New examples of effective formulas for holomorphically contractible functions

by

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Abstract. Let $G \subset \mathbb{C}^n$ and $B \subset \mathbb{C}^m$ be domains and let $\Phi : G \to B$ be a surjective holomorphic mapping. We characterize some cases in which invariant functions and pseudometrics on $G$ can be effectively expressed in terms of the corresponding functions and pseudometrics on $B$.

0. Introduction. It is well known that holomorphically contractible families of functions or pseudometrics give very useful and powerful tools in complex analysis. Recall that a family $(d_G)_G$ of functions $d_G : G \times G \to \mathbb{R}_+$ (where $G$ runs through all domains in $\mathbb{C}^n$ with arbitrary $n$) is called holomorphically contractible if

- $\tanh d_E$ is the hyperbolic distance on the unit disc $E$,
- $d_{G_2}(F(a), F(z)) \leq d_{G_1}(a, z)$ for all $F \in \mathcal{O}(G_1, G_2)$ and $a, z \in G_1$.

A family $(\delta_G)_G$ of pseudometrics $\delta_G : G \times \mathbb{C}^n \to \mathbb{R}_+$ (i.e. $\delta_G(a; \lambda X) = |\lambda|\delta_G(a; X)$, $a \in G \subset \mathbb{C}^n \ni X$, $\lambda \in \mathbb{C}$) is called holomorphically contractible if

- $\delta_E$ is the hyperbolic pseudometric on $E$,
- $\delta_{G_2}(F(a); F'(a)(X)) \leq \delta_{G_1}(a; X)$ for all $F \in \mathcal{O}(G_1, G_2)$ and $a \in G_1 \subset \mathbb{C}^{n_1} \ni X$.

For simplicity, each $d_G$ (resp. $\delta_G$) will be called an invariant function (resp. invariant pseudometric); cf. [Jar-Pfl].

Frequently, the following problem appears. We are given a holomorphically contractible family $(d_G)_G$ of functions (e.g. the family of the pluricomplex Green functions). We want to verify certain holomorphic properties of a domain $G$ via corresponding properties of $d_G$. Consequently, we have to check whether $d_G$ satisfies some conditions, e.g. whether $d_G$ has a restricted

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growth near the boundary. However, for a given domain $G$, it is in general
difficult to describe $d_G$ by an effective formula. Therefore, usually one could
proceed as follows. First, we approximate $G$ by more elementary domains
$G'$ such that $d_{G'}$ can be calculated. Next, using limit procedures, we try to
estimate $d_G$. It is clear that what we need for such an approach is a large (up
to a biholomorphic equivalence) class of “elementary” domains for which at
least some of the invariant functions and pseudometrics can be calculated.

Recall that in the case of one complex variable the formulas are known
only in the case of the unit disc or an annulus (cf. [Jar-Pfl], Ch. V). In the
case of several variables the formulas are known for example for all norm
balls in $\mathbb{C}_n$ with transitive group of automorphisms (i.e. the unit polydisc
$E^n$, the Euclidean ball $\mathbb{B}_n$, and the Lie ball $L_n$; cf. [Jar-Pfl], §8.3). Besides
these classical domains, the only class for which effective descriptions of
invariant functions are known, is the class of elementary Reinhardt domains
of the form

$$G := \{(z_1, \ldots, z_n) \in \mathbb{C}_n : |z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n} < 1\},$$

where $\alpha_1, \ldots, \alpha_n > 0$ (cf. [Jar-Pfl], §4.4, [Pfl-Zwo], [Edi-Zwo 2]).

The aim of the paper is to generalize results of [Jar-Pfl] and to obtain a
larger class of domains for which we can produce effective formulas. More
precisely, suppose that $\Phi : G \to B$ is a surjective holomorphic mapping,
where $B \subset \mathbb{C}_m$ is a domain for which $d_B$ is known. We are interested
(Theorem 1) in those cases in which

$$d_G(a, z) = (d_B(\Phi(a), \Phi(z)))^{1/r(a)},$$

where $r(a) := \text{ord}_a(\Phi - \Phi(a))$ denotes the order of vanishing. In the special
case where $B = B_1 \times \cdots \times B_m$ we also discuss (Proposition 3) some situations
in which

$$d_G(a, z) = \max\{(d_B(\Phi_j(a), \Phi_j(z)))^{1/r_j(a)} : j = 1, \ldots, m\},$$

where $r_j(a) := \text{ord}_a(\Phi_j - \Phi_j(a))$, $j = 1, \ldots, m$.

We also present (Propositions 4 and 5) some characterizations of proper
and biholomorphic mappings between domains from Theorem 1.

