\varepsilon > 0$, there exist factorizations $S_i = B_{i,\varepsilon} T A_{i,\varepsilon}$ such that $\|A_{i,\varepsilon} : E_i \to X\| \le 1 + \varepsilon$ and $\|B_{i,\varepsilon} : Y \to F_i\| \le 1$. Define E to be the direct sum $E_1 \oplus E_2$ equipped with the l_1-norm, while F is the direct sum $F_1 \oplus F_2$ under the l_∞-norm. Put

$$A_\varepsilon := A_{1,\varepsilon} Q_1^E + A_{2,\varepsilon} Q_2^E, \quad B_\varepsilon := J_1^F B_{1,\varepsilon} + J_2^F B_{2,\varepsilon} \quad \text{and} \quad S_\varepsilon := B_\varepsilon T A_\varepsilon,$$

where J_i^E and Q_i^E, J_i^F and Q_i^F are the canonical injections and surjections, respectively. Then $\|A_\varepsilon : E \to X\| \le 1 + \varepsilon$ and $\|B_\varepsilon : Y \to F\| \le 1$. In addition,

$$S_i = B_{i,\varepsilon} T A_{i,\varepsilon} = Q_i^F B_\varepsilon T A_\varepsilon J_i^E = Q_i^F S_\varepsilon J_i^E.$$

By compactness, the operators $S_\varepsilon \in (1+\varepsilon)\mathcal{L}(T \,|\, E, F)$ have a cluster point $S \in \overline{\mathcal{L}}(T \,|\, E, F)$ as $\varepsilon \to 0$. Since $S_i = Q_i^F S_\varepsilon J_i^E$ passes into $S_i = Q_i^F S J_i^E$, we finally obtain $S_i \le S$ for $i = 1, 2$. Conditions (1) and (3) are obviously fulfilled.

7.4. From the philosophical point of view, the following observation is the most important result of this paper. *Operators at hand can be reconstructed from their germs*, *and finite-dimensional pieces can be glued together to form new operators.* Our considerations are based on an *isometric* concept of local equivalence: $T_0 \overset{\text{loc}}{\sim}_1 T_1$ if and only if $\overline{\mathcal{L}}(T_0 \,|\, E, F) = \overline{\mathcal{L}}(T_1 \,|\, E, F)$ for all finite-dimensional test spaces E and F.

Theorem. *There is a one-to-one correspondence between* $\overset{\text{loc}}{\sim}_1$*-equivalence classes of operators and local operator schemes.*

Proof. We only need to show that every local operator scheme can be obtained from an operator $\boldsymbol{T}$, which will be produced by ultraproduct techniques. The underlying index set $\mathbb{I}$ consists of all triples $i = (S : E \to F)$ with $S \in \mathcal{G}(E, F)$. Condition (2) implies that $\mathbb{I}$ is upwards directed. So we can find an ultrafilter $\mathcal{U}$ that contains all sections $U(i_0) := \{i \in \mathbb{I} : i \ge i_0\}$ with $i_0 \in \mathbb{I}$. Construct the ultraproducts $\boldsymbol{X} := (X_i)^{\mathcal{U}}$, $\boldsymbol{Y} := (Y_i)^{\mathcal{U}}$ and $\boldsymbol{T} := (T_i)^{\mathcal{U}}$, where $X_i := E$, $Y_i := F$ and $T_i := S$.

Fix $i_0 = (S_0 : E_0 \to F_0) \in \mathbb{I}$. If $i \in U(i_0)$, then we find operators A_i and B_i such that $S_0 = B_i T_i A_i$, $\|A_i : E_0 \to X_i\| \le 1$ and $\|B_i : Y_i \to F_0\| \le 1$. Put $A_i := O$ and $B_i := O$ whenever $i \notin U(i_0)$. Define

$$\boldsymbol{A} : e \mapsto (A_i e)^{\mathcal{U}} \quad \text{and} \quad \boldsymbol{B} : (y_i)^{\mathcal{U}} \mapsto \mathcal{U}\text{-}\lim_i B_i y_i.$$

Then $\|\boldsymbol{A} : E_0 \to \boldsymbol{X}\| \le 1$ and $\|\boldsymbol{B} : \boldsymbol{Y} \to F_0\| \le 1$. Finally,

$$\boldsymbol{B}\boldsymbol{T}\boldsymbol{A} e = \mathcal{U}\text{-}\lim_i B_i T_i A_i e = S_0 e \quad \text{for all } e \in E_0.$$

This means that $\mathcal{G}(E_0, F_0) \subseteq \mathcal{L}(\boldsymbol{T} \,|\, E_0, F_0)$.

In order to prove the reverse inclusion, let $\|\boldsymbol{A} : E_0 \to \boldsymbol{X}\| \le 1$ and $\|\boldsymbol{B} : \boldsymbol{Y} \to F_0\| \le 1$. Write

$$\boldsymbol{A} = \sum_{h=1}^{m} e_h^* \otimes \boldsymbol{x}_h \quad \text{and} \quad \boldsymbol{B} = \sum_{k=1}^{n} \boldsymbol{y}_k^* \otimes f_k,$$

where $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m \in \boldsymbol{X}$ and $\boldsymbol{y}_1^*, \ldots, \boldsymbol{y}_n^* \in \boldsymbol{Y}^*$ are linearly independent. By 5.11, we find an isomorphism $\boldsymbol{I}_\varepsilon$ from $\boldsymbol{F} := \operatorname{span}\{\boldsymbol{y}_1^*, \ldots, \boldsymbol{y}_n^*\}$ onto some subspace $\boldsymbol{F}_\varepsilon$ of $(Y_i^*)^{\mathcal{U}}$ such that $\|\boldsymbol{I}_\varepsilon\| \le 1 + \varepsilon$ and

$$\langle \boldsymbol{T}\boldsymbol{x}_h, \boldsymbol{y}_k^* \rangle = \langle \boldsymbol{T}\boldsymbol{x}_h, \boldsymbol{I}_\varepsilon \boldsymbol{y}_k^* \rangle \quad \text{for } h = 1, \ldots, m \text{ and } k = 1, \ldots, n.$$

