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New examples of effective formulas for holomorphically contractible functions

by

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Abstract. Let $G \subset \mathbb{C}^n$ and $B \subset \mathbb{C}^m$ be domains and let $\Phi : G \to B$ be a surjective holomorphic mapping. We characterize some cases in which invariant functions and pseudometrics on G can be effectively expressed in terms of the corresponding functions and pseudometrics on B.

0. Introduction. It is well known that holomorphically contractible families of functions or pseudometrics give very useful and powerful tools in complex analysis. Recall that a family $(d_G)_G$ of functions $d_G : G \times G \to \mathbb{R}_+$ (where G runs through all domains in $\mathbb{C}^n$ with arbitrary n) is called *holomorphically contractible* if

- $\tanh d_E =$ the hyperbolic distance on the unit disc E,
- $d_{G_2}(F(a), F(z)) \le d_{G_1}(a, z)$ for all $F \in \mathcal{O}(G_1, G_2)$ and $a, z \in G_1$.

A family $(\delta_G)_G$ of pseudometrics $\delta_G : G \times \mathbb{C}^n \to \mathbb{R}_+$ (i.e. $\delta_G(a; \lambda X) = |\lambda|\delta_G(a; X)$, $a \in G \subset \mathbb{C}^n \ni X$, $\lambda \in \mathbb{C}$) is called *holomorphically contractible* if

- $\delta_E =$ the hyperbolic pseudometric on E,
- $\delta_{G_2}(F(a); F'(a)(X)) \le \delta_{G_1}(a; X)$ for all $F \in \mathcal{O}(G_1, G_2)$ and $a \in G_1 \subset \mathbb{C}^{n_1} \ni X$.

For simplicity, each d_G (resp. δ_G) will be called an *invariant function* (resp. *invariant pseudometric*); cf. [Jar-Pfl].

Frequently, the following problem appears. We are given a holomorphically contractible family $(d_G)_G$ of functions (e.g. the family of the pluricomplex Green functions). We want to verify certain holomorphic properties of a domain G via corresponding properties of d_G. Consequently, we have to check whether d_G satisfies some conditions, e.g. whether d_G has a restricted

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growth near the boundary. However, for a given domain G, it is in general difficult to describe d_G by an effective formula. Therefore, usually one could proceed as follows. First, we approximate G by more elementary domains G' such that $d_{G'}$ can be calculated. Next, using limit procedures, we try to estimate d_G. It is clear that what we need for such an approach is a large (up to a biholomorphic equivalence) class of "elementary" domains for which at least some of the invariant functions and pseudometrics can be calculated.

Recall that in the case of one complex variable the formulas are known only in the case of the unit disc or an annulus (cf. [Jar-Pfl], Ch. V). In the case of several variables the formulas are known for example for all norm balls in $\mathbb{C}^n$ with transitive group of automorphisms (i.e. the unit polydisc E^n, the Euclidean ball $\mathbb{B}_n$, and the Lie ball $\mathbb{L}_n$; cf. [Jar-Pfl], §8.3). Besides these classical domains, the *only* class for which effective descriptions of invariant functions are known, is the class of elementary Reinhardt domains of the form

$$G := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1|^{\alpha_1} \cdot \ldots \cdot |z_n|^{\alpha_n} < 1\},$$

where $\alpha_1, \ldots, \alpha_n > 0$ (cf. [Jar-Pfl], §4.4, [Pfl-Zwo], [Edi-Zwo 2]).

The aim of the paper is to generalize results of [Jar-Pfl] and to obtain a larger class of domains for which we can produce effective formulas. More precisely, suppose that $\Phi : G \to B$ is a surjective holomorphic mapping, where $B \subset \mathbb{C}^m$ is a domain for which d_B is known. We are interested (Theorem 1) in those cases in which

$$d_G(a, z) = (d_B(\Phi(a), \Phi(z)))^{1/r(a)},$$

where $r(a) := \operatorname{ord}_a(\Phi - \Phi(a))$ denotes the order of vanishing. In the special case where $B = B_1 \times \ldots \times B_m$ we also discuss (Proposition 3) some situations in which

$$d_G(a, z) = \max\{(d_{B_j}(\Phi_j(a), \Phi_j(z)))^{1/r_j(a)} : j = 1, \ldots, m\},$$

where $r_j(a) := \operatorname{ord}_a(\Phi_j - \Phi_j(a))$, $j = 1, \ldots, m$.

We also present (Propositions 4 and 5) some characterizations of proper and biholomorphic mappings between domains from Theorem 1.

1. Notation. Let $G \subset \mathbb{C}^n$ be a domain. We will consider the following invariant functions and pseudometrics (cf. [Jar-Pfl]).

