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*Received May 4, 1995*
*Revised version January 25, 1999* (3460)

STUDIA MATHEMATICA 135 (2) (1999)

# Some geometric properties of typical compact convex sets in Hilbert spaces

by

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**Abstract.** An investigation is carried out of the compact convex sets $X$ in an infinite-dimensional separable Hilbert space $\mathbb{E}$, for which the metric antiprojection $q_X(e)$ from $e$ to $X$ has fixed cardinality $n+1$ ($n \in \mathbb{N}$ arbitrary) for every $e$ in a dense subset of $\mathbb{E}$. A similar study is performed in the case of the metric projection $p_X(e)$ from $e$ to $X$ where $X$ is a compact subset of $\mathbb{E}$.

**1. Introduction.** One of the methods used to study the geometry of convex sets is based, as is well known, on the Baire category. This method, which goes back to the fundamental contributions by Klee [14] and, independently, by Gruber [11], has made it possible to discover several elusive and even unexpected properties of convex sets (see Gruber [12], Schneider [19], Schneider and Wieacker [20], Wieacker [23], Zamfirescu [24]). We refer to Gruber [13] and Zamfirescu [26] for a survey about this area of research and for additional bibliography.

In the present paper the Baire category will be used to investigate some geometric properties of typical compact convex sets. Let $\mathbb{E}$ be a real infinite-dimensional separable Hilbert space. In [6] it has been recently shown that, for a typical compact convex set $X \subset \mathbb{E}$ and any $n \in \mathbb{N}$, the metric antiprojection $q_X(e)$ from $e$ to $X$ (that is, the set of all points of $X$ which are farthest from $e$) is such that $\operatorname{card} q_X(e) \geq n+1$ for every $e$ in a dense subset of $\mathbb{E}$. The aim of the present paper is to establish a stronger version of this result. In fact, it is proved (Theorem 5.1) that, for a typical compact convex set $X \subset \mathbb{E}$ and any $n \in \mathbb{N}$, one actually has

$$\operatorname{card} q_X(e) = n+1$$

for every $e$ in a dense subset of $\mathbb{E}$. Similarly, it is shown (Theorem 6.1) that for a typical compact set $X \subset \mathbb{E}$ and any $n \in \mathbb{N}$, the metric projection $p_X(e)$

1991 *Mathematics Subject Classification*: Primary 52A05, 41A65; Secondary 54E52.

from $e$ to $X$ (that is, the set of all points of $X$ which are closest to $e$) satisfies

$$\operatorname{card} p_X(e) = n+1$$

for every $e$ in a dense subset of $\mathbb{E}$. Of course, the latter result fails if $X$ is also convex since, in this case, $p_X(e)$ is single-valued for every $e \in \mathbb{E}$. So far, there are no examples of compact convex or compact subsets of $\mathbb{E}$ with any of the above-mentioned properties. The proofs are based on some ideas due to Klee, Gruber and Zamfirescu, combined with technical details adapted from [6] and [7].

In the case of the metric projection, the study of typical compact sets with properties of the above kind was started by Zamfirescu [25] and pursued by De Blasi and Zamfirescu [7] in finite dimensions, and by Zhivkov [27], [28] in infinite dimensions. It is worthwhile to note that in [7] it has been proved that for a typical compact set $X \subset \mathbb{R}^n$, $n \geq 2$, we have $\operatorname{card} p_X(e) = n+1$ for every $e$ in a dense subset of $\mathbb{R}^n$. For the metric projection, dense multivalued loci with cardinality two have recently been considered by Zhivkov [28] who has established, in a Banach space setting, a sharp existence theorem.

Our paper consists of a total of six sections. Sections 2 and 3 contain, respectively, notation and terminology, and some auxiliary propositions. In Section 4, two technical lemmas are proved. The main result (Theorem 5.1) about typical compact convex sets is proved in Section 5. An analogous result (Theorem 6.1) about typical compact sets is discussed in Section 6.

**2. Notation and terminology.** Throughout this paper $\mathbb{E}$ denotes a real infinite-dimensional Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$, and $\mathcal{C}(\mathbb{E})$ (resp. $\mathcal{K}(E)$) the space of all nonempty compact convex (resp. compact) subsets of $\mathbb{E}$ endowed with the Hausdorff metric $h$. Under this metric the spaces $\mathcal{C}(\mathbb{E})$ and $\mathcal{K}(\mathbb{E})$ are complete.

Our notation is standard. If $X$ is a nonempty subset of $\mathbb{E}$, we denote by $\operatorname{ext} X$, $\exp X$, $\overline{\operatorname{co}}\, X$, and $\operatorname{lin} X$, respectively, the set of extreme points of $X$, the set of strongly exposed points of $X$, the closed convex hull of $X$, and the linear span of $X$. If $X$ is a linear subspace of $\mathbb{E}$, by $\dim X$ we mean the algebraic dimension of $X$. The cardinality of a set $X$ is denoted by $\operatorname{card} X$.

In a metric space $M$, we denote by $U_M(x,r)$ (resp. $\widetilde{U}_M(x,r)$) an open (resp. closed) ball with center $x \in M$ and radius $r > 0$. If $M$ is complete, a set $X \subset M$ is called *residual* in $M$ if $M \setminus X$ is a set of Baire first category in $M$. As is well known (see Oxtoby [17], p. 41), $X$ is residual in $M$ if and only if $X$ contains a dense $G_\delta$-subset of $M$. The elements of a residual subset of $M$ are also called *typical* elements of $M$.

As usual, $\mathbb{N}$ stands for the set of the integers $n \geq 1$, and $\mathbb{Q}^+$ for the set of rationals $r > 0$.



For any nonempty bounded set $X \subset \mathbb{E}$ and $e \in \mathbb{E}$, the *metric antiprojection $q_X(e)$ from $e$ to $X$* is defined by

(2.1) $$q_X(e) = \{x \in X \mid \|x-e\| = \delta(X,e)\},$$

where $\delta(X,e) = \sup\{\|x-e\| \mid x \in X\}$. Furthermore, for fixed $X$, $q_X$ stands for the metric antiprojection mapping given by (2.1). Clearly, $q_X(e) \in \mathcal{K}(\mathbb{E})$ for each $e \in \mathbb{E}$, provided $X \in \mathcal{K}(\mathbb{E})$.

For $X \in \mathcal{K}(\mathbb{E})$ and $n \in \mathbb{N}$, the sets

$$M_{n+1}(X) = \{e \in \mathbb{E} \mid \operatorname{card} q_X(e) = n+1\},$$
$$\widehat{M}_{n+1}(X) = \{e \in \mathbb{E} \mid \operatorname{card} q_X(e) \geq n+1\}$$

are called, respectively, the *multivalued locus of $q_X$ of cardinality* $n+1$, and the *multivalued locus of $q_X$ of cardinality at least* $n+1$.

Analogously, for any nonempty set $X \subset \mathbb{E}$ and $e \in \mathbb{E}$, we define the *metric projection $p_X(e)$ from $e$ to $X$* by

$$p_X(e) = \{x \in X \mid \|x-e\| = \gamma(X,e)\},$$

where $\gamma(X,e) = \inf\{\|x-e\| \mid x \in X\}$, and we denote by $p_X$ the corresponding metric projection mapping. For $X \in \mathcal{K}(\mathbb{E})$ and $n \in \mathbb{N}$, the *multivalued locus of $p_X$ of cardinality* $n+1$ (resp. *of cardinality at least* $n+1$) is defined as in the case of $q_X$.

