STUDIA MATHEMATICA 135 (3) (1999)

Asymptotics for multifractal conservation laws

by

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Abstract. We study asymptotic behavior of solutions to multifractal Burgers-type equation $u_t + f(u)_x = Au$, where the operator A is a linear combination of fractional powers of the second derivative $-\partial^2/\partial x^2$ and f is a polynomial nonlinearity. Such equations appear in continuum mechanics as models with fractal diffusion. The results include decay rates of the L^p-norms, $1 \le p \le \infty$, of solutions as time tends to infinity, as well as determination of two successive terms of the asymptotic expansion of solutions.

1. Motivation and results. The goal of this paper is to study the large-time behavior of solutions of the Cauchy problem for a class of equations, called here *multifractal conservation laws*:

$$u_t + f(u)_x = Au, \tag{1.1}$$

where $x \in \mathbb{R}$, $t \ge 0$, $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, $f(u)$ is a polynomially bounded nonlinear term, and

$$A = c_0 \frac{\partial^2}{\partial x^2} - \sum_{j=1}^{N} c_j \left(- \frac{\partial^2}{\partial x^2} \right)^{\alpha_j/2}, \tag{1.2}$$

with $c_0, c_j \ge 0$, is the diffusion operator including fractional powers of order $\alpha_j/2$, $0 < \alpha_j < 2$, of the square root of the second derivative with respect to x, related to Lévy stochastic processes (see, e.g., [26], [15]). The problem (1.1)–(1.2) is a generalization of the one-dimensional *Burgers equation* (see, e.g., [6])

$$u_t + 2uu_x = u_{xx}. \tag{1.3}$$

The classical Burgers equation (1.3) has been used in various physical contexts, where shock creation is an important phenomenon. These applications vary from growth of molecular interfaces ([14]), through simplified

1991 *Mathematics Subject Classification*: 35K,35B40, 35Q.

Key words and phrases: generalized Burgers equation, fractal diffusion, asymptotics of solutions.

hydrodynamic models ([1]), to the mass distribution in the large scale of the Universe ([18]). For a general overview, see [25], and [29] for the Burgers turbulence problem.

Nonlocal Burgers-type equations similar to (1.1)–(1.2) appeared as model equations simplifying the multidimensional Navier–Stokes system with modified dissipativity ([1]), describing hereditary effects for nonlinear acoustic waves ([27]), and modeling interfacial growth mechanisms which would include trapping surface effects ([17], [28]). A variety of physically motivated linear fractal differential equations with applications to hydrodynamics, statistical physics and molecular biology can be found in [21] and [24]. An introduction to the fractional derivatives calculus and fractal relaxation models are in [20]. Nonlinear wave equations with fractional derivatives terms describing dispersive effects have been used and rigorously studied even earlier than those with fractal diffusion (see, e.g., [22], [19], [5] and [8]).

The well-known Hopf–Cole formula permits one to simplify the classical Burgers equation (1.3) reducing it to the linear heat equation. This leads, e.g., to a rapid determination of large time asymptotics of solutions to (1.3) described by source-type solutions (see comments below Theorem 1.1). Such a simplification is no longer available when nonlocal equations (1.1) are studied.

The recent paper [2] dealt with basic mathematical issues of existence, uniqueness and asymptotics of solutions to multidimensional versions of (1.1) with purely fractal diffusion. The methods used there include weak solutions and energy estimates, mild solutions approach in Morrey and Besov spaces and a self-similar solution analysis.

Our aim in this paper is to describe the long time behavior of solutions to (1.1) in a manner more precise than the one employed in [2]. In particular, we study the influence of various dissipative terms $-(-\partial^2/\partial x^2)^{\alpha_j/2}u$ in (1.1) on time asymptotics of solutions; the latter turns out to be different from that for the usual Brownian diffusion described by the term u_{xx}.

The technical tools used here include those applied in [30] and [7] to parabolic type equations, and those developed in [13] for a completely different class of equations featuring dispersive effects as well as dissipation. We expect that these versatile methods would be useful in a further study of nonlinear Markov processes and propagation of chaos associated with fractal Burgers equation (see [12]). There, as well as in [4] and [3], the reader may find more motivations to study stochastic aspects of nonlocal evolution equations with fractal diffusion, and their finite particle systems approximations.

To focus attention on an equation simpler than (1.1), consider the Cauchy problem for the *fractal Burgers-type equation*

$$u_t - u_{xx} + D^\alpha u + 2uu_x = 0 \tag{1.4}$$

with initial condition

$$u(x,0) = u_0(x), \tag{1.5}$$

where $D^\alpha = (-\partial^2/\partial x^2)^{\alpha/2}$ is the fractional symmetric derivative of order $\alpha \in (0,2)$ defined via the Fourier transform by $(\widehat{D^\alpha v})(\xi) = |\xi|^\alpha \widehat{v}(\xi)$, and the nonlinear term corresponds to $f(u) = u^2$. Note that the fractional derivative $\partial^\alpha/\partial x^\alpha$ studied in [20], [21] has a different meaning than our D^α. Equation (1.4) is, however, representative of the general class of equations (1.1) with Brownian diffusion, one ($N = 1$) fractal diffusion term and a genuinely nonlinear f such that $f(0) = f'(0) = 0, f''(0) \neq 0$. Indeed, scaling u, x, and t, we may get rid of all unimportant constants $c_0, c_1, f''(0)$. Comments on the general case with $N \geq 2$ and polynomially bounded nonlinearities f can be found in the last Section 6.

Our functional framework for (1.1) is that of Lebesgue $L^p(\mathbb{R})$ spaces. However, there are other, more general, function spaces suitable for studying (1.1) (see e.g. [2]), and we refer the reader to [9] for a recent work on the classical Burgers equation (1.3) with irregular initial data.

The results of the paper give the first two terms of the large-time asymptotics for the solutions of (1.4)–(1.5) and can be summarized as follows.

THEOREM 1.1. *Let $0 < \alpha < 2$. Assume that u is a solution to the Cauchy problem (1.4)–(1.5) with $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. For every $p \in [1,\infty]$ there exists a constant C such that*

$$\|u(t) - e^{tA} * u_0\|_p \leq C \begin{cases} t^{-(1-1/p)/\alpha - 2/\alpha + 1} & \textit{for } 1 < \alpha < 2, \\ t^{-(1-1/p)/\alpha - 1/\alpha} \log(1+t) & \textit{for } \alpha = 1, \\ t^{-(1-1/p)/\alpha - 1/\alpha} & \textit{for } 0 < \alpha < 1, \end{cases} \tag{1.6}$$

for all $t > 0$. Here e^{tA} denotes the (integral kernel of the) semigroup generated by the operator $A = \partial^2/\partial x^2 - D^\alpha$, so that $v = e^{tA} * u_0$ solves the linear equation $v_t = v_{xx} - D^\alpha v$ with the initial condition $v(0) = u_0$.

In other words, the first term of the asymptotic expansion of solution is given by the solution to the linear equation. Note (see e.g. [11]) that the asymptotics of solutions to the Cauchy problem for the Burgers equation (1.3) is described by the relation

$$t^{(1-1/p)/2} \|u(\cdot,t) - U_M(\cdot,t)\|_p \to 0 \quad \text{as } t \to \infty,$$

where

$$U_M(x,t) = t^{-1/2} \exp(-x^2/(4t)) \Big(K + \frac{1}{2} \int_0^{x/(2\sqrt{t})} \exp(-\xi^2/4)\, d\xi \Big)^{-1}$$

is the so-called source solution such that $\int_{\mathbb{R}} U_M(x,1)\,dx = M$ (to each $M>0$ there corresponds a constant K). Thus, the long time behavior of solutions to the classical Burgers equation is genuinely nonlinear, i.e., it is not determined by the asymptotics of the linear heat equation.