1. **Notation.** Let $G \subset \mathbb{C}_n$ be a domain. We will consider the following
invariant functions and pseudometrics (cf. [Jar-Pfl]).

The $k$th *Möbius function*:

$$m_G^{(k)}(a, z) := \sup\{|f(z)|^{1/k} : f \in \mathcal{O}(G, E), \text{ord}_a f \geq k\},$$

$$a, z \in G, \ k \in \mathbb{N}.$$
The \(k\)th Reiffen pseudometric:
\[
\gamma_G^{(k)}(a; X) := \sup \left\{ \left| \frac{1}{k!} f^{(k)}(a)(X) \right|^{1/k} : f \in \mathcal{O}(G, E), \ \text{ord}_a f \geq k \right\},
\]
\(a \in G, \ X \in \mathbb{C}^n, \ k \in \mathbb{N}.
\]

The pluricomplex Green function with poles \(P\) and weights \(\nu\) (cf. [Lei]):
\[
g_G(P; \nu; z) := \sup \{ u(z) : \log u \in \mathcal{PSH}(G, [−\infty, 0)) \},
\]
where \(P\) is a finite subset of \(G\), and \(\nu : P \to (0, \infty)\).

The pluricomplex Green function:
\[
g_G(a, z) := g_G(\{a\}; 1; z), \quad a, z \in G.
\]

The Azukawa pseudometric:
\[
A_G(a; X) := \limsup_{0 \neq \lambda \to 0} \frac{g_G(a, a + \lambda X)}{|\lambda|}, \quad a \in G, \ X \in \mathbb{C}^n.
\]

Let \(V\) be an analytic subset of an open set \(\Omega \subset \mathbb{C}^n\). Recall that an upper semicontinuous function \(u : V \to [−\infty, \infty)\) is said to be plurisubharmonic if for any holomorphic mapping \(\phi : E \to V\) the function \(u \circ \phi\) is subharmonic on \(E\) (cf. [For-Nar]).

We say that \(V\) has the plurisubharmonic Liouville property if any function plurisubharmonic and bounded from above on \(V\) is constant.

Observe that if \(V\) has the plurisubharmonic Liouville property, then \(V\) has the Liouville property, i.e., any function holomorphic and bounded on \(V\) is constant.

2. Main results. The main result of the paper is the following theorem.

**Theorem 1.** Let \(\Omega \subset \mathbb{C}^n\) be open, \(\Phi \in \mathcal{O}(\Omega, \mathbb{C}^m)\) \((m < n)\), and let \(B \subset \Phi^{-1}(\Omega)\) be a domain. Put \(G := \Phi^{-1}(B)\). For \(a \in G\) let \(r(a) := \text{ord}_a(\Phi - \Phi(a))\).

Assume that there exists a thin relatively closed subset \(S\) of \(B\) such that for any \(\xi \in B \setminus S\),
\[
(C1) \text{ } \Phi^{-1}(\xi) \text{ has the plurisubharmonic Liouville property,}
\]
\[
(C2) \exists a \in \Phi^{-1}(\xi) : \text{rank} \Phi'(a) = m.
\]

Then \(G\) is a domain and the following formulas hold:
(a) We have
\[
(1) \quad m_G^{(1)}(a, z) = m_B^{(1)}(\Phi(a), \Phi(z)), \quad a, z \in G,
\]
\[
(2) \quad \gamma_G^{(1)}(a; X) = \gamma_B^{(1)}(\Phi(a); \Phi'(a)(X)), \quad a \in G, \ X \in \mathbb{C}^n.
\]
(b) If \(a \in G\) is such that the analytic set dimension satisfies
\[
(\ast) \quad \dim(\{X \in \mathbb{C}^n : \Phi'(a)(X) = 0\}) = n - m,
\]
where \( r := r(a), \) then

\[
m_G^{(k)}(a, z) = (m_B^{(r)}(\Phi(a), \Phi(z)))^{\ell/k}, \quad z \in G,
\]

\[
\gamma_G^{(k)}(a; X) = \begin{cases} 
\left( \gamma_B^{(k/r)}(\Phi(a); \frac{1}{r!} \Phi^{(r)}(a)(X)) \right)^{1/r} & \text{if } k/r \in \mathbb{N}, \\
0 & \text{if } k/r \notin \mathbb{N}, 
\end{cases}
\]

\[ X \in \mathbb{C}^n, \quad k \in \mathbb{N}, \]

where \( \ell = \ell(a, k) := \mu(k/r) \) and \( \mu(t) := -(\text{the integer part of } -t). \)

\[
\gamma^{(k)}(a) = \left( \gamma^{(k/r)}(B(\Phi(a); \Phi(z)))^{1/r} \right)^{1/r}, \quad z \in \mathbb{C}^n,
\]

\[
\gamma^{(k)}(a, z) = \left( \gamma^{(k/r)}(B(\Phi(a); \Phi(z)))^{1/r} \right)^{1/r}, \quad z \in \mathbb{C}^n.
\]

Conditions (C1), (C2), \((*)\) are always satisfied if \( m = 1 \) and \( \Phi \) is a primitive polynomial (cf. Remark 7). Consequently, if \( \Phi \) is a primitive polynomial, then formulas (1)–(7) are true.