Fix representations $\boldsymbol{x}_h = (x_{hi})^{\mathcal{U}}$ and $\boldsymbol{I}_\varepsilon \boldsymbol{y}_k^* = (y_{ki}^*)^{\mathcal{U}}$ with $x_{hi} \in X_i$ and $y_{ki}^* \in Y_i^*$. In view of Lemma 5.12, there exists $U \in \mathcal{U}$ such that

$$\Big\| \sum_{h=1}^{m} \xi_h x_{hi} \Big\| \le (1+\varepsilon) \Big\| \sum_{h=1}^{m} \xi_h \boldsymbol{x}_h \Big\| \quad \text{and} \quad \Big\| \sum_{k=1}^{n} \eta_k y_{ki}^* \Big\| \le (1+\varepsilon) \Big\| \sum_{k=1}^{n} \eta_k \boldsymbol{y}_k^* \Big\|$$

for $\xi_1, \ldots, \xi_m \in \mathbb{K}$, $\eta_1, \ldots, \eta_n \in \mathbb{K}$ and $i \in U$. Letting

$$A_i := \sum_{h=1}^{m} e_h^* \otimes x_{hi} \quad \text{and} \quad B_i := \sum_{k=1}^{n} y_{ki}^* \otimes f_k$$

yields operators with $\|A_i : E_0 \to X_i\| \le 1 + \varepsilon$ and $\|B_i : Y_i \to F_0\| \le 1 + \varepsilon$. Moreover, in view of $\langle \boldsymbol{T}\boldsymbol{x}_h, \boldsymbol{y}_k^* \rangle = \langle \boldsymbol{T}\boldsymbol{x}_h, \boldsymbol{I}_\varepsilon \boldsymbol{y}_k^* \rangle = \mathcal{U}\text{-}\lim_i \langle T_i\, x_{hi}, y_{ki}^* \rangle$, it may be achieved that

$$\|\boldsymbol{B}\boldsymbol{T}\boldsymbol{A} - B_i T_i A_i\| = \Big\| \sum_{h=1}^{m} \sum_{k=1}^{n} (\langle \boldsymbol{T}\boldsymbol{x}_h, \boldsymbol{y}_k^* \rangle - \langle T_i\, x_{hi}, y_{ki}^* \rangle) e_h^* \otimes f_k \Big\|$$

becomes as small as we please. By (1), we have $B_i T_i A_i \in (1+\varepsilon)^2 \mathcal{G}(E_0, F_0)$. So $\boldsymbol{B}\boldsymbol{T}\boldsymbol{A} \in \mathcal{G}(E_0, F_0)$ as $\varepsilon \to 0$, which gives $\mathcal{L}(\boldsymbol{T} \,|\, E_0, F_0) \subseteq \mathcal{G}(E_0, F_0)$.

7.5. Assigning to (E, F) the closed unit ball of $\mathfrak{L}(E, F)$, we get a local operator scheme. Any generating operator $\boldsymbol{T}_0$ is maximal with respect to the preordering $\overset{\text{loc}}{\prec}$; that is, $T \overset{\text{loc}}{\prec} \boldsymbol{T}_0$ for all operators T. There is a reflexive and separable Banach space whose identity map is universal in this sense. Indeed, choosing a dense sequence $E_{m1}, E_{m2}, \ldots$ in the Minkowski compactum of all m-dimensional Banach spaces with $m = 1, 2, \ldots$, we may take the l_2-sum of the double sequences (E_{mn}) so obtained.

8. Injective-local representability

8.1. Let $c \ge 0$. We say that an operator $T_0 \in \mathfrak{L}(X_0, Y_0)$ is *injective-locally c-representable* in an operator $T_1 \in \mathfrak{L}(X_1, Y_1)$ if for $\varepsilon > 0$, for any choice of a finite-dimensional space E and $N = 1, 2, \ldots$, for $A_0 \in \mathfrak{L}(E, X_0)$

and $B_0 \in \mathfrak{L}(Y_0, l_\infty^N)$ there exist $A_1 \in \mathfrak{L}(E, X_1)$ and $B_1 \in \mathfrak{L}(Y_1, l_\infty^N)$ such that

$$B_1 T_1 A_1 = B_0 T_0 A_0 \quad \text{and} \quad \|B_1\| \cdot \|A_1\| \le (c+\varepsilon)\|B_0\| \cdot \|A_0\|.$$

In other words, we assume that

$$\mathcal{L}(T_0 \,|\, E, l_\infty^N) \subseteq (c+\varepsilon)\mathcal{L}(T_1 \,|\, E, l_\infty^N) \quad \text{whenever } \varepsilon > 0$$

or, by 5.10, that

$$\overline{\mathcal{L}}(T_0 \,|\, E, l_\infty^N) \subseteq c\overline{\mathcal{L}}(T_1 \,|\, E, l_\infty^N).$$

This means that, compared with Definition 6.1, we only use a special type of test spaces F. The main point is that l_∞^N has the metric extension property.

8.2. An operator $T_0 \in \mathfrak{L}(X_0, Y_0)$ is said to be *injective-locally representable* in an operator $T_1 \in \mathfrak{L}(X_1, Y_1)$ if the above conditions hold for some $c \ge 0$. In this case, we write $T_0 \overset{\text{injloc}}{\prec} T_1$. Of course, $\overset{\text{injloc}}{\prec}$ is a preordering on $\mathfrak{L}$. Note that $T_0 \overset{\text{loc}}{\prec} T_1$ implies $T_0 \overset{\text{injloc}}{\prec} T_1$.