The second term of the asymptotics of a solution u to (1.4) has a different form in each of the three cases: $1<\alpha<2$, $\alpha=1$ and $0<\alpha<1$.

THEOREM 1.2. *Assume that $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and that $p \in [1,\infty]$.*

(i) *If $1<\alpha<2$ then*

$$t^{(1-1/p)/\alpha+2/\alpha-1}\Big\|u(t)-e^{tA}*u_0+\int_0^t \partial_x p_\alpha(t-\tau)*(Mp_\alpha(\tau))^2\,d\tau\Big\|_p \to 0 \tag{1.7}$$

as $t\to\infty$, where $M=\int_{\mathbb{R}} u_0(x)\,dx$ and

$$p_\alpha(x,t) = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-t|\xi|^\alpha+ix\xi}\,d\xi \tag{1.8}$$

is the kernel of the semigroup solving the linear equation $v_t=-D^\alpha v$.

(ii) *If $\alpha=1$ then*

$$\frac{t^{2-1/p}}{\log t}\|u(t)-e^{tA}*u_0+(\log t)M^2(2\pi)^{-1}\partial_x p_1(t)\|_p \to 0 \tag{1.9}$$

as $t\to\infty$.

(iii) *If $0<\alpha<1$ then*

$$t^{(1-1/p)/\alpha+1/\alpha}\Big\|u(t)-e^{tA}*u_0+\Big(\int_0^\infty\int_{\mathbb{R}} u^2(y,\tau)\,dy\,d\tau\Big)\,\partial_x p_\alpha(t)\Big\|_p \to 0 \tag{1.10}$$

as $t\to\infty$.

Observe that, in general (if $\int_{\mathbb{R}} u_0(x)\,dx \neq 0$), the estimates in Theorems 1.1–1.2 cannot be improved because $\|\partial_x p_\alpha(t)\|_p = Ct^{-(1-1/p)/\alpha-1/\alpha}$.

The subsequent sections deal with an analysis of the linearized equation (Section 2), solvability of the problem (1.4)–(1.5) and decay of solutions (Section 3), and the proofs of Theorem 1.1 (Section 4) and Theorem 1.2 (Section 5). Finally, Section 6 deals with the general multifractal conservation laws (1.1).

Throughout this paper we use the notation $\|u\|_p$ for the Lebesgue $L^p(\mathbb{R})$-norms of functions. The constants independent of solutions considered and of t will be denoted by the same letter C, even if they may vary from line to line. For a variety of facts from the theory of parabolic type equations and interpolation inequalities we refer to [16].

2. Analysis of the linear equation. The goal of this section is to gather several properties of solutions to the linear Cauchy problem

$$u_t - Au \equiv u_t - u_{xx} + D^\alpha u = 0, \tag{2.1}$$

$$u(x,0) = u_0(x). \tag{2.2}$$

These properties will be used in the proof of asymptotic results for the full fractal Burgers equation (1.4).

Using the Fourier transform we immediately deduce that, with sufficiently regular u_0, each solution to (2.1)–(2.2) has the form

$$u(x,t) = p_\alpha(t) * p_2(t) * u_0(x) \equiv e^{tA} * u_0(x), \tag{2.3}$$

where p_α is defined in (1.8). Here we identify the analytic semigroup e^{tA} with its kernel $p_\alpha * p_2$, a smooth function, decaying like $|x|^{-1-\alpha}$ for $|x| \to \infty$ (cf., e.g., [15]). Note that for $\alpha = 2$, the heat kernel

$$p_2(x,t) = (4\pi t)^{-1/2} \exp(-|x|^2/(4t))$$

decays exponentially in x. It is easy to see (by a change of variables in (1.8)) that p_α has the self-similarity property:

$$p_\alpha(x,t) = t^{-1/\alpha} p_\alpha(xt^{-1/\alpha}, 1).$$

Recall that $p_\alpha(x,1), \partial_x p_\alpha(x,1) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for every $0 < \alpha \le 2$. Moreover, $\|p_\alpha(\cdot,t)\|_1 = 1$ for all $t > 0$.

Our first lemma determines exponents γ in the L^q-L^p estimates

$$\|e^{tA} * u_0\|_p \le Ct^{-\gamma} \|u_0\|_q$$

for the semigroup e^{tA} associated with (2.1).

Lemma 2.1. *For every $p \in [1,\infty]$, there exists a positive constant C independent of t such that*

$$\|e^{tA}\|_p = \|p_2(t) * p_\alpha(t)\|_p \le C \min\{t^{-(1-1/p)/2}, t^{-(1-1/p)/\alpha}\} \tag{2.4}$$

and

$$\begin{aligned} \|\partial_x e^{tA}\|_p &= \|\partial_x (p_2(t) * p_\alpha(t))\|_p \\ &\le C \min\{t^{-(1-1/p)/2-1/2}, t^{-(1-1/p)/\alpha-1/\alpha}\} \end{aligned} \tag{2.5}$$

for every $t > 0$.

Proof. The proof is elementary and based on the Young inequality

$$\|h * g\|_p \le \|h\|_q \|g\|_r, \tag{2.6}$$

valid for all $p, q, r \in [1,\infty]$ satisfying $1 + 1/p = 1/q + 1/r$, and all $h \in L^q(\mathbb{R})$, $g \in L^r(\mathbb{R})$. Additionally, one uses the self-similarity of p_α and the well-known property of convolution: $\partial_x(p_2 * p_\alpha) = (\partial_x p_2) * p_\alpha = p_2 * (\partial_x p_\alpha)$. ∎

Next we give a decomposition lemma which can be used for approximations and expansions of e^{tA} (see Corollaries 2.1, 2.2 below).

LEMMA 2.2. *For each nonnegative integer N there exists a constant C such that for every $h \in L^1(\mathbb{R}, (1+|x|)^{N+1}\,dx)$, $g \in C^{N+1}(\mathbb{R}) \cap W^{1,N+1}(\mathbb{R})$, and $p \in [1,\infty]$ we have*

$$\Big\| h * g(\cdot) - \sum_{k=0}^{N} \frac{(-1)^k}{k!} \Big(\int_{\mathbb{R}} h(y) y^k \, dy \Big) \partial_x^k g(\cdot) \Big\|_p \le C \|\partial_x^{N+1} g\|_p \|h\|_{L^1(\mathbb{R}, |x|^{N+1}\,dx)}. \tag{2.7}$$

Proof. The result is obtained by a straightforward application of the Taylor expansion of $g(x-y)$ and the Young inequality (2.6). This lemma is a particular case of a more general result proved in [10]. ∎

COROLLARY 2.1. *For every $p \in [1,\infty]$ there exists $C > 0$, independent of t, such that*

$$\|e^{tA} - p_\alpha(t)\|_p \le \|\partial_x p_\alpha(t)\|_p \|p_2(t)\|_{L^1(|x|\,dx)} \le C t^{-(1-1/p)/\alpha + 1/2 - 1/\alpha} \tag{2.8}$$

and

$$\|\partial_x (e^{tA} - p_\alpha(t))\|_p \le C t^{-(1-1/p)/\alpha + 1/2 - 2/\alpha} \tag{2.9}$$

for all $t > 0$.

