[23] S. A. Vinogradov and V. P. Khavin, *Free interpolation in H^∞ and some other classes of functions II*, ibid. 14 (1980), no. 2, 1027–1065.

UFR Mathématiques/Informatique
Université Bordeaux I
351, cours de la Libération
33405 Talence Cedex, France
E-mail: hartmann@math.u-bordeaux.fr

Received June 10, 1997 (3899)
Revised version December 28, 1998 and February 23, 1999

STUDIA MATHEMATICA 135 (2) (1999)

Eigenvalue problems with indefinite weight

by

ANDRZEJ SZULKIN (Stockholm)
and MICHEL WILLEM (Louvain-la-Neuve)

Abstract. We consider the linear eigenvalue problem $-\Delta u = \lambda V(x)u$, $u \in \mathcal{D}_0^{1,2}(\Omega)$, and its nonlinear generalization $-\Delta_p u = \lambda V(x)|u|^{p-2}u$, $u \in \mathcal{D}_0^{1,p}(\Omega)$. The set Ω need not be bounded, in particular, $\Omega = \mathbb{R}^N$ is admitted. The weight function V may change sign and may have singular points. We show that there exists a sequence of eigenvalues $\lambda_n \to \infty$.

1. Introduction. In this paper we shall be concerned with the linear eigenvalue problem

$$-\Delta u = \lambda V(x)u, \quad u \in \mathcal{D}_0^{1,2}(\Omega), \tag{1}$$

Ω open in $\mathbb{R}^N$, $N \ge 3$, and its nonlinear generalization

$$-\Delta_p u = \lambda V(x)|u|^{p-2}u, \quad u \in \mathcal{D}_0^{1,p}(\Omega), \tag{2}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $1 < p < N$, and Ω is open in $\mathbb{R}^N$. Observe that Ω may be unbounded, and in particular, it may be equal to $\mathbb{R}^N$. We assume that $V \in L^1_{\mathrm{loc}}(\Omega)$, $V = V^+ - V^-$ (as usual, $V^\pm(x) := \max\{\pm V(x), 0\}$) and $V^+ = V_1 + V_2$, where $V_1 \in L^{N/p}(\Omega)$, $|x|^p V_2(x) \to 0$ as $|x| \to \infty$ and for each $y \in \overline{\Omega}$, $|x-y|^p V_2(x) \to 0$ as $x \to y$ (in the linear case (1), $p = 2$ in the conditions on V^+). Under these hypotheses we show that (1) and (2) have a sequence of eigenvalues $\lambda_n \to \infty$. This generalizes several earlier results. In particular, for $\Omega = \mathbb{R}^N$ it was shown in [3, 4] that (1) has a principal eigenvalue λ_1 if V is sufficiently smooth and satisfies an appropriate condition at infinity, and in [1] existence of infinitely many eigenvalues $\lambda_n \to \infty$ of (1) was established under

1991 *Mathematics Subject Classification*: 35J20, 35J70, 35P05, 35P30.
Key words and phrases: eigenvalue problem, Laplacian, p-Laplacian, indefinite weight.
The first author was supported in part by the Swedish Natural Science Research Council.
The second author thanks the members of the Department of Mathematics of Stockholm University for their invitation and hospitality.

the assumption that $V \in L^\infty(\mathbb{R}^N)$ and $V^+ \in L^{N/2}(\mathbb{R}^N)$. In [18] several results on the existence and nonexistence of a principal eigenvalue of (1) were obtained for nonnegative weight functions V of Hardy type. In this case even if a principal eigenvalue exists, one cannot expect to have a sequence of eigenvalues $\lambda_n \to \infty$. Equation (2) for $\Omega = \mathbb{R}^N$ was studied in [2], where it was demonstrated that if $V \in L^\infty(\mathbb{R}^N)$ and $V^+ \in L^{N/p}(\mathbb{R}^N)$, then there is a sequence $\lambda_n \to \infty$ (see also [8, 10]). Furthermore, it was shown in [7] that (2) has a principal eigenvalue whenever $V \in L^{N/p}(\mathbb{R}^N) \cap L^{(N+\delta)/p}(\mathbb{R}^N)$ for some $\delta > 0$. More references concerning (1)–(2), in particular to earlier work on bounded Ω, may be found in the papers cited above.

