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Spectrum for a solvable Lie algebra of operators

by

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Abstract. A new concept of spectrum for a solvable Lie algebra of operators is introduced, extending the Taylor spectrum for commuting tuples. This spectrum has the projection property on any Lie subalgebra and, for algebras of compact operators, it may be computed by means of a variant of the classical Ringrose theorem.

0. Introduction. A first non-commutative version of the Taylor spectrum of commuting tuples of operators ([17]) was studied in [7] for families of operators generating nilpotent Lie algebras. Independently, a spectral theory for solvable Lie algebras of operators was introduced in [3], [2]. Some complements to this theory are made in [10], [11] using the (Taylor type) spectrum $\sigma(\varrho)$, where $\varrho : E \to \mathcal{B}(\mathcal{X})$ is a representation of a Lie algebra E on a Banach space $\mathcal{X}$.

One of the main results of [3] is the projection property of the spectrum on Lie ideals. As remarked in [3], this projection property does not hold on any subalgebra of the given solvable Lie algebra. However, [7] shows that in the case of nilpotent Lie algebras we have:

0.1. PROPOSITION. *If $\varrho : E \to \mathcal{B}(\mathcal{X})$ is a representation of a nilpotent Lie algebra E and F is a Lie subalgebra of E, then*

$$\sigma(\varrho|_F) = \sigma(\varrho)|_F.$$

0.2. COROLLARY. *In the situation of Proposition* 0.1, *if we take an arbitrary element e of E and set $T = \varrho(e)$ then*

$$\sigma(T) = \{\lambda(e) \mid \lambda \in \sigma(\varrho)\}.$$

In the present paper we introduce a new concept of spectrum for a solvable Lie algebra of operators, which agrees in the case of a nilpotent Lie algebra with the spectrum from [7], [3]. Our spectrum has the projection property on *any* Lie subalgebra, exactly as in the nilpotent case (see Proposition 0.1 above), not only on Lie ideals. As a consequence, the spectrum of

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the present paper is distinct, in general, from the spectrum of [3]. Another difference of these spectra is that our construction is *not* based on a Koszul complex; instead, we use Cartan subalgebras to reduce the study of solvable Lie algebras to that of nilpotent Lie algebras made in [7].

The structure of the present paper is the following.

§1 contains some simple facts needed in the sequel. They are concerned mainly with some features of the Cartan decompositions, which are specific to the case of solvable Lie algebras.

In §2 the new spectrum of a solvable Lie algebra of operators is constructed. Then we prove the basic properties of the spectrum: projection property, compactness, non-emptieness, a variant of Corollary 0.2 above. We show that any weight in the sense of [9] belongs to the spectrum. As an application we show that if the operators T, Q generate a solvable Lie algebra and Q is quasinilpotent then $\sigma(T+Q)=\sigma(T)$. At the end of §2 we construct a spectrum for an arbitrary locally solvable Lie algebra of operators.

The aim of §3 is to show that the spectrum of a quasisolvable Lie algebra of compact operators can be computed by means of a maximal nest of invariant subspaces. (This fact can be viewed as a variant of the classical Ringrose theorem [5, Theorem 2.32].) To this end we first extend, to the framework of our spectral theory for Lie algebras, the classical Riesz–Schauder theory of the spectrum of a compact operator.

Finally, I wish to thank Professor Mihai Şabac for the suggestion to extend results of [7] to the case of solvable Lie algebras. Also, I acknowledge the kindness of Professor Ştefan Frunză, who indicated to me the papers [10], [11].

1. Preliminaries. First let us introduce some conventions and notations. In all what follows we use only complex Lie algebras and by a solvable or nilpotent Lie algebra we mean a complex finite-dimensional solvable, respectively nilpotent, Lie algebra. By a *quasisolvable Lie algebra* we mean a Lie algebra which is generated as a vector space by its family of solvable Lie ideals (cf. e.g. [13]). We denote by $\mathcal{X}$ an arbitrary (unless otherwise stated) complex Banach space. For a Lie algebra $\mathcal{G}$ we denote by $r(\mathcal{G})$ the set of regular elements of $\mathcal{G}$ (see [14, Chapter 9, Definition 2]) and by $\mathcal{E}(\mathcal{G})$ the group of automorphisms of $\mathcal{G}$ generated by the set

$$\{e^{\operatorname{ad} G} \mid G\in\mathcal{G} \text{ and } \operatorname{ad} G:\mathcal{G}\to\mathcal{G} \text{ is a nilpotent operator}\}.$$

For Cartan subalgebras, Cartan decompositions, roots and related topics we refer to [14, Chapters 8 and 12]. We denote by $\widehat{\mathcal{G}}$ the set of all characters of $\mathcal{G}$ (i.e. linear functionals on $\mathcal{G}$ which vanish on $[\mathcal{G},\mathcal{G}]$). For Koszul complexes and related topics we refer to [7]. If $\varrho:\mathcal{G}\to\mathcal{B}(\mathcal{X})$ is a representation then we denote by $\sigma(\varrho)$ the Taylor spectrum of ϱ (see [10], [11]). If moreover $\mathcal{G}$ is a Lie subalgebra of $\mathcal{B}(\mathcal{X})$ then we denote by $\operatorname{id}_{\mathcal{G}}:\mathcal{G}\to\mathcal{B}(\mathcal{X})$ the identity representation of $\mathcal{G}$ ($\operatorname{id}_{\mathcal{G}}(G)=G$ for any G in $\mathcal{G}$) and by $\mathcal{Q}_{\mathcal{G}}$ the set of all quasinilpotent operators which belong to $\mathcal{G}$. Finally, if A, B are arbitrary sets such that $A\subseteq B$ and if $\mathcal{F}$ is a set of functions defined on B then $\mathcal{F}|_A$ denotes the set of restrictions to A of the functions from $\mathcal{F}$.

Except for the conjugation theorem for Cartan subalgebras of a finite-dimensional Lie algebra we need just the properties of the group $\mathcal{E}(\mathcal{G})$ asserted in the following lemma.

1.1. Lemma. *Let $\mathcal{G}$ be a Lie algebra, $H\in\mathcal{G}$ and $\varphi\in\mathcal{E}(\mathcal{G})$.*

(i) *We have $H-\varphi(H)\in[\mathcal{G},\mathcal{G}]$.*

(ii) *If moreover $\mathcal{G}$ is a Lie subalgebra of $\mathcal{B}(\mathcal{X})$ then the operators H and $\varphi(H)$ are similar and hence they have the same spectrum.*

Proof. (i) It suffices to verify the assertion for $\varphi=e^{\operatorname{ad} G}$, where $G\in\mathcal{G}$ and $\operatorname{ad} G$ is nilpotent. We have

$$(e^{\operatorname{ad} G})H = H+[G,H]+\frac{1}{2!}[G,[G,H]]+\ldots\in H+[\mathcal{G},\mathcal{G}].$$

(ii) For any $A,B\in\mathcal{B}(\mathcal{X})$ we have (see [12, Theorem 4.1])

$$(e^{\operatorname{ad} A})B=e^{-A}Be^{A}. \ \blacksquare$$

1.2. Proposition. *Let $\mathcal{H}$ be a Cartan subalgebra of a finite-dimensional Lie algebra $\mathcal{G}$. Let $n=\dim\mathcal{H}$ and*

$$\mathcal{G}=\mathcal{H}\oplus\mathcal{C}_{\mathcal{H}} \tag{1}$$

be the Cartan decomposition determined by $\mathcal{H}$ (see [13, Chapter 8, remarks preceding Lemma 2]).