Proof. Apply (2.7) with $N = 0$, $h(x) = p_2(x,t)$ (remember that $\int_{\mathbb{R}} p_2(x,t)\,dx = 1$), and $g(x) = p_\alpha(x,t)$ to show (2.8). To get (2.9), put $g(x) = \partial_x p_\alpha(x,t)$. ∎

COROLLARY 2.2. *Assume that $u_0 \in L^1(\mathbb{R})$ and set $M = \int_{\mathbb{R}} u_0(x)\,dx$. For every $p \in [1,\infty]$ there exists a nonnegative function $\eta \in L^\infty(0,\infty)$ satisfying $\lim_{t\to\infty} \eta(t) = 0$ and such that*

$$\|e^{tA} * u_0 - M p_\alpha(t)\|_p \le t^{-(1-1/p)/\alpha} \eta(t) \quad \text{for all } t > 0. \tag{2.10}$$

Proof. The obvious inequality

$$\|e^{tA} * u_0 - M p_\alpha(t)\|_p \le t^{-(1-1/p)/\alpha} (C\|u_0\|_1 + M)$$

can be improved to show that

$$t^{(1-1/p)/\alpha} \|e^{tA} * u_0 - M p_\alpha(t)\|_p \to 0 \quad \text{as } t \to \infty.$$

Indeed, first consider $u_0 \in L^1(\mathbb{R}, (1+|x|)\,dx)$. Using (2.7) we immediately obtain

$$\|p_\alpha(t) * u_0 - M p_\alpha(t)\|_p \le C t^{-(1-1/p)/\alpha - 1/\alpha} \|u_0\|_{L^1(|x|\,dx)}. \tag{2.11}$$

Now

$$t^{(1-1/p)/\alpha} \|e^{tA} * u_0 - M p_\alpha(t)\|_p \le t^{(1-1/p)/\alpha} \|e^{tA} - p_\alpha(t)\|_p \|u_0\|_1 + t^{(1-1/p)/\alpha} \|p_\alpha(t) * u_0 - M p_\alpha(t)\|_p.$$

Applying (2.11) and (2.8) we can easily see that the right-hand side tends to 0 as $t \to \infty$. By a standard density argument, this result extends to every $u_0 \in L^1(\mathbb{R})$. ■

3. Existence of solutions and preliminary estimates. By a solution to the Cauchy problem for the fractal Burgers equation (1.4)-(1.5) we mean a *mild solution*, i.e., a function $u \in C([0,T];X)$ satisfying the Duhamel formula

$$u(t) = e^{tA} * u_0 - \int_0^t \partial_x e^{(t-\tau)A} * u^2(\tau)\,d\tau \tag{3.1}$$

for each $t \in (0,T)$. Here X is a suitable Banach space such that e^{tA} acts as a strongly continuous semigroup in X. However, our preferred choice is $X = L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, which leads to a small modification of the above definition. Because of poor properties of e^{tA} on $L^\infty(\mathbb{R})$ (cf. a similar situation in [2]), we need u to belong to a larger space $\mathcal{C}([0,T];X)$ of *weakly* continuous functions with values in X.

THEOREM 3.1. *Assume that* $0 < \alpha < 2$. *Given* $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, *there exists a unique mild solution* $u = u(x,t)$ *to the problem* (1.4)–(1.5) *in the space* $\mathcal{C}([0,\infty); L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$. *This solution satisfies the inequalities*

$$\|u(t)\|_1 \le \|u_0\|_1, \tag{3.2}$$

$$\|u(t)\|_2 \le C(1+t)^{-1/(2\alpha)}, \tag{3.3}$$

for all $t > 0$ *and a constant* $C > 0$.

Proof. We define the operator

$$N(u)(t) = e^{tA}u_0 - \int_0^t \partial_x e^{(t-\tau)A} * u^2(\tau)\,d\tau$$

and the Banach space

$$X_T = L^\infty((0,T); L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$$

equipped with the norm $\|u\|_{X_T} = \sup_{0<t<T} \|u(t)\|_1 + \sup_{0<t<T} \|u(t)\|_\infty$. Now the local-in-time mild solution to (1.4)–(1.5) is obtained, *via* the Banach contraction theorem, as a fixed point of N in the ball $B_R = \{u \in X_T : \|u\|_{X_T} \le R\}$, for sufficiently large R and small $T > 0$. This is an immediate consequence of the inequalities

$$\|N(u)\|_{X_T} \le CR^2T^{1/2},$$

$$\|N(u) - N(v)\|_{X_T} \le CRT^{1/2}\|u - v\|_{X_T},$$

valid for any $u, v \in B_R$. Here we used L^1- and L^∞-bounds for the linear semigroup e^{tA} from Lemma 2.1.

The classical regularity result allows us to prove that

$$u \in C((0,T); W^{2,p}(\mathbb{R})) \cap C^1((0,T); L^p(\mathbb{R}))$$

for every $p \in (1,\infty)$. Indeed, we have $u_t - u_{xx} = -D^\alpha u - uu_x \in W^{-\alpha,p}(\mathbb{R}) \cap W^{-1,p}(\mathbb{R})$, so by repeated use of [16, Chapter 3], the regularity of solutions follows. Weak continuity of $u(t)$ at $t = 0$ is a standard consequence of properties of e^{tA} on X.

Note that, since our $L^1 \cap L^\infty$ mild solutions do fit into the framework of Sobolev spaces in [16], from now on we may use energy estimates which are standard for weak solutions.

The local solution constructed above may be extended to a global one provided the following estimate holds:

$$\sup_{t\in[0,T_*)} (\|u(t)\|_1 + \|u(t)\|_\infty) < \infty, \tag{3.4}$$

where T_* is the maximal time of existence of $u(t)$. We are going to prove that this is actually our case.

First, inequality (3.2) is obtained by multiplying (1.4) by $\operatorname{sgn} u$ and integrating over $\mathbb{R}$ with respect to x. Details are given in [2, Thm. 3.1].

After multiplying (1.4) by u, a similar calculation gives

$$\|u(t)\|_2 \le \|u_0\|_2. \tag{3.5}$$

By Lemma 2.1 and (3.5), we have

$$\begin{aligned}
\|e^{tA} * u_0\|_p &\le \|u_0\|_p, \\
\|\partial_x e^{(t-\tau)A} * u^2(\tau)\|_p &\le \|\partial_x e^{(t-\tau)A}\|_p \|u^2(\tau)\|_1 \\
&\le C(t-\tau)^{-(1-1/p)/2-1/2}\|u_0\|_2^2
\end{aligned}$$

for all $p \in [1,\infty]$. Now, computing the L^p-norm of (3.1) for $p \in [1,\infty)$ we obtain

$$\begin{aligned}
\|u(t)\|_p &\le \|e^{tA} * u_0\|_p + \int_0^t \|\partial_x e^{(t-\tau)A} * u^2(\tau)\|_p \, d\tau \\
&\le \|u_0\|_p + Ct^{1/(2p)}\|u_0\|_2^2
\end{aligned} \tag{3.6}$$

for every $t \in [0,T_*)$ (by the way, this inequality will be improved below; cf. the proof of Theorem 1.1). Next, we use (3.6) for $p = 4$ to show that

$$\begin{aligned}
\|u(t)\|_\infty &\le \|e^{tA} * u_0\|_\infty + \int_0^t \|\partial_x e^{(t-\tau)A} * u^2(\tau)\|_\infty \, d\tau \\
&\le \|e^{tA} * u_0\|_\infty + \int_0^t (t-\tau)^{-3/4}(\|u_0\|_4 + C\tau^{1/8}\|u_0\|_2^2)^2 \, d\tau \\
&\le C(1 + t^{1/2}).
\end{aligned} \tag{3.7}$$

Finally, combining (3.2) with (3.7) we get (3.4).