The paper is organized as follows: In Section 2 we prove the existence of infinitely many eigenvalues of (1). Our argument is fairly elementary and is based on a simple minimization procedure. We also show that under an additional assumption on V the principal eigenvalue of (1) is simple. In Section 3 we give a few examples demonstrating that our hypotheses on V are in a sense optimal. Finally, in Section 4 we are concerned with the nonlinear problem (2). Again, a simple minimization argument shows the existence of a principal eigenvalue λ_1. However, since the equation is nonlinear now, it is not clear whether higher eigenvalues can be obtained by minimization. Therefore we use a different approach, based on minimax methods in critical point theory.

NOTATION. $B(x,r)$ and $B[x,r]$ denote respectively the open and the closed ball centered at x and having radius r. $|\cdot|_p$ is the usual norm in $L^p(\Omega)$, $\mathcal{D}(\Omega)$ are the test functions in Ω and $\mathcal{D}_0^{1,p}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in the norm $\|u\| := |\nabla u|_p$. A functional $\chi : X \to \mathbb{R}$ is weakly continuous if $u_n \rightharpoonup u$ implies that $\chi(u_n) \to \chi(u)$.

2. Eigenvalues of the Laplacian. In this section we consider the linear eigenvalue problem

$$-\Delta u = \lambda V(x) u, \quad u \in \mathcal{D}_0^{1,2}(\Omega), \tag{3}$$

where Ω is an open subset of $\mathbb{R}^N$, $N \ge 3$. Possibly $\Omega = \mathbb{R}^N$. Our basic assumption is

(H) $\quad V \in L^1_{\mathrm{loc}}(\Omega)$, $V^+ = V_1 + V_2 \neq 0$, $V_1 \in L^{N/2}(\Omega)$,

$$\lim_{\substack{x\to y\\ x\in\Omega}} |x-y|^2 V_2(x) = 0 \quad \text{for every } y \in \overline{\Omega}, \qquad \lim_{\substack{|x|\to\infty\\ x\in\Omega}} |x|^2 V_2(x) = 0.$$

In order to find the principal eigenvalue of (3) we solve the following minimization problem:

(P_1) $\quad$ minimize $\int_\Omega |\nabla u|^2\,dx$, $u \in \mathcal{D}_0^{1,2}(\Omega)$, $\int_\Omega V u^2\,dx = 1$.

We shall use the following notation:

$$X := \mathcal{D}_0^{1,2}(\Omega), \qquad \varphi(u) := \int_\Omega |\nabla u|^2\,dx, \qquad \psi(u) := \int_\Omega V u^2\,dx.$$

LEMMA 2.1. *Under assumption (H), $\int_\Omega V^+ u^2\,dx$ is weakly continuous.*

Proof. By [20, Lemma 2.13], $\int_\Omega V_1 u^2\,dx$ is weakly continuous.

In order to prove that $\int_\Omega V_2 u^2\,dx$ is weakly continuous, let us recall the Hardy inequality in $\mathcal{D}_0^{1,2}(\mathbb{R}^N)$:

$$\int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\,dx \le \frac{4}{(N-2)^2} \int_{\mathbb{R}^N} |\nabla u|^2\,dx.$$

Let $u_n \rightharpoonup u$ and $\varepsilon > 0$. By assumption, there exists $R > 0$ such that if $x \in \Omega$ and $|x| \ge R$, then $|x|^2 V_2(x) \le \varepsilon$. Define

$$\Omega_1 := \Omega \setminus B[0,R], \qquad \Omega_2 := \Omega \cap B(0,R), \qquad c := \frac{2}{N-2} \sup_n \|u_n\|.$$

The Hardy inequality implies that

$$\int_{\Omega_1} V_2 u_n^2\,dx \le \varepsilon \int_{\Omega_1} \frac{u_n^2}{|x|^2}\,dx \le \varepsilon c^2, \tag{4}$$

and similarly,

$$\int_{\Omega_1} V_2 u^2\,dx \le \varepsilon c^2. \tag{5}$$

By compactness, there is a finite covering of $\overline{\Omega}_2$ by closed balls $B[x_1, r_1], \ldots, B[x_k, r_k]$ such that, for $1 \le j \le k$,

$$|x - x_j| \le r_j \Rightarrow |x - x_j|^2 V_2(x) \le \varepsilon. \tag{6}$$

There exists $r > 0$ such that, for $1 \le j \le k$,

$$|x - x_j| \le r \Rightarrow |x - x_j|^2 V_2(x) \le \varepsilon/k.$$

Define $A := \bigcup_{j=1}^k B[x_j, r]$. Then by the Hardy inequality,

$$\int_A V_2 u_n^2\,dx \le \varepsilon c^2, \qquad \int_A V_2 u^2\,dx \le \varepsilon c^2. \tag{7}$$

