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Free interpolation in Hardy–Orlicz spaces

by

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Abstract. We show that the Carleson condition is necessary and sufficient for free interpolation in Hardy–Orlicz spaces \mathcal{H}_φ on the unit disk \mathbb{D} under certain conditions on φ , and we give a characterization of the trace space $\mathcal{H}_\varphi|_A$ if A is a finite union of Carleson sequences.

Introduction. Let $\text{Hol}(\mathbb{D})$ be the space of holomorphic functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} = \partial\mathbb{D}$. A sequence $A = \{\lambda_n\}_{n \geq 1} \subset \mathbb{D}$ is called *of free interpolation* for a subspace $X \subset \text{Hol}(\mathbb{D})$ if the trace space

$$l = X|_A = \{f|_A : f \in X\}$$

is an *ideal space*, i.e. if $a \in l$ and $b \in \mathbb{C}^A$ are such that $|b(\lambda)| \leq |a(\lambda)|$, $\lambda \in A$, then $b \in l$ (cf. for example [14] or [6]). We will also use the following notation for the associated sequence space:

$$(1) \quad X(A) = \{(f(\lambda_n))_{n \geq 1} : f \in X\}.$$

The description of free interpolation sequences for the space $H^\infty = \{f \in \text{Hol}(\mathbb{D}) : \sup_{z \in \mathbb{D}} |f(z)| < \infty\}$ was given by L. Carleson [1] and it was shown by H. S. Shapiro and A. L. Shields [19] that the free interpolation sequences for H^∞ are exactly the same as for the Hardy spaces

$$H^p(\mathbb{D}) = \left\{ f \in \text{Hol}(\mathbb{D}) : \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^p dt < \infty \right\}, \quad 1 \leq p < \infty$$

(cf. also [10] for the case $0 < p < 1$). It is well known that we may identify $H^p(\mathbb{D})$ and $H^p(\mathbb{T}) = \{f \in L^p(\mathbb{T}) : \widehat{f}(n) = (2\pi)^{-1} \int_{-\pi}^{\pi} f(e^{it}) e^{-int} dt = 0, n < 0\}$.

In the first section we give the definition of Hardy–Orlicz spaces \mathcal{H}_φ and various conditions that one may impose on the defining function φ to get

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interesting results. Some of these conditions are rather classic, others are introduced here for technical reasons.

The second section will be devoted to the proof of the necessity of the Carleson condition for free interpolation in Hardy–Orlicz spaces \mathcal{H}_φ . The basic idea has been taken from the classical case of H^p (cf. Shapiro and Shields [19]). In fact, we interpolate the function $a(\lambda)\chi_\lambda \in \mathbb{C}^A$, where $a(\lambda)$ has to be chosen conveniently (and χ_λ is the characteristic function of the set $\{\lambda\}$), by a function $g_\lambda \in \mathcal{H}_\varphi$. Factorizing this function into $B_\lambda h_\lambda$, where B_λ is the Blaschke product associated with the zero set $A \setminus \{\lambda\}$, we get the desired result if this factorization is compatible with the norm of the space.

In the third section, we will give an explicit description of the trace space $\mathcal{H}_\varphi|_A$ of the Hardy–Orlicz space \mathcal{H}_φ if A is a Carleson sequence. Using the result of [7] (cf. also [5]), we extend the characterization of the trace space $\mathcal{H}_\varphi|_A$ to finite unions of Carleson sequences in the last section.

1. Definitions. It was shown by L. Carleson [1] that for a sequence $A \subset \mathbb{D}$ we have $H^\infty|_A = l^\infty(A) = \{a \in \mathbb{C}^A : \sup_{\lambda \in A} |a(\lambda)| < \infty\}$ if and only if

$$(2) \quad \inf_{\lambda \in A} \prod_{\mu \neq \lambda} |b_\mu(\lambda)| \geq \delta > 0,$$

where

$$b_\lambda = \frac{|\lambda|}{\lambda} \cdot \frac{\lambda - z}{1 - \bar{\lambda}z}$$

is the Blaschke factor. A sequence satisfying condition (2) is called a *Carleson sequence*, and in this case we write $A \in (C)$. It is well known that Carleson sequences satisfy the Blaschke condition $\sum_{\lambda \in A} (1 - |\lambda|) < \infty$. We remark that the condition $H^\infty|_A = l^\infty(A)$ is equivalent to $H^\infty|_A$ being an ideal space.

In order to define Hardy–Orlicz spaces not only in the Banach space case, we need the notion of strongly convex functions (see [16]):

DEFINITION 1.1. A function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is called *strongly convex* if

- (i) φ is convex,
- (ii) φ is nondecreasing,
- (iii) $\varphi \geq 0$,
- (iv) $\lim_{t \rightarrow \infty} \varphi(t)/t = \infty$,
- (v) for all $c > 0$ there exist $M, K \geq 0$ such that $\varphi(t + c) \leq M\varphi(t) + K$, $t \in \mathbb{R}$.