(i) *If $H\in\mathcal{H}\cap r(\mathcal{G})$ then $(\operatorname{ad} H)^n\mathcal{G}=\mathcal{C}_{\mathcal{H}}$.*

(ii) *If $\mathcal{K}$ is another Cartan subalgebra of $\mathcal{G}$ and φ is an automorphism of $\mathcal{G}$ such that $\varphi(\mathcal{H})=\mathcal{K}$ then $\varphi(\mathcal{C}_{\mathcal{H}})=\mathcal{C}_{\mathcal{K}}$.*

(iii) *$\mathcal{C}_{\mathcal{H}}\subseteq[\mathcal{G},\mathcal{G}]$.*

(iv) *If $\mathcal{I}$ is an ideal of $\mathcal{G}$ such that $\dim(\mathcal{G}/\mathcal{I})=1$ then $\mathcal{C}_{\mathcal{H}}\subseteq\mathcal{I}$ and $\mathcal{I}=(\mathcal{H}\cap\mathcal{I})\oplus\mathcal{C}_{\mathcal{H}}$.*

Proof. (i) One uses the decomposition (1) and the following remarks:

(a) $(\operatorname{ad} H)^n(\mathcal{H})=0$ because $H\in\mathcal{H}$ and $\mathcal{H}$ is a nilpotent Lie algebra.

(b) $\operatorname{ad} H:\mathcal{C}_{\mathcal{H}}\to\mathcal{C}_{\mathcal{H}}$ is bijective because $H\in r(\mathcal{G})$ (see the proof of the main theorem of [14, Chapter 12]).

(ii) One uses (i) and the fact that $r(\mathcal{G})$ is invariant for every automorphism of $\mathcal{G}$.

(iii) One uses (i).

(iv) One easily checks that $[\mathcal{G},\mathcal{G}]\subseteq\mathcal{I}$, hence $\mathcal{C}_{\mathcal{H}}\subseteq\mathcal{I}$ by (iii). Now, using the decomposition (1) and the inclusion just proved, it is straightforward to deduce the last equality stated in (iv). ■

1.3. COROLLARY. *If $\mathcal{G}$ is a solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$ and $\mathcal{H}$ is a Cartan subalgebra of $\mathcal{G}$ then $\mathcal{C}_{\mathcal{H}}$ contains only quasinilpotents.* (*With the above notations, $\mathcal{C}_{\mathcal{H}}\subseteq\mathcal{Q}_{\mathcal{G}}$.*)

Proof. One applies Proposition 1.2(iii) and [3, Proposition 8] (see also [10, Corollary 2.8]). ■

1.4. PROPOSITION. *Let $\mathcal{G}$ be a solvable Lie algebra.*

(i) *If C is an element of $\mathcal{G}$ such that $\operatorname{ad}C:\mathcal{G}\to\mathcal{G}$ is a nilpotent operator then $C+r(\mathcal{G})=r(\mathcal{G})$.*

(ii) $[\mathcal{G},\mathcal{G}]+r(\mathcal{G})=r(\mathcal{G})$.

(iii) *If $\mathcal{G}$ is moreover a Lie subalgebra of $\mathcal{B}(\mathcal{X})$ then $\mathcal{Q}_{\mathcal{G}}+r(\mathcal{G})=r(\mathcal{G})$.*

Proof. By the Lie theorem, we choose a basis of $\mathcal{G}$ such that any operator $\operatorname{ad}G:\mathcal{G}\to\mathcal{G}$ ($G\in\mathcal{G}$) is represented by an upper triangular matrix. For G in $\mathcal{G}$, the number of zeros on the diagonal of $\operatorname{ad}G$ equals the multiplicity of 0 as a root of the characteristic polynomial of $\operatorname{ad}G$, hence it finally equals the dimension of the kernel of $(\operatorname{ad}G)^{\dim\mathcal{G}}$ ([14, Chapter 9, the proof of Theorem 1]). We now prove the assertions (i)–(iii).

(i) For any H in $\mathcal{G}$ the matrices of $\operatorname{ad}H$ and $\operatorname{ad}(H+C)$ have the same number of zeros on the diagonal (because the diagonal of $\operatorname{ad}C$ consists only of zeros). Hence, by the remark from the beginning of the proof and the definition of $r(\mathcal{G})$, $H\in r(\mathcal{G})$ if and only if $H+C\in r(\mathcal{G})$.

To prove (ii), (iii), one applies (i) for $C\in[\mathcal{G},\mathcal{G}]$, respectively $C\in\mathcal{Q}_{\mathcal{G}}$. (In the last case one also uses the Rosenblum theorem [12, Corollary 3.3(i)].) ■

1.5. COROLLARY. *Let $\mathcal{G}_0$ be a nilpotent Lie subalgebra of the solvable Lie algebra $\mathcal{G}$.*

(i) *If $\mathcal{G}_0\cap r(\mathcal{G})\neq\emptyset$ then $\mathcal{G}_0$ is contained in a Cartan subalgebra of $\mathcal{G}$.*

(ii) *If $\mathcal{G}_0\cap r(\mathcal{G})=\emptyset$ then $\mathcal{G}_0$ is contained in an ideal $\mathcal{I}$ of $\mathcal{G}$ such that $\mathcal{I}\neq\mathcal{G}$ (more exactly $\mathcal{I}\cap r(\mathcal{G})=\emptyset$).*

Proof. If $\mathcal{G}_0\cap r(\mathcal{G})\neq\emptyset$ then one uses [14, Chapter 8, Theorem 2]. If $\mathcal{G}_0\cap r(\mathcal{G})=\emptyset$ then one considers the ideal $\mathcal{I}=\mathcal{G}_0+[\mathcal{G},\mathcal{G}]$ of $\mathcal{G}$. One has $\mathcal{I}\supseteq\mathcal{G}_0$ and, if $\mathcal{I}\cap r(\mathcal{G})\neq\emptyset$, then it easily follows by Proposition 1.4(ii) that $\mathcal{G}_0\cap r(\mathcal{G})\neq\emptyset$, a contradiction. ■

2. Construction and the general properties of the spectrum

2.1. PROPOSITION. *Let $\mathcal{G}$ be a solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$. Let $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{G}$ and $\mathcal{G}=\mathcal{H}\oplus\mathcal{C}_{\mathcal{H}}$ be the corresponding Cartan*

decomposition. Define

$$\Sigma_{\mathcal{H}}:=\{f:\mathcal{G}\to\mathbb{C}\mid f|_{\mathcal{H}}\in\sigma(\mathrm{id}_{\mathcal{H}})\text{ and } f \text{ vanishes on } \mathcal{C}_{\mathcal{H}}\}.\tag{2}$$

Then for any Cartan subalgebras $\mathcal{H},\mathcal{K}$ of $\mathcal{G}$ we have $\Sigma_{\mathcal{H}}=\Sigma_{\mathcal{K}}\subseteq\widehat{\mathcal{G}}$.

