

**Distributional fractional powers of the Laplacean.
Riesz potentials**

by

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Abstract. For different reasons it is very useful to have at one's disposal a duality formula for the fractional powers of the Laplacean, namely, $((-\Delta)^\alpha u, \phi) = (u, (-\Delta)^\alpha \phi)$, $\alpha \in \mathbb{C}$, for ϕ belonging to a suitable function space and u to its topological dual. Unfortunately, this formula makes no sense in the classical spaces of distributions. For this reason we introduce a new space of distributions where the above formula can be established. Finally, we apply this distributional point of view on the fractional powers of the Laplacean to obtain some properties of the Riesz potentials in a wide class of spaces which contains the L^p -spaces.

1. Introduction. Throughout this paper we consider complex functions defined on \mathbb{R}^n . We denote by \mathcal{D} the space of functions of class C^∞ with compact support and by \mathcal{S} the Schwartz space, both endowed with their usual topologies. Given a topological vector space Y , its topological dual will be denoted by Y' . If $T : D(T) \subset Y \rightarrow Y$ is a linear operator and $X \subset Y$ is a linear subspace of Y , we denote by T_X the operator in X with domain $D(T_X) = \{x \in X \cap D(T) : Tx \in X\}$ and defined by $T_X x = Tx$ for $x \in D(T_X)$. If $X = L^p$ we write T_p instead of T_{L^p} .

It is known that the restriction of the negative distributional Laplacean, $-\Delta$, to L^p -spaces is a non-negative operator. Hence, we can calculate its fractional powers in these spaces. However, just as the duality identity

$$(\Delta u, \phi) = (u, \Delta \phi) \quad \text{for } \phi \in \mathcal{D} \text{ and } u \in \mathcal{D}'$$

gives a meaning to Δf for a non-classically differentiable function, it would be desirable that the fractional power of exponent α , with $\operatorname{Re} \alpha > 0$, of this

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operator satisfied an analogous relation, namely

$$((-\Delta)^\alpha u, \phi) = (u, (-\Delta)^\alpha \phi)$$

for ϕ belonging to a suitable function space \mathcal{T} and u to its topological dual \mathcal{T}' . The distributional space \mathcal{T}' must include the L^p -spaces, $1 \leq p \leq \infty$, and the fractional power $(-\Delta)^\alpha$ must be understood in the sense of the classical theory of fractional powers developed by A. V. Balakrishnan and H. Komatsu in Banach spaces and by C. Martínez, M. Sanz and V. Calvo in locally convex spaces. We solve this problem in Section 3. It is not possible to take $\mathcal{T} = \mathcal{D}$ or $\mathcal{T} = \mathcal{S}$. For this reason we introduce an appropriate function space.

For a complete theory of fractional powers and its applications we refer the reader to [1, 3–5, 8–13, 18], for instance. In Section 2 we establish, in locally convex spaces, some specific facts of this theory that we need later.

In Section 4 we apply this distributional point of view on the fractional powers of $-\Delta$ to the study of Riesz potentials. Given a complex number α such that $0 < \operatorname{Re} \alpha < n/2$, the *Riesz potential* R_α acting on a function f locally integrable on \mathbb{R}^n is defined by

$$(R_\alpha f)(x) = \frac{\Gamma(n/2 - \alpha)}{\pi^{n/2} 2^{2\alpha} \Gamma(\alpha)} \int_{\mathbb{R}^n} |x - y|^{2\alpha - n} f(y) dy$$

whenever this convolution exists. This always happens if $f \in L^p$ with $1 \leq p < n/(2 \operatorname{Re} \alpha)$, since the function

$$\psi_\alpha(x) = |x|^{2\alpha - n}, \quad x \in \mathbb{R}^n, x \neq 0,$$

belongs to $L^1 + L^q$ for $n/(n - 2 \operatorname{Re} \alpha) < q \leq \infty$.

If we take the Fourier transform of the Riesz potential R_α with $0 < \operatorname{Re} \alpha < n/2$, we find that

$$(R_\alpha f)^\wedge(x) = (2\pi|x|)^{-2\alpha} \widehat{f}(x) \quad \text{for } f \in \mathcal{S}.$$

On the other hand, since $((-\Delta)f)^\wedge(x) = (2\pi|x|)^2 \widehat{f}(x)$, it is natural to think that a “good” definition of the fractional power of $-\Delta$ has to satisfy

$$((-\Delta)^\alpha f)^\wedge(x) = (2\pi|x|)^{2\alpha} \widehat{f}(x)$$

for $f \in \mathcal{S}$ and $\operatorname{Re} \alpha \neq 0$. For this reason it is common to identify the operator R_α with the fractional power $(-\Delta)^{-\alpha}$. However, the identity $R_\alpha f = (-\Delta)^{-\alpha} f$ has only been proved for $f \in \mathcal{S}$ and therefore, the identity $R_\alpha = (-\Delta)^{-\alpha}$ (as operators in L^p) has only a “formal” meaning.

In this paper we study the operator $(-\Delta)^{-\alpha}$ in the context of the classical theory of fractional powers and we obtain a relationship between this operator and R_α in the context of the duality $(\mathcal{T}, \mathcal{T}')$.

As a consequence, we deduce some interesting properties of the operator $[R_\alpha]_p$. Moreover, our distributional point of view on Riesz potentials allows us to obtain some properties of R_α in other spaces.

Finally, we introduce the operator $B_{\alpha,\varepsilon}$, $\varepsilon > 0$, by

$$(B_{\alpha,\varepsilon}f)(x) = \frac{1}{(4\pi)^\alpha \Gamma(\alpha)} \int_{\mathbb{R}^n} \left(\int_0^\infty e^{-\pi|y|^2/t} e^{-\varepsilon t/(4\pi)} t^{-n/2+\alpha-1} dt \right) f(x-y) dy$$

which is similar to the Bessel potential (where $\varepsilon = 1$). We prove that

$$s\text{-}\lim_{\varepsilon \rightarrow 0^+} [B_{\alpha,\varepsilon}]_p = [R_\alpha]_p, \quad 1 < p < \frac{n}{2 \operatorname{Re} \alpha}.$$

2. Previous results on fractional powers. In this section, X will be a sequentially complete locally convex space endowed with a directed family \mathfrak{P} of seminorms. The following definition was introduced in [13].

DEFINITION 2.1. We say that a closed linear operator $A : D(A) \subset X \rightarrow X$ is *non-negative* if $] -\infty, 0[$ is contained in the resolvent set $\varrho(A)$ and the set $\{\lambda(\lambda + A)^{-1} : \lambda > 0\}$ is equicontinuous, i.e., for all $\mathfrak{p} \in \mathfrak{P}$ there is a seminorm $\mathfrak{p}_0(\mathfrak{p}) \in \mathfrak{P}$ and a constant $M = M(A, \mathfrak{p}) > 0$ such that

$$\mathfrak{p}(\lambda(A + \lambda)^{-1}x) \leq M\mathfrak{p}_0(x), \quad \lambda > 0, x \in X.$$

We denote by $D(A)$ the domain of A and by $R(A)$ the range of A . From now on, α will be a complex number such that $\operatorname{Re} \alpha > 0$.

It is not hard to show that if A is a non-negative operator then

$$(1) \quad \lim_{\mu \rightarrow 0^+} A^n [(A + \mu)^{-1}]^n x = x \quad \text{for } x \in \overline{R(A)} \text{ and } n \in \mathbb{N}.$$

Consequently, $\overline{R(A)} = \overline{R(A^n)}$. This identity can be extended to exponents $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha > 0$.