The kth *Möbius function*:

$$m_G^{(k)}(a, z) := \sup\{|f(z)|^{1/k} : f \in \mathcal{O}(G, E),\ \operatorname{ord}_a f \geq k\}, \qquad a, z \in G,\ k \in \mathbb{N}.$$

The kth *Reiffen pseudometric*:

$$\gamma_G^{(k)}(a;X) := \sup\left\{\left|\frac{1}{k!}f^{(k)}(a)(X)\right|^{1/k} : f \in \mathcal{O}(G,E),\ \mathrm{ord}_a f \geq k\right\},$$
$$a \in G,\ X \in \mathbb{C}^n,\ k \in \mathbb{N}.$$

The *pluricomplex Green function with poles P and weights ν* (cf. [Lel]):

$$g_G(P;\nu;z) := \sup\{u(z) : \log u \in \mathcal{PSH}(G,[-\infty,0)),$$
$$\exists_{M>0} : u(w) \leq M\|w-a\|^{\nu(a)},\ (a,w) \in P \times G\}, \quad z \in G,$$

where P is a finite subset of G, and $\nu : P \to (0,\infty)$.

The *pluricomplex Green function*:

$$g_G(a,z) := g_G(\{a\};1;z), \quad a,z \in G.$$

The *Azukawa pseudometric*:

$$A_G(a;X) := \limsup_{0\neq\lambda\to 0} \frac{g_G(a,a+\lambda X)}{|\lambda|}, \quad a \in G,\ X \in \mathbb{C}^n.$$

Let V be an analytic subset of an open set $\Omega \subset \mathbb{C}^n$. Recall that an upper semicontinuous function $u : V \to [-\infty,\infty)$ is said to be *plurisubharmonic* if for any holomorphic mapping $\phi : E \to V$ the function $u \circ \phi$ is subharmonic on E (cf. [For-Nar]).

We say that V has the *plurisubharmonic Liouville property* if any function plurisubharmonic and bounded from above on V is constant.

Observe that if V has the plurisubharmonic Liouville property, then V has the *Liouville property*, i.e. any function holomorphic and bounded on V is constant.

2. Main results. The main result of the paper is the following theorem.

THEOREM 1. *Let $\Omega \subset \mathbb{C}^n$ be open, $\Phi \in \mathcal{O}(\Omega,\mathbb{C}^m)$ $(m < n)$, and let $B \subset \Phi(\Omega)$ be a domain. Put $G := \Phi^{-1}(B)$. For $a \in G$ let $r(a) := \mathrm{ord}_a(\Phi - \Phi(a))$. Assume that there exists a thin relatively closed subset S of B such that for any $\xi \in B \setminus S$,*

(C1) *$\Phi^{-1}(\xi)$ has the plurisubharmonic Liouville property*,

(C2) $\exists_{a\in\Phi^{-1}(\xi)} : \mathrm{rank}\,\Phi'(a) = m$.

Then G is a domain and the following formulas hold:

(a) *We have*

$$m_G^{(1)}(a,z) = m_B^{(1)}(\Phi(a),\Phi(z)), \quad a,z \in G, \tag{1}$$

$$\gamma_G^{(1)}(a;X) = \gamma_B^{(1)}(\Phi(a);\Phi'(a)(X)), \quad a \in G,\ X \in \mathbb{C}^n. \tag{2}$$

(b) *If $a \in G$ is such that the analytic set dimension satisfies*

$$\dim(\{X \in \mathbb{C}^n : \Phi^{(r)}(a)(X) = 0\}) = n - m, \tag{$*$}$$

where $r := r(a)$, *then*

$$m_G^{(k)}(a,z) = (m_B^{(\ell)}(\Phi(a),\Phi(z)))^{\ell/k}, \quad z \in G, \tag{3}$$

$$\gamma_G^{(k)}(a;X) = \begin{cases} \left(\gamma_B^{(k/r)}\left(\Phi(a); \frac{1}{r!}\Phi^{(r)}(a)(X)\right)\right)^{1/r} & \text{if } k/r \in \mathbb{N}, \\ 0 & \text{if } k/r \notin \mathbb{N}, \end{cases} \quad X \in \mathbb{C}^n,\ k \in \mathbb{N}, \tag{4}$$

$$g_G(a,z) = (g_B(\Phi(a),\Phi(z)))^{1/r}, \quad z \in G, \tag{5}$$

$$A_G(a;X) = \left(A_B\left(\Phi(a); \frac{1}{r!}\Phi^{(r)}(a)(X)\right)\right)^{1/r}, \quad X \in \mathbb{C}^n, \tag{6}$$

where $\ell = \ell(a,k) := \mu(k/r)$ *and* $\mu(t) := -($*the integer part of* $(-t))$.

(c) *If* $P \subset G$ *is a finite set such that* $(*)$ *is satisfied for every* $a \in P$, *then*

$$g_G(P;\nu;,z) = g_B(\Phi(P);\widetilde{\nu};\Phi(z)), \quad z \in G, \tag{7}$$

where

$$\widetilde{\nu}(\xi) := \max\{\nu(a)/r(a) : a \in P \cap \Phi^{-1}(\xi)\}, \quad \xi \in \Phi(P).$$

Conditions (C1), (C2), $(*)$ *are always satisfied if* $m = 1$ *and* Φ *is a primitive polynomial* (cf. Remark 7). *Consequently*, *if* Φ *is a primitive polynomial*, *then formulas* (1)–(7) *are true.*

The proof of Theorem 1 will be given in §3.