Proof. STEP 1. We show that $\Sigma_{\mathcal{H}}\subseteq\widehat{\mathcal{G}}$. We denote by $\alpha,\beta,\ldots$ the non-zero roots of $\mathcal{G}$ with respect to $\mathcal{H}$, and by $\mathcal{G}^\alpha,\mathcal{G}^\beta,\ldots$ the corresponding root spaces. We have $\mathcal{C}_{\mathcal{H}}=\bigoplus_\alpha\mathcal{G}^\alpha$ and $\mathcal{G}=\mathcal{H}\oplus\mathcal{C}_{\mathcal{H}}$, so

$$[\mathcal{G},\mathcal{G}]=[\mathcal{H},\mathcal{H}]+\sum_\alpha[\mathcal{H},\mathcal{G}^\alpha]+\sum_{\alpha,\beta}[\mathcal{G}^\alpha,\mathcal{G}^\beta]\subseteq[\mathcal{H},\mathcal{H}]+\sum_\alpha[\mathcal{G}^\alpha,\mathcal{G}^{-\alpha}]+\sum_\alpha\mathcal{G}^\alpha.$$

Now let f be in $\Sigma_{\mathcal{H}}$. By (2) the functional f vanishes on $\sum_\alpha\mathcal{G}^\alpha$ ($=\mathcal{C}_{\mathcal{H}}$). Then by [14, Chapter 8, Lemma 1] the set $[\mathcal{H},\mathcal{H}]+\sum_\alpha[\mathcal{G}^\alpha,\mathcal{G}^{-\alpha}]$ is contained in $[\mathcal{G},\mathcal{G}]\cap\mathcal{H}$, so it consists only of quasinilpotent operators from $\mathcal{H}$ (see [3, Proposition 8]). Since $f|_{\mathcal{H}}\in\sigma(\mathrm{id}_{\mathcal{H}})$, by Corollary 0.2 it then follows that f also vanishes on $[\mathcal{H},\mathcal{H}]+\sum_\alpha[\mathcal{G}^\alpha,\mathcal{G}^{-\alpha}]$. Hence f vanishes on all $[\mathcal{G},\mathcal{G}]$.

STEP 2. Now we show that $\Sigma_{\mathcal{H}}=\Sigma_{\mathcal{K}}$. By the conjugation theorem for Cartan subalgebras of $\mathcal{G}$ ([14, Chapter 12]), there exists $\varphi\in\mathcal{E}(\mathcal{G})$ such that $\varphi(\mathcal{H})=\mathcal{K}$. This obviously implies that $\mathrm{id}_{\mathcal{K}}\circ\varphi=\mathrm{id}_{\mathcal{H}}$, hence by [7, Proposition 2.6] one gets $\sigma(\mathrm{id}_{\mathcal{H}})=\{f\circ\varphi\mid f\in\sigma(\mathrm{id}_{\mathcal{K}})\}$. Since $\varphi(\mathcal{C}_{\mathcal{H}})=\mathcal{C}_{\mathcal{K}}$ (Proposition 1.2(ii)) we have even

$$\Sigma_{\mathcal{H}}=\{f\circ\varphi\mid f\in\Sigma_{\mathcal{K}}\}.$$

Moreover for any f in $\Sigma_{\mathcal{K}}$ and G in $\mathcal{G}$, by Lemma 1.1(i) and Step 1 of the present proof we get $f(G-\varphi(G))=0$. It follows that $f\circ\varphi=f$ for any f in $\Sigma_{\mathcal{K}}$, so $\Sigma_{\mathcal{H}}=\Sigma_{\mathcal{K}}$. ■

Now Proposition 2.1 ensures the correctness of the following definition.

2.2. DEFINITION. If $\mathcal{G}$ is a solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$ then using an arbitrary Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$ we define the *spectrum* of $\mathcal{G}$ by

$$\Sigma(\mathcal{G}):=\Sigma_{\mathcal{H}}.$$

Let us remark that by Proposition 2.1 we have $\Sigma(\mathcal{G})\subseteq\widehat{\mathcal{G}}$. Moreover, if $\mathcal{G}$ is a nilpotent Lie algebra then $\Sigma(\mathcal{G})=\sigma(\mathrm{id}_{\mathcal{G}})$.

Now we establish a sequence of auxiliary results and finally we prove that the spectrum $\Sigma(\mathcal{G})$ has the projection property on any Lie subalgebra of $\mathcal{G}$ (Theorem 2.6).

2.3. LEMMA. *If $\mathcal{G}$ is a finite-dimensional Lie subalgebra of $\mathcal{B}(\mathcal{X})$ then $\mathcal{G}$ is solvable if and only if $\mathcal{Q}_{\mathcal{G}}$ is a Lie ideal of $\mathcal{G}$ containing $[\mathcal{G},\mathcal{G}]$.*

Proof. *Necessity.* If $\mathcal{G}$ is a solvable Lie subalgebra then $[\mathcal{G},\mathcal{G}]\subseteq\mathcal{Q}_{\mathcal{G}}$ by [3, Proposition 8], hence it remains to prove that $\mathcal{Q}_{\mathcal{G}}$ is a vector space. Suppose that $Q_1,Q_2\in\mathcal{Q}_{\mathcal{G}}$ and set $\mathcal{L}:=\mathbb{C}Q_1+\mathbb{C}Q_2+[\mathcal{G},\mathcal{G}]$; then $\mathcal{L}$ is an ideal of $\mathcal{G}$. By the Lie theorem, choose a basis in $\mathcal{G}$ such that any operator

$\operatorname{ad} G : \mathcal{G} \to \mathcal{G}$ $(G \in \mathcal{G})$ is represented by an upper triangular matrix. Then, by the Rosenblum theorem [12, Corollary 3.3(i)], any operator $\operatorname{ad} G$ $(G \in \mathcal{G})$ will be represented by the sum of three upper triangular matrices whose diagonals consist only of zeros. So $\operatorname{ad} G : \mathcal{G} \to \mathcal{G}$ is a nilpotent operator for any G in $\mathcal{L}$. Since $(\operatorname{ad} G)(\mathcal{L}) \subseteq \mathcal{L}$ for any G in $\mathcal{L}$, the Engel theorem shows that $\mathcal{L}$ is a nilpotent Lie algebra. Then, by an application of either the spectral mapping theorem [7, Theorem 5.1] or Corollary 0.1, one gets $\sigma(Q_1 + Q_2) = \{0\}$, i.e. $Q_1 + Q_2 \in \mathcal{Q}_{\mathcal{G}}$.

Sufficiency. Since $\mathcal{Q}_{\mathcal{G}}$ is a finite-dimensional Lie algebra containing only quasinilpotent operators, it is nilpotent by the above cited Rosenblum theorem (see [20]). But $[\mathcal{G}, \mathcal{G}] \subseteq \mathcal{Q}_{\mathcal{G}}$, hence $[\mathcal{G}, \mathcal{G}]$ is also a nilpotent Lie algebra; so $\mathcal{G}$ is a solvable Lie algebra. ∎

2.4. COROLLARY. *If $\mathcal{G}$ is a solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$ then for any Q in $\mathcal{Q}_{\mathcal{G}}$ and f in $\Sigma(\mathcal{G})$ one has $f(Q) = 0$.*

Proof. Let $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{G}$. By (1), there exist Q' in $\mathcal{H}$ and Q'' in $\mathcal{C}_{\mathcal{H}}$ such that $Q' + Q'' = Q$. By Corollary 1.3, $Q'' \in \mathcal{Q}_{\mathcal{G}}$. So $Q' = Q - Q'' \in \mathcal{Q}_{\mathcal{G}}$ by Lemma 2.3. Since $\mathcal{H}$ is a nilpotent Lie algebra and Q' is a quasinilpotent operator belonging to $\mathcal{H}$ one gets $f_0(Q') = 0$ for any f_0 in $\sigma(\mathrm{id}_{\mathcal{H}})$. Then by (2) one gets $f(Q') = 0 = f(Q'')$ for any f in $\Sigma_{\mathcal{H}}$ $(= \Sigma(\mathcal{G}))$ and the conclusion follows. ∎

2.5. LEMMA. *Let $\mathcal{G}$ be a solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$. If $\mathcal{I}$ is a Lie ideal of $\mathcal{G}$ then $\Sigma(\mathcal{G})|_{\mathcal{I}} = \Sigma(\mathcal{I})$.*