LEMMA 2.1. *We have the identity*

$$\overline{R(A)} = \overline{R(A^\alpha)}, \quad \operatorname{Re} \alpha > 0.$$

PROOF. It is known (see [8] and [12]) that if $0 < \operatorname{Re} \alpha < n$, n integer, then the fractional power A^α is given by

$$(2) \quad A^\alpha x = \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)} \int_0^\infty \lambda^{\alpha-1} [(\lambda + A)^{-1} A]^n x d\lambda, \quad x \in D(A^n).$$

Moreover, in [12, Theorem 4.1] we proved that

$$D(A^\alpha) = \{x \in X : A^\alpha(1 + A)^{-n}x \in D(A^n)\}$$

and

$$(3) \quad A^\alpha x = (1 + A)^n A^\alpha (1 + A)^{-n} x \quad \text{for } x \in D(A^\alpha).$$

From (2) it follows that $R(A^\alpha[(1 + A)^{-1}]^n) \subset \overline{R(A)}$. Hence, by (3) we conclude that $R(A^\alpha) \subset \overline{R(A)}$.

On the other hand, by additivity (see [13]) we find that $R(A^\alpha) \supset R(A^n)$ and consequently $\overline{R(A^\alpha)} = \overline{R(A)}$. ■

REMARK 2.1. Balakrishnan and Komatsu defined the fractional power of exponent α of A as the closure of the operator given by (2). Therefore, the range of this fractional power is included in $D(A) \cap \overline{R(A)}$, which is a proper subspace of $\overline{R(A)}$ if $D(A)$ is non-dense. So, the property given in the previous lemma is a specific property of the concept of fractional power given by the authors in [11, 13].

From (1) one deduces that if $R(A)$ is dense, then A is one-to-one. Moreover, the operator A_R has dense range in $\overline{R(A)}$ (we write A_R instead of $A_{\overline{R(A)}}$). Since A_R is non-negative in $\overline{R(A)}$ (as $(A + \lambda)^{-1}(\overline{R(A)}) \subset \overline{R(A)}$, $\lambda > 0$), it easily follows that A_R is one-to-one. It is also evident that if A is a one-to-one, non-negative operator (with not necessarily dense range), then A^{-1} is non-negative. In this case, the fractional power $A^{-\alpha}$ is given by $A^{-\alpha} = (A^{-1})^\alpha$. The operator $A^{-\alpha}$ is closed since A^α is (see [12]).

DEFINITION 2.2. Given $n > \text{Re } \alpha > 0$, $x = A^n y \in R(A^n)$, $y \in D(A^n)$, we define

$$(4) \quad A_{-\alpha} x = \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)} \int_0^\infty \lambda^{n-\alpha-1} (\lambda + A)^{-n} x \, d\lambda.$$

From (2) one deduces that $A_{-\alpha} A^n y = A^{n-\alpha} y$. Moreover, with the change $\lambda \mapsto \lambda^{-1}$ in (4), it is very easy to show that if A is one-to-one, then $A_{-\alpha} x = A^{-\alpha} x$ for $x \in R(A^n)$. In this case, A^α is one-to-one and

$$(5) \quad (A^\alpha)^{-1} = A^{-\alpha}.$$

That is because $A^\alpha A^{-\alpha} x = x$ for $x \in R(A^n)$, and $A^{-\alpha} A^\alpha x = x$ for $x \in D(A^n)$. By (3) these identities also hold for $x \in D(A^{-\alpha})$ and $x \in D(A^\alpha)$, respectively.

PROPOSITION 2.2. $A_{-\alpha}$ is closable and its closure is given by

$$(6) \quad \overline{A}_{-\alpha} = (A_R)^{-\alpha}$$

Consequently, if A is one-to-one, then $A^{-\alpha}$ is an extension of $\overline{A}_{-\alpha}$. The identity $A^{-\alpha} = \overline{A}_{-\alpha}$ holds if and only if $R(A)$ is dense.

Proof. Given $x = A^n y \in R(A^n)$ and $\mu > 0$ we get

$$\begin{aligned} A(\mu + A)^{-1} A_{-\alpha} x &= A_{-\alpha} A(\mu + A)^{-1} x \\ &= (A_R)_{-\alpha} A(\mu + A)^{-1} x = (A_R)^{-\alpha} A(\mu + A)^{-1} x \end{aligned}$$

and taking limits as $\mu \rightarrow 0$, as $(A_R)^{-\alpha}$ is closed, we conclude that $x \in D[(A_R)^{-\alpha}]$ and $(A_R)^{-\alpha}x = A_{-\alpha}x$. Hence, $A_{-\alpha}$ is closable and $(A_R)^{-\alpha}$ is an extension of $\overline{A_{-\alpha}}$.

Let now $x \in D[(A_R)^{-\alpha}]$ and $\mu > 0$. As $A^n(A + \mu)^{-n}x \in R[(A_R)^n]$ it follows that

$$A^n(\mu + A)^{-n}(A_R)^{-\alpha}x = (A_R)^{-\alpha}A^n(\mu + A)^{-n}x = A_{-\alpha}A^n(\mu + A)^{-n}x$$

and taking limits as $\mu \rightarrow 0$ we conclude that $x \in D(\overline{A_{-\alpha}})$ and $\overline{A_{-\alpha}}x = (A_R)^{-\alpha}x$. This proves (6).

If A is one-to-one and $R(A)$ is not dense, by choosing $x \notin \overline{R(A)}$, it is evident that $(A^{-1})^n(A^{-1} + 1)^{-n}x \notin \overline{R(A)}$. By additivity one deduces that $A^{-\alpha}A^{-n+\alpha}(A + 1)^{-n}x \notin \overline{R(A)}$. Consequently, $A^{-\alpha}$ is a proper extension of $(A_R)^{-\alpha}$. ■

From (1) it is easy to show that if $x \in \overline{R(A)}$, then

$$(7) \quad \lim_{\mu \rightarrow 0^+} \mu^n(A + \mu)^{-n}x = 0.$$

This result can be improved in this way:

PROPOSITION 2.3. *The operators $A^\alpha(\mu + A)^{-\alpha}$ and $\mu^\alpha(\mu + A)^{-\alpha}$ are uniformly bounded for $\mu > 0$. Moreover, given $x \in X$, the following assertions are equivalent:*

- (i) $x \in \overline{R(A)}$.
- (ii) $\lim_{\mu \rightarrow 0^+} \mu^\alpha(\mu + A)^{-\alpha}x = 0$.
- (iii) $\lim_{\mu \rightarrow 0^+} A^\alpha(\mu + A)^{-\alpha}x = x$.

PROOF. First note that since $(A + \mu)^{-1}$ is bounded, so is $(A + \mu)^{-\alpha}$. Moreover, since $D[(A + \mu)^\alpha] = D(A^\alpha)$ (see [13]) one deduces that $D(A^\alpha(\mu + A)^{-\alpha}) = X$.