Remark 2. (a) The case where Φ is a monomial and $B = E$ has been studied in [Jar-Pfl], §4.4, and [Edi-Zwo 2].

(b) If $\operatorname{rank}\Phi'(a) = m$ or $m = 1$, then $(*)$ is satisfied.

(c) If $(*)$ is not satisfied, then formulas (3)–(7) need not be true (cf. Proposition 3).

Let $\alpha_j = (\alpha_{j,1},\dots,\alpha_{j,n}) \in (\mathbb{Z}_+)^n \setminus \{0\}$, $j = 1,\dots,m$ $(m \geq 2)$, and

$$G := \{z \in \mathbb{C}^n : |z^{\alpha_j}| < 1,\ j = 1,\dots,m\}.$$

Fix an $a \in G$ with $a^{\alpha_j} = 0$, $j = 1,\dots,m$. Assume that

$$a = (a_1,\dots,a_s,0,\dots,0)$$

with $a_1 \dots a_s \neq 0$ and $1 \leq s \leq n-1$. Put

$$A := [\alpha_{j,k}]_{\substack{j=1,\dots,m\\ k=1,\dots,n}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}, \quad \widetilde{A} := [\alpha_{j,k}]_{\substack{j=1,\dots,m\\ k=s+1,\dots,n}} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}.$$

Notice that $r_j := \operatorname{ord}_a z^{\alpha_j} = |\beta_j| > 0$, $j = 1,\dots,m$.

PROPOSITION 3. *The following conditions are equivalent*:

(i) $$\operatorname{rank} A = \operatorname{rank} \widetilde{A};$$

(ii) $$g_G(a,z) = \max\{|z^{\alpha_j}|^{1/r_j} : j = 1, \ldots, m\}, \quad z \in G;$$

(iii) $$g_G(a,z) = \sup\{|z^{\alpha}|^{1/r} : \alpha \in (\mathbb{Z}_+)^n,\ |z^\alpha| < 1 \text{ in } G,\ r = \operatorname{ord}_a z^\alpha > 0\}, \quad z \in G;$$

(iv) $$g_G(a,(z',\lambda z'')) = |\lambda| g_G(a,z), \quad z = (z',z'') \in G \subset \mathbb{C}^s \times \mathbb{C}^{n-s},\ \lambda \in \overline{E};$$

(v) $$\forall_{k\in\mathbb{N}} : \left\{(z',z'') \in G : \limsup_{\theta\to 0+} \frac{1}{\theta} m_G^{(k)}(a,(z',\theta z'')) < \infty\right\} \text{ is not thin.}$$

The proof of Proposition 3 will be given in §4.

Recall that a bounded domain $D \subset \mathbb{C}^n$ is called *hyperconvex* if it admits a negative plurisubharmonic exhaustion function.

PROPOSITION 4. *Let $\Omega_j \subset \mathbb{C}^{n_j}$, $\Phi_j \in \mathcal{O}(\Omega_j, \mathbb{C}^{m_j})$, and $S_j \subset B_j \subset \Phi_j(\Omega_j)$ be such that (Φ_j, B_j, S_j) satisfies* (C1) *and* (C2) *from Theorem* 1 *and B_j is bounded. Put $G_j := \Phi_j^{-1}(B_j)$, $j = 1,2$, and let $F : G_1 \to G_2$ be a holomorphic mapping. Then there exists a holomorphic mapping $\widetilde{F} : B_1 \to B_2$ such that*

$$\Phi_2 \circ F = \widetilde{F} \circ \Phi_1.$$

Moreover,

- *if F is biholomorphic, then so is $\widetilde{F}$*;
- *if $n_1 = n_2 =: n$, $m_1 = m_2 =: m$, B_1 is hyperconvex, and F is proper, then $\widetilde{F}$ is proper.*

The case where Φ_j is a monomial and $B_j = E$, $j = 1,2$, has been studied in [Edi-Zwo 2].