For a nonempty subset $X$ of $\mathbb{E}$ and $n \in \mathbb{N}$, set

$$\mathcal{P}_n(X) = \{A \mid A \text{ is a set of } n \text{ distinct points } a_1,\ldots,a_n \in X\}.$$

If $(a_1,\ldots,a_n)$ and $(e_1,\ldots,e_n)$ are ordered $n$-tuples of points $a_i, e_i \in \mathbb{E}$, $i = 1,\ldots,n$, we put for brevity

$$\det(a_1,\ldots,a_n;e_1,\ldots,e_n) = \det\begin{pmatrix} \langle a_1,e_1\rangle & \langle a_1,e_2\rangle & \ldots & \langle a_1,e_n\rangle \\ \langle a_2,e_1\rangle & \langle a_2,e_2\rangle & \ldots & \langle a_2,e_n\rangle \\ \ldots & \ldots & \ldots & \ldots \\ \langle a_n,e_1\rangle & \langle a_n,e_2\rangle & \ldots & \langle a_n,e_n\rangle \end{pmatrix},$$

where det stands for determinant.

**3. Auxiliary results.** In this section we prove some elementary geometric propositions that we shall need later.

PROPOSITION 3.1. *Let $H_n$, $n \in \mathbb{N}$, be a linear subspace of $\mathbb{E}$ of dimension $n$. Let $V = \{x \in \mathbb{E} \mid \langle x, a_0 - e_0\rangle = 0\}$, where $e_0 \in H_n$ and $a_0 \notin H_n$. Let $K_n$ be the metric projection of $H_n$ onto $V$, that is,*

$$K_n = \{p_V(e) \mid e \in H_n\}.$$

*Then $K_n$ is a linear subspace of $V$ of dimension $n$ and if $(e_1,\ldots,e_n)$ and $(f_1,\ldots,f_n)$ are arbitrary bases for $H_n$ and $K_n$, respectively,*

(3.1) $$\det(f_1,\ldots,f_n;e_1,\ldots,e_n) \neq 0.$$

Proof. Let $(e_1,\ldots,e_n)$ be an arbitrary basis for $H_n$. Every $e\in H_n$ is given by $e=\sum_{i=1}^n t_i e_i$ for some $(t_1,\ldots,t_n)\in\mathbb{R}^n$, thus

$$p_V(e)=\sum_{i=1}^n t_i\, p_V(e_i),$$

which implies $\dim K_n\le n$. Suppose $\dim K_n<n$. Then for some $(t_1,\ldots,t_n)\in\mathbb{R}^n\setminus\{0\}$ we have $\sum_{i=1}^n t_i p_V(e_i)=0$. Clearly, $e=\sum_{i=1}^n t_i e_i$ is nonzero and $p_V(e)=0$. As $e\in V^\perp$, the orthogonal complement of $V$, we have $e=\lambda(a_0-e_0)$ for some $\lambda\in\mathbb{R}\setminus\{0\}$ and so $a_0=e/\lambda+e_0\in H_n$, a contradiction. It follows that $\{p_V(e_1),\ldots,p_V(e_n)\}$ is a basis for $K_n$, proving that $K_n$ is $n$-dimensional.

Let $(e_1,\ldots,e_n)$ and $(f_1,\ldots,f_n)$ be arbitrary bases for $H_n$ and $K_n$, respectively. If (3.1) does not hold, there exists $(t_1,\ldots,t_n)\in\mathbb{R}^n\setminus\{0\}$ such that

$$\langle f_1,e\rangle=\ldots=\langle f_n,e\rangle=0\quad \text{where } e=\sum_{i=1}^n t_ie_i,$$

thus $p_{K_n}(e)=0$. Moreover, $p_V(e)=p_{K_n}(e)$, because $p_V(e)\in K_n$. Since

$$\sum_{i=1}^n t_i\, p_V(e_i)=p_V(e)=p_{K_n}(e)=0,$$

and $(p_V(e_1),\ldots,p_V(e_n))$ is linearly independent, it follows that $(t_1,\ldots,t_n)=(0,\ldots,0)$, a contradiction. Hence (3.1) holds true, completing the proof.

Proposition 3.2. *Let $(e_1,\ldots,e_n)\subset\mathbb{E}$, $n\in\mathbb{N}$, be any linearly independent set, and let $A\in\mathcal{P}_{n+1}(\mathbb{E})$, $A=(a_1,\ldots,a_{n+1})$, and $\varepsilon>0$. Then there exists a set $B\in\mathcal{P}_{n+1}(\mathbb{E})$, $B=(b_1,\ldots,b_{n+1})$, with $b_i\in U_{\mathbb{E}}(a_i,\varepsilon)$, $i=1,\ldots,n+1$, such that*

$$\det(b_1-b_{n+1},\ldots,b_n-b_{n+1};e_1,\ldots,e_n)\neq 0. \tag{3.2}$$

Proof. The statement is obvious for $n=1$. Assume that it holds for $n$. Let $(e_1,\ldots,e_{n+1})\subset\mathbb{E}$ be any linearly independent set, and let $(a_1,\ldots,a_{n+2})\in\mathcal{P}_{n+2}(\mathbb{E})$ and $\varepsilon>0$. By the induction hypothesis, corresponding to $(e_1,\ldots,e_n),(a_1,\ldots,a_{n+1})$ and $\varepsilon$ there is a set $(b_1,\ldots,b_{n+1})\in\mathcal{P}_{n+1}(\mathbb{E})$, with $b_i\in U_{\mathbb{E}}(a_i,\varepsilon)$, $i=1,\ldots,n+1$, such that (3.2) is satisfied.

For $x\in U_{\mathbb{E}}(a_{n+2},\varepsilon)$ we have

$$\begin{aligned}\det(b_1-x,\ldots,b_{n+1}-x;e_1,\ldots,e_{n+1})&=\det(b_1-b_{n+1},\ldots,b_n-b_{n+1},b_{n+1}-x;e_1,\ldots,e_n)\\&=\sum_{i=1}^{n+1}\langle b_{n+1}-x,e_i\rangle D_{n+1,i}=\langle b_{n+1}-x,e\rangle,\end{aligned}\tag{3.3}$$



where $D_{n+1,i}$ is the cofactor of the element $\langle b_{n+1}-x,e_i\rangle$ in the last determinant, and $e=\sum_{i=1}^{n+1}D_{n+1,i}\,e_i$. Observe that $e\neq 0$, because

$$D_{n+1,n+1}=\det(b_1-b_{n+1},\ldots,b_n-b_{n+1};e_1,\ldots,e_n)\neq 0,$$

and $(e_1,\ldots,e_n)$ is linearly independent.

Take now any $\overline{x}\in U_{\mathbb{E}}(a_{n+2},\varepsilon)$ so that $\langle b_{n+1}-\overline{x},e\rangle\neq 0$, and set $b_{n+2}=\overline{x}$. Put $B=(b_1,\ldots,b_{n+2})$. Clearly, $B\in\mathcal{P}_{n+2}(\mathbb{E})$ and $b_i\in U_{\mathbb{E}}(a_i,\varepsilon)$, $i=1,\ldots,n+2$. As $b_{n+2}=\overline{x}$, (3.3) implies

$$\det(b_1-b_{n+2},\ldots,b_{n+1}-b_{n+2};e_1,\ldots,e_{n+1})\neq 0,$$

completing the proof.