By additivity, we can restrict the proof of the first assertion to the case $0 < \text{Re } \alpha < 1$. In this case, given $\mathfrak{p} \in \mathfrak{P}$, as

$$(8) \quad (\mu + A)^{-\alpha}x = \frac{\sin \alpha \pi}{\pi} \int_0^\infty \lambda^{-\alpha}(\lambda + \mu + A)^{-1}x d\lambda, \quad x \in X,$$

we find that

$$(9) \quad \mathfrak{p}[(\mu + A)^{-\alpha}x] \leq \mu^{-\text{Re } \alpha}c(\alpha)M\mathfrak{p}_0(x), \quad x \in X.$$

Hence, the operators $\mu^\alpha(\mu + A)^{-\alpha}$, $\mu > 0$, are uniformly bounded. In a similar way, from (2), with $n = 1$, one deduces that $A^\alpha - (\mu + A)^\alpha$ can be extended to a bounded operator, T , on X which satisfies

$$(10) \quad \mathfrak{p}(Tx) \leq \mu^{\text{Re } \alpha}k(M, \alpha)\mathfrak{p}_1(x), \quad x \in X \text{ and } \mathfrak{p}_1 \in \mathfrak{P}.$$

From (9) and (10) it follows that

$$(11) \quad A^\alpha(\mu + A)^{-\alpha} - 1 = [A^\alpha - (\mu + A)^\alpha](\mu + A)^{-\alpha}, \quad \mu > 0,$$

are uniformly bounded.

Let us now prove that (iii) implies (i). It is evident that (iii) implies that $x \in \overline{R(A^\alpha)} = \overline{R(A)}$, according to Lemma 2.1.

To prove that (i) implies (ii) suppose that $x = Ay \in R(A)$ and $0 < \operatorname{Re} \alpha < 1$. From (8) we obtain

$$(12) \quad \mathfrak{p}[(\mu + A)^{-\alpha}x] \leq h_0(\alpha)M\mathfrak{p}_0(x) + h_1(\alpha)(M + 1)\mathfrak{p}_2(y),$$

where $\mathfrak{p}_0, \mathfrak{p}_2 \in \mathfrak{P}$.

Therefore, $\lim_{\mu \rightarrow 0^+} \mu^\alpha(\mu + A)^{-\alpha}x = 0$. As $\mu^\alpha(\mu + A)^{-\alpha}$ are uniformly bounded for $\mu > 0$, by additivity, this property also holds for $x \in \overline{R(A)}$ and $\operatorname{Re} \alpha \geq 1$.

Let us finally prove that (ii) implies (iii). If $0 < \operatorname{Re} \alpha < 1$, by applying the operator $\mu^{1-\alpha}(\mu + A)^{-1+\alpha}$ we find that $\lim_{\mu \rightarrow 0^+} \mu(\mu + A)^{-1}x = 0$. Therefore $x \in \overline{R(A)}$, since $x = A(\mu + A)^{-1}x + \mu(\mu + A)^{-1}x$. By (10)–(12) it easily follows that (iii) holds for $x \in R(A)$ and by density, also for $x \in \overline{R(A)}$. Finally, if $\operatorname{Re} \alpha \geq 1$ we take $m \in \mathbb{N}$ such that $\beta = \operatorname{Re} \alpha/m < 1$ and from

$$A^\alpha(\mu + A)^{-\alpha}x - x = \left[1 + \sum_{1 \leq j \leq m-1} A^{j\beta}(\mu + A)^{-j\beta}\right][A^\beta(\mu + A)^{-\beta}x - x]$$

we deduce that $\lim_{\varepsilon \rightarrow 0^+} A^\alpha(\mu + A)^{-\alpha}x = x$. ■

PROPOSITION 2.4. *The operator $\overline{A}_{-\alpha}$ satisfies*

$$(13) \quad s\text{-}\lim_{\mu \rightarrow 0^+} (A + \mu)^{-\alpha} = \overline{A}_{-\alpha}.$$

Proof. Set $T = s\text{-}\lim_{\mu \rightarrow 0^+} (A + \mu)^{-\alpha}$. If $x \in D(T)$, then

$$\lim_{\mu \rightarrow 0^+} \mu^\alpha(A + \mu)^{-\alpha}x = 0,$$

and by Proposition 2.3 we conclude that $x \in \overline{R(A)}$. Also by Proposition 2.3, taking into account that $\overline{R(A)} = \overline{R(A_R)}$ and $(A_R + \mu)^{-\alpha}x = (A + \mu)^{-\alpha}x$, we have

$$\lim_{\mu \rightarrow 0^+} (A_R)^\alpha(A_R + \mu)^{-\alpha}x = x.$$

Therefore, as $(A_R)^\alpha$ is closed, we deduce that $Tx \in D[(A_R)^\alpha]$ and $(A_R)^\alpha Tx = x$. Hence, by (5) it follows that $x \in D[(A_R)^{-\alpha} = \overline{A}_{-\alpha}]$ and $\overline{A}_{-\alpha}x = Tx$.

Conversely, if $x \in D(\bar{A}_{-\alpha})$ then $x = (A_R)^\alpha y$ for some $y \in D[(A_R)^\alpha]$. By Proposition 2.3 we obtain

$$\lim_{\mu \rightarrow 0^+} (A + \mu)^{-\alpha} x = \lim_{\mu \rightarrow 0^+} (A_R + \mu)^{-\alpha} (A_R)^\alpha y = y.$$

Therefore, $x \in D(T)$ and the proof is complete. ■

According to [3], if A has dense domain and range and $\tau \in \mathbb{R}$, the imaginary power $A^{i\tau}$ is the closure of the closable operator

$$(14) \quad A_{i\tau} x = A^{1+i\tau} y, \quad x = Ay \in D(A) \cap R(A).$$

It is evident that $A^{i\tau}$ and $(A + 1)^{-1}$ commute.

PROPOSITION 2.5. *Let A be a non-negative operator with dense domain and range and $\tau \in \mathbb{R}$. Then $A^{\alpha+i\tau}$ is an extension of $A^{i\tau} A^\alpha$.*

PROOF. Let $n > \text{Re } \alpha$ be a positive integer. Given $x \in D(A^\alpha)$ such that $A^\alpha x \in D(A^{i\tau})$, by (14) and additivity we have

$$\begin{aligned} A(A + 1)^{-n} A^{i\tau} A^\alpha x &= A^{i\tau} A A^\alpha (A + 1)^{-n} x = A^{1+i\tau} A^\alpha (A + 1)^{-n} x \\ &= A A^{\alpha+i\tau} (A + 1)^{-n} x \end{aligned}$$

and, as A is one-to-one,

$$(A + 1)^{-n} A^{i\tau} A^\alpha x = A^{\alpha+i\tau} (A + 1)^{-n} x.$$

The identity (3) now implies that $x \in D(A^{\alpha+i\tau})$ and $A^{\alpha+i\tau} x = A^{i\tau} A^\alpha x$. ■

As a straightforward consequence, we find that if $A^{i\tau}$ is bounded, then $D(A^{\alpha+i\tau}) = D(A^\alpha)$.

We conclude this section with a result which states that restriction to subspaces commutes with fractional powers.

PROPOSITION 2.6. *Let Y be a sequentially complete locally convex space and $A : D(A) \subset Y \rightarrow Y$ be a non-negative operator. Let $X \subset Y$ be a linear subspace of Y with the same topological properties of Y (but not necessarily a topological subspace of Y) and suppose that the restricted operator A_X is non-negative in X . If there exists a positive integer $n > \text{Re } \alpha$ such that*

$$(15) \quad A^\alpha x = (A_X)^\alpha x \quad \text{for all } x \in D[(A_X)^n],$$

then

$$(16) \quad [A^\alpha]_X = (A_X)^\alpha.$$

In particular, (16) holds if the topology of X has the following property:

- (p) If $(x_n)_{n \in \mathbb{N}} \subset X$ converges to x_0 in the topology of X and also converges to x_1 in the topology induced by Y , then $x_0 = x_1$.