The proof of the L^2-decay estimate (3.3) is based on the Fourier splitting method introduced in [23]; it can be found in [2, Thm. 4.1]. An alternative approach to the L^2-decay estimates for parabolic equations might follow [11]. ■

For $\alpha \geq 1$ we have a better L^∞-bound for u.

PROPOSITION 3.1. *Under the assumptions of Theorem* 3.1 *with* $\alpha \geq 1$ *the maximum principle*

$$\operatorname{ess\,inf} u_0 \leq u(x,t) \leq \operatorname{ess\,sup} u_0 \tag{3.8}$$

holds for the solution u *to the problem* (1.4)–(1.5).

Proof. Set $v_+ = \max\{v,0\}$ and $m = \operatorname{ess\,sup} u_0$. Given $\varepsilon > 0$, we multiply (1.4) by $g = (u-m-\varepsilon)_+$ and integrate over $\mathbb{R}$ to get

$$\int_{\mathbb{R}} u_t g\,dx + \int_{\mathbb{R}} (-u_{xx} + D^\alpha u) g\,dx + \int_{\mathbb{R}} g u u_x\,dx = 0. \tag{3.9}$$

Now note that $u_t = g_t$ and $u_x = g_x$ on the support of g so that $\int_{\mathbb{R}} u_t g\,dx = \int_{\mathbb{R}} g_t g\,dx$ and $\int_{\mathbb{R}} g u u_x\,dx = \int_{\mathbb{R}} g(g+m+\varepsilon) g_x\,dx = 0$. Next, using the relation $D^\alpha 1 = 0$ for $\alpha \geq 1$ and the Plancherel identity we obtain

$$\int_{\mathbb{R}} g D^\alpha u\,dx = \int_{\mathbb{R}} g D^\alpha g\,dx = \int_{\mathbb{R}} (D^{\alpha/2} g)^2\,dx.$$

Inserting this into (3.9) and integrating over $[0,t]$ we get

$$\int_{\mathbb{R}} g^2(x,t)\,dx + 2\int_0^t \int_{\mathbb{R}} ((D^{\alpha/2} g(x,s))^2 + (g_x(x,s))^2)\,dx\,ds \leq 0,$$

so, in particular, $\int_{\mathbb{R}} g^2(x,t)\,dx = 0$ follows. Since ε was arbitrary, we conclude that $(u-m)_+ = 0$. Repeating the argument above with the function $g = (u+m+\varepsilon)_- = \min\{u+m+\varepsilon, 0\}$, where $m = -\operatorname{ess\,inf} u_0$, we obtain (3.8).

Obviously, this proof extends to general nonlinearities f. ■

Note that the above proof does not apply to the case $0 < \alpha < 1$ because, e.g., $D^\alpha 1(x) = C_\alpha |x|^{-\alpha} \not\equiv 0$ (cf. [20] and [21]). In fact, we cannot expect that the maximum principle holds for solutions of arbitrary multifractal conservation laws (1.1) and more general initial conditions $u_0 \in L^\infty(\mathbb{R})$. Indeed, for $c_0 = 0$ in (1.2) the results in [2, Section 5] on the nonexistence of traveling waves suggest that the maximum principle fails for solutions to (1.1) with u_0 merely in $L^\infty(\mathbb{R})$.

These difficulties are connected with the fact that for $0 < \alpha < 1$ and $u_0 \in L^\infty(\mathbb{R})$ the linear problem (2.1)–(2.2) is not, in general, equivalent to (2.3). For instance, $e^{tA} * 1 \equiv 1$ but $u \equiv 1$ does not satisfy (2.1) if $0 < \alpha < 1$.

4. The first order term of asymptotics for fractal Burgers equation. We begin by proving that the asymptotics of solutions to (1.4) is, in the first approximation, linear. First, we formulate

LEMMA 4.1. *Under the assumptions of Theorem* 1.1, *for every* $p\in[1,\infty]$, *there exists a nonnegative function* $\eta\in L^\infty(0,\infty)$ *satisfying* $\lim_{t\to\infty}\eta(t)=0$ *and such that*

$$\|u(t)-e^{tA}*u_0\|_p\le(1+t)^{-(1-1/p)/\alpha}\eta(t)\quad \textit{for all } t>0. \tag{4.1}$$

Proof. From the construction of solutions to (1.4) we have $\sup_{0<t<T}\|u(t)\|_p<\infty$ for every $T<\infty$. Moreover, (2.4) implies that $\|e^{tA}*u_0\|_p\le\|e^{tA}\|_1\|u_0\|_p=\|u_0\|_p$. Hence, to prove Lemma 4.1, it suffices to consider large t and show that

$$t^{(1-1/p)/\alpha}\|u(t)-e^{tA}*u_0\|_p\to0\quad \text{as } t\to\infty. \tag{4.2}$$

To do this, note that by the integral equation (3.1), it remains to estimate the L^p-norm of

$$\int_0^t\partial_xe^{(t-\tau)A}*u^2(\tau)\,d\tau=\int_0^{t/2}\ldots\,d\tau+\int_{t/2}^t\ldots\,d\tau \tag{4.3}$$

for $t\ge1$.

For $\tau\in[0,t/2]$, we use the Young inequality, (2.5) and (3.3) as follows:

$$\begin{aligned}\|\partial_xe^{(t-\tau)A}*u^2(\tau)\|_p&\le\|\partial_xe^{(t-\tau)A}\|_p\|u^2(\tau)\|_1\\&\le C(t-\tau)^{-(1-1/p)/\alpha-1/\alpha}(1+\tau)^{-1/\alpha}.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\Big\|\int_0^{t/2}\partial_xe^{(t-\tau)A}*u^2(\tau)\,d\tau\Big\|_p \tag{4.4}\\
&\quad\le C\int_0^{t/2}(t-\tau)^{-(1-1/p)/\alpha-1/\alpha}(1+\tau)^{-1/\alpha}\,d\tau\\
&\quad\le C(t/2)^{-(1-1/p)-1/\alpha}\int_0^{t/2}(1+\tau)^{-1/\alpha}\,d\tau\\
&\quad\le C\begin{cases}t^{-(1-1/p)/\alpha-2/\alpha+1} & \text{for } 1<\alpha<2,\\ t^{-(1-1/p)/\alpha-1/\alpha}\log(1+t) & \text{for } \alpha=1,\\ t^{-(1-1/p)/\alpha-1/\alpha} & \text{for } 0<\alpha<1.\end{cases}\end{aligned}$$

Now, it is easy to see that $t^{(1-1/p)/\alpha}\|\int_0^{t/2}\ldots\,d\tau\|_p\to0$ as $t\to\infty$.