Proposition 3.3. *Let $H_n=\operatorname{lin}(e_1,\ldots,e_n)$, $n\in\mathbb{N}$, where $(e_1,\ldots,e_n)\subset\mathbb{E}$ is a linearly independent set, and let $B\in\mathcal{P}_{n+1}(\mathbb{E})$, $B=(b_1,\ldots,b_{n+1})$. Then the following are equivalent:*

(i) *there is one and only one sphere $S_{b_1\ldots b_{n+1}}$ with center in $H_n$ containing the points $b_1,\ldots,b_{n+1}$;*

(ii) $\det(b_1-b_{n+1},\ldots,b_n-b_{n+1};e_1,\ldots,e_n)\neq 0$.

Proof. Suppose (i). From the assumption, there is a unique point $c\in H_n$ satisfying

$$\|c-b_j\|=\|c-b_{n+1}\|,\quad j=1,\ldots,n. \tag{3.4}$$

As $(e_1,\ldots,e_n)$ is linearly independent there is a unique point $(x_1,\ldots,x_n)\in\mathbb{R}^n$ such that $c=\sum_{i=1}^n x_ie_i$. If we insert the latter expression in (3.4) and square both sides, we have

$$\sum_{i=1}^n\langle b_j-b_{n+1},e_i\rangle x_i=\frac{\|b_j\|^2-\|b_{n+1}\|^2}{2},\quad j=1,\ldots,n,$$

which implies (ii), by the uniqueness of $c$. The reverse implication can be proved analogously. This completes the proof.

Proposition 3.4. *Let $(e_1,\ldots,e_n)\subset\mathbb{E}$, $n\in\mathbb{N}$, be a linearly independent set. Let $B\in\mathcal{P}_{n+1}(\mathbb{E})$, $B=(b_1,\ldots,b_{n+1})$, be such that*

$$\det(b_1-b_{n+1},\ldots,b_n-b_{n+1};e_1,\ldots,e_n)\neq 0. \tag{3.5}$$

*Then for every $z\in\mathbb{E}$ and $\varepsilon>0$ there exists a point $\overline{x}\in U_{\mathbb{E}}(z,\varepsilon)$ such that*

$$\det(b_1-\overline{x},\ldots,b_n-\overline{x};e_1,\ldots,e_n)\neq 0. \tag{3.6}$$

Proof. For $x \in U_{\mathbb{E}}(z,\varepsilon)$ we have

$$(3.7)\quad \det(b_1 - x, \ldots, b_n - x; e_1, \ldots, e_n) = \det(b_1 - b_n, \ldots, b_{n-1} - b_n, b_n - x; e_1, \ldots, e_n) = \sum_{i=1}^{n} \langle b_n - x, e_i\rangle D_{n,i} = \langle b_n - x, e\rangle,$$

where $D_{n,i}$ is the cofactor of the element $\langle b_n - x, e_i\rangle$ in the last determinant, and $e = \sum_{i=1}^{n} D_{n,i} e_i$. Observe that $e \neq 0$ because, for $i = 1, \ldots, n$, $D_{n,i}$ coincides with the cofactor of the element $\langle b_n - b_{n+1}, e_i\rangle$ of the determinant (3.5), which is nonzero. Take now $\overline{x} \in U_{\mathbb{E}}(z,\varepsilon)$ so that $\langle b_n - \overline{x}, e\rangle \neq 0$. With this choice of $\overline{x}$, (3.6) follows from (3.7), completing the proof.

PROPOSITION 3.5. *Let $H_n$, $n \in \mathbb{N}$, be a linear subspace of $\mathbb{E}$ of dimension $n$. Then, for each $k \in \mathbb{N}$, the following statement* ($\mathrm{P}_k$) *holds true:*

($\mathrm{P}_k$) *For each $A \in \mathcal{P}_{n+k}(\mathbb{E})$, $A = (a_1, \ldots, a_{n+k})$, and $\varepsilon > 0$ there exists $B \in \mathcal{P}_{n+k}(\mathbb{E})$, $B = (b_1, \ldots, b_{n+k})$, with $h(B, A) < \varepsilon$, such that for every set $(b_{i_1}, \ldots, b_{i_{n+1}}) \in \mathcal{P}_{n+1}(B)$ both properties* (i) *and* (ii) *below are satisfied:*

(i) *there is one and only one sphere $S_{b_{i_1},\ldots,b_{i_{n+1}}}$, with center in $H_n$, containing $b_{i_1}, \ldots, b_{i_{n+1}}$;*

(ii) $S_{b_{i_1}\ldots b_{i_{n+1}}} \cap B = (b_{i_1}, \ldots, b_{i_{n+1}})$.

Proof. Let $(e_1, \ldots, e_n)$ be a basis for $H_n$, thus $H_n = \mathrm{lin}(e_1, \ldots, e_n)$. We shall prove ($\mathrm{P}_k$) by induction. Clearly, ($\mathrm{P}_1$) is satisfied by virtue of Propositions 3.2 and 3.1. Assuming now ($\mathrm{P}_k$), we shall prove ($\mathrm{P}_{k+1}$).

Let $A \in \mathcal{P}_{n+k+1}(\mathbb{E})$, $A = (a_1, \ldots, a_{n+k+1})$, and $\varepsilon > 0$. Set $A' = (a_1, \ldots, a_{n+k})$, thus $A = A' \cup \{a_{n+k+1}\}$ and $A' \in \mathcal{P}_{n+k}(\mathbb{E})$. By virtue of ($\mathrm{P}_k$) there exists $B' \in \mathcal{P}_{n+k}(\mathbb{E})$, $B' = (b_1, \ldots, b_{n+k})$, with $h(B', A') < \varepsilon$, such that for every set $(b_{i_1}, \ldots, b_{i_{n+1}}) \in \mathcal{P}_{n+1}(B')$ properties (i) and (ii) are both satisfied.

Fix now $z \in \mathbb{E}$ and $\sigma > 0$ so that

$$(3.8)\quad U_{\mathbb{E}}(z,\sigma) \subset U_{\mathbb{E}}(a_{n+k+1},\varepsilon) \setminus S_{b_{i_1}\ldots b_{i_{n+1}}},$$

for every $(b_{i_1}, \ldots, b_{i_{n+1}}) \in \mathcal{P}_{n+1}(B')$. This is certainly possible for, in view of (i), the spheres $S_{b_{i_1}\ldots b_{i_{n+1}}}$ with center in $H_n$ are uniquely determined by $(b_{i_1}, \ldots, b_{i_{n+1}})$, and so there are a finite number of them.

Denote by $\{B_i\}_{i=1}^{N}$, $B_i = \{b_1^i, \ldots, b_n^i\}$, the finite family of all subsets of $B'$ of cardinality $n$.