Proof. Since $\mathcal{G}$ is a solvable Lie algebra we may suppose that $\dim(\mathcal{G}/\mathcal{I}) = 1$. Let $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{G}$. It follows that $\mathcal{H} \cap \mathcal{I}$ is contained in a certain Cartan subalgebra $\mathcal{K}$ of $\mathcal{I}$ (see [19]). Since $\mathcal{H} \cap \mathcal{I} \subseteq \mathcal{K} \subseteq (\mathcal{H} \cap \mathcal{I}) \oplus \mathcal{C}_{\mathcal{H}}$, it is easy to see that

$$\mathcal{K} = (\mathcal{H} \cap \mathcal{I}) \oplus (\mathcal{K} \cap \mathcal{C}_{\mathcal{H}}). \tag{3}$$

Let $\mathcal{I} = \mathcal{K} \oplus \mathcal{C}_{\mathcal{K}}$ be the Cartan decomposition of $\mathcal{I}$ with respect to $\mathcal{K}$ and set $\Sigma = \Sigma(\mathcal{G})|_{\mathcal{I}}$. To prove that $\Sigma = \Sigma(\mathcal{I})$ we must check that

$$\Sigma|_{\mathcal{C}_{\mathcal{K}}} = 0 \quad \text{and} \quad \Sigma|_{\mathcal{K}} = \sigma(\mathrm{id}_{\mathcal{K}}). \tag{4}$$

The first equality is a consequence of Corollaries 1.3 and 2.4. To prove the second equality we use the decomposition (3). First, using (2) we get

$$\Sigma|_{\mathcal{H} \cap \mathcal{I}} = \sigma(\mathrm{id}_{\mathcal{H}})|_{\mathcal{H} \cap \mathcal{I}}.$$

Now let us remark that $\mathcal{H} \cap \mathcal{I}$ is a subalgebra of the nilpotent Lie algebras $\mathcal{H}$ and $\mathcal{K}$ (see (3)), so by Proposition 0.1 we get

$$\sigma(\mathrm{id}_{\mathcal{K}})|_{\mathcal{H} \cap \mathcal{I}} = \sigma(\mathrm{id}_{\mathcal{H} \cap \mathcal{I}}) = \sigma(\mathrm{id}_{\mathcal{H}})|_{\mathcal{H} \cap \mathcal{I}}.$$

Hence $\Sigma|_{\mathcal{H} \cap \mathcal{I}} = \sigma(\mathrm{id}_{\mathcal{K}})|_{\mathcal{H} \cap \mathcal{I}}$. Moreover the functionals from Σ vanish on $\mathcal{K} \cap \mathcal{C}_{\mathcal{H}}$ (by Corollaries 1.3 and 2.4) as do those from $\sigma(\mathrm{id}_{\mathcal{K}})$ (by Corollary 1.3 and Proposition 0.1). Hence $\Sigma = \sigma(\mathrm{id}_{\mathcal{K}})$ in view of (3). ∎

Now we are ready to prove a variant of Proposition 0.1 for solvable Lie algebras.

2.6. THEOREM. *If $\mathcal{G}$ is a solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$ and $\mathcal{L}$ is a Lie subalgebra of $\mathcal{G}$ then $\Sigma(\mathcal{G})|_{\mathcal{L}} = \Sigma(\mathcal{L})$.*

Proof. We proceed by induction on the dimension of $\mathcal{G}$. The conclusion is obvious if $\dim \mathcal{G} = 1$. Now suppose that the assertion holds for solvable Lie algebras of dimension strictly less than $\dim \mathcal{G}$. Let $\mathcal{K}$ be a Cartan subalgebra of $\mathcal{L}$ and $\mathcal{L} = \mathcal{K} \oplus \mathcal{C}_{\mathcal{K}}$ be the corresponding Cartan decomposition. By Corollaries 1.3 and 2.4, the elements of $\Sigma(\mathcal{G})$ vanish on $\mathcal{C}_{\mathcal{K}}$. Hence we only have to check that $\Sigma(\mathcal{G})|_{\mathcal{K}} = \sigma(\mathrm{id}_{\mathcal{K}})$. By Corollary 1.5 only the following situations can occur.

(i) $\mathcal{K}$ is contained in a certain Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$. Using (2) and Proposition 0.1 we deduce that

$$\Sigma(\mathcal{G})|_{\mathcal{K}} = (\Sigma(\mathcal{G})|_{\mathcal{H}})|_{\mathcal{K}} = \sigma(\mathrm{id}_{\mathcal{H}})|_{\mathcal{K}} = \sigma(\mathrm{id}_{\mathcal{K}}).$$

(ii) $\mathcal{K}$ is contained in a certain Lie ideal $\mathcal{I}$ of $\mathcal{G}$ such that $\mathcal{I} \neq \mathcal{G}$. Then $\dim \mathcal{I} < \dim \mathcal{G}$, so $\Sigma(\mathcal{I})|_{\mathcal{K}} = \Sigma(\mathcal{K}) = \sigma(\mathrm{id}_{\mathcal{K}})$ by the induction hypothesis. Then an application of Lemma 2.5 shows that $\Sigma(\mathcal{G})|_{\mathcal{K}} = \sigma(\mathrm{id}_{\mathcal{K}})$. ∎

Now we state the variant of Corollary 0.2.

2.7. COROLLARY. *If $\mathcal{G}$ is a solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$ then for any element G of $\mathcal{G}$ we have*

$$\sigma(G) = \{f(G) \mid f \in \Sigma(\mathcal{G})\}.$$

More generally, if $G = (G_1, \ldots, G_n)$ is a commuting tuple of elements of $\mathcal{G}$ then one can compute the Taylor spectrum of G in the following way:

$$\sigma(G) = \{(f(G_1), \ldots, f(G_n)) \mid f \in \Sigma(\mathcal{G})\}.$$

Proof. It suffices to consider the nilpotent (actually abelian) Lie subalgebra of $\mathcal{G}$ generated by G and to apply Theorem 2.6 and then [5, Theorem 2.3]. ∎

A straightforward consequence of Corollary 2.7 is the following.

2.8. COROLLARY. *If $\mathcal{G}$ is a solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$ then $\Sigma(\mathcal{G})$ is a compact non-empty set of characters of $\mathcal{G}$.*

Next we give an application of Corollary 2.7 in perturbation theory.

2.9. COROLLARY. *Let $T, Q \in \mathcal{B}(\mathcal{X})$ be operators generating a solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$. If Q is a quasinilpotent operator then $\sigma(T + Q) = \sigma(T)$.*

Proof. Let $\mathcal{G}$ be the solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$ generated by T and Q. Computing the spectra as shown in Corollary 2.7, we get

$$\sigma(T+Q) = \{f(T+Q) \mid f \in \Sigma(\mathcal{G})\} = \{f(T)+f(Q) \mid f \in \Sigma(\mathcal{G})\} = \{f(T) \mid f \in \Sigma(\mathcal{G})\} = \sigma(T). \blacksquare$$

In connection with Corollary 2.9 let us remark that, even if there exists γ in $\mathbb{C}^*$ such that $[T,Q] = \gamma Q$, there exist simple examples of operators on finite-dimensional spaces which show that the operators T and $T+Q$ need not be quasinilpotent equivalent. Hence Corollary 2.9 above and [4, Chapter 1, Theorem 2.2] are distinct generalizations of [6, Chapter XV, §4, Lemma 4] (where $[T,Q]=0$).

Next we show a connection between the spectrum $\Sigma(\bullet)$ introduced above and certain linear functionals whose existence was established in [9]. With the notation of [10, Definition 2.1] we actually show that $\sigma_{\rm ap}({\rm id}_\bullet) \subseteq \Sigma(\bullet)$.