The proof of Theorem 1 will be given in §3.

Remark 2. (a) The case where \( \Phi \) is a monomial and \( B = E \) has been studied in [Jar-Pfl], §4.4, and [Edi-Zwo 2].

(b) If rank \( \Phi'(a) = m \) or \( m = 1 \), then \((*)\) is satisfied.

(c) If \((*)\) is not satisfied, then formulas (3)–(7) need not be true (cf. Proposition 3).

Let \( a_j = (a_{j,1}, \ldots, a_{j,n}) \in (\mathbb{Z}_+)^n \setminus \{0\}, \) \( j = 1, \ldots, m \) \((m \geq 2), \) and

\[
G := \{ z \in \mathbb{C}^n : |z^{a_j}| < 1, \quad j = 1, \ldots, m \}.
\]

Fix an \( a \in G \) with \( a^{a_j} = 0, \) \( j = 1, \ldots, m. \) Assume that

\[
a = (a_1, \ldots, a_s, 0, \ldots, 0)
\]

with \( a_1 \ldots a_s \neq 0 \) and \( 1 \leq s \leq n - 1. \) Put

\[
A := [\alpha_{j,k}]_{j=1,\ldots,m}^{k=1,\ldots,n} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}, \quad \tilde{A} := [\beta_{j,k}]_{j=1,\ldots,m}^{k=s+1,\ldots,n} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}.
\]

Notice that \( r_j := \text{ord}_a z^{a_j} = |\beta_j| > 0, \) \( j = 1, \ldots, m. \)
Proposition 3. The following conditions are equivalent:

(i) \( \text{rank } A = \text{rank } \tilde{A} \);
(ii) \( g_G(a, z) = \max\{|z^\alpha|^{1/r} : j = 1, \ldots, m\}, \quad z \in G; \)
(iii) \( g_G(a, z) = \sup\{|z^\alpha|^{1/r} : \alpha \in (\mathbb{Z}_+)^n, |z^\alpha| < 1 \text{ in } G, \quad r = \text{ord}_z z^\alpha > 0\}, \quad z \in G; \)
(iv) \( g_G(a, (z', \lambda z'')) = |\lambda| g_G(a, z), \quad z = (z', z'') \in G \subset \mathbb{C}^* \times \mathbb{C}^{n-s}, \quad \lambda \in E; \)
(v) \( \forall z \in \mathbb{N}: \left\{ (z', z'') \in G : \limsup_{\theta \to 0+} \frac{1}{dG}(a, (z', \theta z'')) < \infty \right\} \) is not thin.

The proof of Proposition 3 will be given in §4.

Recall that a bounded domain \( D \subset \mathbb{C}^n \) is called hyperconvex if it admits a negative plurisubharmonic exhaustion function.

Proposition 4. Let \( \Omega_j \subset \mathbb{C}^n \), \( \Phi_j \in \mathcal{O}(\Omega_j, \mathbb{C}^{n'}) \), and \( S_j \subset B_j \subset \Phi_j(\Omega_j) \) be such that \( (\Phi_j, B_j, S_j) \) satisfies (C1) and (C2) from Theorem 1 and \( B_j \) is bounded. Put \( G_j := \Phi_j^{-1}(B_j) \), \( j = 1, 2 \), and let \( F: G_1 \to G_2 \) be a holomorphic mapping. Then there exists a holomorphic mapping \( \tilde{F} : B_1 \to B_2 \) such that

\[
\Phi_2 \circ F = \tilde{F} \circ \Phi_1.
\]

Moreover,

- if \( F \) is biholomorphic, then so is \( \tilde{F} \);
- if \( n_1 = n_2 =: n, m_1 = m_2 =: m, B_1 \) is hyperconvex, and \( F \) is proper, then \( \tilde{F} \) is proper.

The case where \( \Phi_j \) is a monomial and \( B_j = E, j = 1, 2 \), has been studied in [Edi-Zwo 2].