Consider $B_1 = (b_1^1, \ldots, b_n^1)$. Take a point, say $b_{n+1}^1$, in the set $B' \setminus B_1$. Since $(b_1^1, \ldots, b_{n+1}^1) \in \mathcal{P}_{n+1}(B')$, by (ii) there is one and only one sphere



$S_{b_1^1\ldots b_{n+1}^1}$, with center in $H_n$, containing $b_1^1, \ldots, b_{n+1}^1$. By Proposition 3.3,

$$\det(b_1^1 - b_{n+1}^1, \ldots, b_n^1 - b_{n+1}^1; e_1, \ldots, e_n) \neq 0$$

and hence, by Proposition 3.4, there exists $\overline{x}_1 \in U_{\mathbb{E}}(z,\sigma)$ such that

$$\det(b_1^1 - \overline{x}_1, \ldots, b_n^1 - \overline{x}_1; e_1, \ldots, e_n) \neq 0.$$

By continuity, there exists a ball $U_{\mathbb{E}}(\overline{x}_1,\sigma_1) \subset U_{\mathbb{E}}(z,\sigma)$ so that

$$\det(b_1^1 - x, \ldots, b_n^1 - x; e_1, \ldots, e_n) \neq 0 \quad \text{for every } x \in U_{\mathbb{E}}(\overline{x}_1,\sigma_1).$$

Consider $B_2 = (b_1^2, \ldots, b_n^2)$. By a similar argument, also for $B_2$ one can construct a ball $U_{\mathbb{E}}(\overline{x}_2,\sigma_2) \subset U_{\mathbb{E}}(\overline{x}_1,\sigma_1)$ such that

$$\det(b_1^2 - x, \ldots, b_n^2 - x; e_1, \ldots, e_n) \neq 0 \quad \text{for every } x \in U_{\mathbb{E}}(\overline{x}_2,\sigma_2).$$

By this procedure one can construct a finite sequence of balls $U_{\mathbb{E}}(\overline{x}_1,\sigma_1), \ldots, U_{\mathbb{E}}(\overline{x}_N,\sigma_N)$ corresponding to $B_1, \ldots, B_N$, respectively, with

$$(3.9)\quad U_{\mathbb{E}}(z,\sigma) \supset U_{\mathbb{E}}(\overline{x}_1,\sigma_1) \supset U_{\mathbb{E}}(\overline{x}_2,\sigma_2) \supset \ldots \supset U_{\mathbb{E}}(\overline{x}_N,\sigma_N),$$

such that, for $i = 1, \ldots, N$,

$$\det(b_1^i - x, \ldots, b_n^i - x; e_1, \ldots, e_n) \neq 0 \quad \text{for every } x \in U_{\mathbb{E}}(\overline{x}_i,\sigma_i).$$

Thus, by Proposition 3.3, for every $x \in U_{\mathbb{E}}(\overline{x}_i,\sigma_i)$, $i = 1, \ldots, N$, there is one and only one sphere $S_{b_1^i\ldots b_n^i x}$, with center in $H_n$, satisfying

$$S_{b_1^i\ldots b_n^i x} \supset (b_1^i, \ldots, b_n^i, x).$$

Fix now a point $\overline{x} \in U_{\mathbb{E}}(\overline{x}_N,\sigma_N)$, and set

$$B = (b_1, \ldots, b_{n+k}, \overline{x}).$$

Clearly, $B \in \mathcal{P}_{n+k+1}(\mathbb{E})$. Furthermore, $h(B, A) < \varepsilon$, since $h(B', A') < \varepsilon$ and $\overline{x} \in U_{\mathbb{E}}(a_{n+k+1},\varepsilon)$, by virtue of (3.8) and (3.9). It remains to show that each set $(b_{i_1}, \ldots, b_{i_{n+1}}) \in \mathcal{P}_{n+1}(B)$ satisfies (i) and (ii).

To this end, consider a $(b_{i_1}, \ldots, b_{i_{n+1}}) \in \mathcal{P}_{n+1}(B)$. Suppose that $\overline{x} \notin (b_{i_1}, \ldots, b_{i_{n+1}})$, thus $(b_{i_1}, \ldots, b_{i_{n+1}}) \in \mathcal{P}_{n+1}(B')$. In this case (i) is trivially satisfied. As $S_{b_{i_1}\ldots b_{i_{n+1}}} \cap B' = (b_{i_1}, \ldots, b_{i_{n+1}})$ and $\overline{x} \notin S_{b_{i_1}\ldots b_{i_{n+1}}}$, by (3.8) and (3.9), also (ii) is satisfied. Suppose now $\overline{x} \in (b_{i_1}, \ldots, b_{i_{n+1}})$, say $\overline{x} = b_{i_{n+1}}$. For some $1 \le r \le N$, we have $(b_{i_1}, \ldots, b_{i_n}) = B_r$. Since $\overline{x} \in U_{\mathbb{E}}(\overline{x}_r,\sigma_r)$, there is one and only one sphere $S_{b_{i_1}\ldots b_{i_n}\overline{x}}$, with center in $H_n$, containing $(b_{i_1}, \ldots, b_{i_n}, \overline{x})$, and hence (i) is satisfied. Suppose that (ii) does not hold. Then there is a point $b \in B \setminus (b_{i_1}, \ldots, b_{i_n}, \overline{x})$ such that

$$b \in S_{b_{i_1}\ldots b_{i_n}\overline{x}}.$$

But $(b_{i_1}, \ldots, b_{i_n}, b) \in \mathcal{P}_{n+1}(B')$, and so there is one and only one sphere $S_{b_{i_1}\ldots b_{i_n} b}$, with center in $H_n$, containing $(b_{i_1}, \ldots, b_{i_n}, b)$. As $S_{b_{i_1}\ldots b_{i_n}\overline{x}}$ has this property, it follows that

$$S_{b_{i_1}\ldots b_{i_n}\overline{x}} = S_{b_{i_1}\ldots b_{i_n} b},$$

which implies $\overline{x} \in S_{b_{i_1}\dots b_{i_n}b}$. On the other hand, $\overline{x} \in U_{\mathbb{E}}(z,\sigma)$ and, by (3.8), $\overline{x} \notin S_{b_{i_1}\dots b_{i_n}b}$, a contradiction. Hence also (ii) is satisfied. It follows that $(\mathrm{P}_{k+1})$ holds true, completing the proof.

**4. Two lemmas on the antiprojection mapping for compact convex sets.** In this section we prove two technical lemmas concerning the antiprojection mapping $q_X : \mathbb{E} \to \mathcal{K}(\mathbb{E})$, where $X \in \mathcal{C}(\mathbb{E})$. The construction follows closely a pattern already used in [6] and [7]. Both lemmas play a fundamental role in the proof of the main results, discussed in the following sections.

Recall that, for $X \in \mathcal{C}(\mathbb{E})$, $\widehat{M}_{n+1}(X)$ denotes the multivalued locus of cardinality at least $n+1$ for the antiprojection mapping $q_X$.

LEMMA 4.1. *Let $H_n$, $n \in \mathbb{N}$, be a linear subspace of $\mathbb{E}$ of dimension $n$. Let $A_0 \in \mathcal{C}(\mathbb{E})$, $e_0 \in H_n$, $\lambda > 0$ and $r > 0$. Then there exist $B \in \mathcal{C}(\mathbb{E})$ and $\sigma > 0$, with $U_{\mathcal{C}(\mathbb{E})}(B,\sigma) \subset U_{\mathcal{C}(\mathbb{E})}(A_0,\lambda)$, such that for every $X \in U_{\mathcal{C}(\mathbb{E})}(B,\sigma)$ we have*

$$\widehat{M}_{n+1}(X) \cap U_{H_n}(e_0, r) \neq \emptyset.$$

Proof. Let $a_0 \in A_0$ be such that $\|a_0 - e_0\| = \delta(A_0, e_0)$. It is sufficient to prove the lemma under the additional assumptions

$$\delta(A_0, e_0) > 0 \quad \text{and} \quad \gamma(H_n, a_0) > 0 \tag{4.1}$$

because, arbitrarily close to any set $A \in \mathcal{C}(\mathbb{E})$, there is some $A_0 \in \mathcal{C}(\mathbb{E})$ satisfying (4.1).