It follows from (6) that $V_2 \in L^\infty(\Omega_2 \setminus A)$. Since $\Omega_2 \setminus A$ is bounded, $V_2 \in L^{N/2}(\Omega_2 \setminus A)$ so that by [20, Lemma 2.13],

$$\int_{\Omega_2\setminus A} V_2 u_n^2\,dx \to \int_{\Omega_2\setminus A} V_2 u^2\,dx. \tag{8}$$

We deduce from (4), (5), (7) and (8) that $\int_\Omega V_2 u_n^2\,dx \to \int_\Omega V_2 u^2\,dx$. ∎

THEOREM 2.2. *Under assumption* (H), *problem* (P_1) *has a solution* $e_1 \geq 0$. *Moreover*, e_1 *is an eigenfunction of* (3) *corresponding to the eigenvalue* $\lambda_1 := \int_\Omega |\nabla e_1|^2\,dx$.

Proof. Let (u_n) be a minimizing sequence for (P_1). Since (u_n) is bounded in X, we may assume that $u_n \rightharpoonup u$. Hence we obtain

$$\int_\Omega |\nabla u|^2\,dx \leq \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^2\,dx = \inf(P_1).$$

Since $\int_\Omega V^- u_n^2\,dx = \int_\Omega V^+ u_n^2\,dx - 1$, the preceding lemma and Fatou's lemma imply that $\int_\Omega V^- u^2\,dx \leq \int_\Omega V^+ u^2\,dx - 1$, i.e., $\int_\Omega V u^2\,dx \geq 1$. It is then clear that u is a solution of (P_1). Moreover, since also $|u|$ is a solution, we may assume $u \geq 0$.

Since for every $v \in \mathcal{D}(\Omega)$,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \frac{\varphi(u+\varepsilon v)}{\psi(u+\varepsilon v)} = 0,$$

u is an eigenfunction of (3) corresponding to the eigenvalue $\int_\Omega |\nabla u|^2\,dx$. ∎

In order to find the other positive eigenvalues of (3) we solve the problems

(P_n) minimize $\int_\Omega |\nabla u|^2\,dx$, $u \in \mathcal{D}_0^{1,2}(\Omega)$,
$\int_\Omega \nabla u \cdot \nabla e_1\,dx = \ldots = \int_\Omega \nabla u \cdot \nabla e_{n-1}\,dx = 0$, $\int_\Omega V u^2\,dx = 1$,

where e_j is the solution of (P_j), $1 \leq j \leq n-1$.

THEOREM 2.3. *Under assumption* (H), *for every* $n \geq 2$, *problem* (P_n) *has a solution* e_n. *Moreover*, e_n *is an eigenfunction of* (3) *corresponding to the eigenvalue* $\lambda_n := \int_\Omega |\nabla e_n|^2\,dx$, *and* $\lambda_n \to \infty$ *as* $n \to \infty$.

Proof. The existence of e_n is proved as in Theorem 2.2. An elementary argument (see [19, Lemma 4.44]) shows that e_n is an eigenfunction of (3) corresponding to the eigenvalue $\lambda_n := \int_\Omega |\nabla e_n|^2\,dx$.

The sequence $f_n := e_n/\sqrt{\lambda_n}$ is orthonormal in X so that $f_n \rightharpoonup 0$. Since $\lambda_n^{-1} = \lambda_n^{-1}\int_\Omega |\nabla f_n|^2\,dx = \int_\Omega V f_n^2\,dx$, Lemma 2.1 implies that $0 \leq \lim_{n\to\infty} \lambda_n^{-1} = \lim_{n\to\infty} \int_\Omega V f_n^2\,dx \leq 0$. ∎

REMARKS 2.4. (a) If $-V$ satisfies (H), then problem (3) has infinitely many negative eigenvalues $0 > \lambda_{-1} \geq \lambda_{-2} \geq \ldots$ Moreover, $\lambda_{-n} \to -\infty$ as $n \to \infty$ and the eigenfunction corresponding to λ_{-1} is nonnegative.

(b) Theorems 2.2 and 2.3 depend only on the weak continuity of $\int_\Omega V^+ u^2\,dx$ and on the weak lower semicontinuity of $\int_\Omega V^- u^2\,dx$. It is easy to formulate an abstract version of these results.

(c) Necessary and sufficient conditions for the weak continuity of $\int_\Omega V^+ u^2\,dx$, in terms of capacities, may be found in [13, Section 2.4.2]. We would like to thank A. Laptev for bringing the reference [13] to our attention.