A list of examples of strongly convex functions may be found in [16]. Note that (v) is equivalent to the famous Δ_2 -condition. If φ is a strongly convex function then the *Hardy–Orlicz space* \mathcal{H}_φ is the set of functions $f \in \text{Hol}(\mathbb{D})$

such that $\varphi(\log^+ |f(z)|)$ has a harmonic majorant on \mathbb{D} . Here we use the notation $\log^+ x = \max(0, \log x)$. In view of the description of subharmonic functions having a least harmonic majorant we get the following more convenient characterization of \mathcal{H}_φ (see [16]).

DEFINITION 1.2. Let $N^+ = \{f \in \text{Hol}(\mathbb{D}) : f = f_1/f_2, f_1, f_2 \in H^\infty \text{ and } f_2 \text{ outer}\}$ be the Smirnov class on \mathbb{D} and φ a strongly convex function. Then

$$\mathcal{H}_\varphi = \left\{ f \in N^+ : \int_{-\pi}^{\pi} \varphi(\log |f(e^{it})|) dt < \infty \right\},$$

where $f(e^{it})$ is the nontangential boundary value of f at $e^{it} \in \mathbb{T}$, which exists almost everywhere. We introduce two functionals on \mathcal{H}_φ :

$$|f|_{\mathcal{H}_\varphi} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(\log |f(e^{it})|) dt$$

and

$$\|f\|_{\mathcal{H}_\varphi} = \inf\{k > 0 : |f/k|_{\mathcal{H}_\varphi} \leq k\}.$$

The space \mathcal{H}_φ equipped with $\|\cdot\|_{\mathcal{H}_\varphi}$ is a complete metric space (cf. [12] or [15]), and if $\varphi \circ \log$ is convex, it is in fact a Banach space. We remark that if $|f|_{\mathcal{H}_\varphi} = c$ with some $c \geq 1$ then $|f/c|_{\mathcal{H}_\varphi} \leq |f|_{\mathcal{H}_\varphi} \leq c$ and hence

$$\|f\|_{\mathcal{H}_\varphi} = \inf \left\{ k > 0 : \int_{-\pi}^{\pi} \varphi(\log |f(e^{it})/k|) dt \leq k \right\} \leq c = |f|_{\mathcal{H}_\varphi}.$$

There exists another approach to the definition of Hardy–Orlicz spaces (we will restrict ourselves to spaces on \mathbb{D} and on \mathbb{T}). We define the so-called Orlicz classes L_Φ by

$$L_\Phi = \left\{ f \in \mathcal{M}(\mathbb{T}) : \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(|f(e^{it})|) dt < \infty \right\},$$

where $\mathcal{M}(\mathbb{T})$ is the space of measurable functions on \mathbb{T} and Φ is a continuous, strictly increasing function on $[0, \infty)$ satisfying $\Phi(0) = 0$. The corresponding Orlicz spaces are given by $L_{\Phi,*} = \{f \in \mathcal{M}(\mathbb{T}) : \exists a > 0, af \in L_\Phi\}$.

The first approach—using the fact that the function $\varphi(\log |f(z)|)$ is subharmonic if f is holomorphic and φ is convex—is useful for complex-analytic arguments and allows enlarging the definition of Hardy–Orlicz spaces to non-locally convex spaces, the second one for geometric Banach space arguments. As we are also interested in the non-locally convex case, we will keep in mind the special structure $\Phi = \varphi \circ \log$ and we will skip between these two definitions writing

$$\Phi = \varphi \circ \log.$$

We may reformulate condition (v) for Φ :

(v') The function Φ satisfies condition Δ_2 if there exist $d > 1$ and $t_0 \geq 0$ such that

$$\Phi(2t) \leq d\Phi(t), \quad t \geq t_0.$$

With this definition, if Φ satisfies Δ_2 (or equivalently φ) then the Orlicz class is just equal to the Orlicz space ([12]). Note that we may then identify ([12])

$$(3) \quad \mathcal{H}_\varphi = L_{\Phi,*} \cap N^+ = L_\Phi \cap N^+.$$

In what follows we use the notations $\mathcal{H}_\Phi = L_\Phi \cap N^+$ and $\|\cdot\|_{\mathcal{H}_\Phi} = \|\cdot\|_{\mathcal{H}_\varphi}$ if $\Phi = \varphi \circ \log$ and φ is strongly convex.

Let us add some supplementary conditions for strongly convex functions φ :

(vi) The function Φ satisfies condition V_2 (cf. [12]) if there exist $d > 1$ and $t_0 \geq 0$ such that

$$2\Phi(t) \leq \Phi(dt), \quad t \geq t_0.$$

(vii) The function Φ satisfies condition ∇_2 (cf. [15] or [12]) if there exist $d > 1$ and $t_0 \geq 0$ such that

$$2\Phi(t) \leq \frac{1}{d}\Phi(dt), \quad t \geq t_0.$$

(viii) The function Φ is s -convex ($0 < s < \infty$) if there exists a convex function ψ such that

$$\Phi(t) = \psi(t^s).$$

All these conditions have their analogous formulation for φ .

NOTATION. If a function Φ (or equivalently φ) satisfies condition Δ_2 (V_2 , ∇_2) then we write $\Phi \in (\Delta_2)$ ($\Phi \in (V_2)$, $\Phi \in (\nabla_2)$, and $\varphi \in (\Delta_2)$, etc.).