PROPOSITION 5. *Let $\Phi_j = (Q_{j,1}, \ldots, Q_{j,m}) : \mathbb{C}^n \to \mathbb{C}^m$ be a homogeneous polynomial with $\deg Q_{j,1} = \ldots = \deg Q_{j,m} =: d_j \geq 2$. Let $S_j \subset B_j \subset \Phi_j(\mathbb{C}^n)$ be such that (Φ_j, B_j, S_j) satisfies* (C1) *and* (C2) (*for example*, $m = 1$ *and Φ_j is a primitive homogeneous polynomial*), *B_j is bounded*, *and $0 \in B_j$. Put $G_j := \Phi_j^{-1}(B_j)$, $j = 1,2$.*

(a) *If G_1 and G_2 are biholomorphic*, *then the following conditions are equivalent*:

(i) *for any biholomorphic mapping $F : G_1 \to G_2$ we have $F(0) = 0$*;
(ii) $\operatorname{ord}_a(\Phi_2 - \Phi_2(a)) < d_2$ *for any $a \neq 0$.*

(b) *Assume additionally that B_1 and B_2 are balanced* (*consequently*, *G_1 and G_2 are balanced*). *Then the following conditions are equivalent*:

(i) *G_1, G_2 are biholomorphic*;
(ii) *there is a linear isomorphism $L : \mathbb{C}^n \to \mathbb{C}^n$ such that $L(G_1) = G_2$.*

In particular, if $\Phi_1(z) = z^\alpha$, $\Phi_2(z) = z^\beta$ *are primitive monomials,* $G_1 := \{|z^\alpha| < 1\}$, *and* $G_2 := \{|z^\beta| < 1\}$, *then* G_1 *and* G_2 *are biholomorphic iff* $\alpha = \beta$ *up to permutation* (cf. [Edi-Zwo 2]).

The proofs of Propositions 4 and 5 will be given in §5.

3. Proof of Theorem 1

PROPOSITION 6. (a) *Let* $D \subset \mathbb{C}^d$ *be a domain having the plurisubharmonic Liouville property and let* V *be a connected pure d-dimensional analytic subset of* $D \times \mathbb{C}^{n-d}$ *such that the natural projection*

$$V \ni (z, w) \overset{\pi}{\mapsto} z \in D$$

is proper. Then V *has the plurisubharmonic Liouville property.*

(b) *Any connected pure d-dimensional algebraic subset of* $\mathbb{C}^n$ *has the plurisubharmonic Liouville property.*

Proof. (a) We may assume that V is irreducible. Let u be plurisubharmonic on V with $c_0 := \sup_V u < \infty$. Define

$$\widetilde{u}(z) := \max\{u(z, w) : (z, w) \in V\}, \qquad z \in D.$$

Using the standard methods (cf. [For-Nar], the proof of Lemma 5.1), we prove that $\widetilde{u} \in \mathcal{PSH}(D)$. Since D has the plurisubharmonic Liouville property, $\widetilde{u} = \text{const} = c_0$.

To prove that $u \equiv \text{const}$ it suffices to show that $u = c_0$ on a dense subset of V. Let $\Delta \subset D$ be an analytic set such that $\pi : V \setminus \pi^{-1}(\Delta) \to D \setminus \Delta$ is a holomorphic covering. We show that $u = c_0$ on $V_0 := V \setminus \pi^{-1}(\Delta)$. Notice that $V_0 = \mathrm{Reg}(V) \setminus \pi^{-1}(\Delta)$, where $\mathrm{Reg}(V)$ denotes the set of all regular points of V. We know that $\mathrm{Reg}(V)$ is connected (because V is irreducible). Thus V_0 is connected.

Let $\widetilde{V}_0 := \{(z, w) \in V_0 : u(z, w) = c_0\}$. Then $\widetilde{V}_0 \neq \emptyset$ and $\widetilde{V}_0$ is closed in V_0. Moreover, by the maximum principle, $\widetilde{V}_0$ is open. Thus $\widetilde{V}_0 = V_0$, i.e. $u = c_0$ on V_0.

(b) follows from (a) and the fact that for an algebraic subset V of $\mathbb{C}^n$, after a linear change of coordinates, the projection π is proper; cf. [Chi]. ∎

REMARK 7. (a) Recall that a polynomial P of n complex variables is primitive iff P cannot be represented in the form $P = f(Q)$, where f is a polynomial of one complex variable of degree ≥ 2 and Q is a polynomial of n complex variables (cf. [Cyg]).

In particular, a homogeneous polynomial P is primitive iff P cannot be written as $P = Q^p$, where $p \geq 2$ and Q is a homogeneous polynomial.

A monomial z^α, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $n \geq 2$, is primitive iff the numbers $\alpha_1, \ldots, \alpha_n$ are relatively prime.

(b) It is known (cf. [Cyg]) that if P is a primitive polynomial, then the fibers $P^{-1}(\xi)$ are connected except for a finite number of ξ.

(c) If P is a polynomial, then the set $P(\{z \in \mathbb{C}^n : P'(z) = 0\})$ is finite.

(d) Properties (b) and (c) and Proposition 6(b) show that if Φ is a primitive polynomial, then (C1), (C2) are satisfied with a finite set $S \subset B$.