8.3. In the following, we need a folklore result.

LEMMA. *Let* $\varepsilon > 0$, $A \in \mathfrak{L}(E, X)$ *and* $B \in \mathfrak{L}(X, l_\infty(\mathbb{I}))$. *Then there exist* $B_0 \in \mathfrak{L}(X, l_\infty^N)$ *and* $V \in \mathfrak{L}(l_\infty^N, l_\infty(\mathbb{I}))$ *such that* $\|B_0\| \le \|B\|$, $\|V\| \le 1+\varepsilon$ *and*

$$\begin{tikzcd}
E \arrow[r, "A"] & X \arrow[rr, "B"] \arrow[dr, "B_0"'] & & l_\infty(\mathbb{I}) \\
& & l_\infty^N \arrow[ur, "V"'] &
\end{tikzcd}$$

In addition, we may arrange that $N \le (3+2/\varepsilon)^n$, *with* $n := \dim(E)$ *replaced by* $2n$ *in the complex case.*

P r o o f. Choose functionals $x_1^*, \ldots, x_N^* \in X^*$ of norm 1 whose restrictions to $M := A(E)$ constitute a minimal finite δ-net in the unit sphere of M^*, where $\delta := \varepsilon/(1+\varepsilon)$. Since $(1-\delta)\|x\| \le \max_i |\langle x, x_i^*\rangle|$ for all $x \in M$, the map $B_0 : x \mapsto \|B\|(\langle x, x_i^*\rangle)$ is invertible on $B_0(M)$. Finally, V can be obtained as a norm-preserving extension of BB_0^{-1}. The estimate of N follows from entropy theory.

8.4. The symbol $\overset{\text{injloc}}{\prec}$ is justified by the next criterion.

PROPOSITION. *Let* $c \ge 0$. *Then the following are equivalent*:

(1) T_0 *is injective-locally c-representable in* T_1.

(2) T_0^{inj} *is locally c-representable in* T_1^{inj}.

P r o o f. (1)⇒(2). Look at the commutative diagram

$$
\begin{tikzcd}
X_0 \arrow[r, "T_0"] & Y_0 \arrow[r, "J^{\mathrm{inj}}_{Y_0}"] \arrow[d, "B_0"'] & Y_0^{\mathrm{inj}} \arrow[d, "C_0"] \\
E \arrow[u, "A_0"] \arrow[d, "A_1"'] & l_\infty^N \arrow[ur, "V_0"'] & G \\
X_1 \arrow[r, "T_1"'] & Y_1 \arrow[r, "J^{\mathrm{inj}}_{Y_1}"'] \arrow[u, "B_1"] & Y_1^{\mathrm{inj}} \arrow[u, "C_1"'] \arrow[ul, "V_1"']
\end{tikzcd}
$$

which is obtained as follows: Let A_0 and C_0 be given. Applying Lemma 8.3 to T_0A_0 and $J^{\mathrm{inj}}_{Y_0}$, we find B_0 and V_0 such that $\|B_0\| \le 1$ and $\|V_0\| \le 1+\varepsilon$. Choose A_1 and B_1 with $\|B_1\|\cdot\|A_1\| \le (c+\varepsilon)\|B_0\|\cdot\|A_0\|$. Finally, let V_1 be any norm-preserving extension of B_1 and put $C_1 := C_0V_0V_1$. Then

$$
\begin{aligned}
\|C_1\|\cdot\|A_1\| &\le \|C_0\|\cdot\|V_0\|\cdot\|V_1\|\cdot\|A_1\| \le (1+\varepsilon)\|C_0\|\cdot\|B_1\|\cdot\|A_1\| \\
&\le (1+\varepsilon)(c+\varepsilon)\|C_0\|\cdot\|A_0\|.
\end{aligned}
$$

(2)⇒(1). Now we use the commutative diagram

$$
\begin{tikzcd}
X_0 \arrow[r, "T_0"] & Y_0 \arrow[r, "J^{\mathrm{inj}}_{Y_0}"] \arrow[d, "B_0"'] & Y_0^{\mathrm{inj}} \arrow[dl, "C_0"] \\
E \arrow[u, "A_0"] \arrow[d, "A_1"'] & l_\infty^N & \\
X_1 \arrow[r, "T_1"'] & Y_1 \arrow[r, "J^{\mathrm{inj}}_{Y_1}"'] \arrow[u, "B_1"] & Y_1^{\mathrm{inj}} \arrow[ul, "C_1"']
\end{tikzcd}
$$

Let A_0 and B_0 be given. Fix any norm-preserving extension C_0 of B_0. Choose A_1 and C_1. Finally, let $B_1 := C_1J^{\mathrm{inj}}_{Y_1}$.

8.5. Next, we state a counterpart of 6.8.

Theorem. *A property is injective-local if and only if it is injective, ultrapower-stable and global.*

8.6. In the case of injective-local representability the list of locally equivalent operators, given in 6.3, can be extended by T^{inj}.

Theorem. *The operators T, T^{reg}, T^{inj}, T^{**} and $T^{\mathcal{U}}$ are injective-locally equivalent.*

P r o o f. Note that $\mathcal{L}(T^{\mathrm{inj}} \mid E, l_\infty^N) = \mathcal{L}(T \mid E, l_\infty^N)$.

8.7. We now establish an analogue of 6.5.

PROPOSITION. *Let $c \geq 0$. Then the following are equivalent*:

(1) T_0 *is injective-locally c-representable in* T_1.

(2) *For* $\varepsilon > 0$ *and for* $x_{01}, \ldots, x_{0m} \in X_0$ *we can find* $x_{11}, \ldots, x_{1m} \in X_1$ *such that*

$$\Big\| \sum_{h=1}^{m} \xi_h x_{1h} \Big\| \leq \Big\| \sum_{h=1}^{m} \xi_h x_{0h} \Big\| \quad \textit{and} \quad \Big\| \sum_{h=1}^{m} \xi_h T_0 x_{0h} \Big\| \leq (c+\varepsilon) \Big\| \sum_{h=1}^{m} \xi_h T_1 x_{1h} \Big\|$$

whenever $\xi_1, \ldots, \xi_m \in \mathbb{K}$.