To deal with the integrand over $[t/2,t]$, we consider first $p\in[1,\infty)$. Using (2.5) and (3.3) we have

$$\begin{aligned}\|\partial_x e^{(t-\tau)A}*u^2(\tau)\|_p &\le \|\partial_x e^{(t-\tau)A}\|_p\|u^2(\tau)\|_1\\ &\le C(t-\tau)^{-(1-1/p)/2-1/2}(1+\tau)^{-1/\alpha}.\end{aligned}$$

Hence, as before,

$$\begin{aligned}\Big\|\int\limits_{t/2}^{t}\partial_x e^{(t-\tau)A}*u^2(\tau)\,d\tau\Big\|_p &\le C\int\limits_{t/2}^{t}(t-\tau)^{-(1-1/p)/2-1/2}(1+\tau)^{-1/\alpha}\,d\tau\\ &\le Ct^{1/(2p)-1/\alpha}.\end{aligned}$$

Now, for $p\in[1,\infty)$, using the assumption $0<\alpha<2$, we see that

$$t^{(1-1/p)/\alpha}\Big\|\int\limits_{t/2}^{t}\ldots\,d\tau\Big\|_p\to 0\qquad\text{as } t\to\infty.$$

To handle the case $p=\infty$, note that it follows from (4.2) (already proved for $p\in[1,\infty)$) and from (2.4) that

$$\|u(t)\|_p\le\|u(t)-e^{tA}*u_0\|_p+\|e^{tA}*u_0\|_p\le C(1+t)^{-(1-1/p)/\alpha}\tag{4.5}$$

for $1\le p<\infty$. Using this estimate for $p=4$ and applying (2.5) we prove that

$$\begin{aligned}\Big\|\int\limits_{t/2}^{t}\partial_x e^{(t-\tau)A}*u^2(\tau)\,d\tau\Big\|_\infty &\le \int\limits_{t/2}^{t}\|\partial_x e^{(t-\tau)A}\|_2\|u^2(\tau)\|_2\,d\tau\\ &\le C\int\limits_{t/2}^{t}(t-\tau)^{-3/4}(1+\tau)^{-3/(2\alpha)}\,d\tau\\ &\le Ct^{1/4-3/(2\alpha)}.\end{aligned}$$

Since $1/4-3/(2\alpha)<-1/\alpha$, this concludes the proof of Lemma 4.1. ∎

Proof of Theorem 1.1. To get (1.6), we use the decomposition (4.3). Note that the integral over $[0,t/2]$ is already estimated in (4.4). For the second integral, we use (4.5) as follows:

$$\begin{aligned}\Big\|\int\limits_{t/2}^{t}\partial_x e^{(t-\tau)A)}*u^2(\tau)\,d\tau\Big\|_p &\le \int\limits_{t/2}^{t}\|\partial_x e^{(t-\tau)A}\|_1\|u(\tau)\|_{2p}^2\,d\tau\\ &\le C(t/2)^{-(1-1/p)/\alpha-1/\alpha}\int\limits_{t/2}^{t}\|\partial_x e^{(t-\tau)A}\|_1\,d\tau.\end{aligned}$$

Now, for $1<\alpha<2$, we have immediately

$$\int\limits_{t/2}^{t}\|\partial_x e^{(t-\tau)A}\|_1\,d\tau\le C\int\limits_{t/2}^{t}(t-\tau)^{-1/\alpha}\,d\tau\le Ct^{1-1/\alpha}.$$

For $0 < \alpha \le 1$, using (2.5), we obtain

$$\int_{t/2}^{t} \|\partial_x e^{(t-\tau)A)}\|_1 \, d\tau \le C \int_{t/2}^{t-1} (t-\tau)^{-1/\alpha} \, d\tau + C \int_{t-1}^{t} (t-\tau)^{-1/2} \, d\tau.$$

It is easy to see that, if $0 < \alpha < 1$, both integrals on the right-hand side are uniformly bounded for $t \ge 1$. However, for $\alpha = 1$, the first of them grows as $\log(1+t)$. ■

REMARK 4.1. In the sequel, we will often use the estimate

$$\begin{aligned}\text{(4.6)}\quad \|u^2(t) - (e^{tA} * u_0)^2\|_p &\le C\|u(t) - e^{tA} * u_0\|_p (\|u(t)\|_\infty + \|e^{tA} * u_0\|_\infty) \\ &\le C(1+t)^{-(1-1/p)/\alpha - 1/\alpha} \eta(t),\end{aligned}$$

where η satisfies the assumptions in Lemma 4.1. It follows immediately from (4.1) and (4.5). Moreover, if we use (2.10), then (4.6) may be reformulated as

$$\|u^2(t) - (Mp_\alpha(t))^2\|_p \le Ct^{-(1-1/p)/\alpha - 1/\alpha} \eta(t). \tag{4.7}$$

5. The second order term of asymptotics for fractal Burgers equation. In this section we find the second term of the asymptotic expansion of solutions to (1.4). This term reflects nonlinear effects.

Proof of Theorem 1.2(i). Let $1 < \alpha < 2$. By the integral representation (3.1) of solutions to (1.4)–(1.5), it suffices to estimate the L^p-norm of the difference

$$\begin{aligned}\int_0^t \partial_x e^{(t-\tau)A} * u^2(\tau) \, d\tau - \int_0^t \partial_x p_\alpha(t-\tau) * (Mp_\alpha(\tau))^2 \, d\tau \\ = \int_0^t \partial_x e^{(t-\tau)A} * (u^2(\tau) - (Mp_\alpha(\tau))^2) \, d\tau \\ + \int_0^t \partial_x (e^{(t-\tau)A} - p_\alpha(t-\tau)) * (Mp_\alpha(\tau))^2 \, d\tau \\ \equiv I_1(t) + I_2(t).\end{aligned}$$

Now, as in the proof of Theorem 1.1, we split the range of integration in $I_1(t)$ and in $I_2(t)$ into two parts: $[0, t/2]$ and $[t/2, t]$. Next, the Young inequality, combined with (2.4), (2.5) and (4.7), is applied to obtain appropriate estimates of the integrands in $I_1(t)$ and $I_2(t)$.

First we estimate $I_1(\cdot)$. By (4.7), the L^p-norm of the integrand can be bounded either by

$$\|\partial_x e^{(t-\tau)A}\|_p \|u^2(\tau) - (Mp_\alpha(\tau))^2\|_1 \le C(t-\tau)^{-(1-1/p)/\alpha - 1/\alpha} \tau^{-1/\alpha} \eta(\tau),$$

or by

$$\|\partial_x e^{(t-\tau)A}\|_1 \|u^2(\tau) - (Mp_\alpha(\tau))^2\|_p \le C(t-\tau)^{-1/\alpha}\tau^{-(1-1/p)/\alpha-1/\alpha}\eta(\tau).$$

Consequently, by the change of variables $\tau = st$, it follows that for $p \in [1,\infty]$,

$$t^{(1-1/p)/\alpha-(1-2/\alpha)}\|I_1(t)\|_p \le C\int_0^{1/2}(1-s)^{-(1-1/p)/\alpha-1/\alpha}s^{-1/\alpha}\eta(st)\,ds + C\int_{1/2}^{1}(1-s)^{-1/\alpha}s^{-(1-1/p)/\alpha-1/\alpha}\eta(st)\,ds.$$

By the Lebesgue Dominated Convergence Theorem, the right-hand side of the above expression tends to 0 as $t \to \infty$.