(e) Let $\Phi : \mathbb{C}^n \to \mathbb{C}^m$ $(m < n)$,

$$\Phi(z_1, \ldots, z_m, z_{m+1}, \ldots, z_n) = \Phi(z_1, \ldots, z_m, z') := (z_1(z')^{\beta_1}, \ldots, z_m(z')^{\beta_m}),$$

where $\beta_1, \ldots, \beta_m \in (\mathbb{Z}_+)^{n-m}$. Then for any $\xi = (\xi_1, \ldots, \xi_m) \in (\mathbb{C}_*)^m$ $(\mathbb{C}_* := \mathbb{C} \setminus \{0\})$ we have the global parametrization

$$(\mathbb{C}_*)^{n-m} \ni \lambda \mapsto (\xi_1/\lambda^{\beta_1}, \ldots, \xi_m/\lambda^{\beta_m}, \lambda) \in \Phi^{-1}(\xi).$$

Hence, (C1) and (C2) are satisfied with $S := \{(\xi_1, \ldots, \xi_m) : \xi_1 \cdot \ldots \cdot \xi_m = 0\}$.

Proof of Theorem 1. First we prove that G is a domain. Observe that

$$G = \Phi^{-1}(B \setminus S) \cup \Phi^{-1}(S) =: G_0 \cup S_0.$$

Since S_0 is a thin relatively closed subset of G, it suffices to prove that G_0 is a domain. Suppose that $G_0 = U_1 \cup U_2$, where U_1, U_2 are open, disjoint, and non-empty. Let $B_j := \{\xi \in B \setminus S : \Phi^{-1}(\xi) \subset U_j\}$, $j = 1, 2$. By (C1) we have $B_1 \cup B_2 = B \setminus S$. Obviously, B_1, B_2 are disjoint and non-empty. Fix $\xi^0 \in B_j$. By (C2) there exists $a \in \Phi^{-1}(\xi^0)$ such that $\operatorname{rank} \Phi'(a) = m$. We may assume that $\operatorname{rank}[\partial\Phi_j/\partial z_{n-m+k}(a)]_{j,k=1,\ldots,m} = m$. By the implicit mapping theorem, the equation $\Phi(z', z'') = \xi$ (where $(z', z'') \in \mathbb{C}^{n-m} \times \mathbb{C}^m$) is equivalent in a neighborhood of (a, ξ^0) to $z'' = \phi(z', \xi)$, where ϕ is holomorphic. In particular, $(a', \phi(a', \xi)) \in U_j \cap \Phi^{-1}(\xi)$ for ξ in a neighborhood W of ξ_0. Hence, by (C1), $W \subset B_j$. Consequently, B_j is open, $j = 1, 2$; a contradiction.

Formulas (2), (4), and (6) follow from (1), (3) and (5), respectively, and from properties of $\gamma_G^{(k)}$ and A_G (cf. [Jar-Pfl], §4.2). Formula (5) follows from (7). In formulas (1), (3), and (7) the inequalities $\geq$ follow directly from the definitions. Therefore we only need to prove the opposite inequalities.

Let $f \in \mathcal{O}(G, E)$, $\operatorname{ord}_a f \geq k$. By (C1) there exists a function $\widetilde{f} : B \setminus S \to E$ such that $f = \widetilde{f} \circ \Phi$ on $G \setminus \Phi^{-1}(S)$. Condition (C2) and the implicit mapping theorem imply that $\widetilde{f} \in \mathcal{O}(B \setminus S)$. Now, by the Riemann theorem, $\widetilde{f}$ extends holomorphically to B (we denote the extension by the same symbol) and $f = \widetilde{f} \circ \Phi$ in G. Obviously, $\widetilde{f}(\Phi(a)) = 0$. In particular, if $k = 1$, then we get (1).

Assume that $(*)$ is satisfied.

Let L be an m-dimensional vector subspace of $\mathbb{C}^n$ such that

$$L \cap \{X \in \mathbb{C}^n : \Phi^{(r)}(a)(X) = 0\} = \{0\}.$$

It is clear that there exist $\varrho > 0$ and $C > 0$ such that

$$(+) \qquad \|\Phi(a+X) - \Phi(a)\| \ge C\|X\|^r, \quad X \in L \cap B(\varrho).$$

Now one can easily prove that there exist neighborhoods $U \subset B(\varrho)$ of 0 and $V \subset \mathbb{C}^m$ of $\Phi(a)$ such that the mapping $L \cap U \ni X \mapsto \Phi(a+X) \in V$ is proper (in particular, surjective).

For $\xi \in V$ let $X(\xi) \in L \cap U$ be such that $\Phi(a + X(\xi)) = \xi$. Then, by (+), we get

$$\begin{aligned} |\widetilde{f}(\xi)| = |f(a+X(\xi))| &\le \text{const}\,\|X(\xi)\|^k \le \text{const}\,\|\Phi(a+X(\xi)) - \Phi(a)\|^{k/r} \\ &= \text{const}\,\|\xi - \Phi(a)\|^{k/r}, \quad \xi \in V. \end{aligned}$$

Hence $\operatorname{ord}_{\Phi(a)} \widetilde{f} \ge \mu(k/r)$, and therefore,

$$m_B^{(\ell)}(\Phi(a), \Phi(z)) \ge |\widetilde{f}(\Phi(z))|^{1/\ell} = |f(z)|^{1/\ell},$$

which implies that

$$m_B^{(\ell)}(\Phi(a), \Phi(z)) \ge (m_G^{(k)}(a,z))^{k/\ell},$$

and so the proof of (1) and (3) is complete.