(3) *For* $\varepsilon > 0$, *for any finite-dimensional space* E, *and for* $A_0 \in \mathfrak{L}(E, X_0)$ *there exists* $A_1 \in \mathfrak{L}(E, X_1)$ *such that*

$$\|A_1 e\| \leq \|A_0 e\| \quad \textit{and} \quad \|T_0 A_0 e\| \leq (c+\varepsilon)\|T_1 A_1 e\| \quad \textit{whenever } e \in E.$$

Proof. (1)$\Rightarrow$(2). Let $A_0 := J_M^{X_0}$ with $M := \operatorname{span}\{x_{01}, \ldots, x_{0m}\}$. Applying 8.3 to $T_0 A_0$ and $J_{Y_0}^{\mathrm{inj}}$, we find B_0 and V with $J_{Y_0}^{\mathrm{inj}} T_0 A_0 = V B_0 T_0 A_0$, $\|B_0\| \leq 1$ and $\|V\| \leq 1 + \varepsilon$. Choose $A_1 \in \mathfrak{L}(M, X_1)$ and $B_1 \in \mathfrak{L}(Y_1, l_\infty^N)$ such that

$$B_1 T_1 A_1 = B_0 T_0 A_0, \quad \|A_1\| \leq \|A_0\| = 1 \quad \text{and} \quad \|B_1\| \leq (c+\varepsilon)\|B_0\| \leq c+\varepsilon.$$

Putting $x_{1h} := A_1 x_{0h}$, we finally arrive at

$$\Big\| \sum_{h=1}^{m} \xi_h x_{1h} \Big\| \leq \Big\| \sum_{h=1}^{m} \xi_h x_{0h} \Big\|$$

and

$$\begin{aligned} \Big\| \sum_{h=1}^{m} \xi_h T_0 x_{0h} \Big\| &= \Big\| \sum_{h=1}^{m} \xi_h J_{Y_0}^{\mathrm{inj}} T_0 A_0 x_{0h} \Big\| \leq \|V\| \cdot \Big\| \sum_{h=1}^{m} \xi_h B_0 T_0 A_0 x_{0h} \Big\| \\ &\leq (1+\varepsilon) \Big\| \sum_{h=1}^{m} \xi_h B_1 T_1 A_1 x_{0h} \Big\| \\ &\leq (1+\varepsilon)(c+\varepsilon) \Big\| \sum_{h=1}^{m} \xi_h T_1 x_{1h} \Big\|. \end{aligned}$$

(2)$\Rightarrow$(3). Write $A_0 \in \mathfrak{L}(E, X_0)$ in the form $A_0 = \sum_{h=1}^{m} e_h^* \otimes x_{0h}$, and let $A_1 := \sum_{h=1}^{m} e_h^* \otimes x_{1h}$.

(3)$\Rightarrow$(1). Given $A_0 \in \mathfrak{L}(E, X_0)$ and $B_0 \in \mathfrak{L}(Y_0, l_\infty^N)$, we choose an operator $A_1 \in \mathfrak{L}(E, X_1)$ such that $\|A_1 e\| \leq \|A_0 e\|$ and $\|T_0 A_0 e\| \leq (c + \varepsilon)\|T_1 A_1 e\|$ whenever $e \in E$. Of course, $\|A_1\| \leq \|A_0\|$. It follows from

$$\|B_0 T_0 A_0 e\| \leq \|B_0\| \cdot \|T_0 A_0 e\| \leq (c+\varepsilon)\|B_0\| \cdot \|T_1 A_1 e\|$$

that $T_1 A_1 e \mapsto B_0 T_0 A_0 e$ yields a well-defined operator from $T_1 A_1(E)$ into l_∞^N. If B_1 is any norm-preserving extension, then $\|B_1\| \leq (c+\varepsilon)\|B_0\|$.

Remark. In the case when T_0 and T_1 are identity maps, the inequalities from (2) pass into

$$\Big\|\sum_{h=1}^{m}\xi_h x_{1h}\Big\| \le \Big\|\sum_{h=1}^{m}\xi_h x_{0h}\Big\| \le (c+\varepsilon)\Big\|\sum_{h=1}^{m}\xi_h x_{1h}\Big\|.$$

This yields an upper estimate of the Banach–Mazur distance between $\operatorname{span}\{x_{01},\dots,x_{0m}\}$ and $\operatorname{span}\{x_{11},\dots,x_{1m}\}$. More precisely, it turns out that the relation $\overset{\text{injloc}}{\prec}$ extends the concept of crudely finite representability to the setting of operators.

8.8. The next criterion is an analogue of 6.6, from which it could be derived via 8.4. We prefer, however, to give a direct proof.

Proposition. *Let $c \ge 0$. Then the following are equivalent*:

(1) *T_0 is injective-locally c-representable in T_1.*

(2) *There exist operators $\boldsymbol{A} \in \mathfrak{L}(X_0, X_1^{\mathcal{U}})$ and $\boldsymbol{B} \in \mathfrak{L}(Y_1^{\mathcal{U}}, Y_0^{\text{inj}})$, where $\mathcal{U}$ is an ultrafilter on a suitable index set $\mathbb{I}$, such that*

$$T_0^{\text{inj}} = \boldsymbol{B}T_1^{\mathcal{U}}\boldsymbol{A} \quad \textit{and} \quad \|\boldsymbol{B}\|\cdot\|\boldsymbol{A}\| \le c.$$

Proof. If $T_0^{\text{inj}} = \boldsymbol{B}T_1^{\mathcal{U}}\boldsymbol{A}$, then

$$\begin{aligned}\mathcal{L}(T_0 \mid E, l_\infty^N) = \mathcal{L}(T_0^{\text{inj}} \mid E, l_\infty^N) &\subseteq \|\boldsymbol{B}\|\cdot\|\boldsymbol{A}\|\mathcal{L}(T_1^{\mathcal{U}} \mid E, l_\infty^N)\\ &\subseteq (1+\varepsilon)c\mathcal{L}(T_1 \mid E, l_\infty^N).\end{aligned}$$

This proves that (2)⇒(1).