REMARK. 1) It is sufficient to assume (i)–(iii) only for $t \geq t_0$. Indeed, by (iv) there exists $t_1 \geq t_0$ such that $\varphi(t_1)/t_1 > 1$. Let now m_1 be the right hand derivative of φ at t_1 , which is greater than 1 (remember that φ is convex for $t \geq t_0$). Extend φ by $\varphi(t_1)e^{(m_1/\varphi(t_1))(t-t_1)}$ for $t < t_1$ and call this new strongly convex function $\hat{\varphi}$. In view of Corollary 4.1 of [8], we have $\mathcal{H}_\varphi = \mathcal{H}_{\hat{\varphi}}$ (cf. also [11]) and $\hat{\varphi}$ satisfies the cited conditions on the whole real line. We may also suppose that φ is strictly increasing for if this not the case, choose in the previous construction $t_0 \in \mathbb{R}$ such that the right hand derivative of φ at t_0 is different from zero. Finally, this justifies that we can assume that $\Phi = \varphi \circ \log$ is (right-) continuous at zero and $\lim_{t \rightarrow 0} \Phi(t) = 0$.

2) It is not hard to see that if $\Phi(t^s)$ is convex, then there exists s_1 (for example $s_1 = 2s$) such that $\Phi(t^{s_1})$ satisfies V_2 . Let us show a kind of reverse implication. A function $f : [0, \infty) \rightarrow [0, \infty)$ is called *almost increasing* if there exist $C > 0$ and $t_0 \geq 0$ such that $f(t_2) \geq Cf(t_1)$ for all $t_2 \geq t_1 \geq t_0$. If we set $p = \log 2 / \log d$, it is easily seen that the condition $\Phi \in (V_2)$ implies

that for all $\lambda \geq 1$ we have $\Phi(\lambda t) \geq (\lambda/d)^p \Phi(t)$. Take now $f(t) = \Phi(t^{1/p})$ and verify that $f(t)/t$ is almost increasing. But this condition now guarantees that the function $\gamma(t) = \sup\{g(t) : g \leq f, g \text{ convex}\}$, which exists and is convex, will be comparable to f . We deduce that Φ is comparable to a p -convex function: $\gamma(t^p)/c \leq \Phi(t) \leq c\gamma(t^p)$. This in view of [11] implies that the corresponding Hardy-Orlicz spaces are equal, that is, if $\tilde{\Phi}(t) = \gamma(t^p)$ we have

$$\mathcal{H}_\Phi = \mathcal{H}_{\tilde{\Phi}}.$$

(Note that, a priori, this result is in [11] for the case of Φ convex, but it remains valid if $\Phi \in (V_2)$ and φ is strongly convex). Set $\tilde{\varphi}(t) = \tilde{\Phi}(e^t)$. It is clear that $\tilde{\varphi}$ is still strongly convex (in particular we conserve the Δ_2 -property). In what follows we suppose that Φ is an s -convex function satisfying Δ_2 .

3) Let $\Phi \in (V_2)$. If we choose $n \in \mathbb{N}$ such that $2^{n-1} \geq d$, then

$$2\Phi(t^n) \leq 2 \frac{1}{2^n} \Phi((dt)^n) = \frac{1}{2^{n-1}} \Phi((dt)^n) \leq \frac{1}{d} \Phi((dt)^n),$$

from which we deduce that $\Phi(t^s) \in (\nabla_2)$ with $s = n$ and this remains true also for $s \geq n$.

4) In general, $s_1 = 1/p$ constructed in Remark 2 for convexity and $s_2 = n$ constructed in Remark 3 for ∇_2 do not coincide. But if we set $s = \max(1/p, n)$ we in fact obtain a convex function $\Phi(t^s)$ satisfying ∇_2 .

EXAMPLES. The following examples give a list of candidates for the application of Corollary 2.3, Theorem 3.1 and Corollary 4.3 below.

1) The function $\Phi(x) = (1+x)\log(1+x) - x$ is a convex function (and $\varphi(t) = \Phi(e^t)$ is strongly convex) satisfying Δ_2 but not ∇_2 (cf. [15]). As it is convex, we have $\Phi(t^2) \in (\nabla_2)$. The corresponding Orlicz class L_Φ is just the classical Zygmund space $L \log L$.

2) The functions $\Phi(t) = t^p, t \geq 0, 0 < p < \infty$, satisfy (i)–(vi).

3) The functions $\Phi(t) = t^p (\log t)^{r_1} \dots (\log \log \dots \log t)^{r_n}, t \geq t_0, 0 < p < \infty, r_i \geq 0 (i = 1, \dots, n), r_n \neq 0$, satisfy (i)–(vi) (for appropriate t_0 , cf. also Remark 1 above). Note that for any two different sequences p, r_1, \dots, r_n and $\tilde{p}, \tilde{r}_1, \dots, \tilde{r}_m$ we get two different Orlicz spaces.