2.10. Corollary. *Let $\mathcal{G}$ be a solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$. If $\{x_n\}_{n\geq 1}$ is a sequence of unit vectors from $\mathcal{X}$ and $f:\mathcal{G}\to\mathbb{C}$ is a functional such that*

$$\lim_{n\to\infty}(Gx_n - f(G)x_n) = 0$$

for any G in $\mathcal{G}$, then $f \in \Sigma(\mathcal{G})$.

Proof. Let $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{G}$. We have $f(G)\in\sigma(G)$ for any G in $\mathcal{G}$ so, by Corollary 1.3, f vanishes on $\mathcal{C}_{\mathcal{H}}$. Moreover $f|_{\mathcal{H}} \in \sigma({\rm id}_{\mathcal{H}})$ (see [7, Remarks 2.9(2)]) hence $f \in \Sigma_{\mathcal{H}} = \Sigma(\mathcal{G})$. ■

Our next result is concerned with the notion of spectrum for a locally solvable Lie algebra of operators (see [16]). The extension of the concept introduced in Definition 2.2 above is obtained by means of projective limits (a method used in [15, Theorem 2.5] and [16]).

2.11. Theorem. *Let $\mathcal{G}$ be a locally solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$. There exists a unique map $\Sigma(\bullet)$ which associates with any Lie subalgebra $\mathcal{L}$ of $\mathcal{G}$ a compact non-empty subset $\Sigma(\mathcal{L})$ of $\widehat{\mathcal{L}}$ such that the following conditions are satisfied.*

(a) *If $\mathcal{L}$ is a nilpotent Lie subalgebra of $\mathcal{G}$ then $\Sigma(\mathcal{L}) = \sigma({\rm id}_{\mathcal{L}})$.*

(b) *If $\mathcal{L}$ and $\mathcal{M}$ are Lie subalgebras of $\mathcal{G}$ such that $\mathcal{L}\subseteq\mathcal{M}$ then $\Sigma(\mathcal{M})|_{\mathcal{L}} = \Sigma(\mathcal{L})$.*

Proof. *Existence.* Let $\{\mathcal{L}_i\}_{i\in I}$ be the local system of all solvable Lie subalgebras of $\mathcal{G}$, partially ordered by inclusion. For each i in I, $\Sigma(\mathcal{L}_i)$ (introduced in Definition 2.2) is a compact non-empty subset of the finite-dimensional vector space $\widehat{\mathcal{L}}_i$, as follows from Corollary 2.8. Hence, endowing $\{\Sigma(\mathcal{L}_i)\}_{i\in I}$ with the restriction maps $p_{ij}:\Sigma(\mathcal{L}_j)\to\Sigma(\mathcal{L}_i)$ ($i\leq j$), we get a projective limit system of compact spaces whose maps are onto (cf. Theorem 2.6). In these conditions it is well known that the corresponding projective limit $\text{proj}\lim\Sigma(\mathcal{L}_i)$ is a compact topological space and the natural projections $p_j : \text{proj}\lim\Sigma(\mathcal{L}_i)\to\Sigma(\mathcal{L}_j)$ are onto. Obviously, $\text{proj}\lim\Sigma(\mathcal{L}_i)$ can be identified with a subset $\Sigma(\mathcal{G})$ of $\widehat{\mathcal{G}}$, and $\Sigma(\mathcal{G})|_{\mathcal{L}_j} = \Sigma(\mathcal{L}_j)$ for any j in I. Moreover if $\mathcal{L}$ is an arbitrary Lie subalgebra of $\mathcal{G}$ then $\{\mathcal{L}\cap\mathcal{L}_i\}_{i\in I}$ is the local system of all solvable Lie subalgebras of $\mathcal{L}$. Define, as above,

$$\Sigma(\mathcal{L}) := \text{proj}\lim\Sigma(\mathcal{L}\cap\mathcal{L}_i). \tag{5}$$

Then condition (a) is obviously satisfied (see the remark after Definition 2.2). To check (b), we remark that $\mathcal{L}\cap\mathcal{L}_i\subseteq\mathcal{M}\cap\mathcal{L}_i$ so, by Theorem 2.6, we have $\Sigma(\mathcal{L}\cap\mathcal{L}_i) = \Sigma(\mathcal{M}\cap\mathcal{L}_i)|_{\mathcal{L}\cap\mathcal{L}_i}$. This fact together with (5) (applied for $\mathcal{L}$ and $\mathcal{M}$) easily implies that $\Sigma(\mathcal{L}) = \Sigma(\mathcal{M})|_{\mathcal{L}}$.

Uniqueness. Let $\Sigma'(\bullet)$ be another map with the stated properties. The property (b) of $\Sigma'(\bullet)$ easily implies a formula similar to (5), so $\Sigma'(\bullet)$ is uniquely determined by its restriction to the set of all solvable Lie subalgebras of $\mathcal{G}$. Hence it suffices to suppose that $\mathcal{G}$ is solvable. Let $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{G}$. In view of the properties (a) and (b) of $\Sigma'(\bullet)$ we have

$$\Sigma'(\mathcal{G})|_{\mathcal{H}} = \Sigma'(\mathcal{H}) = \sigma({\rm id}_{\mathcal{H}}). \tag{6}$$

Now let Q be arbitrary in $\mathcal{C}_{\mathcal{H}}$ and denote by $\mathcal{S}$ the one-dimensional Lie algebra generated by Q. Then $\mathcal{S}$ is an abelian (hence nilpotent) Lie subalgebra of $\mathcal{G}$, so $\Sigma'(\mathcal{G})|_{\mathcal{S}} = \Sigma'(\mathcal{S}) = \sigma({\rm id}_{\mathcal{S}})$. But $Q\in\mathcal{J}_{\mathcal{G}}$ (cf. Corollary 1.3) so, obviously, $\sigma({\rm id}_{\mathcal{S}}) = \{0\}$. Hence

$$\Sigma'(\mathcal{G})|_{\mathcal{C}_{\mathcal{H}}} = \{0\}. \tag{7}$$

Now in view of (6), (7) and (2) we get $\Sigma'(\mathcal{G}) = \Sigma_{\mathcal{H}} = \Sigma(\mathcal{G})$. This fact, together with the property (b) enjoyed by both $\Sigma'(\bullet)$ and $\Sigma(\bullet)$, implies that $\Sigma'(\mathcal{L}) = \Sigma(\mathcal{L})$ for any Lie subalgebra $\mathcal{L}$ of $\mathcal{G}$. ■

3. Lie algebras of compact operators. We begin with some simple facts about tuples of arbitrary operators generating nilpotent Lie algebras.

3.1. Lemma. *If $C,P\in\mathcal{B}(\mathcal{X})$, $P^2=P$ and $[[C,P],P]=0$ then $[C,P]=0$.*

Proof. The relation $[[C,P],P]=0$ is equivalent to $CP+PC=2PCP$. If we multiply this last equality on the left, respectively on the right, by P then we get $PC=PCP$, respectively $CP=PCP$. Hence $PC=CP$. ■

The following result may be viewed as a complement to [8, Corollary 4.5].

3.2. Proposition. *Suppose that $A\in\mathcal{B}(\mathcal{X})$ and $B=(B_1,\ldots,B_q)$ is a commuting tuple from $\mathcal{B}(\mathcal{X})$. Assume that there exists a positive integer N*

such that for any $i_1, \dots, i_N \in \{1, \dots, q\}$ we have

$$(\operatorname{ad} B_{i_1}) \dots (\operatorname{ad} B_{i_N})A = 0.$$

Then for every function f holomorphic on a neighbourhood of $\sigma(B)$ and such that $f^2 = f$, we have $[A, f(B)] = 0$.