STEP 1: Construction of $B$. Let $(e_1, \dots, e_n)$ be an orthonormal basis for $H_n$. Let $V = \{x \in \mathbb{E} \mid \langle x, a_0 - e_0\rangle = 0\}$, and let $K_n$ be the metric projection of $H_n$ onto $V$, that is,

$$K_n = \{p_V(e) \mid e \in H_n\}.$$

By Proposition 3.1, $K_n$ is a linear subspace of $V$ with $\dim K_n = n$. Let $(u_1, \dots, u_n)$ be an orthonormal basis for $K_n$ and observe that, by Proposition 3.1,

$$D = \det(u_1, \dots, u_n; e_1, \dots, e_n) \neq 0. \tag{4.2}$$

Furthermore, set

$$D_1 = \det\left(u_1 - u_n, \dots, u_{n-1} - u_n, \frac{a_0 - e_0}{\|a_0 - e_0\|}; e_1, \dots, e_n\right),$$

and take $\varepsilon_0 > 0$ such that

$$\sqrt{\varepsilon_0}\,|D_1| < |D|. \tag{4.3}$$



Fix now $\gamma$ and $\beta$ satisfying

$$1 < \gamma < \min\left\{2, 1 + \frac{\lambda}{4\|a_0 - e_0\|}\right\},$$

$$\gamma > \beta > \max\left\{1, \gamma - \frac{\lambda^2}{64\|a_0 - e_0\|^2}, \gamma - \varepsilon_0\right\}.$$

Let $b_0, b_1, \dots, b_n$ be given by

$$\begin{aligned} b_0 &= e_0 + \gamma(a_0 - e_0),\\ b_k &= e_0 + \beta(a_0 - e_0) + v_k, \end{aligned} \tag{4.4}$$

where

$$v_k = \sqrt{\gamma^2 - \beta^2}\,\|a_0 - e_0\| u_k, \qquad k = 1, \dots, n.$$

Clearly, we have

$$\|b_k - b_0\| = \sqrt{2\gamma(\gamma - \beta)}\,\|a_0 - e_0\|, \qquad k = 1, \dots, n, \tag{4.5}$$

$$\|b_k - b_h\| = \sqrt{2(\gamma^2 - \beta^2)}\,\|a_0 - e_0\|, \qquad k, h = 1, \dots, n,\ k \neq h, \tag{4.6}$$

$$\|b_k - e_0\| = \gamma\|a_0 - e_0\|, \qquad k = 0, 1, \dots, n. \tag{4.7}$$

Set now $B = \overline{\mathrm{co}}\{b_0, b_1, \dots, b_n, A_0\}$, and observe that $B \in \mathcal{C}(\mathbb{E})$, by Mazur's theorem. From (4.5)–(4.7) it follows that the points $b_0, b_1, \dots, b_n$ are distinct and lie on the sphere $S_{\mathbb{E}}(e_0, \gamma\|a_0 - e_0\|)$. Since $A_0$ is contained in the corresponding open ball $U_{\mathbb{E}}(e_0, \gamma\|a_0 - e_0\|)$, it follows that

$$q_B(e_0) = (b_0, b_1, \dots, b_n).$$

Furthermore,

$$h(B, A_0) < \lambda/2. \tag{4.8}$$

In fact, from (4.5), as $\gamma < 2$ and $\gamma - \beta < \lambda^2/(64\|a_0 - e_0\|^2)$, we have $\|b_k - b_0\| < \lambda/4$, $k = 1, \dots, n$. Moreover, $\|b_0 - a_0\| < \lambda/4$, for $\|b_0 - a_0\| = (\gamma - 1)\|a_0 - e_0\|$ and $\gamma - 1 < \lambda/(4\|a_0 - e_0\|)$. Thus $\|b_k - a_0\| \le \|b_k - b_0\| + \|b_0 - a_0\| < \lambda/2$, $k = 0, 1, \dots, n$, and (4.8) is proved.

STEP 2: $\Delta = \det(b_0 - b_1, \dots, b_0 - b_n; e_1, \dots, e_n) \neq 0$.

In fact, by virtue of (4.4), we have

$$\begin{aligned}
\Delta &= \det((\gamma - \beta)(a_0 - e_0) + v_1, \dots, (\gamma - \beta)(a_0 - e_0) + v_n; e_1, \dots, e_n)\\
&= \det(v_1 - v_n, \dots, v_{n-1} - v_n, (\gamma - \beta)(a_0 - e_0) + v_n; e_1, \dots, e_n)\\
&= \det(v_1 - v_n, \dots, v_{n-1} - v_n, (\gamma - \beta)(a_0 - e_0); e_1, \dots, e_n)\\
&\quad + \det(v_1 - v_n, \dots, v_{n-1} - v_n, v_n; e_1, \dots, e_n)\\
&= (\sqrt{\gamma^2 - \beta^2}\|a_0 - e_0\|)^{n-1}\\
&\quad \times \det(u_1 - u_n, \dots, u_{n-1} - u_n, (\gamma - \beta)(a_0 - e_0); e_1, \dots, e_n)\\
&\quad + \det(v_1, \dots, v_n; e_1, \dots, e_n)
\end{aligned}$$

$$= (\sqrt{\gamma^2-\beta^2}\|a_0-e_0\|)^n$$
$$\times \left[\sqrt{\frac{\gamma-\beta}{\gamma+\beta}}\det\left(u_1-u_n,\dots,u_{n-1}-u_n,\frac{a_0-e_0}{\|a_0-e_0\|};e_1,\dots,e_n\right)\right.$$
$$\left. + \det(u_1,\dots,u_n;e_1,\dots,e_n)\right]$$
$$= (\sqrt{\gamma^2-\beta^2}\|a_0-e_0\|)^n\left[\sqrt{\frac{\gamma-\beta}{\gamma+\beta}}D_1+D\right].$$

But $\gamma-\beta<\varepsilon_0$ and $\gamma+\beta>2$, thus

$$\left|\sqrt{\frac{\gamma-\beta}{\gamma+\beta}}D_1+D\right| \geq |D| - \sqrt{\frac{\gamma-\beta}{\gamma+\beta}}|D_1| \geq |D| - \sqrt{\varepsilon_0}\,|D_1|,$$

which is strictly positive, by (4.3). Hence $\Delta\neq 0$, and Step 2 is proved.

For $X\in\mathcal{C}(\mathbb{E})$ and $\eta>0$, set

$$(4.9)\qquad X_k^\eta = X\cap\widetilde{U}_{\mathbb{E}}(b_k,\eta),\quad k=0,1,\dots,n,\qquad \widetilde{X}^\eta = X\setminus\bigcup_{k=0}^n U_{\mathbb{E}}(b_k,\eta).$$

STEP 3: There exist $\eta>0$ and $0<\varrho<\min\{\eta,r\}$ such that:

(i) the balls $\widetilde{U}_{\mathbb{E}}(b_k,\eta)$, $k=0,1,\dots,n$, are pairwise disjoint;

(ii) $q_{B_k^\eta}(e)=b_k$ for every $e\in U_{\mathbb{E}}(e_0,\eta)$, $k=0,1,\dots,n$;

(iii) for every $X\in U_{\mathcal{C}(\mathbb{E})}(B,\varrho)$ the sets $X_0^\eta, X_1^\eta,\dots,X_n^\eta$, $\widetilde{X}^\eta$ are compact nonempty and satisfy

$$\delta(X_k^\eta,e)>\delta(\widetilde{X}^\eta,e)\quad \text{for every } e\in U_{\mathbb{E}}(e_0,\varrho),\ k=0,1,\dots,n.$$

The proof is as in [6] (see Steps 2 and 3 of the Lemma) and so it is omitted.