4) The function $\Phi(t) = t^{1+(\log t)^{-1/2}}$ also satisfies (i)–(vi). The corresponding Orlicz space L_Φ is strictly contained between $L^1 = L_{\Phi_1}$ where $\Phi_1(t) = t$ and all the spaces L_{Φ_2} where Φ_2 is as in Example 3 with $p = 1$ and any finite sequence $r_1, \dots, r_n, n \in \mathbb{N}$.

2. The necessity of the Carleson condition. Throughout this section, we suppose that $\Lambda = \{\lambda_n\}_{n \geq 1}$ satisfies the Blaschke condition $\sum_{\lambda \in \Lambda} (1 - |\lambda|) < \infty$ and hence the Blaschke product $B = \prod_{\lambda \in \Lambda} b_\lambda$ exists.

To prove the necessity of the Carleson condition, we use similar arguments to the case of classical Hardy spaces H^p (cf. [19]). Observe that we use a slightly more general definition of free interpolation (based on that of [13] and [20]). As the method of Shapiro and Shields involves functional analysis methods, we will have to study bounded maps. So let us first recall that a *bounded set* in a topological space X is a set $E \subset X$ such that for each neighbourhood V of 0 in X , there exists $s > 0$ such that $E \subset sV$. We need a property for Φ that guarantees that bounded sets are exactly $\|\cdot\|_{\mathcal{H}_\Phi}$ -bounded sets (i.e. $E \subset \mathcal{H}_\Phi$ is bounded if and only if there exists $R < \infty$ such that $\sup_{f \in E} \|f\|_{\mathcal{H}_\Phi} \leq R$). In order to have this property in \mathcal{H}_Φ it is necessary and sufficient that $\Phi \in (V_2)$ (cf. [12]).

In view of the definition of the metric on \mathcal{H}_Φ , multiplication by an inner function θ , i.e. $\theta \in H^\infty$ and $|\theta| = 1$ a.e. on \mathbb{T} , is an isometry on \mathcal{H}_Φ . In particular $B\mathcal{H}_\Phi$ is a closed subspace of \mathcal{H}_Φ and consequently $\mathcal{H}_\Phi/B\mathcal{H}_\Phi$ is a complete metric space. Set $l_{(\Phi)} = \mathcal{H}_\Phi|_\Lambda$ (which, for the moment, will be distinguished from the Orlicz sequence space l_Φ that will be introduced in Section 3) and define an operator $R: \mathcal{H}_\Phi \rightarrow l_{(\Phi)}$ by

$$R(f) = f|_\Lambda \quad \text{for } f \in \mathcal{H}_\Phi.$$

The space $l_{(\Phi)}$ may be identified in a natural manner with $\mathcal{H}_\Phi/B\mathcal{H}_\Phi$, and thus we may endow $l_{(\Phi)}$ with the quotient metric of $\mathcal{H}_\Phi/B\mathcal{H}_\Phi$. This metric is given by $|a|_{l_{(\Phi)}} = \inf_{f \in \mathcal{H}_\Phi, f|_\Lambda = a} \|f\|_{\mathcal{H}_\Phi}$, and $(l_{(\Phi)}, |\cdot|_{l_{(\Phi)}})$ is a complete metric space. Observe that if $\Phi \in (V_2)$ then the quotient space $l_{(\Phi)}$ inherits also the important property from \mathcal{H}_Φ that the bounded sets in $l_{(\Phi)}$ are exactly the $|\cdot|_{l_{(\Phi)}}$ -bounded ones.

The following result generalizes the setting of Theorem 0.1 of [13].

LEMMA 2.1. *Let φ satisfy (i)–(vi). If $\mathcal{H}_\Phi|_\Lambda$ is an ideal space, then for all $a \in l_{(\Phi)}$ and $b \in \mathbb{C}^\Lambda$ we have the following implication:*

$$|b(\lambda)| \leq |a(\lambda)|, \quad \lambda \in \Lambda \Rightarrow |b|_{l_{(\Phi)}} \leq c(|a|_{l_{(\Phi)}}),$$

where $c: [0, \infty) \rightarrow [0, \infty)$ is an increasing map.

Proof. For any $\mu \in l^\infty = l^\infty(\Lambda)$, define an operator $T_\mu: l_{(\Phi)} \rightarrow l_{(\Phi)}$ by

$$T_\mu(a) = \mu a \quad \text{for } a \in l_{(\Phi)},$$

where the product is defined pointwise: $(\mu a)(\lambda) = \mu(\lambda)a(\lambda)$, $\lambda \in \Lambda$. By the ideal property, T_μ is well defined. And as the evaluation map $a \mapsto a(\lambda)$, $\lambda \in \Lambda$, is continuous (by the continuity of $f \mapsto f(\lambda)$, $\mathcal{H}_\Phi \rightarrow \mathbb{C}$, [12]), the operator T_μ is closed and hence bounded by the closed graph theorem.