We turn to the proof of (7). Let $u : G \to [0,1)$ be such that $\log u \in \mathcal{PSH}(G)$ and $u(w) \le M\|w - a\|^{\nu(a)}$ for any $(a,w) \in P \times G$. By (C1) there exists a function $\widetilde{u} : B \setminus S \to [0,1)$ such that $u = \widetilde{u} \circ \Phi$ on $G \setminus \Phi^{-1}(S)$. Condition (C2) and the implicit mapping theorem imply that $\log \widetilde{u} \in \mathcal{PSH}(B \setminus S)$. Now, by the Riemann type theorem for plurisubharmonic functions, $\widetilde{u}$ extends to a log-plurisubharmonic function on B (we denote the extension by the same symbol). By the identity principle for plurisubharmonic functions we get $u = \widetilde{u} \circ \Phi$ in G.

Fix $a \in P$, let $r := r(a)$, $\nu := \nu(a)$, and let $X(\xi)$, $\xi \in V$, be as above. Then

$$\begin{aligned} \widetilde{u}(\xi) = u(a+X(\xi)) &\le M\|X(\xi)\|^\nu \le \text{const}\,\|\Phi(a+X(\xi)) - \Phi(a)\|^{\nu/r} \\ &= \text{const}\,\|\xi - \Phi(a)\|^{\nu/r}, \quad \xi \in V. \end{aligned}$$

Hence

$$\widetilde{u}(\xi) \le \widetilde{M}|\xi - \xi_0|^{\widetilde{\nu}(\xi_0)}, \quad (\xi_0, \xi) \in \Phi(P) \times B.$$

Thus

$$g_B(\Phi(P); \widetilde{\nu}; \Phi(z)) \ge \widetilde{u}(\Phi(z)) = u(z),$$

which implies that

$$g_B(\Phi(P); \widetilde{\nu}; \Phi(z)) \ge g_G(P; \nu; z).$$

The last part of the theorem follows from Remark 7(d). ■

4. Proof of Proposition 3. (i)⇒(ii). Let

$$L(z) := g_G(a,z), \quad R(z) := \max\{|z^{\alpha_j}|^{1/r_j} : j = 1,\dots,m\}, \quad z \in G.$$

The inequality $L \ge R$ follows from the definition of g_G. To prove that $L \le R$ it suffices to show that $L(z) \le R(z)$ for any $z \in G_0 := G \cap ((\mathbb{C}_*)^s \times \mathbb{C}^{n-s})$.

By (i), for any $k = 1,\dots,s$, the system of equations

$$\alpha_{j,s+1}x_{s+1} + \dots + \alpha_{j,n}x_n = -\alpha_{j,k}, \quad j = 1,\dots,m,$$

has a rational solution $(Q_{s+1,k}/\mu_k,\dots,Q_{n,k}/\mu_k)$ with $Q_{s+1,k},\dots,Q_{n,k} \in \mathbb{Z}$, $\mu_k \in \mathbb{N}$. Put $Q_{k,k} := \mu_k$ and $Q_{j,k} := 0$, $j,k = 1,\dots,s$, $j \ne k$. Then

$$(\dagger) \qquad \alpha_{j,1}Q_{1,k} + \dots + \alpha_{j,n}Q_{n,k} = 0, \quad j = 1,\dots,m,\ k = 1,\dots,s.$$

Let

$$Q_j := (Q_{j,1},\dots,Q_{j,s}) \in \mathbb{Z}^s, \quad j = 1,\dots,n.$$

Define $F : (\mathbb{C}_*)^s \times \mathbb{C}^{n-s} \to (\mathbb{C}_*)^s \times \mathbb{C}^{n-s}$ by

$$\begin{aligned} F(\xi,\eta) &:= (\xi^{Q_1},\dots,\xi^{Q_s},\xi^{Q_{s+1}}\eta_1,\dots,\xi^{Q_n}\eta_{n-s}) \\ &= (\xi_1^{\mu_1},\dots,\xi_s^{\mu_s},\xi^{Q_{s+1}}\eta_1,\dots,\xi^{Q_n}\eta_{n-s}), \\ &\qquad (\xi,\eta) = (\xi_1,\dots,\xi_s,\eta_1,\dots,\eta_{n-s}) \in (\mathbb{C}_*)^s \times \mathbb{C}^{n-s}. \end{aligned}$$

Observe that F is surjective. Indeed, for $z = (z_1,\dots,z_n) \in (\mathbb{C}_*)^s \times \mathbb{C}^{n-s}$, take an arbitrary $\xi_j \in (z_j)^{1/\mu_j}$, $j = 1,\dots,s$, and define $\eta_j := z_{s+j}/\xi^{Q_{s+j}}$, $j = 1,\dots,n-s$.