By Step 2, the $n\times n$ matrix $(a_{ij})_{i,j=1}^n$, where $a_{ij}=\langle b_0-b_i,e_j\rangle$, is nonsingular, hence it has inverse $A^{-1}$, say $A^{-1}=(a'_{ij})_{i,j=1}^n$. Fix now

$$(4.10)\qquad 0<\theta<\frac{\varrho}{n\|A^{-1}\|},$$

where $\varrho$ is as in Step 3 and $\|A^{-1}\|=\sum_{i,j=1}^n|a'_{ij}|$. Furthermore, set

$$Q_n^\theta = [-\theta,\theta]\times\dots\times[-\theta,\theta]\quad (n \text{ times}),$$
$$L_k^{\pm\theta} = \{(t_1,\dots,t_n)\in Q_n^\theta \mid t_k=\pm\theta\}.$$

For any $t\in Q_n^\theta$, $t=(t_1,\dots,t_n)$, consider the following $n\times n$ linear system



in the unknown $(\lambda_1,\dots,\lambda_n)$:

$$(4.11)\qquad \begin{cases} \sum_{j=1}^n\langle b_1-b_0,e_j\rangle\lambda_j = t_1,\\ \sum_{j=1}^n\langle b_2-b_0,e_j\rangle\lambda_j = t_2,\\ \dots\dots\dots\dots\\ \sum_{j=1}^n\langle b_n-b_0,e_j\rangle\lambda_j = t_n.\end{cases}$$

By Step 2, $\det A=(-1)^n\Delta\neq 0$. It follows that for each $t\in Q_n^\theta$ the system (4.11) has one and only one solution, say $(\lambda_1(t),\dots,\lambda_n(t))$, and

$$(4.12)\qquad |\lambda_j(t)|\leq\theta\|A^{-1}\|,\quad j=1,\dots,n.$$

Define now a map $w: Q_n^\theta\to\mathbb{E}$ by

$$w(t)=e_0+\sum_{i=1}^n\lambda_i(t)e_i.$$

Observe that, from (4.10) and (4.12), $\sum_{i=1}^n|\lambda_i(t)|\leq n\theta\|A^{-1}\|<\varrho$, thus

$$(4.13)\qquad w(t)\in U_{\mathbb{E}}(e_0,\varrho)\quad \text{for every } t\in Q_n^\theta.$$

STEP 4: Let $\eta$ and $\varrho$ be as in Step 3, and let $\theta$ satisfy (4.10); then there exists $0<\sigma<\varrho$ such that, for every $X\in U_{\mathcal{C}(\mathbb{E})}(B,\sigma)$ and $k=1,\dots,n$, we have

$$(4.14)\qquad \delta(X_0^\eta,w(t))-\delta(X_k^\eta,w(t))<0\quad \text{for every } t\in L_k^{-\theta},$$
$$(4.15)\qquad \delta(X_0^\eta,w(t))-\delta(X_k^\eta,w(t))>0\quad \text{for every } t\in L_k^{+\theta}.$$

Let $1\leq k\leq n$. We first prove that (4.14) holds when $X=B$. Let $t\in L_k^{-\theta}$. Since $\|w(t)-e_0\|<\varrho<\eta$, by Step 3(ii) we have

$$\delta(B_0^\eta,w(t))-\delta(B_k^\eta,w(t)) = \|b_0-w(t)\|-\|b_k-w(t)\|.$$

On the other hand,

$$\|b_0-w(t)\|^2-\|b_k-w(t)\|^2$$
$$= \left\|(b_0-e_0)-\sum_{i=1}^n\lambda_i(t)e_i\right\|^2 - \left\|(b_k-e_0)-\sum_{i=1}^n\lambda_i(t)e_i\right\|^2$$
$$= 2\left\langle b_k-b_0,\sum_{i=1}^n\lambda_i(t)e_i\right\rangle = 2\sum_{i=1}^n\langle b_k-b_0,e_i\rangle\lambda_i(t) = 2t_k = -2\theta,$$

by virtue of (4.11). This shows that $\delta(B_0^\eta,w(t))-\delta(B_k^\eta,w(t))<0$ for every $t\in L_k^{-\theta}$. Analogously one can prove that $\delta(B_0^\eta,w(t))-\delta(B_k^\eta,w(t))>0$ for every $t \in L_k^{+\theta}$. As $1 \le k \le n$ is arbitrary, both (4.14) and (4.15) are satisfied whenever $X = B$.

Consider now the general case. For $k = 1, \ldots, n$ the sets $L_k^{-\theta}$, $L_k^{+\theta}$ are compact, thus by a property of continuous functions, there is $\mu > 0$ (independent of $k$) such that for $k = 1, \ldots, n$ we have

$$(4.16)\qquad \delta(B_0^\eta, w(t)) - \delta(B_k^\eta, w(t)) < -\mu \quad \text{for every } t \in L_k^{-\theta},$$

$$(4.17)\qquad \delta(B_0^\eta, w(t)) - \delta(B_k^\eta, w(t)) > \mu \quad \text{for every } t \in L_k^{+\theta}.$$

But for $k = 0, 1, \ldots, n$ also the maps $X \to X_k^\eta$, $X \in \mathcal{C}(\mathbb{E})$, are continuous at $X = B$. Hence there exists $\sigma$ (independent of $k$), where

$$(4.18)\qquad 0 < \sigma < \min\{\varrho, \lambda/2\},$$

such that for all $X \in U_{\mathcal{C}(\mathbb{E})}(B, \sigma)$ we have $h(X_k^\eta, B_k^\eta) < \mu/2$, $k = 0, 1, \ldots, n$.

With this choice of $\sigma$, Step 4 is satisfied. In fact, let $X \in U_{\mathcal{C}(\mathbb{E})}(B, \sigma)$ and let $1 \le k \le n$. For every $t \in L_k^{-\theta}$, we have

$$\delta(X_0^\eta, w(t)) \le \delta(B_0^\eta, w(t)) + h(X_0^\eta, B_0^\eta) < \delta(B_0^\eta, w(t)) + \mu/2,$$
$$\delta(X_k^\eta, w(t)) \ge \delta(B_k^\eta, w(t)) - h(X_k^\eta, B_k^\eta) > \delta(B_k^\eta, w(t)) - \mu/2,$$

thus, in view of (4.16),

$$\delta(X_0^\eta, w(t)) - \delta(X_k^\eta, w(t)) < \delta(B_0^\eta, w(t)) - \delta(B_k^\eta, w(t)) + \mu < 0$$

and (4.14) is proved. By a similar argument, using (4.17), one can show (4.15), completing the proof of Step 4.

STEP 5: With $B$ as in Step 1 and $\sigma$ as in Step 4, satisfying (4.18), the statement of the lemma holds true. Clearly, $U_{\mathcal{C}(\mathbb{E})}(B, \sigma) \subset U_{\mathcal{C}(\mathbb{E})}(A_0, \lambda)$ as $h(B, A_0) < \lambda/2$ by (4.8), and $\sigma < \lambda/2$ by (4.18). Let $X \in U_{\mathcal{C}(\mathbb{E})}(B, \sigma)$.