We consider the family $\{T_\mu\}_{\|\mu\|_{l^\infty} \leq 1}$ of continuous mappings. For $a \in l_{(\Phi)}$, we may define $T_a: l^\infty \rightarrow l_{(\Phi)}$, $\mu \mapsto a\mu$. As before the closed graph theorem shows that this operator is bounded. Hence the image of the unit ball of l^∞ under T_a is bounded in $l_{(\Phi)}$. But this implies that the orbit

$\{T_\mu a\}_{\|\mu\|_{l^\infty} \leq 1} = \{T_a \mu\}_{\|\mu\|_{l^\infty} \leq 1}$ is bounded in $l_{(\Phi)}$. Applying the Banach–Steinhaus theorem, we get the equicontinuity of the family $\{T_\mu\}_{\|\mu\|_{l^\infty} \leq 1}$. We have already mentioned that if $\Phi \in (V_2)$, then the bounded subsets in $l_{(\Phi)}$ are exactly the $|\cdot|_{l_{(\Phi)}}$ -bounded ones. Take now a $|\cdot|_{l_{(\Phi)}}$ -bounded set $B \subset \{a \in l_{(\Phi)} : |a|_{l_{(\Phi)}} \leq R\}$ for some $R < \infty$. Its images under the operators of the family $\{T_\mu\}_{\|\mu\|_{l^\infty} \leq 1}$ are contained in one and the same bounded subset of $l_{(\Phi)}$, i.e.

$$\sup_{\|\mu\|_{l^\infty} \leq 1, |a|_{l_{(\Phi)}} \leq R} |\mu a|_{l_{(\Phi)}} \leq c(R) \quad \text{for all } R < \infty.$$

Hence if $|b(\lambda)| \leq |a(\lambda)|$, $\lambda \in \Lambda$, and $|a|_{l_{(\Phi)}} \leq R$, then $|b|_{l_{(\Phi)}} \leq c(R)$. ■

We now obtain the following

THEOREM 2.2. *Let $\Lambda = \{\lambda_n\}_{n \geq 1} \subset \mathbb{D}$ and let φ be a function satisfying (i)–(vi). If $\mathcal{H}_\Phi|_\Lambda$ is an ideal space then $\Lambda \in (C)$.*

Let us give the general proof of this result which was mentioned for the Banach case in [14] (p. 188).

Proof. Let $E_\lambda: \mathcal{H}_\Phi \rightarrow \mathbb{C}$ be the point evaluation functional $f \mapsto f(\lambda)$ and set $\gamma(\lambda) = \|E_\lambda\| = \sup\{|f(\lambda)| : f \in \mathcal{H}_\Phi\} > 0$. Clearly for every $\lambda \in \Lambda$ there is a function $f_\lambda \in \mathcal{H}_\Phi$ such that $|f_\lambda(\lambda)| \geq \gamma(\lambda)/2$ and $\|f_\lambda\|_{\mathcal{H}_\Phi} \leq 1$. As Λ is an interpolating sequence we get in view of Lemma 2.1 a function $g_\lambda \in \mathcal{H}_\Phi$ such that $g_\lambda(\mu) = 0$, $\mu \neq \lambda$, $g_\lambda(\lambda) = f_\lambda(\lambda)$, and $\|g_\lambda\|_{\mathcal{H}_\Phi/B\mathcal{H}_\Phi} \leq c(\|f_\lambda\|_{\mathcal{H}_\Phi/B\mathcal{H}_\Phi}) \leq c(1)$. In particular we may choose g_λ such that $\|g_\lambda\| \leq 2c(1)$. Set $B_\lambda = \prod_{\mu \neq \lambda} b_\mu$. As $g_\lambda|_{\Lambda \setminus \{\lambda\}} = 0$ we may divide $g_\lambda/B_\lambda = h_\lambda \in \mathcal{H}_\Phi$ conserving the metric. Hence

$$\frac{1}{2}\gamma(\lambda) \leq |f_\lambda(\lambda)| = |g_\lambda(\lambda)| = |B_\lambda(\lambda)h_\lambda(\lambda)| = |B_\lambda(\lambda)| \cdot |E_\lambda(h_\lambda)|.$$

As the defining function Φ was supposed to be s -convex, the metric $\|\cdot\|_{\mathcal{H}_\Phi}$ is equivalent to an s -homogeneous metric (cf. [12]). This implies $\|rf\|_{\mathcal{H}_\Phi} \leq Ar^s\|f\|_{\mathcal{H}_\Phi}$, $r \geq 0$, for some $A < \infty$, and in particular $\|(2Ac(1))^{-1/s}h_\lambda\|_{\mathcal{H}_\Phi} \leq 1$. We deduce

$$|E_\lambda(h_\lambda)| = |(2Ac(1))^{1/s}E_\lambda((2Ac(1))^{-1/s}h_\lambda)| \leq (2Ac(1))^{1/s}\gamma(\lambda).$$

Hence

$$\frac{1}{2}\gamma(\lambda) \leq |B_\lambda(\lambda)|(2Ac(1))^{1/s}\gamma(\lambda).$$

This obviously gives the Carleson condition. ■

As $\Phi \in (\Delta_2)$ we have $H^\infty\mathcal{H}_\Phi \subset \mathcal{H}_\Phi$ (cf. [8]). This and the fact that $H^\infty|_\Lambda = l^\infty(\Lambda)$ provided that $\Lambda \in (C)$ imply that $\mathcal{H}_\Phi|_\Lambda$ is an ideal space if $\Lambda \in (C)$ (cf. also [14] and [6]). We now get