PROOF. It is evident that $(1 + A_X)^{-n}x = (1 + A)^{-n}x$ for all $x \in X$. Therefore, given $x \in D([A^\alpha]_X)$ we have

$$(A_X)^\alpha(1 + A_X)^{-n}x = A^\alpha(1 + A)^{-n}x = (1 + A)^{-n}A^\alpha x.$$

Taking into account that $A^\alpha x \in X$ one deduces that $(A_X)^\alpha(1 + A_X)^{-n}x \in D[(A_X)^\alpha]$. Hence, by (3) we conclude that $x \in D[(A_X)^\alpha]$ and $(A_X)^\alpha x = A^\alpha x$.

In a similar way, given $x \in D[(A_X)^\alpha]$, as

$$A^\alpha(1 + A)^{-n}x = (A_X)^\alpha(1 + A_X)^{-n}x = (1 + A_X)^{-n}(A_X)^\alpha x$$

it follows easily that $x \in D(A^\alpha)$ and $A^\alpha x = (A_X)^\alpha x \in X$.

If X has the property (p), then it is evident that (15) holds. Therefore (16) also holds. ■

3. Distributional fractional powers of $-\Delta$. From now on, if Y is a vector space included in the general space of distributions, we denote by Δ_Y the restriction to Y of the distributional Laplacean, i.e., $\Delta_Y u = \Delta u$ for $u \in D(\Delta_Y) = \{u \in Y : \Delta u \in Y\}$. If $Y = L^p$, we write Δ_p instead of Δ_{L^p} .

PROPOSITION 3.1. *Neither $-\Delta_{\mathcal{D}}$ nor $-\Delta_{\mathcal{S}}$ are non-negative.*

PROOF. Let $\phi : \mathbb{R}^n \rightarrow]0, \infty[$, $\phi \in \mathcal{D}$, be non-identically vanishing. Given $\lambda > 0$, if the operator $\lambda - \Delta_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{D}$ were surjective, then the function

$$\begin{aligned} [(\lambda - \Delta_{\mathcal{D}})^{-1}\phi](x) &= \int_0^\infty e^{-\lambda t}(K_t * \phi)(x) dt \\ &= \int_{\mathbb{R}^n} \left(\int_0^\infty e^{-\lambda t} K_t(x - y) dt \right) \phi(y) dy \end{aligned}$$

would vanish outside a compact set. However, for all $x \in \mathbb{R}^n$ this function is positive and therefore $(\lambda - \Delta_{\mathcal{D}})^{-1}\phi \notin C_0^\infty$. Here we have denoted by K_t the heat kernel, i.e.,

$$K_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)}, \quad x \in \mathbb{R}^n, \quad t > 0.$$

On the other hand, also by means of the Fourier transform, it is very easy to show that, for $\lambda > 0$, the operator $\lambda - \Delta_{\mathcal{S}} : \mathcal{S} \rightarrow \mathcal{S}$ is bijective and its inverse $(\lambda - \Delta_{\mathcal{S}})^{-1}$ is continuous. Moreover, if $\phi \in \mathcal{S}$ and $x \in \mathbb{R}^n$ then

$$[(\lambda - \Delta_{\mathcal{S}})^{-1}\phi]^\wedge(x) = \frac{1}{\lambda + 4\pi^2|x|^2} \widehat{\phi}(x).$$

Consequently, if $-\Delta_{\mathcal{S}}$ were a non-negative operator, then given $\alpha \in \mathbb{C}$ such that $0 < \operatorname{Re} \alpha < 1$, by (2) we would obtain

$$\begin{aligned} [(-\Delta_{\mathcal{S}})^{\alpha} \phi]^{\wedge}(x) &= \left(\frac{\sin \alpha \pi}{\pi} \int_0^{\infty} \lambda^{\alpha-1} (-\Delta_{\mathcal{S}})(\lambda - \Delta_{\mathcal{S}})^{-1} \phi \, d\lambda \right)^{\wedge}(x) \\ &= \frac{\sin \alpha \pi}{\pi} \int_0^{\infty} \lambda^{\alpha-1} [(-\Delta_{\mathcal{S}})(\lambda - \Delta_{\mathcal{S}})^{-1} \phi]^{\wedge}(x) \, d\lambda \\ &= \left(\frac{\sin \alpha \pi}{\pi} \int_0^{\infty} \lambda^{\alpha-1} \frac{4\pi^2 |x|^2}{\lambda + 4\pi^2 |x|^2} \, d\lambda \right) \widehat{\phi}(x) \\ &= (4\pi^2 |x|^2)^{\alpha} \widehat{\phi}(x), \end{aligned}$$

where we have used the fact that the Fourier transform is a continuous operator from \mathcal{S} to itself and that the convergence in the usual topology of \mathcal{S} implies uniform convergence.

Therefore, the function $(4\pi^2 |x|^2)^{\alpha} \widehat{\phi}(x)$ would belong to \mathcal{S} , which in general is not true. ■

This proposition justifies the introduction of a new space, to study the Laplacean, instead of the spaces \mathcal{D} or \mathcal{S} .

DEFINITION 3.1. We denote by \mathcal{T} the space of functions $\phi : \mathbb{R}^n \rightarrow \mathbb{C}$ of class C^{∞} such that any partial derivative belongs to $L^1 \cap L^{\infty}$. We endow this space with the natural topology defined by the seminorms

$$|\phi|_m = \max\{\|D^{\beta} \phi\|_1, \|D^{\beta} \phi\|_{\infty} : \beta \in \mathbb{N}^n, |\beta| \leq m\}, \quad \phi \in \mathcal{T}, \quad m \in \mathbb{N}.$$

REMARK 3.1. It is very easy to show that \mathcal{T} endowed with the increasing countable family $\{|\cdot|_m : m \in \mathbb{N}\}$ of seminorms is a Fréchet space. However, this space is non-normable. If it were normable, there would be an index m_0 and a constant $k_{m_0} \geq 0$ such that $|\phi|_{m_0+1} \leq k_{m_0} |\phi|_{m_0}$ for all $\phi \in \mathcal{T}$. Thus, if we take a multi-index β such that $|\beta| = m_0 + 1$ and a function $\psi \in \mathcal{T}$ with $D^{\beta} \psi$ non-identically vanishing, for $\phi(x) = \psi(rx)$ ($r > 1$ constant) we have

$$r^{m_0+1} \|D^{\beta} \psi\|_{\infty} = \|D^{\beta} \phi\|_{\infty} \leq k_{m_0} |\phi|_{m_0} \leq k_{m_0} r^{m_0} |\psi|_{m_0},$$

and taking limits as $r \rightarrow \infty$ we conclude that $\|D^{\beta} \psi\|_{\infty} = 0$, which is a contradiction.

REMARK 3.2. It is evident that $\mathcal{T} \subset L^p$ ($1 \leq p \leq \infty$) and also that $\mathcal{S} \subset \mathcal{T}$. Moreover, it is very easy to show that the induced topology of \mathcal{S} is weaker than the usual topology of this space, and that \mathcal{D} is dense in $(\mathcal{T}, |\cdot|_m, m \in \mathbb{N})$. It is also easy to prove that, for all multi-indices β , $\lim_{|x| \rightarrow \infty} D^{\beta} \phi(x) = 0$ for $\phi \in \mathcal{T}$.

LEMMA 3.2. If $f \in L^p$ ($1 \leq p \leq \infty$) and $\phi \in \mathcal{T}$, then the convolution $f * \phi$ exists, belongs to C^{∞} and satisfies $D^{\beta}(f * \phi) = f * D^{\beta} \phi$ for all multi-

indices β . In particular, the convolution $R_\alpha\phi = \psi_\alpha * \phi$ is well defined if $0 < \operatorname{Re}\alpha < n/2$ and $\phi \in \mathcal{T}$. Moreover, $R_\alpha\phi \in C^\infty$. If $f \in L^1$, then $f * \phi \in \mathcal{T}$ and the operator $\phi \mapsto f * \phi$ is continuous in \mathcal{T} .