Proposition 5. Let \( \Phi_j = (Q_{j,1}, \ldots, Q_{j,m}) : \mathbb{C}^n \to \mathbb{C}^m \) be a homogeneous polynomial with \( \deg Q_{j,1} = \ldots = \deg Q_{j,m} =: d_j \geq 2 \). Let \( S_j \subset B_j \subset \Phi_j(\mathbb{C}^n) \) be such that \( (\Phi_j, B_j, S_j) \) satisfies (C1) and (C2) (for example, \( m = 1 \) and \( \Phi_j \) is a primitive homogeneous polynomial), \( B_j \) is bounded, and \( 0 \in B_j \).

Put \( G_j := \Phi_j^{-1}(B_j) \), \( j = 1, 2 \).

(a) If \( G_1 \) and \( G_2 \) are biholomorphic, then the following conditions are equivalent:

(i) for any biholomorphic mapping \( F : G_1 \to G_2 \) we have \( F(0) = 0 \);
(ii) \( \text{ord}_a (\Phi_2 - \Phi_2(a)) < d_2 \) for any \( a \neq 0 \).

(b) Assume additionally that \( B_1 \) and \( B_2 \) are balanced (consequently, \( G_1 \) and \( G_2 \) are balanced). Then the following conditions are equivalent:

(i) \( G_1, G_2 \) are biholomorphic;
(ii) there is a linear isomorphism \( L : \mathbb{C}^n \to \mathbb{C}^n \) such that \( L(G_1) = G_2. \)
In particular, if \( \Phi_1(z) = z^\alpha \), \( \Phi_2(z) = z^\beta \) are primitive monomials, 
\( G_1 := \{ |z^\alpha| < 1 \} \), and \( G_2 := \{ |z^\beta| < 1 \} \), then \( G_1 \) and \( G_2 \) are biholomorphic if \( \alpha = \beta \) up to permutation (cf. [Edi-Zwo 2]).

The proofs of Propositions 4 and 5 will be given in §5.

3. Proof of Theorem 1

Proposition 6. (a) Let \( D \subset \mathbb{C}^d \) be a domain having the plurisubharmonic Liouville property and let \( V \) be a connected pure \( d \)-dimensional analytic subset of \( D \times \mathbb{C}^{n-d} \) such that the natural projection

\[ V \ni (z, w) \mapsto z \in D \]

is proper. Then \( V \) has the plurisubharmonic Liouville property.

(b) Any connected pure \( d \)-dimensional algebraic subset of \( \mathbb{C}^n \) has the plurisubharmonic Liouville property.

Proof. (a) We may assume that \( V \) is irreducible. Let \( u \) be plurisubharmonic on \( V \) with \( c_0 := \sup_V u < \infty \). Define

\[ \tilde{u}(z) := \max\{u(z, w) : (z, w) \in V\}, \quad z \in D. \]

Using the standard methods (cf. [For-Nar], the proof of Lemma 5.1), we prove that \( \tilde{u} \in \mathcal{P}SH(D) \). Since \( D \) has the plurisubharmonic Liouville property, \( \tilde{u} = \text{const} = c_0 \).

To prove that \( u = \text{const} \) it suffices to show that \( u = c_0 \) on a dense subset of \( V \). Let \( \Delta \subset D \) be an analytic set such that \( \pi : V \setminus \pi^{-1}(\Delta) \to D \setminus \Delta \) is a holomorphic covering. We show that \( u = c_0 \) on \( V_0 := V \setminus \pi^{-1}(\Delta) \). Notice that \( V_0 = \text{Reg}(V) \setminus \pi^{-1}(\Delta) \), where \( \text{Reg}(V) \) denotes the set of all regular points of \( V \). We know that \( \text{Reg}(V) \) is connected (because \( V \) is irreducible). Thus \( V_0 \) is connected.

Let \( \tilde{V}_0 := \{(z, w) \in V_0 : u(z, w) = c_0\} \). Then \( \tilde{V}_0 \neq \emptyset \) and \( \tilde{V}_0 \) is closed in \( V_0 \). Moreover, by the maximum principle, \( \tilde{V}_0 \) is open. Thus \( \tilde{V}_0 = V_0 \), i.e. \( u = c_0 \) on \( V_0 \).

(b) follows from (a) and the fact that for an algebraic subset \( V \) of \( \mathbb{C}^n \), after a linear change of coordinates, the projection \( \pi \) is proper; cf. [Chi].

Remark 7. (a) Recall that a polynomial \( P \) of \( n \) complex variables is primitive iff \( P \) cannot be represented in the form \( P = f(Q) \), where \( f \) is a polynomial of one complex variable of degree \( \geq 2 \) and \( Q \) is a polynomial of \( n \) complex variables (cf. [Cyg]).

In particular, a homogeneous polynomial \( P \) is primitive iff \( P \) cannot be written as \( P = Q^p \), where \( p \geq 2 \) and \( Q \) is a homogeneous polynomial.