A similar argument applies to $I_2(\cdot)$. Indeed, for $\tau \in [0, t/2]$, we use (2.9) to bound the integrand in $I_2(t)$ by

$$\|\partial_x(e^{(t-\tau)A} - p_\alpha(t-\tau))\|_p \|(Mp_\alpha(\tau))^2\|_1 \le C(t-\tau)^{-(1-1/p)/\alpha+1/2-2/\alpha}\tau^{-1/\alpha}.$$

When $\tau \in [t/2, t]$, the integrand is bounded by

$$\|e^{(t-\tau)A} - p_\alpha(t-\tau)\|_1 \|\partial_x((Mp_\alpha(\tau))^2)\|_p \le C(t-\tau)^{1/2-1/\alpha}(1+\tau)^{-(1-1/p)/\alpha-2/\alpha}.$$

Using these two estimates, it is easy to prove that

$$t^{(1-1/p)/\alpha+2/\alpha-1}\|I_2(t)\|_p \to 0 \quad \text{as } t \to \infty,$$

for every $p \in [1,\infty]$. Now Theorem 1.2(i) is proved. ■

Proof of Theorem 1.2(ii). Let $\alpha = 1$. Recall that $p_1(x,1)$ is the well-known Cauchy distribution $p_1(x,1) = (\pi(1+x^2))^{-1}$. We begin the proof of Theorem 1.2(ii) by showing that

$$\lim_{t\to\infty}\frac{1}{\log t}\int_0^{t-1}\int_{\mathbb{R}} u^2(y,\tau)\,dy\,d\tau = \frac{M^2}{2\pi}. \tag{5.1}$$

Indeed, since by (4.5),

$$\int_0^1\int_{\mathbb{R}} u^2(y,\tau)\,dy\,d\tau + \int_{t-1}^{t}\int_{\mathbb{R}} u^2(y,\tau)\,dy\,d\tau \le C\Big(\int_0^1 + \int_{t-1}^{t}\Big)(1+\tau)^{-1}\,d\tau \le C$$

with C independent of t, it is sufficient to prove that

$$\frac{1}{\log t}\int_1^t\int_{\mathbb{R}} u^2(y,\tau)\,dy\,d\tau \to \frac{M^2}{2\pi} \quad \text{as } t\to\infty.$$

Moreover, using the self-similarity of $p_1(x,t) = t^{-1}p_1(xt^{-1},1)$, one can easily show that

$$\frac{1}{\log t}\int_1^t\int_{\mathbb{R}} p_1^2(y,\tau)\,dy\,d\tau = \int_{\mathbb{R}} p_1^2(y,1)\,dy = \frac{1}{2\pi}.$$

Hence, applying (4.7), we immediately obtain

$$\frac{1}{\log t}\int_1^t\int_{\mathbb{R}} |u^2(y,\tau)-(Mp_1(y,\tau))^2|\,dy\,d\tau \le C\frac{1}{\log t}\int_1^t \tau^{-1}\eta(\tau)\,d\tau \to 0$$

as $t\to\infty$. The proof of (5.1) is complete.

Now we are ready to prove (1.9). Analogously to the proof of (1.7), it suffices to show that

$$\frac{t^{2-1/p}}{\log t}\Big\|\int_0^t \partial_x e^{(t-\tau)A} * u^2(\tau)\,d\tau - (\log t)M^2(2\pi)^{-1}\partial_x p_1(t)\Big\|_p \to 0 \tag{5.2}$$

as $t\to\infty$. To do this, note first that, by (2.5) and (4.5),

$$\Big\|\int_{t-1}^t \partial_x e^{(t-\tau)A} * u^2(\tau)\,d\tau\Big\|_p \le C\int_{t-1}^t (t-\tau)^{-1/2}\tau^{-2+1/p}\,d\tau \le Ct^{-2+1/p}.$$

Moreover, by (2.9), we have

$$\begin{aligned}\Big\|\int_0^{t-1} &\partial_x(e^{(t-\tau)A}-p_1(t-\tau)) * u^2(\tau)\,d\tau\Big\|_p\\ &\le C\int_0^{t-1}(t-\tau)^{-2+1/p-1/2}(1+\tau)^{-1}\,d\tau \le Ct^{-2+1/p-1/2}\log(1+t).\end{aligned}$$

Hence, we may replace $\int_0^t \partial_x e^{(t-\tau)A} * u^2(\tau)\,d\tau$ by $\int_0^{t-1}\partial_x p_1(t-\tau) * u^2(\tau)\,d\tau$ in (5.2). Moreover, it follows immediately from (5.1) that

$$\frac{t^{2-1/p}}{\log t}\Big\|\Big(\int_0^{t-1}\int_{\mathbb{R}} u^2(y,\tau)\,dy\,d\tau - (\log t)M^2(2\pi)^{-1}\Big)\partial_x p_1(t)\Big\|_p \to 0$$

as $t\to\infty$. Therefore the proof of (5.2) will be completed by showing that

$$\frac{t^{2-1/p}}{\log t}\Big\|\int_0^{t-1}\int_{\mathbb{R}}(\partial_x p_1(\cdot-y,t-\tau)-\partial_x p_1(\cdot,t))u^2(y,\tau)\,dy\,d\tau\Big\|_p \to 0 \tag{5.3}$$

as $t\to\infty$. To prove (5.3), we fix $\delta>0$ and we decompose the integration range into three parts: $[0,t-1]\times\mathbb{R} = \Omega_1\cup\Omega_2\cup\Omega_3$, where

$$\begin{aligned}\Omega_1 &= [0,\delta t]\times[-\delta t,\delta t],\\ \Omega_2 &= [0,\delta t]\times((-\infty,-\delta t]\cup[\delta t,\infty)),\\ \Omega_3 &= [\delta t,t-1]\times\mathbb{R}.\end{aligned}$$

First we estimate the integral over Ω_1. The change of variables, the self-similarity of $p_1(x,t)$, and continuity of translations in L^p imply that given $\varepsilon > 0$ there is $0 < \delta < 1$ such that

$$
\begin{aligned}
t^{2-1/p} \sup_{\substack{|y|\le t\delta \\ \tau\le t\delta}} &\|\partial_x p_1(\cdot - y, t-\tau) - \partial_x p_1(\cdot, t)\|_p \\
&= \sup_{\substack{|z|\le \delta \\ s\le\delta}} \|\partial_x p_1(\cdot - z, 1-s) - \partial_x p_1(\cdot, 1)\|_p \le \varepsilon.
\end{aligned}
$$

In view of this remark, we obtain

$$
\begin{aligned}
\frac{t^{2-1/p}}{\log t} &\Big\| \iint_{\Omega_1} (\partial_x p_1(\cdot - y, t-\tau) - \partial_x p_1(\cdot, t)) u^2(y,\tau)\, dy\, d\tau \Big\|_p \\
&\le \frac{C\varepsilon}{\log t} \int_0^{\delta t} \|u(\tau)\|_2^2\, d\tau \le \frac{C\varepsilon}{\log t} \int_0^{\delta t} (1+\tau)^{-1}\, d\tau \le C\varepsilon.
\end{aligned}
$$

To deal with the integral over Ω_2, we first prove that

$$
\frac{1}{\log t} \iint_{\Omega_2} u^2(y,\tau)\, dy\, d\tau \to 0 \quad \text{as } t\to\infty. \tag{5.4}
$$

Indeed, as in the proof of (5.1), it may be assumed that the integration in (5.4) with respect to τ is only over $[1, \delta t]$. Moreover, we have $\int_{\mathbb{R}} |y| p_1^2(y,\tau)\, dy = C$, with C independent of τ. Hence there is a number C independent of t such that

$$
\int_1^{\delta t} \int_{|y|\ge \delta t} p_1^2(y,\tau)\, dy\, d\tau \le \int_1^{\delta t}\int_{\mathbb{R}} \left|\frac{y}{\delta t}\right| p_1^2(y,\tau)\, dy\, d\tau = Ct^{-1}\int_1^{\delta t} d\tau \le C.
$$

Moreover, using (4.7), it may be concluded that

$$
\frac{1}{\log t}\int_1^{\delta t}\int_{|y|\ge\delta t} |u^2(y,\tau) - (Mp_1(y,\tau))^2|\, dy\, d\tau \le \frac{C}{\log t}\int_1^{\delta t} \tau^{-1}\eta(\tau)\, d\tau \to 0
$$

as $t\to\infty$. Combining these facts, we obtain (5.4).