Proof. This is an immediate consequence of the Hölder and Young inequalities. The convolution $R_\alpha\phi = \psi_\alpha * \phi$ exists since ψ_α can be decomposed as $\psi_\alpha = \mu_\alpha + v_\alpha$, where $\mu_\alpha = \psi_\alpha \chi_{B(0,1)} \in L^1$ ($\chi_{B(0,1)}$ denotes the characteristic function of the ball of radius 1, centered at the origin) and $v_\alpha = \psi_\alpha - \mu_\alpha \in L^q$ with $n/(n - 2\operatorname{Re}\alpha) < q \leq \infty$. ■

THEOREM 3.3. *The operator $\Delta_{\mathcal{T}}$, restriction of the Laplacean to the space \mathcal{T} , is continuous and it is also the infinitesimal generator of the heat semigroup, which is a contractive semigroup of class C_0 . Consequently, $-\Delta_{\mathcal{T}}$ is a non-negative operator.*

Proof. It is evident that $\Delta_{\mathcal{T}}$ is continuous. On the other hand, as $K_t \in L^1$ and $\|K_t\|_1 = 1$, from the preceding lemma one deduces that $P_t\phi = K_t * \phi \in \mathcal{T}$ for all $\phi \in \mathcal{T}$, and

$$|P_t\phi|_m \leq \|K_t\|_1 |\phi|_m = |\phi|_m, \quad m = 0, 1, 2, \dots$$

Now, by the theorem on approximations to the identity we conclude that $\mathcal{T}\text{-}\lim_{t \rightarrow 0} P_t\phi = \phi$.

A simple calculation shows that, for $t, s > 0$,

$$P_t P_s \phi = K_t * (K_s * \phi) = (K_t * K_s) * \phi = K_{t+s} * \phi = P_{t+s} \phi.$$

Hence, we conclude that $\{P_t : t > 0\}$ is a contractive semigroup of class C_0 .

Let A be its infinitesimal generator. We now prove that $A = \Delta_{\mathcal{T}}$. Given $t_0 > 0$ and $\phi \in D(A)$ we have

$$\mathcal{T}\text{-}\lim_{\delta \rightarrow 0} [\delta^{-1} (P_{t_0+\delta}\phi - P_{t_0}\phi) - P_{t_0}A\phi] = 0$$

and hence,

$$\left. \frac{\partial}{\partial t} \right|_{t=t_0} [(P_t\phi)(x)] = (P_{t_0}A\phi)(x) \quad \text{for all } x \in \mathbb{R}^n.$$

On the other hand, as the function $u(x, t) = (P_t\phi)(x)$ is a solution of the heat equation,

$$\left. \frac{\partial}{\partial t} \right|_{t=t_0} [(P_t\phi)(x)] = (\Delta P_{t_0}\phi)(x) = (P_{t_0}\Delta\phi)(x).$$

So, we deduce that $P_{t_0}\Delta\phi = P_{t_0}A\phi$, and taking limits as $t_0 \rightarrow 0$ we conclude that $\Delta\phi = A\phi$.

As in Banach spaces, it is not hard to show (see [20, Th. 1, p. 240]) that if A is the infinitesimal generator of an equicontinuous semigroup of class C_0 , then $-A$ is a non-negative operator.

Finally, we prove that $D(A) = \mathcal{T}$. To do this it is sufficient to prove that $1 - \Delta_{\mathcal{T}}$ is one-to-one. Indeed, for every $\phi \in \mathcal{T}$ there exists $\psi \in D(A)$ such that $(1 - A)\psi = (1 - \Delta_{\mathcal{T}})\phi$, since $1 - A$ is surjective, due to the fact that $-A$ is non-negative. Since $\psi \in D(A)$,

$$(1 - A)\psi = (1 - \Delta_{\mathcal{T}})\psi = (1 - \Delta_{\mathcal{T}})\phi$$

and as $1 - \Delta_{\mathcal{T}}$ is one-to-one we conclude that $\phi = \psi \in D(A)$.

To prove that $1 - \Delta_{\mathcal{T}}$ is one-to-one it is sufficient to take Fourier transforms since if $(1 - \Delta_{\mathcal{T}})\phi = 0$ then

$$[(1 - \Delta)\phi]^{\wedge}(x) = (1 + 4\pi^2|x|^2)\widehat{\phi}(x) = 0 \quad \text{for all } x \in \mathbb{R}^n$$

and hence $\widehat{\phi} = 0$, which implies $\phi = 0$.

REMARK 3.3. By means of Fourier transforms it is also very easy to show that $\Delta_{\mathcal{T}}$ is one-to-one. However, this operator has non-dense range. To prove this, consider the linear form $u : \phi \mapsto \int_{\mathbb{R}^n} \phi(x) dx$ which is continuous and non-identically vanishing. However, $(u, \Delta\phi) = 0$ for all $\phi \in \mathcal{T}$, by the density of \mathcal{D} in \mathcal{T} .

Now consider the topological dual space of \mathcal{T} , denoted by \mathcal{T}' . Note that as the topology that \mathcal{T} induces on \mathcal{S} is weaker than the usual topology of this space, we find that if $u \in \mathcal{T}'$ then u can be identified as a tempered distribution. Moreover, as \mathcal{S} is dense in \mathcal{T} , u is completely determined by its restriction to \mathcal{S} .

We endow \mathcal{T}' with the topology of uniform convergence on bounded subsets of \mathcal{T} , i.e., the topology defined by the seminorms

$$|u|_B = \sup_{\phi \in B} |(u, \phi)|, \quad u \in \mathcal{T}', \quad B \subset \mathcal{T} \text{ a bounded set.}$$

In \mathcal{T}' the two main requirements that we need hold: the negative of the Laplacean is a non-negative operator and the spaces L^p ($1 \leq p \leq \infty$) are included in \mathcal{T}' .

REMARK 3.4. Since \mathcal{T} is non-normable, no countable family of bounded sets exists such that every bounded set in \mathcal{T} is contained in this family. Hence, \mathcal{T}' is non-metrizable. However, by the Banach–Steinhaus theorem (see [16, p. 86]), this space is sequentially complete. So, we have a non-trivial example of a sequentially complete locally convex space where it will be very useful to apply the theory of fractional powers developed in [12, 13].

PROPOSITION 3.4. *For all $1 \leq p \leq \infty$, $L^p \subset \mathcal{T}'$ and the induced topology of L^p is weaker than the usual topology of this space.*

Proof. Consider $f \in L^p$ and a bounded set $B \subset \mathcal{T}$, and

$$k = \sup_{\phi \in B} \{\|\phi\|_1, \|\phi\|_{\infty}\},$$

which is finite since B is a bounded set. From the Hölder inequality, if q is the conjugate exponent of p , it follows that

$$\sup_{\phi \in B} \left| \int_{\mathbb{R}^n} f(x)\phi(x) dx \right| \leq \sup_{\phi \in B} \{ \|\phi\|_q, \|f\|_p \} \leq k \|f\|_p,$$

and thus $f \in \mathcal{T}'$ and $|f|_B \leq k \|f\|_p$. ■

Derivation and convolution in \mathcal{T}' . Given $u \in \mathcal{T}'$ and a multi-index β , the distributional derivative $D^\beta u$ can be extended to an element (that we also denote by $D^\beta u$) that belongs to the dual space \mathcal{T}' and which is defined by

$$(D^\beta u, \phi) = (u, (-1)^{|\beta|} D^\beta \phi), \quad \phi \in \mathcal{T}.$$

In particular, the Laplacean operator in \mathcal{T}' , $\Delta_{\mathcal{T}'}$, acts as

$$(\Delta_{\mathcal{T}'} u, \phi) = (u, \Delta \phi), \quad \phi \in \mathcal{T}.$$

Given $f \in L^1$ and $u \in \mathcal{T}'$, we define the convolution $u * f$ as the linear form

$$\phi \mapsto (u, \tilde{f} * \phi), \quad \phi \in \mathcal{T},$$

where $\tilde{f}(x) = f(-x)$. From Lemma 3.2 it follows that $u * f \in \mathcal{T}'$.