A monomial \( z^\alpha \), where \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n, n \geq 2 \), is primitive iff the numbers \( \alpha_1, \ldots, \alpha_n \) are relatively prime.
(b) It is known (cf. [Cyg]) that if \( P \) is a primitive polynomial, then the fibers \( P^{-1}(\xi) \) are connected except for a finite number of \( \xi \).

c) If \( P \) is a polynomial, then the set \( P(\{z \in \mathbb{C}^n : P'(z) = 0\}) \) is finite.

d) Properties (b) and (c) and Proposition 6(b) show that if \( \Phi \) is a primitive polynomial, then (C1), (C2) are satisfied with a finite set \( S \subset B \).

e) Let \( \Phi : \mathbb{C}^n \to \mathbb{C}^m \) \((m < n)\),

\[ \Phi(z_1, \ldots, z_m, z_{m+1}, \ldots, z_n) = \Phi(z_1, \ldots, z_m, z') := (z_1(z')^{\beta_1}, \ldots, z_m(z')^{\beta_m}), \]

where \( \beta_1, \ldots, \beta_m \in (\mathbb{Z}_+)^{n-m} \). Then for any \( \xi = (\xi_1, \ldots, \xi_m) \in (\mathbb{C}_\ast)^m \)

\[(\mathbb{C}_\ast)^{n-m} \ni \lambda \mapsto (\xi_1/\lambda^{\beta_1}, \ldots, \xi_m/\lambda^{\beta_m}, \lambda) \in \Phi^{-1}(\xi). \]

Hence, (C1) and (C2) are satisfied with \( S := \{ (\xi_1, \ldots, \xi_m) \times (0) : \xi_1 \cdot \ldots \cdot \xi_m = 0 \} \).

**Proof of Theorem 1.** First we prove that \( G \) is a domain. Observe that

\[ G = \Phi^{-1}(B \setminus S) \cup \Phi^{-1}(S) =: G_0 \cup S_0. \]

Since \( S_0 \) is a thin relatively closed subset of \( G \), it suffices to prove that \( G_0 \)

is a domain. Suppose that \( G_0 = U_1 \cup U_2 \), where \( U_1, U_2 \) are open, disjoint, and non-empty. Let \( B_j := \{ \xi \in B \setminus S : \Phi^{-1}(\xi) \subset U_j \}, j = 1, 2 \). By (C1) we have \( B_1 \cup B_2 = B \setminus S \). Obviously, \( B_1, B_2 \) are disjoint and non-empty.

Fix \( \xi^0 \in B_j \). By (C2) there exists \( a \in \Phi^{-1}(\xi^0) \) such that \( \text{rank} \Phi'(a) = m \). We may assume that \( \text{rank} \Phi'(a) = m \). By the implicit mapping theorem, the equation \( \Phi(z', z^m) = \xi \) (where \( (z', z^m) \in \mathbb{C}^{n-m} \times \mathbb{C}^m \)) is equivalent in a neighborhood of \( (a, \xi^0) \) to \( z^m = \phi(z', \xi) \), where \( \phi \) is holomorphic. In particular, \( (a', \phi(a', \xi)) \in U_j \cap \Phi^{-1}(\xi) \) for \( \xi \) in a neighborhood \( W \) of \( \xi_0 \). Hence, by (C1), \( W \subset B_j \). Consequently, \( B_j \) is open, \( j = 1, 2 \); a contradiction.

Formulas (2), (4), and (6) follow from (1), (3) and (5), respectively, and from properties of \( \gamma_G^{(k)} \) and \( A_G \) (cf. [Jar-Pfl], §4.2). Formula (5) follows from (7). In formulas (1), (3), and (7) the inequalities \( \geq \) follow directly from the definitions. Therefore we only need to prove the opposite inequalities.

Let \( f \in \mathcal{O}(G, E) \), \( \text{ord}_a f \geq k \). By (C1) there exists a function \( \tilde{f} : B \setminus S \to E \) such that \( f = \tilde{f} \circ \Phi \) on \( G \setminus \Phi^{-1}(S) \). Condition (C2) and the implicit mapping theorem imply that \( f \in \mathcal{O}(B \setminus S) \). Now, by the Riemann theorem, \( \tilde{f} \) extends holomorphically to \( B \) (we denote the extension by the same symbol) and \( f = \tilde{f} \circ \Phi \) in \( G \). Obviously, \( \tilde{f}(\Phi(a)) = 0 \). In particular, if \( k = 1 \), then we get (1).