In order to prove the simplicity of λ_1 which we mentioned in the introduction, we need the following additional assumption:

(H_1) There exists $p > N/2$ and a closed subset S of measure 0 in $\mathbb{R}^N$ such that $\Omega \setminus S$ is connected and $V \in L^p_{\mathrm{loc}}(\Omega \setminus S)$.

THEOREM 2.5. *Under assumptions* (H) *and* (H_1), λ_1 *is a simple eigenvalue of* (3).

Proof. Let u be an eigenfunction corresponding to λ_1 such that $\int_\Omega V u^2\,dx = 1$. Since $|u|$ is a solution of (P_1), $|u|$ is also an eigenfunction. Hence u^+ and u^- are eigenfunctions.

By regularity theory [12, Theorem 11.7], any eigenfunction belongs to $W^{2,q}_{\mathrm{loc}}(\Omega \setminus S) \cap C^{0,\alpha}_{\mathrm{loc}}(\Omega \setminus S)$, $q = 2N/(N+2)$, $0 < \alpha < 2 - N/p$. The unique continuation theorem of Jerison and Kenig [11] implies that $u = u^+$ or $u = -u^-$. It follows immediately that λ_1 is simple. ∎

3. Examples and counterexamples. We assume in this section that $\Omega = \mathbb{R}^N$. The following result, due to Tertikas, is contained in Proposition 4.5 of [18]:

THEOREM 3.1. *Let* $V \in L^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N \setminus \{0\})$. *If* u *is an eigenfunction of* (3), *then*

$$\int_{\mathbb{R}^N} (2V(x) + x \cdot \nabla V(x)) u^2(x)\,dx = 0. \tag{9}$$

REMARK 3.2. Theorem 3.1 has a simple formal explanation. An eigenvalue of (3) is a stationary point of φ/ψ. If $T(\varrho)u(x) := u(x/\varrho)$, then

$$\frac{d}{d\varrho}\bigg|_{\varrho=1} \frac{\varphi(T(\varrho)u)}{\psi(T(\varrho)u)} = 0$$

implies (9) (see [20, Appendix B]).

EXAMPLE 3.3. As observed by Tertikas, if $W_1(x) := 1/(1+|x|^2)$, then for all $x \in \mathbb{R}^N$, $2W_1(x) + x \cdot \nabla W_1(x) > 0$, and if $W_2(x) := 1/(|x|^2(1+|x|^2))$, then for all $x \in \mathbb{R}^N \setminus \{0\}$, $2W_2(x) + x \cdot \nabla W_2(x) < 0$. By Theorem 3.1, (3) has no eigenvalue if $V = W_1$ or $V = W_2$.

Now observe that $W_1 \in L^q(\mathbb{R}^N)$ for all $q > N/2$, $W_2 \in L^q(\mathbb{R}^N)$ for all $q \in (N/4, N/2)$ but neither W_1 nor W_2 is in $L^{N/2}(\mathbb{R}^N)$.

EXAMPLE 3.4. Define

$$W_3(x) := \frac{1}{(1+|x|^2)[\log(2+|x|^2)]^{2/N}},$$

$$W_4(x) := \frac{1}{|x|^2(1+|x|^2)[\log(2+1/|x|^2)]^{2/N}}.$$

By Theorem 2.3, (3) has infinitely many positive eigenvalues if $V = W_3$ or W_4 although W_3, W_4 are not in $L^{N/2}(\mathbb{R}^N)$ (W_3, W_4 are in the same L^q-spaces as respectively W_1 and W_2).

THEOREM 3.5. *If $|x|^2V(x) \to \infty$ as $|x| \to \infty$ or $|x-y|^2V(x) \to \infty$ as $x \to y$ for some y, then the infimum in (P_1) is* 0 (*and is not achieved*).

Proof. We only consider the case of $|x|^2V(x) \to \infty$ as $x \to 0$, the other cases being similar. Let $u \in \mathcal{D}(\mathbb{R}^N)$ and set $u_r(x) := u(x/r)$. Then

$$\frac{\int_{\mathbb{R}^N} |\nabla u_r(x)|^2\,dx}{\int_{\mathbb{R}^N} V(x)u_r(x)^2\,dx} = \frac{\int_{\mathbb{R}^N} |\nabla u(x)|^2\,dx}{\int_{\mathbb{R}^N} (r|x|)^2 V(rx)\frac{u(x)^2}{|x|^2}\,dx}.$$

Since u has compact support and $u^2/|x|^2 \in L^1(\mathbb{R}^N)$, it follows easily that the right-hand side above tends to 0 as $r \to 0$.