We claim that

$$\widehat{M}_{n+1}(X) \cap U_{H_n}(e_0, r) \neq \emptyset.$$

By Step 4, for each $k = 1, \ldots, n$, the continuous function $t \mapsto \delta(X_0^\eta, w(t)) - \delta(X_k^\eta, w(t))$ defined for $t \in Q_n^\theta$ satisfies (4.14) and (4.15). By a theorem of Brouwer–Miranda ([2], [12]) there exists $\widehat{t} \in Q_n^\theta$ such that, setting $\widehat{w} = w(\widehat{t})$, we have

$$(4.19)\qquad \delta(X_k^\eta, \widehat{w}) = \delta(X_0^\eta, \widehat{w}) \quad k = 1, \ldots, n.$$

Observe that $\widehat{w} \in H_n$ and $\|\widehat{w} - e_0\| < \varrho$, by (4.13). As $\varrho < r$, it follows that

$$(4.20)\qquad \widehat{w} \in U_{H_n}(e_0, r).$$

It remains to show that $\widehat{w} \in \widehat{M}_{n+1}(X)$. Clearly, $X = X_0^\eta \cup \ldots \cup X_n^\eta \cup \widetilde{X}^\eta$, thus

$$\delta(X, \widehat{w}) = \max\{\delta(X_0^\eta, \widehat{w}), \ldots, \delta(X_n^\eta, \widehat{w}), \delta(\widetilde{X}^\eta, \widehat{w})\}.$$



Since $h(X, B) < \sigma < \varrho$ and $\|\widehat{w} - e_0\| < \varrho$, by Step 3(iii) we have $\delta(X_k^\eta, \widehat{w}) > \delta(\widetilde{X}^\eta, \widehat{w})$, $k = 0, 1, \ldots, n$. The latter inequality and (4.19) imply

$$\delta(X, \widehat{w}) = \delta(X_k^\eta, \widehat{w}), \quad k = 0, 1, \ldots, n.$$

Consequently, $q_X(\widehat{w}) \cap X_k^\eta \neq \emptyset$, $k = 0, 1, \ldots, n$, hence in each ball $\widetilde{U}_{\mathbb{E}}(b_k, \eta)$, $k = 0, 1, \ldots, n$, there are points of $q_X(\widehat{w})$. By Step 3(i), these $n+1$ balls are pairwise disjoint, therefore $\operatorname{card} q_X(\widehat{w}) \ge n+1$, proving that $\widehat{w} \in \widehat{M}_{n+1}(X)$. By (4.20), we have $\widehat{M}_{n+1}(X) \cap U_{\mathbb{E}}(e_0, r) \neq \emptyset$. Since $X \in U_{\mathcal{C}(\mathbb{E})}(B, \sigma)$ is arbitrary, the proof of Lemma 4.1 is complete.

For the next lemma we need to introduce some further notation.

Let $X$ be a nonempty subset of $\mathbb{E}$ and let $n \in \mathbb{N}$. Following [7], for any $(x_1, \ldots, x_n) \in \widetilde{\mathcal{P}}_n(X)$, where $\widetilde{\mathcal{P}}_n(X) = X \times \ldots \times X$ ($n$ times), define

$$m(x_1, \ldots, x_n) = \inf\{\|x_i - x_j\| \mid i, j = 1, \ldots, n,\ i \neq j\}.$$

Observe that $\operatorname{card} X \le n$ if and only if $m(x_1, \ldots, x_{n+1}) = 0$ for every $(x_1, \ldots, x_{n+1}) \in \widetilde{\mathcal{P}}_{n+1}(X)$.

Let $H_n$, $n \in \mathbb{N}$, be linear subspaces of $\mathbb{E}$ of dimension $n$. For $e \in H_n$, $r > 0$ and $p \in \mathbb{N}$, set

$$\mathcal{N}_{e,r,p}^{H_n} = \{X \in \mathcal{C}(\mathbb{E}) \mid X \text{ satisfies (j) and (jj)}\},$$

where (j) and (jj) are given by:

(j) there exists $e' \in \widetilde{U}_{H_n}(e, r)$ with $\operatorname{card} q_X(e') \ge n+2$;

(jj) there exists $(x_1, \ldots, x_{n+2}) \in \widetilde{\mathcal{P}}_{n+2}(q_X(e'))$ such that
$$m(x_1, \ldots, x_{n+2}) > 1/p.$$

LEMMA 4.2. *The set $\mathcal{N}_{e,r,p}^{H_n}$ is nowhere dense in $\mathcal{C}(\mathbb{E})$.*

Proof. The argument is adapted from [7].

Let $A_0 \in \mathcal{C}(\mathbb{E})$ and $\lambda > 0$. It suffices to show that there exist $C \in \mathcal{C}(\mathbb{E})$ and $\sigma > 0$ so that

$$(4.21)\qquad U_{\mathcal{C}(\mathbb{E})}(C, \sigma) \subset U_{\mathcal{C}(\mathbb{E})}(A_0, \lambda) \setminus \mathcal{N}_{e,r,p}^{H_n}.$$

$A_0$ is compact, thus there is $A \in \mathcal{P}_{n+k}(\mathbb{E})$, for some $k \in \mathbb{N}$, such that $h(A, A_0) < \lambda/4$. By Proposition 3.5, there exists $B \in \mathcal{P}_{n+k}(\mathbb{E})$, with $h(B, A) < \lambda/4$, such that for every $(b_{i_1}, \ldots, b_{i_{n+1}}) \in \mathcal{P}_{n+1}(B)$ properties (i) and (ii) of Proposition 3.5 are satisfied. As a consequence we have

$$(4.22)\qquad \operatorname{card} q_B(u) \le n+1 \quad \text{for every } u \in H_n.$$

Set $C = \overline{\operatorname{co}}\, B$, and observe that

$$h(C, A_0) \le h(\overline{\operatorname{co}}\, B, \overline{\operatorname{co}}\, A) + h(\overline{\operatorname{co}}\, A, A_0) \le h(B, A) + h(A, A_0) < \lambda/2.$$

On the other hand, the map $(u,X)\mapsto q_X(u)$ from $H_n\times\mathcal{C}(\mathbb{E})$ to $\mathcal{K}(\mathbb{E})$ is upper semicontinuous. Thus, for every $u\in\widetilde{U}_{H_n}(e,r)$ there is $\delta(u)>0$ such that

$$v\in U_{H_n}(u,\delta(u))\ \&\ X\in U_{\mathcal{C}(\mathbb{E})}(C,\delta(u))\Rightarrow q_X(v)\subset q_C(u)+\frac{1}{2p}U, \tag{4.23}$$

where $U=U_{\mathbb{E}}(0,1)$. Since $\widetilde{U}_{H_n}(e,r)$ is compact, there is a finite family of balls, $\{U_{H_n}(u_j,\delta(u_j))\}_{j=1}^s$ say, with centers $u_j\in H_n$, $j=1,\dots,n$, whose union covers $\widetilde{U}_{H_n}(e,r)$. Fix now