Assume that (\#) is satisfied.
Let $L$ be an $m$-dimensional vector subspace of $\mathbb{C}^n$ such that 

$$L \cap \{X \in \mathbb{C}^n : \Phi^{(1)}(a)(X) = 0\} = \{0\}.$$ 

It is clear that there exist $\rho > 0$ and $C > 0$ such that 

$$\|\Phi(a + X) - \Phi(a)\| \geq C\|X\|^r, \quad X \in L \cap B(\rho).$$ 

Now one can easily prove that there exist neighborhoods $U \subset B(\rho)$ of 0 and $V \subset \mathbb{C}^m$ of $\Phi(a)$ such that the mapping $L \cap U \ni X \mapsto \Phi(a + X) \in V$ is proper (in particular, surjective).

For $\xi \in V$ let $X(\xi) \in L \cap U$ be such that $\Phi(a + X(\xi)) = \xi$. Then, by $(\cdot)$, we get 

$$|\tilde{f}(\xi)| = |f(a + X(\xi))| \leq \text{const} \|X(\xi)\|^k \leq \text{const} \|\Phi(a + X(\xi)) - \Phi(a)\|^k/r
= \text{const} \|\xi - \Phi(a)\|^{k/r}, \quad \xi \in V.$$ 

Hence $\text{ord}_{\Phi(a)} \tilde{f} \geq \mu(k/r)$, and therefore, 

$$m_B^{(\ell)}(\Phi(a), \Phi(z)) \geq |\tilde{f}(\Phi(z))|^{1/\ell} = |f(z)|^{1/\ell},$$ 

which implies that 

$$m_B^{(\ell)}(\Phi(a), \Phi(z)) \geq (m_G^{(k)}(a, z))^{k/\ell},$$ 

and so the proof of (1) and (3) is complete.

We turn to the proof of (7). Let $u : G \to [0, 1]$ be such that $\log u \in \mathcal{PSh}(G)$ and $u(w) \leq M\|w - a\|^\nu(a)$ for any $(a, w) \in P \times G$. By (C1) there exists a function $\tilde{u} : B \setminus S \to [0, 1]$ such that $u = \tilde{u} \circ \Phi$ on $G \setminus \Phi^{-1}(S)$. Condition (C2) and the implicit mapping theorem imply that $\log \tilde{u} \in \mathcal{PSh}(B \setminus S)$.

Now, by the Riemann type theorem for plurisubharmonic functions, $\tilde{u}$ extends to a log-plurisubharmonic function on $B$ (we denote the extension by the same symbol). By the identity principle for plurisubharmonic functions we get $u = \tilde{u} \circ \Phi$ in $G$.

Fix $a \in P$, let $r := r(a)$, $\nu := \nu(a)$, and let $X(\xi), \xi \in V$, be as above. Then 

$$\tilde{u}(\xi) = u(a + X(\xi)) \leq M\|X(\xi)\|^\nu \leq \text{const} \|\Phi(a + X(\xi)) - \Phi(a)\|^\nu/r
= \text{const} \|\xi - \Phi(a)\|^\nu/r, \quad \xi \in V.$$ 

Hence 

$$\tilde{u}(\xi) \leq M|\xi - \xi_0|^\nu(\xi_0), \quad (\xi_0, \xi) \in \Phi(P) \times B.$$ 

Thus 

$$g_B(\Phi(P); \tilde{u}; \Phi(z)) \geq \tilde{u}(\Phi(z)) = u(z),$$ 

which implies that 

$$g_B(\Phi(P); \tilde{u}; \Phi(z)) \geq g_G(P; \nu; z).$$ 

The last part of the theorem follows from Remark 7(d).
4. Proof of Proposition 3. (i)⇒(ii). Let
\[ L(z) := g_G(a, z), \quad R(z) := \max\{|z^\alpha_j|^{1/r_j} : j = 1, \ldots, m\}, \quad z \in G. \]

The inequality \( L \geq R \) follows from the definition of \( g_G \). To prove that \( L \leq R \) it suffices to show that \( L(z) \leq R(z) \) for any \( z \in G_0 := G \cap ((\mathbb{C}_*)^s \times \mathbb{C}^n_k) \).

By (i), for any \( k = 1, \ldots, s \), the system of equations
\[ \alpha_{j,s+1}x_{s+1} + \ldots + \alpha_{j,n}x_n = -\alpha_{j,k}, \quad j = 1, \ldots, m, \]
has a rational solution \((Q_{s+1,k}/\mu_k, \ldots, Q_{n,k}/\mu_k)\) with \( Q_{s+1,k}, \ldots, Q_{n,k} \in \mathbb{Z}, \mu_k \in \mathbb{N} \). Put \( Q_{k,k} := \mu_k \) and \( Q_{k,j} := 0, j, k = 1, \ldots, s, j \neq k \). Then
\[ (\xi, \eta) = (\xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_n) \in (\mathbb{C}_*)^s \times \mathbb{C}^n_k. \]

Moreover, if \( z = F(\xi, \eta) \), then by (i) we get
\[ z^\alpha_j = \xi^{\alpha_{j,1}Q_1, \ldots, \alpha_{j,n}Q_n} \eta^\beta_j, \quad j = 1, \ldots, m. \]

Let
\[ D := \{ \eta \in \mathbb{C}^n_k : |\eta^\beta_j| < 1, j = 1, \ldots, m \}. \]

Using (i) we get the equality \( F((\mathbb{C}_*)^s) = G_0 \).