Proof. We show by descending induction that for $k = N, N-1, \dots, 0$ the property $P(k)$ holds, where

$P(k)$: For any $i_1, \dots, i_k \in \{1, \dots, q\}$ we have

$$[(\operatorname{ad} B_{i_k}) \dots (\operatorname{ad} B_{i_1})A, f(B)] = 0.$$

(Here $P(0)$ denotes just $[A, f(B)] = 0$.) The assertion $P(N)$ immediately follows from the hypothesis. Now suppose that $P(k)$ holds for a certain k in $\{N, \dots, 1\}$. To prove $P(k-1)$ we fix for the moment the indices $i_1, \dots, i_{k-1}$ from $\{1, \dots, q\}$ and define $C := (\operatorname{ad} B_{i_{k-1}}) \dots (\operatorname{ad} B_{i_1})A$. (Of course, if $k = 1$ we do not fix anything and set $C := A$.) From $P(k)$ it follows that for any i in $\{1, \dots, q\}$ we have $[[B_i, C], f(B)] = 0$, hence $[[C, f(B)], B_i] = 0$. It then follows that $[[C, f(B)], f(B)] = 0$ because $f(B)$ belongs to the bicommutant of the tuple B. The last equality shows (in view of the assumption about f and of Lemma 3.1) that $[C, f(B)] = 0$. Remembering the definition of C we get $P(k-1)$. ∎

3.3. Corollary. *Suppose that $C = (A_1, \dots, A_p, B_1, \dots, B_q) \in \mathcal{B}(\mathcal{X})^{p+q}$, the Lie algebra generated by C is nilpotent and $B = (B_1, \dots, B_q)$ is a commuting tuple. Denote by $\pi : \sigma(C) \to \mathbb{C}^q$ the projection on the last q coordinates and assume that $\sigma(B) = F_1 \cup F_2$, where F_1 and F_2 are closed non-empty and disjoint subsets of $\mathbb{C}^q$. Moreover, denote by V_1 an open neighbourhood of F_1 whose closure does not intersect F_2, and by f the characteristic function of V_1. Then f is holomorphic on a neighbourhood of $\sigma(B)$. Set $\mathcal{X}_1 := f(B)\mathcal{X}$ and $\mathcal{X}_2 := (1 - f(B))\mathcal{X}$. Then we have:*

(i) *$\mathcal{X}_1, \mathcal{X}_2 \in \operatorname{Lat} C$ and $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$;*
(ii) *$\sigma(C, \mathcal{X}_j) = \pi^{-1}(F_j)$ for $j = 1, 2$.*

Proof. Since C generates a nilpotent Lie algebra $E(C)$, by Proposition 3.2 we get $[A_i, f(B)] = 0$ for any i in $\{1, \dots, p\}$ and the assertion (i) follows.

Then from (i) we deduce

$$\operatorname{Kos}(C - \lambda, \mathcal{X}) = \begin{array}{c}\operatorname{Kos}(C - \lambda, \mathcal{X}_1) \\ \oplus \\ \operatorname{Kos}(C - \lambda, \mathcal{X}_2)\end{array} \quad \text{for any } \lambda \text{ in } \mathbb{C}^{p+q}. \tag{8}$$

(Here we use the notations of [7, Definition 2.1].) Hence

$$\sigma(C, \mathcal{X}) = \sigma(C, \mathcal{X}_1) \cup \sigma(C, \mathcal{X}_2). \tag{9}$$

From the projection property [7, Consequence 5.5] and the classical result about commuting tuples (see e.g. [18, Chapter II, Theorem 9.6]) it follows that $\pi(\sigma(C, \mathcal{X}_j)) = \sigma(B, \mathcal{X}_j) = F_j$, so

$$\sigma(C, \mathcal{X}_j) \subseteq \pi^{-1}(F_j) \quad (j = 1, 2). \tag{10}$$

We deduce $\pi(\sigma(C)) = \sigma(B) = F_1 \cup F_2$ by the projection property again, so

$$\sigma(C) = \pi^{-1}(F_1) \cup \pi^{-1}(F_2). \tag{11}$$

Now we can deduce (ii) from (9)–(11). ∎

The above corollary is a weak non-commutative variant of the classical result about commuting tuples of operators with the Taylor spectrum consisting of two closed, non-empty and disjoint sets. However, it suffices to get the geometric properties of the Taylor spectrum for a tuple of compact operators generating a nilpotent Lie algebra. (See Theorem 3.7 below.)

3.4. Lemma. *Let $K = (K_1, \dots, K_n)$ be a tuple of operators on a finite-dimensional space $\mathcal{Y}$. If K generates a nilpotent Lie algebra and $\lambda = (\lambda_1, \dots, \lambda_n) \in \sigma(K)$ then one can find a vector y in $\mathcal{Y} \setminus \{0\}$ such that $K_j y = \lambda_j y$ for $1 \le j \le n$.*

Proof. See e.g. [1, Theorem 1]. ∎

3.5. Corollary. *Let $K = (K_1, \dots, K_n)$ be a tuple of compact operators on a Banach space $\mathcal{X}$. If K generates a nilpotent Lie algebra and $\lambda = (\lambda_1, \dots, \lambda_n) \in \sigma(K) \setminus \{0\}$ then λ is an isolated point of $\sigma(K)$ and a joint eigenvalue of the tuple K, i.e. one can find a vector x in $\mathcal{X} \setminus \{0\}$ such that $K_j x = \lambda_j x$ for $1 \le j \le n$.*

Proof. If the dimension of $\mathcal{X}$ is finite then the conclusion follows immediately from the projection property [7, Consequence 5.5] and from Lemma 3.4. Now we assume that $\mathcal{X}$ is an infinite-dimensional space; moreover we can assume that $\lambda_n \neq 0$, because $\lambda \neq 0$. Setting $F_1 := \{\lambda_n\}$ and $F_2 := \sigma(K_n) \setminus \{\lambda_n\}$ we get two closed, non-empty and disjoint sets such that $\sigma(K_n) = F_1 \cup F_2$. Let us remark that we can apply Corollary 3.3 for $C = (K_1, \dots, K_n)$ and $B = K_n$. With the notations of Corollary 3.3 we remark that $\mathcal{X}_1$ is a finite-dimensional subspace because K_n is a compact operator. Since $\sigma(K) = \pi^{-1}(F_1) \cup \pi^{-1}(F_2)$, it easily follows that

$$\sigma(K) \cap (\mathbb{C}^{n-1} \times V_1) = \pi^{-1}(F_1) = \sigma(K, \mathcal{X}_1).$$

But this final set is finite because $\dim \mathcal{X}_1$ is finite. Since $\mathbb{C}^{n-1} \times V_1$ is a neighbourhood of λ, it follows that λ is an isolated point of $\sigma(K)$. We also get $\lambda \in \sigma(K, \mathcal{X}_1)$ so λ is a joint eigenvalue for K, as a consequence of Lemma 3.4. ∎

3.6. LEMMA. *Let $K = (K_1, \ldots, K_n)$ be a tuple of compact operators on a Banach space $\mathcal{X}$. If K generates a nilpotent Lie algebra and $0 \notin \sigma(K)$ then $\mathcal{X}$ is finite-dimensional.*

Proof. If $n = 1$ then the result is well known. Assume now that the conclusion holds for $n-1$. The set $\sigma(K)$ is compact (see [7, Theorem 2.3] and [3, Theorem 2]) and has no accumulation points (by Corollary 3.5), hence it is finite. If $0 \notin \sigma(K_n)$ then $\mathcal{X}$ is finite-dimensional in view of the case $n = 1$. Assume now that $0 \in \sigma(K_n)$. The set $\sigma(K_n)$ is finite by the projection property, hence Corollary 3.3 applied for $C = (K_1, \ldots, K_n)$, $B = K_n$, $F_1 = \{0\}$, $F_2 = \sigma(K_n) \setminus \{0\}$ gives

$$\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2, \quad \sigma(K, \mathcal{X}_j) = \pi^{-1}(F_j) \quad (j = 1, 2).$$

(If $\sigma(K_n) = \{0\}$ then we take $F_2 = \emptyset$ and $\mathcal{X}_2 = \{0\}$.) From Corollary 3.3(ii) it follows that $0 \notin \sigma(K_n, \mathcal{X}_2)$, so $\mathcal{X}_2$ is finite-dimensional.