THEOREM 3.5. *$-\Delta_{\mathcal{T}'}$ is a continuous and non-negative operator but it is not one-to-one.*

PROOF. Given a bounded set $B \subset \mathcal{T}$ and $u \in \mathcal{T}'$, as the set $E = \{\Delta \phi : \phi \in B\}$ is also bounded, from

$$|\Delta_{\mathcal{T}'} u|_B = \sup_{\phi \in B} |(\Delta_{\mathcal{T}'} u, \phi)| = \sup_{\phi \in B} |(u, \Delta \phi)| = |u|_E$$

it follows that $\Delta_{\mathcal{T}'}$ is continuous.

Let now $\lambda > 0$ and $u \in \mathcal{T}'$. It is very easy to prove that the linear form $v : \psi \mapsto (u, (\lambda - \Delta_{\mathcal{T}'})^{-1} \psi)$ is continuous and $(\lambda - \Delta_{\mathcal{T}'})v = u$. Therefore, $\lambda - \Delta_{\mathcal{T}'}$ is surjective.

On the other hand, let $u \in \mathcal{T}'$ be such that $(\lambda - \Delta_{\mathcal{T}'})u = 0$. Then, for all $\phi \in \mathcal{T}$,

$$((\lambda - \Delta_{\mathcal{T}'})u, \phi) = (u, (\lambda - \Delta_{\mathcal{T}'})\phi) = 0,$$

and thus $u = 0$ (as $R(\lambda - \Delta_{\mathcal{T}'}) = \mathcal{T}$, due to the fact that $-\Delta_{\mathcal{T}'}$ is a non-negative operator).

For every bounded set $B \subset \mathcal{T}$ and $u \in \mathcal{T}'$, since $-\Delta_{\mathcal{T}'}$ is non-negative, the set $F = \{\mu(\mu - \Delta_{\mathcal{T}'})^{-1} \phi : \phi \in B, \mu > 0\}$ is also bounded and thus, for $\lambda > 0$,

$$\begin{aligned} |\lambda(\lambda - \Delta_{\mathcal{T}'})^{-1} u|_B &= \sup_{\phi \in B} |(\lambda(\lambda - \Delta_{\mathcal{T}'})^{-1} u, \phi)| \\ &= \sup_{\phi \in B} |(u, \lambda(\lambda - \Delta_{\mathcal{T}'})^{-1} \phi)| \leq |u|_F. \end{aligned}$$

We now conclude that $-\Delta_{\mathcal{T}'}$ is a non-negative operator.

Finally, as the constant functions belong to \mathcal{T}' and obviously their Laplacean is null we find that $\Delta_{\mathcal{T}'}$ is not one-to-one. ■

In the next theorem we point out a dual relationship between the operators $(-\Delta_{\mathcal{T}})^\alpha$ and $(-\Delta_{\mathcal{T}'})^\alpha$.

THEOREM 3.6. *For $\phi \in \mathcal{T}$, $u \in \mathcal{T}'$ and $\text{Re } \alpha > 0$, we have the duality formula*

$$((-\Delta_{\mathcal{T}'})^\alpha u, \phi) = (u, (-\Delta_{\mathcal{T}})^\alpha \phi).$$

Proof. Let $m > \text{Re } \alpha > 0$ be a positive integer, $\phi \in \mathcal{T}$ and $u \in \mathcal{T}'$. Since $\Delta_{\mathcal{T}'}$ is continuous,

$$\begin{aligned} \frac{\Gamma(\alpha)\Gamma(m-\alpha)}{\Gamma(m)}((-\Delta_{\mathcal{T}'})^\alpha u, \phi) &= \left(\int_0^\infty \lambda^{\alpha-1} [(-\Delta_{\mathcal{T}'}) (\lambda - \Delta_{\mathcal{T}'})^{-1}]^m u \, d\lambda, \phi \right) \\ &= \int_0^\infty (\lambda^{\alpha-1} [(-\Delta_{\mathcal{T}'}) (\lambda - \Delta_{\mathcal{T}'})^{-1}]^m u, \phi) \, d\lambda \\ &= \int_0^\infty (u, \lambda^{\alpha-1} [(-\Delta_{\mathcal{T}}) (\lambda - \Delta_{\mathcal{T}})^{-1}]^m \phi) \, d\lambda \\ &= \left(u, \int_0^\infty \lambda^{\alpha-1} [(-\Delta_{\mathcal{T}}) (\lambda - \Delta_{\mathcal{T}})^{-1}]^m \phi \, d\lambda \right) \\ &= \frac{\Gamma(\alpha)\Gamma(m-\alpha)}{\Gamma(m)}(u, (-\Delta_{\mathcal{T}})^\alpha \phi), \end{aligned}$$

where the first and last identities follow from (2); the second one is a consequence of the fact that the convergence in \mathcal{T}' implies weak convergence; the third one can be justified by the duality relations between $(\lambda - \Delta_{\mathcal{T}'})^{-1}$ and $(\lambda - \Delta_{\mathcal{T}})^{-1}$; and, finally, the fourth one is an immediate consequence of the continuity of u . ■

4. Riesz potentials. In this section we obtain a relationship between the Riesz potentials and the fractional powers of the negative of the Laplacean operator in the spaces \mathcal{T} and \mathcal{T}' . As a consequence, we deduce some interesting properties of the operator R_α .

LEMMA 4.1. *If $0 < \text{Re } \alpha < n/2$ and $\phi \in \mathcal{T}$, then*

$$(17) \quad (-\Delta_{\mathcal{T}})^{n-\alpha} \phi = R_\alpha (-\Delta)^n \phi = (-\Delta)^n R_\alpha \phi.$$

Proof. By (2),

$$(-\Delta_{\mathcal{T}})^{n-\alpha} \phi = \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)} \int_0^\infty \lambda^{n-\alpha-1} [(\lambda - \Delta_{\mathcal{T}})^{-1}]^n (-\Delta_{\mathcal{T}})^n \phi \, d\lambda.$$

On the other hand, as $(\lambda - \Delta_{\mathcal{T}})^{-1}$ is the Laplace transform of the heat semigroup P_t , it easily follows (see [20, p. 242]) that

$$[(\lambda - \Delta_{\mathcal{T}})^{-1}]^n \psi = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-\lambda t} (K_t * \psi) dt, \quad \psi \in \mathcal{T}.$$