Fix a \( \xi_0 \in (\mathbb{C}_*)^s \) such that \( a = F(\xi_0, 0) \). Then, for any \( z = F(\xi, \eta) \in G_0 \), we have
\[ g_G(a, z) = g_G(F(0, z), F(\xi, \eta)) \leq g(\xi_0, z) \leq \max\{|\eta^\beta_j|^{1/r_j} : j = 1, \ldots, m\} = \max\{|z^\alpha_j|^{1/r_j} : j = 1, \ldots, m\}. \]

The implications (ii)⇒(iii)⇒(iv)⇒(v) are trivial.

(v)⇒(i). Suppose that rank \( \tilde{\Lambda} < \text{rank} \Lambda \). We may assume that
\[ 2 \leq t := \text{rank} \tilde{\Lambda} = \text{rank} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_t \end{bmatrix}, \quad \text{rank} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_t \end{bmatrix} < t. \]
Then there exist $c_1, \ldots, c_t \in \mathbb{Z}$ such that $c_1 \beta_1 + \ldots + c_t \beta_t = 0$ and $|c_1| + \ldots + |c_t| > 0$. We may assume that $c_1, \ldots, c_n \geq 0$, $c_{n+1}, \ldots, c_t < 0$ for some $1 \leq u \leq t - 1$. Let
\[
d := a^{c_1 \alpha_1 + \ldots + c_t \alpha_t},
\]
\[
r := c_1 r_1 + \ldots + c_u r_u = -(c_{u+1} r_{u+1} + \ldots + c_t r_t),
\]
\[
f(z) := \frac{z^{c_1 \alpha_1 + \ldots + c_u \alpha_u} - \frac{dz}{1 + |d|} - (c_{u+1} \alpha_{u+1} + \ldots + c_t \alpha_t)}{1 + |d|}, \quad z \in G.
\]
Observe that $f \in \mathcal{O}(G, E)$, $\text{ord}_a f \geq r + 1$, and $f \neq 0$ (because $\alpha_1, \ldots, \alpha_t$ are linearly independent). Fix $b = (b', b'') \in G \subseteq \mathbb{C}^s \times \mathbb{C}^{n-t}$ with $f(b) \neq 0$.

Observe that $f(b', \theta b'') = \theta f(b)$, $0 \leq \theta \leq 1$. Thus we get
\[
\frac{1}{\theta} m_G^{(r+1)}(a, (b', \theta b'')) \geq \frac{1}{\theta} |f(b', \theta b'')|^{1/(r+1)}
\]
\[
= \theta^{-1/(r+1)} |f(b)|^{1/(r+1)} \xrightarrow{\theta \to 0 \pm} \infty;
\]
a contradiction. ■

5. Proofs of Propositions 4 and 5

Proof of Proposition 4. By (C1) (for $(\Phi_1, B_1, S_1)$) there exists a mapping $\tilde{F} : B_1 \setminus S_1 \to B_2$ such that $\Phi_2 \circ \tilde{F} = \tilde{F} \circ \Phi_1$. By (C2), $\tilde{F}$ is holomorphic. The Riemann extension theorem implies that $\tilde{F}$ extends holomorphically to a mapping $\tilde{F} : B_1 \to \overline{B_2}$. (we use the same symbol for the extension). By the identity principle we have $\Phi_2 \circ F = \tilde{F} \circ \Phi_1$ on $B_1$. In particular, $\tilde{F} : B_1 \to B_2$.

It is clear that if $F$ is biholomorphic, then so is $\tilde{F}$.