Moreover, if $z = F(\xi,\eta)$, then by (†) we get

$$(\ddagger) \qquad z^{\alpha_j} = \xi^{\alpha_{j,1}Q_1+\dots+\alpha_{j,n}Q_n}\eta^{\beta_j} = \eta^{\beta_j}, \quad j = 1,\dots,m.$$

Let

$$D := \{\eta \in \mathbb{C}^{n-s} : |\eta^{\beta_j}| < 1,\ j = 1,\dots,m\}.$$

Using (‡) we get the equality $F((\mathbb{C}_*)^s \times D) = G_0$.

Fix a $\xi_0 \in (\mathbb{C}_*)^s$ such that $a = F(\xi_0,0)$. Then, for any $z = F(\xi,\eta) \in G_0$, we have

$$\begin{aligned} g_G(a,z) &= g_G(F(\xi_0,0),F(\xi,\eta)) \\ &\le g_{(\mathbb{C}_*)^s\times D}((\xi_0,0),(\xi,\eta)) = g_D(0,\eta) \\ &= \max\{|\eta^{\beta_j}|^{1/r_j} : j = 1,\dots,m\} \\ &= \max\{|z^{\alpha_j}|^{1/r_j} : j = 1,\dots,m\}. \end{aligned}$$

The implications (ii)⇒(iii)⇒(iv)⇒(v) are trivial.

(v)⇒(i). Suppose that $\operatorname{rank}\widetilde{A} < \operatorname{rank} A$. We may assume that

$$2 \le t := \operatorname{rank} A = \operatorname{rank}\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_t\end{bmatrix}, \quad \operatorname{rank}\begin{bmatrix}\beta_1\\ \vdots\\ \beta_t\end{bmatrix} < t.$$

Then there exist $c_1, \ldots, c_t \in \mathbb{Z}$ such that $c_1\beta_1 + \ldots + c_t\beta_t = 0$ and $|c_1| + \ldots + |c_t| > 0$. We may assume that $c_1, \ldots, c_u \geq 0$, $c_{u+1}, \ldots, c_t < 0$ for some $1 \leq u \leq t-1$. Let

$$d := a^{c_1\alpha_1+\ldots+c_t\alpha_t},$$

$$r := c_1r_1 + \ldots + c_ur_u = -(c_{u+1}r_{u+1} + \ldots + c_tr_t),$$

$$f(z) := \frac{z^{c_1\alpha_1+\ldots+c_u\alpha_u} - dz^{-(c_{u+1}\alpha_{u+1}+\ldots+c_t\alpha_t)}}{1+|d|}, \quad z \in G.$$

Observe that $f \in \mathcal{O}(G, E)$, $\operatorname{ord}_a f \geq r+1$, and $f \not\equiv 0$ (because $\alpha_1, \ldots, \alpha_t$ are linearly independent). Fix $b = (b', b'') \in G \subset \mathbb{C}^s \times \mathbb{C}^{n-s}$ with $f(b) \neq 0$. Observe that $f(b', \theta b'') = \theta^r f(b)$, $0 \leq \theta \leq 1$. Thus we get

$$\begin{aligned}\frac{1}{\theta} m_G^{(r+1)}(a, (b', \theta b'')) &\geq \frac{1}{\theta}|f(b', \theta b'')|^{1/(r+1)}\\ &= \theta^{-1/(r+1)}|f(b)|^{1/(r+1)} \xrightarrow[\theta\to 0+]{} \infty;\end{aligned}$$

a contradiction. ∎

5. Proofs of Propositions 4 and 5

Proof of Proposition 4. By (C1) (for (Φ_1, B_1, S_1)) there exists a mapping $\widetilde{F} : B_1 \setminus S_1 \to B_2$ such that $\Phi_2 \circ F = \widetilde{F} \circ \Phi_1$. By (C2), $\widetilde{F}$ is holomorphic. The Riemann extension theorem implies that $\widetilde{F}$ extends holomorphically to a mapping $\widetilde{F} : B_1 \to \overline{B}_2$ (we use the same symbol for the extension). By the identity principle we have $\Phi_2 \circ F = \widetilde{F} \circ \Phi_1$ on B_1. In particular, $\widetilde{F} : B_1 \to B_2$.

It is clear that if F is biholomorphic, then so is $\widetilde{F}$.

Now, assume that B_1 is hyperconvex and F is proper. Since F is proper, there exists $b \in G_2$ such that

- $\Phi_2'(b) \neq 0$,
- $\operatorname{rank} F'(a) = n$ for any $a \in P := F^{-1}(b)$ (note that P is finite),
- $\operatorname{rank} \Phi_1'(a) = m$ for any $a \in P$.