If $\psi = (-\Delta_{\mathcal{T}})^n \phi$, as the \mathcal{T} -convergence implies pointwise convergence, we have

$$\begin{aligned} & ((-\Delta_{\mathcal{T}})^{n-\alpha} \phi)(x) \\ &= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)} \int_0^\infty \lambda^{n-\alpha-1} \left(\int_0^\infty t^{n-1} e^{-\lambda t} (K_t * \psi)(x) dt \right) d\lambda \end{aligned}$$

for all $x \in \mathbb{R}^n$. Interchanging the order of integration gives

$$((-\Delta_{\mathcal{T}})^{n-\alpha} \phi)(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} (K_t * \psi)(x) dt,$$

since $\Gamma(n-\alpha)^{-1} \int_0^\infty \lambda^{n-\alpha-1} e^{-\lambda t} d\lambda = t^{\alpha-n}$. Note that we can apply the Tonelli–Hobson theorem since $0 < \operatorname{Re} \alpha < n/2$ and

$$\begin{aligned} |(K_t * \psi)(x)| &\leq \|K_t\|_\infty \|\psi\|_1 = \frac{1}{(4\pi t)^{n/2}} \|\psi\|_1, \\ |(K_t * \psi)(x)| &\leq \|K_t\|_1 \|\psi\|_\infty = \|\psi\|_\infty. \end{aligned}$$

In a similar fashion

$$\int_0^\infty t^{\alpha-1} (K_t * \psi)(x) dt = \int_{\mathbb{R}^n} \left(\int_0^\infty t^{\alpha-1} K_t(y) dt \right) \psi(x-y) dy$$

since

$$\int_0^\infty t^{\alpha-1} K_t(y) dt = \frac{\Gamma(n/2 - \alpha)}{2^{2\alpha} \pi^{n/2}} |y|^{2\alpha-n}.$$

This proves the first identity. The second one follows from Lemma 3.2. ■

It is known that if $1 < p < \infty$, then $-\Delta_p$ is non-negative, with dense domain and range.

PROPOSITION 4.2. *If $1 \leq p < n/(2 \operatorname{Re} \alpha)$, then $R_\alpha f \in \overline{R(\Delta_{\mathcal{T}'})}$ for all $f \in L^p$. Moreover*

$$(18) \quad (-\Delta_{\mathcal{T}'})^{n-\alpha} f = (-\Delta)^n R_\alpha f.$$

PROOF. Let us first prove that $L^p \subset \overline{R(\Delta_{\mathcal{T}'})}$ for $1 \leq p < \infty$.

If $1 < p < \infty$, we know that L^p is the L^p -closure of $R(\Delta_p)$ which by Proposition 3.4 is included in the \mathcal{T}' -closure. Thus, $L^p \subset \overline{R(\Delta_{\mathcal{T}'})}$. Moreover, as L^p is dense in L^1 one also deduces that $L^1 \subset \overline{R(\Delta_{\mathcal{T}'})}$.

Let $f \in L^p$. By the Young inequality, the condition $1 \leq p < n/(2 \operatorname{Re} \alpha)$ implies that $R_\alpha f$ can be decomposed as $R_\alpha f = g + h$, where $g \in L^p$, $h \in L^r$ and $r > 0$ is such that $1/r < 1/p - 2 \operatorname{Re} \alpha/n$. Therefore $R_\alpha f \in \overline{R(\Delta_{\mathcal{T}'})}$.

By Theorem 3.6 and (17), the proof of (18) reduces to proving that for $f \in L^p$,

$$\int_{\mathbb{R}^n} (R_\alpha f)(x)\phi(x) dx = \int_{\mathbb{R}^n} f(x)(R_\alpha \phi)(x) dx \quad \text{for all } \phi \in \mathcal{T},$$

and this identity easily follows from the Tonelli–Hobson theorem. ■

In the following result, Δ_R denotes the restriction of the distributional Laplacean to $\overline{R(\Delta_{\mathcal{T}'})}$.

THEOREM 4.3. *If $1 \leq p < n/(2 \operatorname{Re} \alpha)$, then $L^p \subset D[(-\Delta_R)^{-\alpha}]$ and*

$$(19) \quad (-\Delta_R)^{-\alpha} f = R_\alpha f \quad \text{for all } f \in L^p.$$

PROOF. Let $f \in L^p$. By applying $(\lambda - \Delta_{\mathcal{T}'})^{-n}$ ($\lambda > 0$) to both sides of (18) and taking into account that $(-\Delta_{\mathcal{T}'})^{n-\alpha}$ commutes with this operator we obtain

$$\begin{aligned} (\lambda - \Delta_{\mathcal{T}'})^{-n}(-\Delta_{\mathcal{T}'})^{n-\alpha} f &= (-\Delta_{\mathcal{T}'})_{-\alpha}(-\Delta_{\mathcal{T}'})^n(\lambda - \Delta_{\mathcal{T}'})^{-n} f \\ &= (\lambda - \Delta_{\mathcal{T}'})^{-n}(-\Delta_{\mathcal{T}'})^n R_\alpha f. \end{aligned}$$

Since f and $R_\alpha f$ belong to $\overline{R(-\Delta_{\mathcal{T}'})}$, taking limits as $\lambda \rightarrow 0$ we conclude that $f \in D((-\Delta_{\mathcal{T}'})_{-\alpha})$ and $(-\Delta_{\mathcal{T}'})_{-\alpha} f = R_\alpha f$. Finally, from (6) one deduces (19). ■

COROLLARY 4.4 (Additivity). *If $\operatorname{Re} \alpha > 0$, $\operatorname{Re} \beta > 0$ and $\operatorname{Re}(\alpha + \beta) < n/(2p)$, then*

$$(20) \quad R_\beta R_\alpha f = R_{\alpha+\beta} f \quad \text{for all } f \in L^p.$$

PROOF. Let $f \in L^p$. The existence of $R_{\alpha+\beta} f$ and $R_\alpha f$ is evident. Moreover, from Theorem 4.3 one deduces that $R_\alpha f = (-\Delta_R)^{-\alpha} f$ and $R_{\alpha+\beta} f = (-\Delta_R)^{-\alpha-\beta} f$.

As we have already seen in the proof of Proposition 4.2, $R_\alpha f = g + h$, with $g \in L^p$ and $h \in L^r$, for all $r > 0$ such that $1/r < 1/p - 2 \operatorname{Re} \alpha/n$. It is clear that $R_\beta g$ exists and, if we take $1/r > 2 \operatorname{Re} \beta/n$, so does $R_\beta h$. Hence, $R_\beta R_\alpha f$ exists.

Theorem 4.3 implies that $R_\beta R_\alpha f = (-\Delta_R)^{-\beta} R_\alpha f$. From the additivity of the fractional powers we now deduce (20). ■

COROLLARY 4.5. *Let $X \subset L^1 + L^p$ ($1 \leq p < \infty$) be a sequentially complete locally convex space which has property (p) of Proposition 2.6 with $Y = \mathcal{T}'$. Then, if the operator $-\Delta_X$ is non-negative, the identity*

$$(21) \quad (-\Delta_X)^{-\alpha} = [R_\alpha]_X, \quad 0 < \operatorname{Re} \alpha < \frac{n}{2p},$$

holds. In particular,

$$(22) \quad (-\Delta_p)^{-\alpha} = [R_\alpha]_p.$$

PROOF. This is an immediate consequence of Theorem 4.3 and Proposition 2.6. ■

REMARK 4.1. The identity (21) can be applied in some interesting spaces such as

$$X = L^r + L^s, \quad 1 \leq r \leq s \leq p,$$

with its usual norm, or

$$X = \{f \in L^{r_1} + L^{s_1} : \Delta f \in L^{r_2} + L^{s_2}\}, \quad 1 \leq r_k \leq s_k \leq p, \quad k = 1, 2,$$

with the graph norm.

Another consequence of (21) is that R_α is one-to-one in $L^1 + L^p$.