Now, it follows immediately from (5.4) that

$$
\frac{t^{2-1/p}}{\log t}\iint_{\Omega_2} \|\partial_x p_1(\cdot, t)\|_p |u(y,\tau)|^2\, dy\, d\tau \le \frac{C}{\log t}\iint_{\Omega_2} |u(y,\tau)|^2\, dy\, d\tau \to 0
$$

and

$$\frac{t^{2-1/p}}{\log t}\iint_{\Omega_2}\|\partial_x p_1(\cdot-y,t-\tau)\|_p|u(y,\tau)|^2\,dy\,d\tau$$
$$\le\frac{t^{2-1/p}}{\log t}\iint_{\Omega_2}(t-\tau)^{-2+1/p}|u(y,\tau)|^2\,dy\,d\tau$$
$$\le\frac{C(1-\delta)^{-2+1/p}}{\log t}\iint_{\Omega_2}|u(y,\tau)|^2\,dy\,d\tau\to 0$$

as $t\to\infty$. This proves (5.3) with $[0,t-1]\times\mathbb{R}$ replaced by Ω_2.

For $p\in[1,\infty]$, the L^p-norm of the integral in (5.3) over Ω_3 is estimated in a straightforward way by the following quantity:

$$\int_{\delta t}^{t-1}\|\partial_x p_1(t-\tau)*u^2(\tau)\|_p\,d\tau+\|\partial_x p_1(t)\|_p\int_{\delta t}^{t-1}\|u(\tau)\|_2^2\,d\tau. \tag{5.5}$$

By (4.5), we immediately see that the second term in (5.5) is bounded by

$$Ct^{-2+1/p}\int_{\delta t}^{t}\tau^{-1}d\tau\le Ct^{-2+1/p}.$$

To estimate the first term in (5.5) we first note that it follows from the properties of e^{tA} that

$$\begin{aligned}\int_{\delta t}^{t-1}&\|\partial_x p_1(t-\tau)*(e^{\tau A}*u_0)^2\|_p\,d\tau \\ &\le\int_{\delta t}^{t-1}\|p_1(t-\tau)\|_1\|e^{\tau A}*u_0\|_p\|\partial_x e^{\tau A}*u_0\|_\infty\,d\tau \\ &\le C\int_{\delta t}^{t-1}\tau^{-3+1/p}\,d\tau\le Ct^{-2+1/p}.\end{aligned}\tag{5.6}$$

Moreover, by (4.6), we have

$$\begin{aligned}\int_{\delta t}^{t-1}&\|\partial_x p_1(t-\tau)*(u^2(\tau)-(e^{\tau A}*u_0)^2)\|_p\,d\tau \\ &\le\int_{\delta t}^{t-1}\|\partial_x p_1(t-\tau)\|_1\|u^2(\tau)-(e^{\tau A}*u_0)^2\|_p\,d\tau \\ &\le C\int_{\delta t}^{t-1}(t-\tau)^{-1}\tau^{-2+1/p}\eta(\tau)\,d\tau\le Ct^{-2+1/p}(\log t)\overline{\eta}(t),\end{aligned}\tag{5.7}$$

where $\overline{\eta}(t)\equiv\sup_{\tau\in[\delta t,t-1]}|\eta(t)|\to 0$ as $t\to\infty$. Combining (5.6) with (5.7) we obtain

$$\frac{t^{2-1/p}}{\log t}\int_{\delta t}^{t-1}\|\partial_x p_1(t-\tau)*u^2(\tau)\|_p\,d\tau\to 0\quad \text{ as } t\to\infty.$$

Now the proof of Theorem 1.2(ii) is complete. ■

Proof of Theorem 1.2(iii). Let $0<\alpha<1$. As in the proof of (i) and (ii), we need to show that

$$t^{(1-1/p)/\alpha+1/\alpha}\Big\|\int_0^t \partial_x e^{(t-\tau)A}*u^2(\tau)\,d\tau - \Big(\int_0^\infty\int_{\mathbb{R}} u^2(y,\tau)\,dy\,d\tau\Big)\partial_x p_\alpha(t)\Big\|_p\to 0\quad \text{ as } t\to\infty. \tag{5.8}$$

Our first step is to prove that

$$t^{(1-1/p)/\alpha+1/\alpha}\Big\|\int_{t/2}^t \partial_x e^{(t-\tau)A}*u^2(\tau)\,d\tau\Big\|_p\to 0\quad \text{ as } t\to\infty. \tag{5.9}$$

Indeed, by (1.6) in Theorem 1.1 and (4.5), we have

$$\begin{aligned}\|u^2(\tau)-(e^{\tau A}*u_0)^2\|_p &\le C\|u(\tau)-e^{\tau A}*u_0\|_p(\|u(\tau)\|_\infty+\|e^{\tau A}*u_0\|_\infty)\\ &\le C\tau^{-(1-1/p)/\alpha-2/\alpha}\end{aligned}$$

for all $\tau>0$, and a constant $C>0$ independent of τ. Hence

$$\begin{aligned}&\Big\|\int_{t/2}^t \partial_x e^{(t-\tau)A}*(u^2(\tau)-(e^{\tau A}*u_0)^2)\,d\tau\Big\|_p\\ &\quad\le \int_{t/2}^t \|\partial_x e^{(t-\tau)A}\|_1\|u^2(\tau)-(e^{\tau A}*u_0)^2\|_p\,d\tau\\ &\quad\le C\int_{t/2}^t (t-\tau)^{-1/2}\tau^{-(1-1/p)/\alpha-2/\alpha}\,d\tau = Ct^{-(1-1/p)/\alpha-2/\alpha+1/2}.\end{aligned} \tag{5.10}$$

Moreover, a similar argument gives

$$\begin{aligned}&\Big\|\int_{t/2}^t \partial_x e^{(t-\tau)A}*(e^{\tau A}*u_0)^2\,d\tau\Big\|_p\\ &\qquad\le 2\int_{t/2}^t \|e^{(t-\tau)A}\|_1\|\partial_x e^{\tau A}*u_0\|_\infty\|e^{\tau A}*u_0\|_p\,d\tau\\ &\qquad\le C\int_{t/2}^t \tau^{-(1-1/p)/\alpha-2/\alpha}\,d\tau = Ct^{-(1-1/p)/\alpha-2/\alpha+1}.\end{aligned} \tag{5.11}$$

Since $0<\alpha<1$, combining (5.10) with (5.11), we immediately obtain (5.9).

In the next step, using (4.5), we obtain

$$
(5.12)\qquad \Big\|\Big(\int_{t/2}^{\infty}\int_{\mathbb{R}} u^2(y,\tau)\,dy\,d\tau\Big)\partial_x p_\alpha(t)\Big\|_p \le Ct^{-(1-1/p)/\alpha-1/\alpha}\int_{t/2}^{\infty}\|u(\tau)\|_2^2\,d\tau \le Ct^{-(1-1/p)/\alpha-2/\alpha+1}.
$$

Moreover, applying Corollary 2.2, we see that

$$
(5.13)\qquad \Big\|\int_0^{t/2}\partial_x(e^{(t-\tau)A}-p_\alpha(t-\tau))*u^2(\tau)\,d\tau\Big\|_p \le C\int_0^{t/2}(t-\tau)^{-(1-1/p)/\alpha-1/\alpha}\eta(t-\tau)\|u(\tau)\|_2^2\,d\tau \le Ct^{-(1-1/p)/\alpha-1/\alpha}\overline{\eta}(t)\int_0^{\infty}\|u(\tau)\|_2^2\,d\tau.
$$

Here $\overline{\eta}(t)=\sup_{s\in[t/2,t]}|\eta(s)|$, and it is clear that $\lim_{t\to\infty}\overline{\eta}(t)=0$.