In the case of $|x| \to \infty$ the function $u \in \mathcal{D}(\mathbb{R}^N)$ should be chosen so that $0 \notin \operatorname{supp} u$. ∎

4. The p-Laplacian. Our purpose here is to extend the results of Section 2 to the nonlinear eigenvalue problem

$$(10) \qquad -\Delta_p u = \lambda V(x)|u|^{p-2}u, \qquad u \in \mathcal{D}_0^{1,p}(\Omega),$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $1 < p < N$ and Ω is an open, in general unbounded, subset of $\mathbb{R}^N$. The assumption (H) of Section 2 now reads:

(H_p) $\quad V \in L^1_{\mathrm{loc}}(\Omega)$, $V^+ = V_1 + V_2 \neq 0$, $V_1 \in L^{N/p}(\Omega)$,

$$\lim_{\substack{x\to y\\ x\in\Omega}} |x-y|^p V_2(x) = 0 \quad \text{for every } y \in \overline{\Omega}, \qquad \lim_{\substack{|x|\to\infty\\ x\in\Omega}} |x|^p V_2(x) = 0.$$

Consider the problem

(Q_1) $\quad$ minimize $\int_\Omega |\nabla u|^p\,dx$, $u \in \mathcal{D}_0^{1,p}(\Omega)$, $\int_\Omega V|u|^p\,dx = 1$.

It is easy to show that $\int_\Omega V^+|u|^p\,dx$ is weakly continuous in $\mathcal{D}_0^{1,p}(\Omega)$. The proof parallels that of Lemma 2.1 except that now we use the Hardy inequality

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p}\,dx \le \left(\frac{p}{N-p}\right)^p \int_{\mathbb{R}^N} |\nabla u|^p\,dx, \qquad u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N)$$

(see [9] for a simple proof).

THEOREM 4.1. *Under assumption (H_p), problem (Q_1) has a solution $e_1 \ge 0$. Moreover, e_1 is an eigenfunction of* (10) *corresponding to the eigenvalue $\lambda_1 := \int_\Omega |\nabla e_1|^p\,dx$.*

Proof. Repeat the argument of Theorem 2.2. ∎

Since equation (10) is nonlinear (unless $p = 2$), it is not possible to obtain higher eigenvalues by the method of Section 2. Instead we shall use critical point theory. Let

$$\varphi(u) := \int_\Omega |\nabla u|^p\,dx \quad \text{and} \quad \psi(u) := \int_\Omega V|u|^p\,dx.$$

Since the set $\{u \in \mathcal{D}_0^{1,p}(\Omega) : \psi(u) = 1\}$ is not a manifold unless further assumptions are made on V^-, we introduce a new space $X := \{u \in \mathcal{D}_0^{1,p}(\Omega) : \|u\|_X < \infty\}$, where

$$\|u\|_X^p := \int_\Omega |\nabla u|^p\,dx + \int_\Omega V^-|u|^p\,dx.$$

Then $M := \{u \in X : \psi(u) = 1\}$ is a C^1-manifold, critical points of $\varphi|_M$ are eigenfunctions and the corresponding critical values are eigenvalues of (10).

Let $\psi^\pm(u) := \int_\Omega V^\pm|u|^p\,dx$.

LEMMA 4.2. *If V satisfies (H_p), then:*

(i) *The Fréchet derivative of ψ^+ is completely continuous as a mapping from X to X^*.*

(ii) *$\psi^+(u) \le c\varphi(u)$ for some $c > 0$ and all $u \in X$.*

Proof. (i) Let $u_n \rightharpoonup u$. By the Hölder and Sobolev inequalities,

$$\int_\Omega V_1(|u_n|^{p-2}u_n - |u|^{p-2}u)v\,dx$$
$$\le \Big(\int_\Omega V_1\big||u_n|^{p-2}u_n - |u|^{p-2}u\big|^{p/(p-1)}\,dx\Big)^{(p-1)/p}\Big(\int_\Omega V_1|v|^p\,dx\Big)^{1/p}$$
$$\le d_1\|v\|_X\Big(\int_\Omega V_1\big||u_n|^{p-2}u_n - |u|^{p-2}u\big|^{p/(p-1)}\,dx\Big)^{(p-1)/p}.$$

It is easy to see that $\big||u_n|^{p-2}u_n - |u|^{p-2}u\big|^{p/(p-1)} \rightharpoonup 0$ in $L^{N/(N-p)}(\Omega)$ (indeed, otherwise there would exist a subsequence going weakly to some $v \neq 0$ and a.e. to 0, a contradiction to [19, Theorem 10.36]). Since $V_1 \in L^{N/p}(\Omega)$, the right-hand side above tends to 0 uniformly for $\|v\|_X \le 1$. This shows the complete continuity of the V_1-part.