We consider the index set $\mathbb{I}$ formed by all pairs $i = (M, \varepsilon)$. Here M is any finite-dimensional subspace of X_0 and $\varepsilon > 0$. For $i = (M,\varepsilon)$ and $i_0 = (M_0, \varepsilon_0)$, we write $i \ge i_0$ if $M \supseteq M_0$ and $0 < \varepsilon \le \varepsilon_0$. Furthermore, fix some ultrafilter $\mathcal{U}$ on $\mathbb{I}$ that contains all sections $\{i \in \mathbb{I} : i \ge i_0\}$ with $i_0 \in \mathbb{I}$. Choose $A_i \in \mathfrak{L}(M, X_1)$ such that

$$\|A_i x\| \le \|x\| \quad \text{and} \quad \|T_0 x\| \le (c+\varepsilon)\|T_1 A_i x\| \quad \text{for } x \in M.$$

Let

$$x_i := \begin{cases} A_i x & \text{if } x \in M,\\ o & \text{if } x \notin M,\end{cases} \qquad \text{and} \qquad y_i := \begin{cases} T_1 A_i x & \text{if } x \in M,\\ o & \text{if } x \notin M.\end{cases}$$

Clearly, $\boldsymbol{A} : x \mapsto (x_i)^{\mathcal{U}}$ defines an operator $\boldsymbol{A} \in \mathfrak{L}(X_0, X_1^{\mathcal{U}})$ with $\|\boldsymbol{A}\| \le 1$. Next, it follows from

$$\|T_0^{\text{inj}} x\| = \|T_0 x\| \le \mathcal{U}\text{-}\lim_i (c+\varepsilon)\|T_1 A_i x\| = c\|(y_i)^{\mathcal{U}}\|$$

that $\boldsymbol{B}_0 : (y_i)^{\mathcal{U}} \mapsto T_0^{\text{inj}} x$ yields a map from the range of $T_1^{\mathcal{U}}\boldsymbol{A}$ into Y_0^{inj}. Choosing any norm-preserving extension, we obtain the required operator $\boldsymbol{B} \in \mathfrak{L}(Y_1^{\mathcal{U}}, Y_0^{\text{inj}})$ with $\|\boldsymbol{B}\| \le c$. Hence (1)⇒(2).

9. Local and injective-local distances. In the definition of local representability the dimensions of the test spaces E and F are arbitrary. Now we introduce a gradation.

9.1. For $n = 1, 2, \ldots$, the nth *local distance* between T_0 and T_1 is defined by

$$\boldsymbol{l}_n(T_0, T_1) := \inf\left\{ c \geq 0 : \mathcal{L}(T_0 \,|\, E, F) \subseteq c\mathcal{L}(T_1 \,|\, E, F), \begin{array}{c} \dim(E) \leq n \\ \dim(F) \leq n \end{array} \right\}.$$

If such a constant $c \geq 0$ does not exist, we let $\boldsymbol{l}_n(T_0, T_1) := \infty$. This happens if and only if $\mathrm{rank}(T_0) \geq n > \mathrm{rank}(T_1)$. In view of 5.10, we have

$$\boldsymbol{l}_n(T_0, T_1) := \min\left\{ c \geq 0 : \overline{\mathcal{L}}(T_0 \,|\, E, F) \subseteq c\overline{\mathcal{L}}(T_1 \,|\, E, F), \begin{array}{c} \dim(E) \leq n \\ \dim(F) \leq n \end{array} \right\}.$$

9.2. Note that

$$\|T_0\| \cdot \|T_1\|^{-1} = \boldsymbol{l}_1(T_0, T_1) \leq \boldsymbol{l}_2(T_0, T_1) \leq \ldots \leq \boldsymbol{l}_n(T_0, T_1) \leq \ldots$$

The growth of this sequence measures the deviation of T_1 from T_0.

For operators T_0, T_1 and T_2 between arbitrary couples of Banach spaces, we have a *multiplicative triangle inequality*:

$$\boldsymbol{l}_n(T_0, T_2) \leq \boldsymbol{l}_n(T_0, T_1)\boldsymbol{l}_n(T_1, T_2).$$

Moreover, 5.6 yields

$$\boldsymbol{l}_n(T_0^*, T_1^*) = \boldsymbol{l}_n(T_0, T_1).$$

9.3. For $n = 1, 2, \ldots$, the nth *injective-local distance* between T_0 and T_1 is defined by

$$\boldsymbol{i}_n(T_0, T_1) := \inf\left\{ c \geq 0 : \mathcal{L}(T_0 \,|\, E, l_\infty^N) \subseteq c\mathcal{L}(T_1 \,|\, E, l_\infty^N), \begin{array}{c} \dim(E) \leq n \\ N = 1, 2, \ldots \end{array} \right\}.$$

If such a constant $c \geq 0$ does not exist, we let $\boldsymbol{i}_n(T_0, T_1) := \infty$. This happens if and only if $\mathrm{rank}(T_0) \geq n > \mathrm{rank}(T_1)$. In view of 5.10, we have

$$\boldsymbol{i}_n(T_0, T_1) := \min\left\{ c \geq 0 : \overline{\mathcal{L}}(T_0 \,|\, E, l_\infty^N) \subseteq c\overline{\mathcal{L}}(T_1 \,|\, E, l_\infty^N), \begin{array}{c} \dim(E) \leq n \\ N = 1, 2, \ldots \end{array} \right\}.$$

9.4. In analogy with 9.2,

$$\|T_0\| \cdot \|T_1\|^{-1} = \boldsymbol{i}_1(T_0, T_1) \leq \boldsymbol{i}_2(T_0, T_1) \leq \ldots \leq \boldsymbol{i}_n(T_0, T_1) \leq \ldots$$

and

$$\boldsymbol{i}_n(T_0, T_2) \leq \boldsymbol{i}_n(T_0, T_1)\boldsymbol{i}_n(T_1, T_2).$$

However, $\boldsymbol{i}_n(T_0^*, T_1^*)$ and $\boldsymbol{i}_n(T_0, T_1)$ may behave quite differently.

9.5. We now compare the local distances with the injective-local distances.

PROPOSITION. *For every $\varepsilon > 0$ there exists a natural number $a > 1$ such that*

$$\boldsymbol{i}_n(T_0, T_1) \le (1+\varepsilon)\boldsymbol{l}_{a^n}(T_0, T_1) \quad \textit{for } n = 1, 2, \ldots$$

Proof. Put $a := 4 + [2/\varepsilon]$ (real case) and $a := (4 + [2/\varepsilon])^2$ (complex case). Note that

$$c_n := \inf \left\{ c \ge 0 : \mathcal{L}(T_0 \,|\, E, l_\infty^{N_0}) \subseteq c\mathcal{L}(T_1 \,|\, E, l_\infty^{N_0}), \begin{array}{l} \dim(E) \le n \\ N_0 \le a^n \end{array} \right\}$$
$$\le \boldsymbol{l}_{a^n}(T_0, T_1).$$