Now it follows immediately from (5.9), (5.12) and (5.13) that (5.8) will be proved provided

$$
(5.14)\qquad t^{(1-1/p)/\alpha+1/\alpha}\Big\|\int_0^{t/2}\int_{\mathbb{R}}(\partial_x p_\alpha(\cdot-y,t-\tau) - \partial_x p_\alpha(\cdot,t))u^2(y,\tau)\,dy\,d\tau\Big\|_p \to 0
$$

as $t\to\infty$. From now on the reasoning is completely analogous to that in the proof of Theorem 1.2(ii); therefore we omit the details. Given $\delta>0$ we decompose the integration range $[0,t/2]\times\mathbb{R}=\Omega_1\cup\Omega_2\cup\Omega_3$, where

$$
\begin{aligned}
\Omega_1 &= [0,\delta t]\times[-\delta t^{1/\alpha},\delta t^{1/\alpha}],\\
\Omega_2 &= [0,\delta t]\times((-\infty,-\delta t^{1/\alpha}]\cup[\delta t^{1/\alpha},\infty)),\\
\Omega_3 &= [\delta t,t/2]\times\mathbb{R}.
\end{aligned}
$$

Now a slight change in the proof of (5.3) gives (5.14). In fact, we only need to replace the estimate $\|u(\tau)\|_2^2\le C(1+\tau)^{-1}$ by $\|u(\tau)\|_2^2\le C(1+\tau)^{-1/\alpha}$ in that reasoning, and we should remember that $0<\alpha<1$. Moreover, the counterpart of (5.4) is

$$
\iint_{\Omega_2} u^2(y,\tau)\,dy\,d\tau \le \int_0^{\infty}\int_{|y|\ge\delta t^{1/\alpha}} u^2(y,\tau)\,dy\,d\tau.
$$

The integral on the right-hand side tends to 0 as $t\to\infty$, because the integral $\int_0^\infty\int_{\mathbb{R}}u^2(y,\tau)\,dy\,d\tau$ converges. Theorem 1.2(iii) is now proved. ∎

6. General multifractal conservation law asymptotics. Here we indicate how results proved in the preceding sections can be extended to the case of a general multifractal conservation law (1.1).

First, observe that the decay estimates from Lemma 2.1 play a central rôle in the proof of the results formulated in Section 1. Analogous results can be proved in the case of the semigroup of linear operators generated by (1.2), which is given by convolution with the kernel

$$e^{tA} = p_2(c_0t) * p_{\alpha_1}(c_1t) * \ldots * p_{\alpha_N}(c_Nt).$$

Indeed, a repeated use of the Young inequality (2.6), combined with the property $\|p_{\alpha_j}(c_jt)\|_1 = 1$ for $j = 0, 1, \ldots, N$ (here $\alpha_0 = 2$), gives immediately inequalities (2.4) and (2.5) for the operator A of the form (1.2) and for

$$\alpha = \min\{\alpha_1, \ldots, \alpha_N\}.$$

Now the proof of the existence of a global-in-time mild solution to the problem

$$u_t + f(u)_x = Au, \tag{6.1}$$

$$u(x, 0) = u_0(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \tag{6.2}$$

for either $0 < \alpha < 1$ and $f(u) = u^2$, or $1 \le \alpha < 2$ and f of polynomial growth, is completely analogous to that given in Section 2. It is important that such a solution satisfies preliminary estimates (3.2), (3.3), and the maximum principle (3.8). Moreover, assuming that $f(0) = f'(0) = 0$, we immediately obtain the estimate

$$|f(u(x,t))| \le u^2(x,t) \sup_{|y|\le\|u_0\|_\infty} |f''(y)|/2,$$

which is the core of the proof of the fact that the first order term of the asymptotic expansion of solutions to (6.1)–(6.2) is given by $e^{tA} * u_0$. Indeed, repeating the argument used in the proof of Lemma 4.1, one can show that

$$t^{(1-1/p)/\alpha}\|u(t) - e^{tA} * u_0\|_p \to 0 \quad \text{as } t \to \infty.$$

The second order term depends essentially on the behavior of the integral $\int_{\mathbb{R}} f(u(y,t))\,dy$ as $t \to \infty$. If $f''(0) \ne 0$, it is an immediate consequence of the Taylor expansion that

$$t^{1/\alpha}\Big|\int_{\mathbb{R}} f(u(y,t))\,dy - \frac{1}{2}f''(0)\int_{\mathbb{R}} u^2(y,t)\,dy\Big| \to 0 \quad \text{as } t \to \infty. \tag{6.3}$$

Applying (6.3), and repeating the reasoning from the proof of Theorem 1.2(i), (ii), one can easily show that for $1 < \alpha \le 2$,

$$t^{(1-1/p)/\alpha+2/\alpha-1}\Big\|u(t) - e^{tA} * u_0 + \frac{1}{2}f''(0)\int_0^t \partial_x p_\alpha(t-\tau) * (Mp_\alpha(\tau))^2\,d\tau\Big\|_p \to 0$$

as $t \to \infty$. However, for $\alpha = 1$ we have

$$\frac{t^{2-1/p}}{\log t}\left\|u(t) - e^{tA} * u_0 + (\log t)\frac{1}{2}f''(0)M^2(2\pi)^{-1}\partial_x p_1(t)\right\|_p \to 0 \quad \text{as } t \to \infty.$$

Now let us look more closely at the case when $f''(0) = 0$. By the Taylor expansion, we have

$$|f(u(y,\tau))| \le |u(y,\tau)|^3 \sup_{|y|\le\|u_0\|_\infty} |f'''(y)|/3,$$

which implies

$$\int_0^\infty \int_{\mathbb{R}} |f(u(y,\tau))|\,dy\,d\tau \le C \int_0^\infty \int_{\mathbb{R}} |u(y,\tau)|^3\,dy\,d\tau \le C\int_0^\infty (1+\tau)^{-2/\alpha}\,d\tau < \infty \tag{6.4}$$

for every $\alpha \in [1,2)$. Now, as a straightforward consequence we obtain the second order term of the asymptotic expansion of solutions to (6.1)–(6.2). Indeed, it follows from (6.4) and from the reasoning in the proof of Theorem 1.2(iii) that

$$t^{(1-1/p)/\alpha+1/\alpha}\left\|u(t) - e^{tA} * u_0 + \int_0^\infty \int_{\mathbb{R}} f(u(y,\tau))\,dy\,d\tau\,\partial_x p_\alpha(t)\right\|_p \to 0$$

as $t \to \infty$ for every $\alpha \in [1,2)$. This asymptotics is different from that in the case $f(u) = u^2$ (cf. Theorem 1.2(i), (ii)).

Acknowledgements. Part of this research was done while the first and the second authors were research scholars at the CWRU Center for Stochastic and Chaotic Processes in Science and Technology, Cleveland (Summer 1998). Grant support from NSF 97-04231, KBN 324/P03/97/12, NATO OUTR.CRG 961130, and the University of Wrocław funds 2204/W/IM/98 is gratefully acknowledged.

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Received September 24, 1998 (4181)