From the general properties of the fractional powers of $-\Delta_p$ we deduce the following results:

COROLLARY 4.6. *The following properties hold:*

(i) *If $\alpha, \beta \in \mathbb{C}$ are such that $0 < \operatorname{Re} \alpha < \operatorname{Re} \beta < n/(2p)$, then*

$$D([R_\beta]_p) \subset D([R_\alpha]_p).$$

(ii) *If $1 < p < n/(2 \operatorname{Re} \alpha)$, $\beta \in \mathbb{C}$ and $\operatorname{Re} \alpha = \operatorname{Re} \beta$, then*

$$(23) \quad D([R_\alpha]_p) = D([R_\beta]_p).$$

PROOF. The first assertion follows from (22) and the additivity of the fractional powers.

On the other hand, it is known (see [15]) that if $1 < p < \infty$ and $\tau \in \mathbb{R}$, then $(-\Delta_p)^{i\tau}$ is bounded. Therefore, Proposition 2.5 yields (23). ■

Following [7] we introduce the notion of ω -sectoriality. Given $\omega \in]0, \pi]$, we say that a closed linear operator $A : D(A) \subset X \rightarrow X$ is ω -sectorial if the spectrum of A satisfies

$$\sigma(A) \subset S_\omega = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \omega\} \cup \{0\}$$

and the operators $z(z - A)^{-1}$ are uniformly bounded for $z \notin S_\omega$. Kato and Hille proved (see [6, p. 384] and [7]) that if A is ω -sectorial, $0 < \omega < \pi/2$, then $-A$ is the infinitesimal generator of an analytic semigroup of amplitude $\pi/2 - \omega$. Conversely, if $\{T(z) : z \in S_\tau \setminus \{0\}\}$, $0 < \tau \leq \pi/2$, is an analytic semigroup and $-A$ is its infinitesimal generator, then A is $(\pi/2 - \tau + \varepsilon)$ -sectorial for $0 < \varepsilon < \tau$.

It is known (see [2]) that if $1 \leq p < \infty$, then the operator Δ_p is the infinitesimal generator of the heat semigroup, which is analytic. Hence, $-\Delta_p$ is $(\pi/2 - \delta + \varepsilon)$ -sectorial for $\delta = \arctan \frac{1}{ne}$ and $0 < \varepsilon < \delta$. If $1 < p < \infty$,

by means of the Mikhlin multiplier theorem it can be proved (see [15]) that $-\Delta_p$ is ε -sectorial for all $\varepsilon > 0$.

COROLLARY 4.7 (Sectoriality). *If $1 < p < \infty$ and $0 < \alpha < n/(2p)$, then $[R_\alpha]_p$ is ε -sectorial for all $\varepsilon > 0$. Moreover*

$$\sigma([R_\alpha]_p) = [0, \infty[.$$

Consequently, $-[R_\alpha]_p$ is the infinitesimal generator of an analytic semigroup of amplitude $\pi/2$.

If $0 < \varepsilon < \delta = \arctan \frac{1}{ne}$ and $0 < \alpha < \min \{ \frac{n}{2}, \frac{\pi}{\pi/2-\delta+\varepsilon} \}$, then $-[R_\alpha]_1$ is a non-negative operator.

PROOF. It is known (see [7, Th. 2]) that if A is ω -sectorial and $0 < \alpha < \pi/\omega$, then A^α is $\alpha\omega$ -sectorial. On the other hand, from the identity $z(z + A)^{-1} = A(z^{-1} + A)^{-1}$ it follows that if A is a one-to-one, ω -sectorial operator, then A^{-1} is also ω -sectorial. Hence, by (22) we deduce the sectoriality properties of $[R_\alpha]_p$.

The identity $\sigma([R_\alpha]_p) = [0, \infty[$ follows from (22) and the spectral mapping theorem for fractional powers (see, for instance, [12]). Finally, from [7] one deduces that $-[R_\alpha]_p$ is the infinitesimal generator of an analytic semigroup of amplitude $\pi/2$. ■

REMARK 4.2. Note that $-[R_\alpha]_1$ does not generate any strongly C_0 -semigroup since its domain is not dense.

COROLLARY 4.8 (Multiplicativity). *If $1 < p < \infty$, $0 < \alpha < n/(2p)$, $\beta \in \mathbb{C}$ and $0 < \alpha \operatorname{Re} \beta < n/(2p)$, then*

$$([R_\alpha]_p)^\beta = [R_{\alpha\beta}]_p.$$

If $0 < \varepsilon < \delta = \arctan \frac{1}{ne}$, $0 < \alpha < \min \{ \frac{n}{2}, \frac{\pi}{\pi/2-\delta+\varepsilon} \}$, $\beta \in \mathbb{C}$ and $0 < \alpha \operatorname{Re} \beta < n/2$, then

$$([R_\alpha]_1)^\beta = [R_{\alpha\beta}]_1.$$

PROOF. The proof is an immediate consequence of (22) and the multiplicativity of the fractional powers (see, for instance, [19] and [14]). ■

Given $\alpha \in \mathbb{C}_+$ and $\varepsilon > 0$ we consider the function

$$G_{\alpha,\varepsilon}(x) = \frac{1}{(4\pi)^\alpha \Gamma(\alpha)} \int_0^\infty e^{-\pi|x|^2/t} e^{-\varepsilon t/(4\pi)} t^{-n/2+\alpha-1} dt, \quad x \in \mathbb{R}^n, x \neq 0.$$

It is easy to check that $G_{\alpha,\varepsilon} \in L^1$ and that its Fourier transform is $\widehat{G}_{\alpha,\varepsilon}(x) = (\varepsilon + 4\pi^2|x|^2)^{-\alpha}$ (see, for instance, [17, p. 131]).

The Bessel potential of degree α acting on a function f locally integrable on \mathbb{R}^n is defined by the convolution $B_{\alpha,\varepsilon}f = G_{\alpha,\varepsilon} * f$, if this convolution

exists. As a consequence of the Young inequality, the operator

$$[B_{\alpha,\varepsilon}]_p : L^p \rightarrow L^p, \quad f \mapsto G_{\alpha,\varepsilon} * f,$$

is bounded.

THEOREM 4.9. *If $1 \leq p < \infty$, $\varepsilon > 0$ and $\operatorname{Re} \alpha > 0$, then*

$$(\varepsilon - \Delta_p)^{-\alpha} = [B_{\alpha,\varepsilon}]_p.$$

Moreover, if $1 < p < n/(2 \operatorname{Re} \alpha)$, then

$$(24) \quad s\text{-}\lim_{\varepsilon \rightarrow 0^+} [B_{\alpha,\varepsilon}]_p = [R_\alpha]_p.$$

The operator $[R_\alpha]_1$ is a proper extension of $s\text{-}\lim_{\varepsilon \rightarrow 0^+} [B_{\alpha,\varepsilon}]_1$.

Proof. The Fourier transforms satisfy

$$(25) \quad [(\varepsilon - \Delta_p)^{-\alpha} f]^\wedge(x) = (\varepsilon + 4\pi^2|x|^2)^{-\alpha} \widehat{f}(x), \quad \text{a.e. } x \in \mathbb{R}^n, \quad f \in \mathcal{S}.$$

Since $(\varepsilon - \Delta_p)^{-\alpha}$ and $[B_{\alpha,\varepsilon}]_p$ are both bounded, by density, from (25) one deduces that $(\varepsilon - \Delta_p)^{-\alpha} = [B_{\alpha,\varepsilon}]_p$.

Finally, (24) is an immediate consequence of Propositions 2.2 and 2.4, taking into account (22). ■

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