Using the notation of Lemma 2.1 and the Hölder, Hardy and Sobolev inequalities, we see that

$$\int_{\Omega_1} V_2(|u_n|^{p-2}u_n - |u|^{p-2}u)v\,dx \le d_2\varepsilon\|v\|_X(\|u_n\|_X^{p-1} + \|u\|_X^{p-1}) \le d_3\varepsilon\|v\|_X.$$

Similarly, the above integral taken over A is $\le d_4\varepsilon\|v\|_X$ (the d_i's are independent of ε). Since $\Omega_2 \setminus A$ is bounded and $V_2 \in L^\infty(\Omega_2 \setminus A)$, it follows from the continuity of the superposition operator [14, 20] that $|u_n|^{p-2}u_n \to |u|^{p-2}u$ in $L^{p/(p-1)}(\Omega_2 \setminus A)$ and

$$\int_{\Omega_2\setminus A} V_2(|u_n|^{p-2}u_n - |u|^{p-2}u)v\,dx \to 0.$$

(ii) By the Hölder and Sobolev inequalities,

$$\int_\Omega V_1|u|^p\,dx \le d_5 \int_\Omega |\nabla u|^p\,dx.$$

Fixing some $\varepsilon > 0$ and using the Hölder, Hardy and Sobolev inequalities again, it is easy to see that

$$\int_{\Omega_1} V_2|u|^p\,dx \le d_6 \int_\Omega |\nabla u|^p\,dx,$$

and similar inequalities hold on A and $\Omega_2 \setminus A$. The conclusion now follows by recalling the definitions of ψ^+ and φ. ∎

Let $\mu > 0$ and let $A_\mu : X \to X^*$ be the operator given by

$$\langle A_\mu(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx + \mu\int_\Omega V^-|u|^{p-2}uv\,dx$$

($\langle\cdot,\cdot\rangle$ denotes the duality pairing).

Lemma 4.3. *If $u_n \rightharpoonup u$ and $\langle A_\mu(u_n), u_n - u\rangle \to 0$, then $u_n \to u$ in X.*

Proof. Our argument is borrowed from [6] where it appears in the proof of Lemma 3.3. Clearly, $\langle A_\mu(u_n) - A_\mu(u), u_n - u\rangle \to 0$. By the Hölder inequality,

$$\begin{aligned}
&\int_\Omega V^-(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\\
&\quad= \int_\Omega V^-(|u_n|^p + |u|^p - |u_n|^{p-2}u_nu - |u|^{p-2}uu_n)\\
&\quad\ge \int_\Omega V^-(|u_n|^p + |u|^p) - \Big(\int_\Omega V^-|u_n|^p\Big)^{(p-1)/p}\Big(\int_\Omega V^-|u|^p\Big)^{1/p}\\
&\qquad - \Big(\int_\Omega V^-|u|^p\Big)^{(p-1)/p}\Big(\int_\Omega V^-|u_n|^p\Big)^{1/p}\\
&\quad= \Big[\Big(\int_\Omega V^-|u_n|^p\Big)^{(p-1)/p} - \Big(\int_\Omega V^-|u|^p\Big)^{(p-1)/p}\Big]\\
&\qquad\times\Big[\Big(\int_\Omega V^-|u_n|^p\Big)^{1/p} - \Big(\int_\Omega V^-|u|^p\Big)^{1/p}\Big] \ge 0.
\end{aligned}$$

Since the left-hand side above tends to 0, $\int_\Omega V^-|u_n|^p\,dx \to \int_\Omega V^-|u|^p\,dx$. Similarly, $\int_\Omega |\nabla u_n|^p\,dx \to \int_\Omega |\nabla u|^p\,dx$. Hence $\|u_n\|_X \to \|u\|_X$ and therefore $u_n \to u$ in X. ∎