We now construct a commutative diagram:

$$\begin{tikzcd}
X_0 \arrow[r, "T_0"] & Y_0 \arrow[d, "B_0"'] \arrow[dr, "C_0"] & \\
E \arrow[u, "A_0"] \arrow[d, "A_1"'] & l_\infty^N & l_\infty^{N_0} \arrow[l, "V"'] \\
X_1 \arrow[r, "T_1"'] & Y_1 \arrow[u, "B_1"] \arrow[ur, "C_1"'] &
\end{tikzcd}$$

Let $\delta > 0$, A_0 and B_0 be given. Applying Lemma 8.3 to T_0A_0 and B_0, we find C_0 and V such that $\|C_0\| \le \|B_0\|$ and $\|V\| \le 1+\varepsilon$. Choose A_1 and C_1 with $\|C_1\| \cdot \|A_1\| \le (c_n + \delta)\|C_0\| \cdot \|A_0\|$. Finally, putting $B_1 := VC_1$, we arrive at $\|B_1\| \cdot \|A_1\| \le (1+\varepsilon)(c_n+\delta)\|B_0\| \cdot \|A_0\|$, which proves that

$$\boldsymbol{i}_n(T_0, T_1) \le (1+\varepsilon)(c_n + \delta) \le (1+\varepsilon)(\boldsymbol{l}_{a^n}(T_0, T_1) + \delta).$$

REMARK. An inequality of the form $\boldsymbol{i}_n(T_0, T_1) \le c\boldsymbol{l}_{\varphi(n)}(T_0, T_1)$ can only hold if $\varphi : \mathbb{N} \to \mathbb{N}$ grows exponentially: $\varphi(n) \succ a^n$ with $a > 1$; see 11.6.

9.6. Here is another consequence of Theorem 5.14.

THEOREM. *The $\boldsymbol{l}_n$-distances between T, T^{reg}, T^{**} and $T^{\mathcal{U}}$ equal* 1. *In the case of $\boldsymbol{i}_n$-distances the same is true for T, T^{reg}, T^{inj}, T^{**} and $T^{\mathcal{U}}$.*

9.7. Distances between spaces X_0 and X_1 are obtained as the distances between the associated identity maps I_{X_0} and I_{X_1}. In the infinite-dimensional case, we have $\boldsymbol{l}_n(X_0, X_1) \le n$ and $\boldsymbol{i}_n(X_0, X_1) \le \sqrt{n}$. It turns out that $\boldsymbol{i}_n(X_0, X_1)$ is the infimum of all constants $c \ge 1$ with the following property: For every n-dimensional subspace M_0 of X_0 there exists an n-dimensional subspace M_1 of X_1 such that the Banach–Mazur distance $d(M_0, M_1)$ is less than or equal to c.

9.8. For $n = 1, 2, \ldots$ and $T \in \mathfrak{L}(X, Y)$, the nth *Bernstein number* $b_n(T)$ is the supremum of all constants $b \ge 0$ such that $\|Tx\| \ge b\|x\|$ for every x in some n-dimensional subspace of X. In the case when $\dim(X) < n$, we let $b_n(T) := 0$. Obviously,

$$\|T\| = b_1(T) \ge b_2(T) \ge \ldots \ge b_n(T) \ge \ldots \ge 0.$$

9.9. Roughly speaking, the following estimate shows that the decay of the sequence $(b_n(T))$ depends *continuously* on T.

Proposition. $b_n(T_0) \le \boldsymbol{i}_n(T_0, T_1) b_n(T_1)$.

Proof. If $b_n(T_0) > b > 0$, then there exists an n-dimensional subspace E_0 of X_0 such that

$$\|T_0 x\| \ge b\|x\| \quad \text{for all } x \in E_0.$$

Let $A_0 \in \mathfrak{L}(E_0, X_0)$ denote the embedding map from E_0 into X_0. Choose $A_1 \in \mathfrak{L}(E_0, X_1)$ such that

$$\|A_1 x\| \le \|A_0 x\| \quad \text{and} \quad \|T_0 A_0 x\| \le (\boldsymbol{i}_n(T_0, T_1) + \varepsilon)\|T_1 A_1 x\|$$

for all $x \in E_0$. Note that $A_1 x = o$ implies $x = o$. So $E_1 := A_1(E_0)$ is an n-dimensional subspace of X_1. Hence it follows from

$$(\boldsymbol{i}_n(T_0, T_1) + \varepsilon)\|T_1 A_1 x\| \ge \|T_0 A_0 x\| \ge b\|A_0 x\| \ge b\|A_1 x\|$$

that

$$\frac{b}{\boldsymbol{i}_n(T_0, T_1) + \varepsilon} \le b_n(T_1).$$

Finally, letting $b \to b_n(T_0)$ and $\varepsilon \to 0$ completes the proof.

10. Operators in Hilbert spaces

10.1. Throughout this section, we let $T \in \mathfrak{L}(H, K)$, $T_0 \in \mathfrak{L}(H_0, K_0)$ and $T_1 \in \mathfrak{L}(H_1, K_1)$, where H, H_0, H_1 and K, K_0, K_1 denote Hilbert spaces. Since in this setting all s-numbers coincide, we may work with the Bernstein numbers whose local nature is evident.

10.2. Theorem. *All non-compact operators between Hilbert spaces are locally equivalent.*

Proof. If $\|A : E \to H\| \le 1$ and $\|B : K \to F\| \le 1$, then there exist factorizations $A : E \xrightarrow{A_0} l_2 \xrightarrow{U} H$ and $B : K \xrightarrow{V} l_2 \xrightarrow{B_0} F$ such that $\|A_0\| = \|A\|$ and $\|U\| = 1$, $\|B_0\| = \|B\|$ and $\|V\| = 1$. Putting $T_0 := VTU$, we get the diagram

$$\begin{tikzcd}
H \arrow[rrr, "T"] \arrow[d, "A"'] & & & K \arrow[d, "B"] \arrow[dl, "V"'] \\
E \arrow[r, "A_0"'] & l_2 \arrow[ul, "U"] \arrow[r, "T_0"'] & l_2 \arrow[r, "B_0"'] & F
\end{tikzcd}$$

That is, $BTA = B_0T_0A_0 = (B_0)I(VTUA_0)$, where I denotes the identity map of l_2. This proves that every operator T is locally $\|T\|$-representable in I.