COROLLARY 2.3. *Let $\Lambda = \{\lambda_n\}_{n \geq 1} \subset \mathbb{D}$ and let φ satisfy (i)–(vi). Then $\mathcal{H}_\Phi|_\Lambda$ is an ideal space if and only if Λ satisfies the Carleson condition.*

3. Characterization of the trace space $\mathcal{H}_\Phi|_A$, $A \in (C)$. Let us first introduce a natural candidate for the trace space of \mathcal{H}_Φ :

$$(4) \quad l_\Phi = \left\{ a \in \mathbb{C}^A : |a|_\Phi = \sum_{\lambda \in A} (1 - |\lambda^2|) \Phi(|a(\lambda)|) < \infty \right\},$$

where $A = \{\lambda_n\}_{n \geq 1} \subset \mathbb{D}$. This is just the Orlicz class $L_\Phi(\mathbb{D}, \mu)$ for the measure space (\mathbb{D}, μ) where $\mu = \sum_{\lambda \in A} (1 - |\lambda^2|) \delta_\lambda$ and δ_λ is the point mass at $\lambda \in \mathbb{D}$. Again, as $\varphi \in (\Delta_2)$, we identify $L_\Phi(\mathbb{D}, \mu) = L_{\Phi,*}(\mathbb{D}, \mu)$.

THEOREM 3.1. *Let φ satisfy (i)-(vi) and $A = \{\lambda_n\}_{n \geq 1} \subset \mathbb{D}$. If $A \in (C)$ then*

$$\mathcal{H}_\varphi|_A = \mathcal{H}_\Phi|_A = l_\Phi.$$

REMARK. 1) As the proof is based on interpolation between L^p -spaces, and for the inclusion $H^p|_A \subset l^p(1 - |\lambda^2|) = \{a \in \mathbb{C}^A : \sum_{\lambda \in A} (1 - |\lambda^2|) |a(\lambda)|^p < \infty\}$ to hold, it is sufficient for A to be a finite union of Carleson sequences, the same holds true for Hardy-Orlicz spaces.

2) Note that we do not suppose that the function Φ is convex and that hence the theorem remains true for non-locally convex spaces (for instance H^p , $0 < p < 1$, cf. [10]).

Proof (of Theorem 3.1) For convenience, let us return to the initial condition $\Phi \in (V_2)$. In view of Remark 4 of the first section, there exists $s > 0$ such that $\Phi(t^s)$ is convex and satisfies ∇_2 . Suppose first $s = 1$. But now, as Φ is a convex function satisfying Δ_2 and ∇_2 we may apply Theorem 4.1 of [2]. In fact, if (X, μ) is a σ -finite measure space, the cited theorem shows the existence of $1 < p_1 < p_0 < \infty$ such that $L_\Phi(X, \mu) = \{f \text{ measurable} : \int_X \Phi(|f|) d\mu < \infty\}$ is an interpolation space between $L^{p_1}(X, \mu)$ and $L^{p_0}(X, \mu)$. We have already mentioned that $L_\Phi(\mathbb{T}, m) \cap N^+ = \mathcal{H}_\Phi$, where m denotes the normalized Lebesgue measure on \mathbb{T} (cf. [12]). Remember that $l_\Phi = L_\Phi(\mathbb{D}, \mu)$.

Let $P_+ : \sum_{n=-N}^N a_n z^n \mapsto \sum_{n=0}^N a_n z^n$ be the usual Riesz projection which is bounded on the spaces $L^p(\mathbb{T}, m)$, $1 < p < \infty$. For a function $f \in H^p(\mathbb{T})$ (the usual Hardy space on \mathbb{T} , see the introduction) write \tilde{f} for its holomorphic extension to the disk \mathbb{D} . Define now, for $1 < p < \infty$,

$$A : L^p(\mathbb{T}, m) \rightarrow l^p(1 - |\lambda^2|), \quad f \mapsto (\widetilde{P_+ f})|_A.$$

This operator is bounded for $1 < p_1 < p_0 < \infty$, and hence (Theorem 4.1 of [2], see also [18]) it is bounded from $L_\Phi(\mathbb{T}, m)$ to l_Φ . But $\mathcal{H}_\Phi = L_\Phi \cap N^+ = \{\widetilde{P_+ f} : f \in L_\Phi(\mathbb{T}, m)\}$ and thus we get $\mathcal{H}_\Phi|_A \subset l_\Phi$.

Consider the inverse inclusion. For a function f in the classical Hardy space $H^p(\mathbb{D})$ on the unit disk, let bf be the boundary function in $H^p(\mathbb{T}) \subset L^p(\mathbb{T})$. There exists a continuous interpolation operator Int from $l^p(1 - |\lambda^2|)$

to $H^p(\mathbb{D})$ (cf. e.g. [9]). Hence, the operator $T : l^p(1 - |\lambda^2|) \rightarrow L^p(\mathbb{T}, m)$ defined by $T(a) = b(\text{Int}(a))$ is continuous. Again [2] shows that T is continuous from l_Φ to $L_\Phi(\mathbb{T}, m)$ and hence it is continuous from l_Φ to \mathcal{H}_Φ , which shows the inclusion $l_\Phi \subset \mathcal{H}_\Phi|_A$.