Let A be a closed subet of M such that $A = -A$. Recall [14, 16] that the *Krasnosel'skiĭ genus* $\gamma(A)$ is by definition the smallest integer k for which there exists an odd mapping $A \to \mathbb{R}^k \setminus \{0\}$. If there is no such mapping for any k, then $\gamma(A) := +\infty$. Moreover, $\gamma(\emptyset) := 0$. Let

$$\lambda_n := \inf_{\gamma(A)\ge n}\sup_{u\in A}\varphi(u), \qquad n = 1, 2, \ldots$$

Since $\{x \in \mathbb{R}^N : V(x) > 0\}$ has positive measure, for each n there is a set $A \subset M$ which is homeomorphic to the unit sphere $S^{n-1} \subset \mathbb{R}^n$ by an odd homeomorphism. Since $\gamma(S^{n-1}) = n$, there exist sets of arbitrarily large genus and all λ_n are well defined. Moreover, $\lambda_1 = \inf_{u\in M}\varphi(u)$. Hence λ_1 coincides with the first eigenvalue obtained in Theorem 4.1 and $\lambda_n \ge \lambda_1 > 0$ for all n. If M is of class C^2 (which is the case for $p \ge 2$) and $\varphi|_M$ satisfies the Palais–Smale condition, then classical critical point theory [16, Section II.5] implies that the λ_n's are critical values. If $1 < p < 2$, then M is only of class C^1; however, the same conclusion remains valid as follows from the results contained in [5] and [17].

As λ_n is a critical value of $\varphi|_M$, there exists a critical point e_n with $\varphi(e_n) = \lambda_n$. Hence $\varphi'(e_n) = \mu\psi'(e_n)$, where μ is a Lagrange multiplier, and (2) is satisfied with $u = e_n$ and $\lambda = \mu$. Since $p\varphi(e_n) = \langle\varphi'(e_n), e_n\rangle = \mu\langle\psi'(e_n), e_n\rangle = p\mu$, we have $\mu = \varphi(e_n) = \lambda_n$, so λ_n is an eigenvalue and e_n is a corresponding eigenfunction.

Theorem 4.4. *Under assumption (H_p), $\varphi|_M$ has a sequence of critical points (e_n) with corresponding critical values $\lambda_n = \int_\Omega |\nabla e_n|^p\,dx$. Moreover, each e_n is an eigenfunction of (10), λ_n is an associated eigenvalue, and $\lambda_n \to \infty$ as $n \to \infty$.*

Proof. Let (u_k) be a Palais–Smale sequence. Then there exist $\mu_k \in \mathbb{R}$ such that

$$\varphi'(u_k) - \mu_k\psi'(u_k) \to 0 \tag{11}$$

(cf. [20, Proposition 5.12]). Since $\varphi(u_k)$ is bounded, so is $\psi^+(u_k)$ according to Lemma 4.2(ii), and therefore also

$$\psi^-(u_k) = \psi^+(u_k) - 1 \tag{12}$$

is bounded. Hence $\|u_k\|_X^p \equiv \varphi(u_k) + \psi^-(u_k)$ is bounded and we may assume passing to a subsequence that $u_k \rightharpoonup u$. Since $(\psi^+)'$ is completely continuous, $\psi^+(u_k) \to \psi^+(u)$ and it follows from (12) that $u \ne 0$. By (11),

$$p(\varphi(u_k) - \mu_k) = \langle\varphi'(u_k), u_k\rangle - \mu_k\langle\psi'(u_k), u_k\rangle \to 0.$$

Therefore (μ_k) is bounded and we may assume $\mu_k \to \mu$. Moreover, taking the limit above we obtain $0 < \varphi(u) \le \mu$, so $\mu > 0$. We may rewrite (11) as

$$A_{\mu_k}(u_k) - \mu_k(\psi^+)'(u_k) \to 0.$$

Since $A_{\mu_k}(u_k) - A_\mu(u_k) \to 0$ as is easily seen from the definition of A_μ and since $(\psi^+)'(u_k) \to (\psi^+)'(u)$, it follows that $A_\mu(u_k)$ is strongly convergent. So $\langle A_\mu(u_k), u_k - u\rangle \to 0$ and $u_k \to u$ according to Lemma 4.3.

We have shown that $\varphi|_M$ satisfies the Palais–Smale condition. It follows from our earlier discussion that each λ_n is a critical value of $\varphi|_M$ and an eigenvalue of the problem (10). Moreover, if $\lambda_n = \ldots = \lambda_{n+m}$ for some $m \ge 1$, then the set of critical points corresponding to λ_n has genus $\ge m+1$ [16, Lemma II.5.6] and is therefore infinite. Hence the eigenfunctions e_n may be chosen so that $e_n \ne e_j$ whenever $n \ne j$. Finally, a well known argument [14, Proposition 9.33] shows that the critical values λ_n must necessarily tend to infinity. ∎

REMARK. 4.5. It was shown in [7] that if $\Omega = \mathbb{R}^N$ and $V \in L^{N/p}(\mathbb{R}^N) \cap L^{(N+\delta)/p}(\mathbb{R}^N)$ for some $\delta > 0$, then the principal eigenvalue λ_1 of (10) is simple.