Conversely, I is even globally representable in all non-compact operators, since these operators fail to be strictly singular.

Remark. In separable Hilbert spaces, all non-compact operators are globally equivalent.

10.3. As just observed, in the setting of Hilbert spaces there is almost no difference between local and global properties. This assertion will be confirmed by the next result.

Theorem. *Let $c \ge 0$. Then, for compact operators T_0 and T_1, the following are equivalent*:

(1) *T_0 is globally representable in T_1.*
(2) *T_0 is locally representable in T_1.*
(3) *T_0 is injective-locally representable in T_1.*
(4) *There exists $c \ge 0$ such that $b_n(T_0) \le cb_n(T_1)$ for $n = 1, 2, \ldots$*

Proof. The implications (1)⇒(2) and (2)⇒(3) are trivial, and (3)⇒(4) follows from 9.9.

In order to verify (4)⇒(1), we assume that the underlying spaces are infinite-dimensional. Consider Schmidt representations

$$T_0 = \sum_{n=1}^{\infty} b_n(T_0) x_{0n}^* \otimes y_{0n} \quad \text{and} \quad T_1 = \sum_{n=1}^{\infty} b_n(T_1) x_{1n}^* \otimes y_{1n},$$

where (x_{0n}), (y_{0n}), (x_{1n}) and (y_{1n}) are orthonormal sequences. Moreover, define $x_{0n}^* : x \mapsto (x|x_{0n})$ and $x_{1n}^* : x \mapsto (x|x_{1n})$. Since $b_n(T_0) \le cb_n(T_1)$, we find a sequence (β_n) such that $b_n(T_0) = \beta_n b_n(T_1)$ and $0 \le \beta_n \le c$. Finally, letting

$$A_1 := \sum_{n=1}^{\infty} x_{0n}^* \otimes x_{1n} \quad \text{and} \quad B_1 := \sum_{n=1}^{\infty} \beta_n y_{1n}^* \otimes y_{0n}$$

yields the factorization $T_0 = B_1T_1A_1$.

10.4. The following formulas can be proved by standard techniques:

$$\overline{\mathcal{L}}(T \mid l_2^n, l_2^n) = \{S \in \mathfrak{L}(l_2^n, l_2^n) : b_1(S) \le b_1(T), \ldots, b_n(S) \le b_n(T)\}$$

and

$$\boldsymbol{l}_n(T_0, T_1) = \boldsymbol{i}_n(T_0, T_1) = \max\{b_1(T_0)b_1(T_1)^{-1}, \ldots, b_n(T_0)b_n(T_1)^{-1}\}.$$

11. Examples

11.1. Obviously, belonging to an operator ideal is a *global* property.

Every maximal quasi-Banach ideal $\mathfrak{A}$ can be obtained as the collection of all operators $T \in \mathfrak{L}$ for which $\sup\{\boldsymbol{\alpha}(S) : S \in \mathcal{L}(T \mid E, F),\ E, F \in \mathsf{F}\}$ is finite. So $\mathfrak{A}$ is uniquely determined if we know the underlying quasi-ideal norm $\boldsymbol{\alpha}$ only for operators acting between finite-dimensional spaces. This observation implies that belonging to a maximal quasi-Banach operator ideal is a *local* property.

11.2. *Global, but non-local classes*:

- completely continuous operators,
- weakly compact operators,
- operators with the Radon–Nikodým property,
- strictly singular operators,
- operators with separable range.

11.3. *Global and ultrapower-stable, but non-regular classes*:

- nuclear operators.

11.4. *Local, but non-injective classes*:

- approximable operators,
- operators with p-summable approximation numbers,
- p-integral operators, $p \neq 2$,
- L_p-factorable operators, $p \neq 2$.

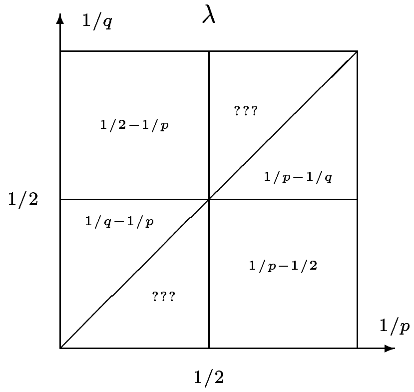
11.5. *Injective-local classes*:

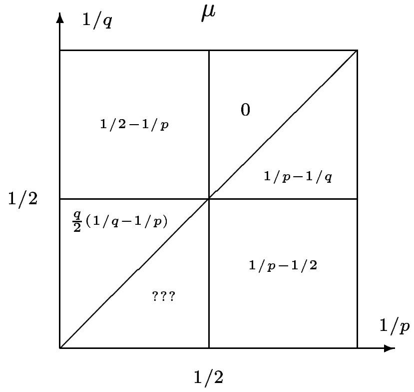
- compact operators,
- super weakly compact operators,
- 2-integral operators,
- L_2-factorable operators,
- p-summing operators,
- operators of Rademacher type p,
- operators of Rademacher cotype q,
- UMD-operators.

11.6. For positive sequences (α_n) and (β_n), the asymptotic relation $\alpha_n \asymp \beta_n$ means that $\alpha_n = O(\beta_n)$ and $\beta_n = O(\alpha_n)$.

We now describe the asymptotic behaviour of the distances $\boldsymbol{l}_n(L_p, L_q)$ and $\boldsymbol{i}_n(L_p, L_q)$ as $n \to \infty$. Since infinite-dimensional L_p spaces are locally equivalent, it is enough to think of $L_p[0,1]$ or l_p. In all cases that are settled until now, we have $\boldsymbol{l}_n(L_p, L_q) \asymp n^{\lambda}$ and $\boldsymbol{i}_n(L_p, L_q) \asymp n^{\mu}$.

The following diagrams show the values of the exponents λ and μ if $1 < p, q < \infty$. On the border-lines $p = 1$, $q = 1$, $p = \infty$ and $q = \infty$, discontinuities may occur. We see that the sequences $(\boldsymbol{l}_n(L_p, L_q))$ and $(\boldsymbol{i}_n(L_p, L_q))$ behave differently for $2 < q \le p < \infty$.