It remains to show the result for an arbitrary function Φ satisfying (i)-(vi). Let s be as at the beginning of the proof. Define $\tilde{\Phi}(t) = \Phi(t^s)$. In view of what has been proved before we get $\mathcal{H}_{\tilde{\Phi}} = l_{\tilde{\Phi}}$. Let now $f \in \mathcal{H}_\Phi$. We have to show that $\sum_{\lambda \in A} (1 - |\lambda^2|) \Phi(|f(\lambda)|) < \infty$. It is sufficient to do this for the outer part F of f . Note that F has no zeros in \mathbb{D} and thus we can define $g = F^{1/s}$. As $g \in \mathcal{H}_{\tilde{\Phi}}$ we obtain $g|_A \in l_{\tilde{\Phi}}$ and hence

$$\sum_{\lambda \in A} (1 - |\lambda^2|) \Phi(|f(\lambda)|) \leq \sum_{\lambda \in A} (1 - |\lambda^2|) \Phi(|F(\lambda)|) = \sum_{\lambda \in A} (1 - |\lambda^2|) \tilde{\Phi}(|g(\lambda)|) < \infty.$$

Conversely, suppose that $a \in l_\Phi$. Define $b(\lambda) = |a(\lambda)|^{1/s}$, $\lambda \in A$. Then $b \in l_{\tilde{\Phi}} = \mathcal{H}_{\tilde{\Phi}}|_A$. Hence there exists $g \in \mathcal{H}_{\tilde{\Phi}}$ such that $g|_A = b$. Let G be the outer part of g , which is still in $\mathcal{H}_{\tilde{\Phi}}$. This implies $h = G^s \in \mathcal{H}_\Phi$ and

$$|a(\lambda)| = |b(\lambda)|^s = |g(\lambda)|^s \leq |G(\lambda)|^s = |h(\lambda)|, \quad \lambda \in A.$$

In view of Corollary 2.3 we see that $\mathcal{H}_\Phi|_A$ is an ideal space, and this implies the existence of $f \in \mathcal{H}_\Phi$ such that $f|_A = a$. ■

It is now clear that we may identify $l_\Phi = l_{(\Phi)}$.

Using again Corollary 2.3 and the fact that l_Φ is an ideal space, we get the following result.

COROLLARY 3.2. *Let φ satisfy (i)-(vi) and $A = \{\lambda_n\}_{n \geq 1} \subset \mathbb{D}$. Then A is of free interpolation for \mathcal{H}_Φ if and only if $A \in (C)$, and in this case we have $\mathcal{H}_\Phi|_A = l_\Phi$.*

REMARK. The usual method (using the closed graph theorem) shows that the description of the trace space $\mathcal{H}_\Phi|_A = l_\Phi$ directly implies the norm estimates of Lemma 2.1, from which one deduces the necessity of the Carleson condition. It is also clear that $\mathcal{H}_\Phi|_A = l_\Phi$ implies that the trace space $\mathcal{H}_\Phi|_A$ is ideal. Corollary 3.2 actually asserts that the a priori weaker condition for $\mathcal{H}_\Phi|_A$ to be an ideal space is already sufficient to have (C) and hence $\mathcal{H}_\Phi|_A = l_\Phi$.

4. Finite unions of Carleson sequences. Let $A = \bigcup_{i=1}^N A_i$, $A_i \in (C)$. In [7] we have introduced the following notion of stability.

DEFINITION 4.1. Let $X \subset \text{Hol}(\mathbb{D})$. The space X is called (C)-stable if for all pairs of Carleson sequences $A = \{\lambda_n\}_{n \geq 1}$ and $\tilde{A} = \{\tilde{\lambda}_n\}_{n \geq 1}$ satisfying

$$(5) \quad \sup_{n \geq 1} |b_{\lambda_n}(\tilde{\lambda}_n)| < 1,$$

we have

$$(6) \quad X(\Lambda) = X(\tilde{\Lambda}).$$

A sequence $\tilde{\Lambda}$ satisfying (5) with $\eta = \sup_{n \geq 1} |b_{\lambda_n}(\tilde{\lambda}_n)|$ will be called η -shifted with respect to Λ (cf. also [22]).

It is a simple observation (cf. [23]) that

$$|b_{\lambda}(\mu)| < \eta \Rightarrow \frac{1-\eta}{1+\eta} \leq \frac{1-|\lambda|}{1-|\mu|} \leq \frac{1+\eta}{1-\eta}.$$

Hence if (5) is satisfied, then the weights $(1-|\lambda_n^2|)_{n \geq 1}$ and $(1-|\tilde{\lambda}_n^2|)_{n \geq 1}$ are equivalent and we get

$$\begin{aligned} \mathcal{H}_{\Phi}(\Lambda) &= \left\{ a \in \mathbb{C}^{\mathbb{N}} : \sum_{n \geq 1} (1-|\lambda_n^2|) \Phi(|a_n|) < \infty \right\} \\ &= \left\{ a \in \mathbb{C}^{\mathbb{N}} : \sum_{n \geq 1} (1-|\tilde{\lambda}_n^2|) \Phi(|a_n|) < \infty \right\} = \mathcal{H}_{\Phi}(\tilde{\Lambda}). \end{aligned}$$

In order to describe the trace space of \mathcal{H}_{Φ} on a finite union of Carleson sequences we need divided differences with respect to the pseudohyperbolic metric.