In [15] Rozenblum and Solomyak studied the existence of the principal eigenvalue of (1) in $\mathbb{R}^N$ under weak conditions on V. While our hypotheses (on V_2) were formulated in terms of pointwise limits, those in [15] involved capacities and conditions on integrals. We would like to thank the referee for pointing out this reference.

References

[1] W. Allegretto, *Principal eigenvalues for indefinite-weight elliptic problems in R^n*, Proc. Amer. Math. Soc. 116 (1992), 701–706.

[2] W. Allegretto and Y. X. Huang, *Eigenvalues of the indefinite-weight p-Laplacian in weighted spaces*, Funkc. Ekvac. 38 (1995), 233–242.

[3] K. J. Brown, C. Cosner and J. Fleckinger, *Principal eigenvalues for problems with indefinite weight function on R^N*, Proc. Amer. Math. Soc. 109 (1990), 147–155.

[4] K. J. Brown and A. Tertikas, *The existence of principal eigenvalues for problems with indefinite weight function on $\mathbb{R}^k$*, Proc. Roy. Soc. Edinburgh Sect. A 123 (1993), 561–569.

[5] J.-N. Corvellec, M. Degiovanni and M. Marzocchi, *Deformation properties for continuous functionals and critical point theory*, Topol. Methods Nonlinear Anal. 1 (1993), 151–171.

[6] P. Drábek and Y. X. Huang, *Bifurcation problems for the p-Laplacian in R^N*, Trans. Amer. Math. Soc. 349 (1997), 171–188.

[7] P. Drábek, Z. Moudan and A. Touzani, *Nonlinear homogeneous eigenvalue problem in R^N: nonstandard variational approach*, Comm. Math. Univ. Carolin. 38 (1997), 421–431.

[8] J. Fleckinger, R. F. Manásevich, N. M. Stavrakakis and F. de Thélin, *Principal eigenvalues for some quasilinear elliptic equations on $\mathbb{R}^N$*, Adv. Differential Equations 2 (1997), 981–1003.

[9] J. P. Garcia Azorero and I. Peral Alonso, *Hardy inequalities and some critical elliptic and parabolic problems*, J. Differential Equations 144 (1998), 441–476.

[10] Y. X. Huang, *Eigenvalues of the p-Laplacian in R^N with indefinite weight*, Comm. Math. Univ. Carolin. 36 (1995), 519–527.

[11] D. Jerison and C. E. Kenig, *Unique continuation and absence of positive eigenvalues for Schrödinger operators*, Ann. of Math. 121 (1985), 463–494.

[12] E. H. Lieb and M. Loss, *Analysis*, Amer. Math. Soc., Providence, R.I., 1997.

[13] V. G. Maz'ja, *Sobolev Spaces*, Springer, Berlin, 1985.

[14] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conf. Ser. in Math. 65, Amer. Math. Soc., Providence, R.I., 1986.

[15] G. Rozenblum and M. Solomyak, *On principal eigenvalues for indefinite problems in Euclidean space*, Math. Nachr. 192 (1998), 205–223.

[16] M. Struwe, *Variational Methods*, Springer, Berlin, 1990.

[17] A. Szulkin, *Ljusternik–Schnirelmann theory on C^1-manifolds*, Ann. Inst. H. Poincaré Anal. Non Linéaire 5 (1988), 119–139.

[18] A. Tertikas, *Critical phenomena in linear elliptic problems*, J. Funct. Anal. 154 (1998), 42–66.

[19] M. Willem, *Analyse Harmonique Réelle*, Hermann, Paris, 1995.

[20] —, *Minimax Theorems*, Birkhäuser, Boston, 1996.

Department of Mathematics
Stockholm University
106 91 Stockholm, Sweden
E-mail: andrzejs@matematik.su.se

Institut de Mathématique Pure et Appliquée
Université Catholique de Louvain
1348 Louvain-la-Neuve, Belgium
E-mail: Willem@amm.ucl.ac.be

Received June 4, 1998
Revised version November 26, 1998
(4118)