On the other hand, since $\sigma(K_n, \mathcal{X}_1) = F_1 = \{0\}$, from the projection property it follows that

$$\sigma(K, \mathcal{X}_1) = \sigma((K_1, \ldots, K_{n-1}), \mathcal{X}_1) \times \{0\}.$$

But $0 \notin \sigma(K, \mathcal{X})$ and $\sigma(K, \mathcal{X}_1) \subseteq \sigma(K, \mathcal{X})$ (cf. Corollary 3.3(ii))), so $0 \notin \sigma((K_1, \ldots, K_{n-1}), \mathcal{X}_1)$. Then we deduce, by the induction hypothesis, that $\mathcal{X}_1$ is finite-dimensional. Since $\mathcal{X}_2$ is also finite-dimensional, it follows that so is $\mathcal{X}$. ■

From the above facts one can deduce the following result.

3.7. THEOREM. *Let K in $\mathcal{B}(\mathcal{X})^n$ be a tuple of compact operators generating a nilpotent Lie algebra. Then $\sigma(K) \setminus \{0\}$ is an at most countable set of joint eigenvalues for K, whose only possible accumulation point is 0. If $\mathcal{X}$ is infinite-dimensional then $0 \in \sigma(K)$.*

3.8. COROLLARY. *Let $\varrho : E \to \mathcal{B}(\mathcal{X})$ be a representation of the nilpotent Lie algebra E by compact operators. Let f be in $\sigma(\varrho)$ and assume that either $f \neq 0$ or $\mathcal{X}$ is finite-dimensional. Then one can find a vector x in $\mathcal{X} \setminus \{0\}$ such that $\varrho(a)x = f(a)x$ for any a in E.*

Proof. We use [7, Theorem 2.3] and either Corollary 3.5 or Lemma 3.4. ■

The following theorem is a variant of Theorem 3.7 for solvable Lie algebras.

3.9. THEOREM. *Let $\mathcal{G}$ be a solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$ containing only compact operators. Then $\Sigma(\mathcal{G})$ is a compact, at most countable subset of $\widehat{\mathcal{G}}$, whose only possible accumulation point is 0. If $\mathcal{X}$ is infinite-dimensional then $0 \in \Sigma(\mathcal{G})$.*

Proof. First we remark that, if $\mathcal{G}$ is nilpotent, then the conclusion follows from Theorem 3.7 and [7, Theorem 2.3].

Returning to the general case, consider a Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$. We have $\Sigma(\mathcal{G}) = \Sigma_{\mathcal{H}}$ and from (2) it follows that the map

$$\Sigma_{\mathcal{H}} \to \sigma(\mathrm{id}_{\mathcal{H}}), \quad f \mapsto f|_{\mathcal{H}},$$

is a homeomorphism which takes 0 to 0. This fact immediately implies the conclusion, in view of the remark from the beginning and the fact that $\sigma(\mathrm{id}_{\mathcal{H}}) = \Sigma(\mathcal{H})$. ■

In the following we shall need a few facts about nests of invariant subspaces. Let $\mathcal{G}$ be a Lie algebra of operators, $\operatorname{Lat} \mathcal{G}$ be the lattice of all closed subspaces which are invariant for $\mathcal{G}$ and $\mathcal{N}$ be a nest in $\operatorname{Lat} \mathcal{G}$. For M in $\mathcal{N}$ we denote by M^- the closed subspace spanned by those elements of $\mathcal{N}$ which are contained in M. Then $M^- \in \operatorname{Lat} \mathcal{G}$ and, if M^- is distinct from M we say that M *determines the atom* M/M^-. There exists a natural representation $\varphi : \mathcal{G} \to \mathcal{B}(M/M^-)$ constructed by restriction and then factorization of operators. If moreover $\dim(M/M^-) = 1$ then φ can be viewed as a character of $\mathcal{G}$; in this case φ is called the *character determined by the atom* M/M^- and it has the property

$$(T - \varphi(T))M \subseteq M^- \tag{12}$$

for any T in $\mathcal{G}$.

We shall need the following result which is a straightforward consequence of [13, Section 3, Theorem 7′].

3.10. LEMMA. *Let $\mathcal{R}$ be a quasisolvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$ consisting only of compact operators. If $\mathcal{N}$ is a maximal nest in the lattice $\operatorname{Lat} \mathcal{R}$ then any atom of $\mathcal{N}$ has dimension 1.*

We shall also need the following fact, which is used in the proof of the Ringrose theorem about the spectrum of a compact operator (see [5, Chapter 2, Lemma 2.29]).

3.11. LEMMA. *Let T be a compact operator, $T \in \mathcal{B}(\mathcal{X})$, and $\mathcal{N}$ be a maximal nest in $\operatorname{Lat} T$. Let ϱ in $\sigma(T) \setminus \{0\}$ and x in $\mathcal{X} \setminus \{0\}$ be such that $Tx = \varrho x$. If M denotes the intersection of those L in $\mathcal{N}$ with the property $x \in L$ then $M \in \mathcal{N}$, $M = \mathbb{C}x \oplus M^-$ and $(T - \varrho)M \subseteq M^-$.*

The following result is a first step towards a variant (concerning solvable Lie algebras) of the Ringrose theorem.

3.12. LEMMA. *Let $\mathcal{L}$ be a nilpotent Lie subalgebra of $\mathcal{B}(\mathcal{X})$ consisting only of compact operators. Let $\mathcal{N}$ be a maximal nest in $\operatorname{Lat} \mathcal{L}$ and $\{\varphi_\alpha \mid \alpha \in A\}$ be the set of characters of $\mathcal{L}$ determined by the atoms of $\mathcal{N}$. Then $\sigma(\mathrm{id}_{\mathcal{L}}) \setminus \{0\} = \{\varphi_\alpha \mid \alpha \in A\} \setminus \{0\}$.*

Proof. For α in A let $\mathcal{Y}_\alpha$ be the element of $\mathcal{N}$ which determines the character φ_α. Assume that $\varphi_\alpha \neq 0$ and for the representation $\varrho_\alpha : \mathcal{L} \to \mathcal{B}(\mathcal{Y}_\alpha)$, $\varrho_\alpha(L) := L|_{\mathcal{Y}_\alpha}$, consider the Koszul complex

$$\operatorname{Kos}(\mathcal{L}, \varrho_\alpha - \varphi_\alpha, \mathcal{Y}_\alpha): \quad 0 \leftarrow \mathcal{Y}_\alpha \xleftarrow{\delta_1} \mathcal{Y}_\alpha \otimes \mathcal{L} \xleftarrow{\delta_2} \ldots$$

The relation (12) (applied for φ_α instead of φ) shows that $\operatorname{Ran}\delta_1 \neq \mathcal{Y}_\alpha$, so the above complex is not exact. Hence $\varphi_\alpha \in \sigma(\varrho_\alpha)$. But $\varphi_\alpha \neq 0$ so, by Corollary 3.8, we can find an x in $\mathcal{Y}_\alpha \setminus \{0\}$ such that $Lx = \varphi_\alpha(L)x$ for any L in $\mathcal{L}$. This implies $\varphi_\alpha \in \sigma(\mathrm{id}_{\mathcal{L}})$ (see Corollary 2.9).