DEFINITION 4.2 ([21]). Let $\sigma = \{\lambda_k\}_{k \geq 1} \subset \mathbb{D}$ and $f : \sigma \rightarrow \mathbb{C}$. Set $\lambda^{(k)} = (\lambda_1, \dots, \lambda_k)$ and $\lambda^{(k+1)} = (\lambda^{(k)}, \lambda_{k+1})$. We define

$$\begin{aligned} \Delta^0 f(\lambda^{(1)}) &= f(\lambda_1), \\ \Delta^1 f(\lambda^{(2)}) &= \frac{f(\lambda_2) - f(\lambda_1)}{b_{\lambda_1}(\lambda_2)}, \\ \Delta^k f(\lambda^{(k+1)}) &= \Delta^1(\Delta^{k-1} f(\lambda^{(k-1)}, \cdot))(\lambda_k, \lambda_{k+1}). \end{aligned}$$

By Δ_2 we have $H^\infty \mathcal{H}_{\Phi} \subset \mathcal{H}_{\Phi}$ ([8]). But now all the conditions of Theorem 1.4 of [7] are satisfied and we get the following Sobolev space type description of $\mathcal{H}_{\Phi}|_{\Lambda}$.

COROLLARY 4.3. *If φ satisfies the conditions (i)–(vi) and $\Lambda = \bigcup_{i=1}^N \Lambda_i \subset \mathbb{D}$, $\Lambda_i \in (C)$, then there exists a decomposition $\Lambda = \bigcup_{n \geq 1} \sigma_n$, $\sigma_n = \{\lambda_{n,k}\}_{k=1}^{|\sigma_n|}$, $|\sigma_n| \leq N$ ($|\sigma_n|$ is the cardinality of σ_n), such that*

$$\mathcal{H}_{\Phi}|_{\Lambda} = \left\{ a \in \mathbb{C}^{\Lambda} : \sum_{n \geq 1} (1-|\lambda_{n,1}^2|) \Phi \left(\sup_{k=1, \dots, |\sigma_n|} |\Delta^{k-1} a(\lambda_n^{(k)})| \right) < \infty \right\},$$

where $\lambda_n^{(k)} = (\lambda_{n,1}, \dots, \lambda_{n,k})$.

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Eigenvalue problems with indefinite weight

by

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Abstract. We consider the linear eigenvalue problem $-\Delta u = \lambda V(x)u$, $u \in \mathcal{D}_0^{1,2}(\Omega)$, and its nonlinear generalization $-\Delta_p u = \lambda V(x)|u|^{p-2}u$, $u \in \mathcal{D}_0^{1,p}(\Omega)$. The set Ω need not be bounded, in particular, $\Omega = \mathbb{R}^N$ is admitted. The weight function V may change sign and may have singular points. We show that there exists a sequence of eigenvalues $\lambda_n \rightarrow \infty$.

1. Introduction. In this paper we shall be concerned with the linear eigenvalue problem

$$(1) \quad -\Delta u = \lambda V(x)u, \quad u \in \mathcal{D}_0^{1,2}(\Omega),$$

Ω open in \mathbb{R}^N , $N \geq 3$, and its nonlinear generalization

$$(2) \quad -\Delta_p u = \lambda V(x)|u|^{p-2}u, \quad u \in \mathcal{D}_0^{1,p}(\Omega),$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p -Laplacian, $1 < p < N$, and Ω is open in \mathbb{R}^N . Observe that Ω may be unbounded, and in particular, it may be equal to \mathbb{R}^N . We assume that $V \in L_{\text{loc}}^1(\Omega)$, $V = V^+ - V^-$ (as usual, $V^\pm(x) := \max\{\pm V(x), 0\}$) and $V^+ = V_1 + V_2$, where $V_1 \in L^{N/p}(\Omega)$, $|x|^p V_2(x) \rightarrow 0$ as $|x| \rightarrow \infty$ and for each $y \in \bar{\Omega}$, $|x - y|^p V_2(x) \rightarrow 0$ as $x \rightarrow y$ (in the linear case (1), $p = 2$ in the conditions on V^+). Under these hypotheses we show that (1) and (2) have a sequence of eigenvalues $\lambda_n \rightarrow \infty$. This generalizes several earlier results. In particular, for $\Omega = \mathbb{R}^N$ it was shown in [3, 4] that (1) has a principal eigenvalue λ_1 if V is sufficiently smooth and satisfies an appropriate condition at infinity, and in [1] existence of infinitely many eigenvalues $\lambda_n \rightarrow \infty$ of (1) was established under

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