By (7) and [Lár-Sig] (see also [Edi-Zwo 1]) we get

$$\begin{aligned}\prod_{\xi_0 \in \Phi_1(P)} g_{B_1}(\xi_0, \Phi_1(z)) &\leq g_{B_1}(\Phi_1(P); 1; \Phi_1(z))\\ &= g_{G_1}(P; 1; z) = g_{G_2}(\{b\}; 1; F(z))\\ &= g_{G_2}(b, F(z)) = g_{B_2}(\Phi_2(b), \Phi_2(F(z)))\\ &= g_{B_2}(\Phi_2(b), \widetilde{F}(\Phi_1(z))), \quad z \in G_1.\end{aligned}$$

Hence, since B_1 is hyperconvex, we obtain

$$\liminf_{\xi\to\partial B_1} g_{B_2}(\Phi_2(b), \widetilde{F}(\xi)) \geq 1,$$

which implies that $\widetilde{F}$ is proper. ∎

REMARK 8. Under the assumptions of Proposition 4 let $F : G_1 \to G_2$ be biholomorphic. Let $\xi \in B_1$ and let $\eta := \widetilde{F}(\xi)$. Then $\Phi_1^{-1}(\xi)$ satisfies (C1) (resp. (C2)) iff $\Phi^{-1}(\eta)$ satisfies (C1) (resp. (C2)).

In particular, a set $S_1 \subset B_1$ is singular for (Φ_1, B_1) iff $\widetilde{F}(S_1)$ is singular for (Φ_2, B_2).

Moreover, for any $a \in G_1$ we have

$$\operatorname{ord}_a(\Phi_1 - \Phi_1(a)) = \operatorname{ord}_{F(a)}(\Phi_2 - \Phi_2(F(a))).$$

REMARK 9. Observe that if Q is a homogeneous polynomial of n complex variables and $Q^{(k)}(a) = 0$, then by the Euler identity, $Q^{(k-1)}(a) = 0, \ldots, Q(a) = 0$.

Let $Q = (Q_1, \ldots, Q_m) : \mathbb{C}^n \to \mathbb{C}^m$ be a homogeneous polynomial mapping with $\deg Q_1 = \ldots = \deg Q_m =: d \geq 2$,

$$Q(z) = \sum_{|\alpha|=d} a_\alpha z^\alpha.$$

Then the following conditions are equivalent:

(i) $$\exists_{a\in\mathbb{C}^n\setminus\{0\}} : Q(z + a) = Q(z),\ z \in \mathbb{C}^n;$$

(ii) $$\exists_{a\in\mathbb{C}^n\setminus\{0\}} : \operatorname{ord}_a(Q - Q(a)) = d;$$

(iii) $$\exists_{a\in\mathbb{C}^n\setminus\{0\}} : Q^{(d-1)}(a) = 0;$$

(iv) $$\operatorname{rank}[(\beta + e_k)!a_{j,\beta+e_k}]_{\substack{|\beta|=d-1,\, j=1,\ldots,m\\ k=1,\ldots,n}} < n,$$

where

$$e_k := (0, \ldots, 0, \underset{k\text{th position}}{1}, 0, \ldots, 0), \qquad k = 1, \ldots, n.$$

Moreover, conditions (i), (ii), and (iii) are also equivalent when the point a is fixed.

Proof of Proposition 5. (a) Suppose that $F_0 : G_1 \to G_2$ is a fixed biholomorphic mapping.

Assume that (i) holds (in particular, $F_0(0) = 0$) and suppose that $\operatorname{ord}_a(\Phi_2 - \Phi_2(a)) = d_2$ for some $a \neq 0$. We may assume that $a \in G_2$. Hence, by Remark 9, the translation $z \overset{T}{\mapsto} z + a$ maps G_2 onto G_2. Then $F := T \circ F_0 : G_1 \to G_2$ is a biholomorphic mapping with $F(0) = T(0) = a \neq 0$; a contradiction.

Now, assume that (ii) holds and suppose that $F : G_1 \to G_2$ is a biholomorphic mapping with $a := F(0) \neq 0$. Then, by Remark 8, we get

$$\begin{aligned}\operatorname{ord}_a(\Phi_2 - \Phi_2(a)) &= \operatorname{ord}_0 \Phi_1 = d_1\\ &\geq \operatorname{ord}_{F^{-1}(0)}(\Phi_1 - \Phi_1(F^{-1}(0))) = \operatorname{ord}_0 \Phi_2 = d_2;\end{aligned}$$

a contradiction.

(b) It is known (cf. [Jar-Pfl], Corollary 3.5.7) that if $F(0) = 0$, then the mapping $L := e^{i\vartheta}F'(0)$ (with a suitable $\vartheta \in \mathbb{R}$) satisfies the required condition.

Assume that $a := F(0) \neq 0$. Then by (a) we have $\operatorname{ord}_a(\Phi_2 - \Phi_2(a)) = d_2$. Hence, by Remark 9, the translation $z \overset{T}{\mapsto} z + a$ maps G_2 onto G_2. Taking $T \circ F$, we reduce the problem to the case $F(0) = 0$. ∎

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