Now, assume that $B_1$ is hyperconvex and $F$ is proper. Since $F$ is proper, there exists $b \in G_2$ such that
\begin{itemize}
  \item $\Phi_2(b) \neq 0$,
  \item $\text{rank} F'(a) = n$ for any $a \in P := F^{-1}(b)$ (note that $P$ is finite),
  \item $\text{rank} \Phi'_1(a) = m$ for any $a \in P$.
\end{itemize}

By (7) and [Lár-Sig] (see also [Edi-Zwo 1]) we get
\[
\prod_{\xi_0 \in \Phi_1(P)} g_{B_1}(\xi_0, \Phi_1(z)) \leq g_{B_1}(\Phi_1(P); 1; \Phi_1(z)) = g_{G_1}(P; 1; z) = g_{G_2}(\{b\}; 1; F(z)) = g_{B_2}(b, F(z)) = g_{B_2}(\Phi_2(b), \Phi_2(F(z))) = g_{B_2}(\Phi_2(b), \tilde{F}(\Phi_1(z))), \quad z \in G_1.
\]
Hence, since $B_1$ is hyperconvex, we obtain
\[
\liminf_{\xi \to \partial B_1} g_{B_2}(\Phi_2(b), \tilde{F}(\xi)) \geq 1,
\]
which implies that $\tilde{F}$ is proper. ■
Remark 8. Under the assumptions of Proposition 4 let $F : G_1 \to G_2$ be biholomorphic. Let $\xi \in B_1$ and let $\eta := \tilde{F}(\xi)$. Then $\Phi^{-1}_1(\xi)$ satisfies (C1) (resp. (C2)) iff $\Phi^{-1}(\eta)$ satisfies (C1) (resp. (C2)).

In particular, a set $S_1 \subset B_1$ is singular for $(\Phi_1, B_1)$ iff $\tilde{F}(S_1)$ is singular for $(\Phi_2, B_2)$.

Moreover, for any $a \in G_1$ we have

$$\text{ord}_a(\Phi_1 - \Phi_1(a)) = \text{ord}_F(\Phi_2 - \Phi_2(F(a))).$$

Remark 9. Observe that if $Q$ is a homogeneous polynomial of $n$ complex variables and $Q(k)(a) = 0$, then by the Euler identity, $Q^{(k-1)}(a) = 0, \ldots, Q(a) = 0$.

Let $Q = (Q_1, \ldots, Q_m) : \mathbb{C}^n \to \mathbb{C}^m$ be a homogeneous polynomial mapping with $\text{deg } Q_1 = \ldots = \text{deg } Q_m =: d \geq 2$,

$$Q(z) = \sum_{|\alpha|=d} a_{\alpha} z^{\alpha}.$$ 

Then the following conditions are equivalent:

(i) $\exists a \in \mathbb{C}^n \setminus \{0\} : Q(z + a) = Q(z), z \in \mathbb{C}^n$;

(ii) $\exists a \in \mathbb{C}^n \setminus \{0\} : \text{ord}_a(Q - Q(a)) = d$;

(iii) $\exists a \in \mathbb{C}^n \setminus \{0\} : Q^{(d-1)}(a) = 0$;

(iv) $\text{rank}((\beta + e_k)!a_{j,\beta + e_k})_{|\beta|=d-1, j=1,\ldots,m} < n$,

where

$$e_k := (0, \ldots, 0, 1_{k\text{th position}}, 0, \ldots, 0), \quad k = 1, \ldots, n.$$ 

Moreover, conditions (i), (ii), and (iii) are also equivalent when the point $a$ is fixed.

Proof of Proposition 5. (a) Suppose that $F_0 : G_1 \to G_2$ is a fixed biholomorphic mapping.

Assume that (i) holds (in particular, $F_0(0) = 0$) and suppose that $\text{ord}_a(\Phi_2 - \Phi_2(a)) = d_2$ for some $a \neq 0$. We may assume that $a \in G_2$.

Hence, by Remark 9, the translation $z \mapsto z + a$ maps $G_2$ onto $G_2$. Then $F := T \circ F_0 : G_1 \to G_2$ is a biholomorphic mapping with $F(0) = T(0) = a \neq 0$; a contradiction.

Now, assume that (ii) holds and suppose that $F : G_1 \to G_2$ is a biholomorphic mapping with $a := F(0) \neq 0$. Then, by Remark 8, we get

$$\text{ord}_a(\Phi_2 - \Phi_2(a)) = \text{ord}_0 \Phi_1 = d_1$$

$$\geq \text{ord}_{F^{-1}(0)}(\Phi_1 - \Phi_1(F^{-1}(0))) = \text{ord}_0 \Phi_2 = d_2;$$

a contradiction.
(b) It is known (cf. [Jar-Pfl], Corollary 3.5.7) that if $F(0) = 0$, then the mapping $L := e^{i\vartheta} F'(0)$ (with a suitable $\vartheta \in \mathbb{R}$) satisfies the required condition.

Assume that $a := F(0) \neq 0$. Then by (a) we have $\text{ord}_a(\Phi_2 - \Phi_2(a)) = d_2$. Hence, by Remark 9, the translation $z \mapsto z + a$ maps $G_2$ onto $G_2$. Taking $T \circ F$, we reduce the problem to the case $F(0) = 0$.

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