The λ-diagram is due to A. Hinrichs (unpublished), while the μ-diagram was established by A. Hinrichs and T. Kaufhold and by Y. Gordon and M. Junge. In particular, we have

$$\sqrt{n/q} \le \boldsymbol{i}_n(L_\infty, L_q) \quad \text{and} \quad \boldsymbol{l}_n(L_\infty, L_q) \le n^{1/q} \quad \text{if } 2 \le q < \infty,$$

which implies the remark in 9.5 by letting $q := n/(4c^2)$.

12. Final remarks. By 6.5 and 8.7 we have the following

CRITERIA. *An operator T_0 is locally c-representable in T_1 if and only if for $\varepsilon > 0$, for $x_{01}, \ldots, x_{0m} \in X_0$ and for $y_{01}^*, \ldots, y_{0n}^* \in Y_0^*$ we can find $x_{11}, \ldots, x_{1m} \in X_1$ and $y_{11}^*, \ldots, y_{1n}^* \in Y_1^*$ such that*

$$\langle T_1 x_{1h}, y_{1k}^* \rangle = \langle T_0 x_{0h}, y_{0k}^* \rangle \quad \text{for } h = 1, \ldots, m \text{ and } k = 1, \ldots, n,$$

$$\Big\| \sum_{h=1}^m \xi_h x_{1h} \Big\| \le \Big\| \sum_{h=1}^m \xi_h x_{0h} \Big\| \quad \text{and} \quad \Big\| \sum_{k=1}^n \eta_k y_{1k}^* \Big\| \le (c + \varepsilon) \Big\| \sum_{k=1}^n \eta_k y_{0k}^* \Big\|$$

whenever $\xi_1, \ldots, \xi_m \in \mathbb{K}$ and $\eta_1, \ldots, \eta_n \in \mathbb{K}$.

An operator T_0 is injective-locally c-representable in T_1 if and only if for $\varepsilon > 0$ and for $x_{01}, \ldots, x_{0m} \in X_0$ we can find $x_{11}, \ldots, x_{1m} \in X_1$ such that

$$\Big\| \sum_{h=1}^m \xi_h x_{1h} \Big\| \le \Big\| \sum_{h=1}^m \xi_h x_{0h} \Big\| \quad \text{and} \quad \Big\| \sum_{h=1}^m \xi_h T_0 x_{0h} \Big\| \le (c + \varepsilon) \Big\| \sum_{h=1}^m \xi_h T_1 x_{1h} \Big\|$$

whenever $\xi_1, \ldots, \xi_m \in \mathbb{K}$.

So the concept of *local respresentability* is based on *elements* and *functionals*, while *injective-local representability* is formulated in terms of *elements* only. This means that the latter concept works on a lower level. Clearly, there is a third concept: *surjective-local representability*. But this can be obtained just by duality, and nothing really new happens.

Since the local theory reduces infinite-dimensional statements to finite-dimensional ones, it requires a deep knowledge about n-dimensional normed linear spaces and about operators acting between them, where the dimension n is understood to be large. Consequently, the finite-dimensional theory is often viewed as a part, or even the main part, of local theory.

References

[bea 1] B. Beauzamy, *Opérateurs uniformément convexifiants*, Studia Math. 57 (1976), 103–139.

[bea 2] —, *Quelques propriétés des opérateurs uniformément convexifiants*, ibid. 60 (1977), 211–222.

[b–s] A. Brunel and L. Sucheston, *On J-convexity and some ergodic super-properties of Banach spaces*, Trans. Amer. Math. Soc. 204 (1975), 79–90.

[d–k] D. Dacunha-Castelle et J. L. Krivine, *Applications des ultraproduits à l'étude des espaces et des algèbres de Banach*, Studia Math. 41 (1972), 315–344.

[g–j] Y. Gordon and M. Junge, *Volume ratios in L_p-spaces*, Studia Math., to appear.

[gro] A. Grothendieck, *Sur certaines classes de suites dans les espaces de Banach, et le théorème de Dvoretzky–Rogers*, Bol. Soc. Mat. São Paulo 8 (1956), 81–110.

[hei 1] S. Heinrich, *Finite representability and super-ideals of operators*, Dissertationes Math. 172 (1980).

[hei 2] —, *Ultraproducts in Banach space theory*, J. Reine Angew. Math. 313 (1980), 72–104.

[h–k] A. Hinrichs and T. Kaufhold, *Distances of finite dimensional subspaces of L_p-spaces*, to appear.

[jam] R. C. James, *Super-reflexive Banach spaces*, Canad. J. Math. 24 (1972), 896–904.

[kue] K.-D. Kürsten, *On some problems of A. Pietsch, II*, Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. 29 (1978), 61–73 (in Russian).

[l–m] J. Lindenstrauss and V. D. Milman, *The local theory of normed spaces and its applications to convexity*, in: Handbook of Convex Geometry, P. M. Gruber and J. M. Wills (eds.), Elsevier, 1993, 1150–1220.

[p–r] A. Pełczyński and H. P. Rosenthal, *Localization techniques in L^p spaces*, Studia Math. 52 (1975), 263–289.

[pie] A. Pietsch, *Ultraprodukte von Operatoren in Banachräumen*, Math. Nachr. 61 (1974), 123–132.

[PIE] —, *Operator Ideals*, Deutscher Verlag Wiss., Berlin, 1978; North-Holland, Amsterdam, 1980.

[P–W] A. Pietsch and J. Wenzel, *Orthonormal Systems and Banach Space Geometry*, Cambridge Univ. Press, 1998.

[ste] J. Stern, *Propriétés locales et ultrapuissances d'espaces de Banach*, Sém. Maurey–Schwartz 1974/75, Exposés 7 et 8, École Polytechnique, Paris.
[TOM] N. Tomczak-Jaegermann, *Banach–Mazur Distances and Finite-Dimensional Operator Ideals*, Longman, Harlow, 1989.

Mathematisches Institut
FSU Jena
D-07740 Jena, Germany
E-mail: pietsch@minet.uni-jena.de

Received February 22, 1999 (4268)