Conversely, let φ be in $\sigma(\mathrm{id}_{\mathcal{L}}) \setminus \{0\}$. By Corollary 2.9 there exists x in $\mathcal{X} \setminus \{0\}$ such that

$$Lx = \varphi(L)x \quad \text{for any } L \text{ in } \mathcal{L}. \tag{13}$$

Let M be the intersection of those L in $\mathcal{N}$ with $x \in L$. An application of Lemma 3.11 (for any T in $\mathcal{L}$ such that $\varphi(T) \neq 0$) shows that

$$M \in \mathcal{N}, \quad M = \mathbb{C}x \oplus M^- \tag{14}$$

and moreover

$$(T - \varphi(T))M \subseteq M^-. \tag{15}$$

If T is an element of $\mathcal{L}$ such that $\varphi(T) = 0$ then (15) also holds (by (13) and (14)). Now (14) shows that M is an element of $\mathcal{N}$ which generates an atom and (15) shows that φ is the character determined by the atom M/M^-. ■

3.13. COROLLARY. *In the situation of Lemma* 3.12 *the following assertions hold.*

(i) *If $\mathcal{X}$ is finite-dimensional then* $\sigma(\mathrm{id}_{\mathcal{L}}) = \{\varphi_\alpha \mid \alpha \in A\}$.
(ii) *If $\mathcal{X}$ is infinite-dimensional then* $\sigma(\mathrm{id}_{\mathcal{L}}) = \{\varphi_\alpha \mid \alpha \in A\} \cup \{0\}$.

Proof. (ii) immediately follows from Lemma 3.12 and from the last sentence of Theorem 3.9.

In view of Lemma 3.12, to prove (i) it suffices to show that $0 \in \sigma(\mathrm{id}_{\mathcal{L}})$ if and only if there exists an α in A such that $\varphi_\alpha = 0$. If such an α exists then, as in the first part of the proof of Lemma 3.12, we get an x in $\mathcal{Y}_\alpha \setminus \{0\}$ such that $Lx = 0$ for any L in $\mathcal{L}$. So, by Corollary 2.9, we have $0 \in \sigma(\mathrm{id}_{\mathcal{L}})$. Conversely, suppose that $0 \in \sigma(\mathrm{id}_{\mathcal{L}})$. First we remark that, if $\mathcal{U}$ is a nilpotent Lie algebra on the finite-dimensional space $\mathcal{Z}$ such that the complex $\operatorname{Kos}(\mathcal{U}, \mathrm{id}_{\mathcal{U}}, \mathcal{Z})$ is not exact, then this complex is not exact at the first term. (See e.g. [1, Theorem 1].) We apply this remark for the nilpotent Lie algebra $\mathcal{U}_\alpha := \{L|_{\mathcal{Y}_\alpha} \mid L \in \mathcal{L}\}$, where α in $\{1, \ldots, N\}$ is the lowest index with $0 \in \sigma(\mathrm{id}_{\mathcal{U}_\alpha})$. (Here $N := \dim \mathcal{X}$ and, obviously, $A = \{1, \ldots, N\}$.) Then $0 \notin \sigma(\mathrm{id}_{\mathcal{U}_{\alpha-1}})$ and hence

$$\mathcal{Y}_{\alpha-1} = \mathcal{L}\mathcal{Y}_{\alpha-1} \subseteq \mathcal{L}\mathcal{Y}_\alpha \neq \mathcal{Y}_\alpha.$$

But $\dim(\mathcal{Y}_\alpha/\mathcal{Y}_{\alpha-1}) = 1$, so $\mathcal{L}\mathcal{Y}_\alpha = \mathcal{Y}_{\alpha-1}$. It follows that $L\mathcal{Y}_\alpha \subseteq \mathcal{Y}_{\alpha-1}$ for any L in $\mathcal{L}$, hence $\varphi_\alpha = 0$. ■

3.14. THEOREM. *Let $\mathcal{G}$ be a quasisolvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$ consisting only of compact operators. Let $\mathcal{N}$ be a maximal nest in* $\operatorname{Lat}\mathcal{G}$ *and Γ be the set of characters of $\mathcal{G}$ determined by the atoms of $\mathcal{N}$. Then*

(i) *If $\mathcal{X}$ is finite-dimensional then $\Sigma(\mathcal{G}) = \Gamma$.*
(ii) *If $\mathcal{X}$ is infinite-dimensional then $\Sigma(\mathcal{G}) = \Gamma \cup \{0\}$.*

Proof. For any Lie subalgebra $\mathcal{L}$ of $\mathcal{G}$ define $\Sigma'(\mathcal{L}) := \Gamma|_{\mathcal{L}}$, respectively $\Sigma'(\mathcal{L}) := \Gamma|_{\mathcal{L}} \cup \{0\}$, if $\mathcal{X}$ is finite-, respectively infinite-dimensional. Then the map $\Sigma'(\bullet)$ obviously satisfies condition (b) of Theorem 2.11; moreover, condition (a) of the same theorem is satisfied by Corollary 3.13. Hence in view of the uniqueness assertion of Theorem 2.11 we deduce that $\Sigma(\mathcal{L}) = \Sigma'(\mathcal{L})$ for any Lie subalgebra $\mathcal{L}$ of $\mathcal{G}$. In particular we have $\Sigma(\mathcal{G}) = \Sigma'(\mathcal{G})$. ■

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Free interpolation in Hardy–Orlicz spaces

by

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Abstract. We show that the Carleson condition is necessary and sufficient for free interpolation in Hardy–Orlicz spaces $\mathcal{H}_\varphi$ on the unit disk $\mathbb{D}$ under certain conditions on φ, and we give a characterization of the trace space $\mathcal{H}_\varphi|_\Lambda$ if Λ is a finite union of Carleson sequences.

Introduction. Let $\mathrm{Hol}(\mathbb{D})$ be the space of holomorphic functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} = \partial\mathbb{D}$. A sequence $\Lambda = \{\lambda_n\}_{n\ge1} \subset \mathbb{D}$ is called *of free interpolation* for a subspace $X \subset \mathrm{Hol}(\mathbb{D})$ if the trace space

$$l = X|_\Lambda = \{f|_\Lambda : f \in X\}$$

is an *ideal space*, i.e. if $a \in l$ and $b \in \mathbb{C}^\Lambda$ are such that $|b(\lambda)| \le |a(\lambda)|$, $\lambda \in \Lambda$, then $b \in l$ (cf. for example [14] or [6]). We will also use the following notation for the associated sequence space:

$$X(\Lambda) = \{(f(\lambda_n))_{n\ge1} : f \in X\}. \tag{1}$$

The description of free interpolation sequences for the space $H^\infty = \{f \in \mathrm{Hol}(\mathbb{D}) : \sup_{z\in\mathbb{D}} |f(z)| < \infty\}$ was given by L. Carleson [1] and it was shown by H. S. Shapiro and A. L. Shields [19] that the free interpolation sequences for H^∞ are exactly the same as for the Hardy spaces

$$H^p(\mathbb{D}) = \Big\{ f \in \mathrm{Hol}(\mathbb{D}) : \sup_{0\le r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^p\,dt < \infty \Big\}, \quad 1 \le p < \infty$$

(cf. also [10] for the case $0 < p < 1$). It is well known that we may identify $H^p(\mathbb{D})$ and $H^p(\mathbb{T}) = \{f \in L^p(\mathbb{T}) : \widehat{f}(n) = (2\pi)^{-1}\int_{-\pi}^{\pi} f(e^{it})e^{-int}\,dt = 0,\ n < 0\}$.

In the first section we give the definition of Hardy–Orlicz spaces $\mathcal{H}_\varphi$ and various conditions that one may impose on the defining function φ to get

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