$$0<\sigma<\min\{\delta(u_1),\dots,\delta(u_s),\lambda/2\}.$$

With $C$ and $\sigma$ defined above, (4.21) is satisfied. In fact, it is evident that $U_{\mathcal{C}(\mathbb{E})}(C,\sigma)\subset U_{\mathcal{C}(\mathbb{E})}(A_0,\lambda)$, for $h(C,A_0)<\lambda/2$ and $\sigma<\lambda/2$. Further, let $X\in U_{\mathcal{C}(\mathbb{E})}(C,\sigma)$. If $\operatorname{card} q_X(e')\le n+1$ for every $e'\in\widetilde{U}_{H_n}(e,r)$, then (j) fails and hence $X\notin\mathcal{N}^{H_n}_{e,r,p}$. Suppose that there is $e'\in\widetilde{U}_{H_n}(e,r)$ such that $\operatorname{card} q_X(e')\ge n+2$, and let $e'\in U_{H_n}(e_j,\delta(u_j))$ for some $i\le j\le s$. Since $h(X,C)<\sigma<\delta(u_j)$, by virtue of (4.23) we have

$$q_X(e')\subset q_C(u_j)+\frac{1}{2p}U. \tag{4.24}$$

As $C=\overline{\mathrm{co}}\,B$ and $B$ is a finite subset of the Hilbert space $\mathbb{E}$, we have

$$q_C(u_j)=q_B(u_j).$$

In view of (4.22), there is a subset $(b_{i_1},\dots,b_{i_d})$ of $B$ of cardinality $1\le d\le n+1$ such that $q_C(u_j)=(b_{i_1},\dots,b_{i_d})$. Hence, by (4.24),

$$q_X(e')\subset\bigcup_{r=1}^{d}U_{\mathbb{E}}\Big(b_{i_r},\frac{1}{2p}\Big).$$

Let $(x_1,\dots,x_{n+2})\in\widetilde{\mathcal{P}}_{n+2}(q_X(e'))$. Since there are at most $n+1$ balls $U_{\mathbb{E}}(b_{i_r},1/(2p))$, at least one of them must contain two of the $x_{i_r}$'s. This implies $m(x_1,\dots,x_{n+2})<1/p$, hence (jj) fails and $X\notin\mathcal{N}^{H_n}_{e,r,p}$. Thus $U_{\mathcal{C}(\mathbb{E})}(C,\sigma)\cap\mathcal{N}^{H_n}_{e,r,p}=\emptyset$ for every $X\in U_{\mathcal{C}(\mathbb{E})}(C,\sigma)$, and (4.21) is satisfied. This completes the proof.

**5. Typical compact convex sets.** In this section we prove a geometric property of some compact convex subsets of $\mathbb{E}$. Namely, if $\mathbb{E}$ is separable, then for a typical $X\in\mathcal{C}(\mathbb{E})$ the multivalued locus $M_{n+1}(X)$ of $q_X$ of cardinality $n+1$ is dense in $\mathbb{E}$, for every $n\in\mathbb{N}$.

THEOREM 5.1. *Let $\mathbb{E}$ be a real infinite-dimensional separable Hilbert space. Let $n\in\mathbb{N}$. Then the set*

$$\mathcal{C}_{n+1}=\{X\in\mathcal{C}(\mathbb{E})\mid M_{n+1}(X)\text{ is dense in }\mathbb{E}\}$$

*is residual in $\mathcal{C}(\mathbb{E})$.*



Proof. Let $n\in\mathbb{N}$. Denote by $\mathcal{F}_n$ a countable family of $n$-dimensional linear subspaces $H\subset\mathbb{E}$ whose union is dense in $\mathbb{E}$. For $H\in\mathcal{F}_n$, set

$$\mathcal{M}^H_{n+1}=\{X\in\mathcal{C}(\mathbb{E})\mid M_{n+1}(X)\cap H\text{ is dense in }H\},$$
$$\widehat{\mathcal{M}}^H_{n+1}=\{X\in\mathcal{C}(\mathbb{E})\mid \widehat{M}_{n+1}(X)\cap H\text{ is dense in }H\}.$$

Denote by $E$ a countable subset of $H$ dense in $H$.

CLAIM 1. *The set $\widehat{\mathcal{M}}^H_{n+1}$ is residual in $\mathcal{C}(\mathbb{E})$.*

For $e\in E$ and $r\in Q^+$ set

$$\widehat{\mathcal{N}}^H_{e,r}=\{X\in\mathcal{C}(\mathbb{E})\mid \widehat{M}_{n+1}(X)\cap U_H(e,r)=\emptyset\}.$$

By Lemma 4.1, $\widehat{\mathcal{N}}^H_{e,r}$ is nowhere dense in $\mathcal{C}(\mathbb{E})$. Since

$$\mathcal{C}(\mathbb{E})\setminus\bigcup_{r\in Q^+}\bigcup_{e\in E}\widehat{\mathcal{N}}^H_{e,r}\subset\widehat{\mathcal{M}}^H_{n+1},$$

the set $\widehat{\mathcal{M}}^H_{n+1}$ is residual in $\mathcal{C}(\mathbb{E})$, proving Claim 1.

CLAIM 2. *The set $\mathcal{M}^H_{n+1}$ is residual in $\mathcal{C}(\mathbb{E})$.*

It suffices to show that

$$\widehat{\mathcal{M}}^H_{n+1}\cap\Big(\mathcal{C}(\mathbb{E})\setminus\bigcup_{p\in\mathbb{N}}\bigcup_{r\in Q^+}\bigcup_{e\in E}\mathcal{N}^H_{e,r,p}\Big)\subset\mathcal{M}^H_{n+1}, \tag{5.1}$$

because $\widehat{\mathcal{M}}^H_{n+1}$ is residual in $\mathcal{C}(\mathbb{E})$, by Claim 1, and $\mathcal{N}^h_{e,r,p}$ is nowhere dense in $\mathcal{C}(\mathbb{E})$, by Lemma 4.2.

Let $X$ be in the set on the left side of (5.1). Let $u\in H$ and $s>0$. Take $e\in E$ and $r>0$ so that $\widetilde{U}_H(e,r)\subset U_H(u,s)$. Since $X\in\widehat{\mathcal{M}}^H_{n+1}$, there is $e'\in\widetilde{U}_H(e,r)$ such that $\operatorname{card} q_X(e')\ge n+1$. We have

$$\operatorname{card} q_X(e')=n+1. \tag{5.2}$$

Suppose otherwise, that is, $\operatorname{card} q_X(e')\ge n+2$. Let $(x_1,\dots,x_{n+2})\in\widetilde{\mathcal{P}}_{n+2}(q_X(e'))$. Let $p\in\mathbb{N}$. As $X\notin\mathcal{N}^H_{e,r,p}$ we have $m(x_1,\dots,x_{n+2})\le 1/p$, which implies $\operatorname{card}(x_1,\dots,x_{n+2})\le n+1$, a contradiction. Therefore (5.2) is satisfied, and so $e'\in M_{n+1}(X)\cap H$. Since $e'\in U_H(u,s)$, $u\in H$ and $s>0$ are arbitrary, the set $M_{n+1}(X)\cap H$ is dense in $H$. Hence $X\in\mathcal{M}^H_{n+1}$. Thus (5.1) holds, and Claim 2 is proved.

CLAIM 3. *The set $\mathcal{C}_{n+1}$ is residual in $\mathcal{C}(\mathbb{E})$.*

It suffices to show that

$$\bigcap_{H\in\mathcal{F}_n}\mathcal{M}^H_{n+1}\subset\mathcal{C}_{n+1}, \tag{5.3}$$

because $\mathcal{F}_n$ is countable and $\mathcal{M}^H_{n+1}$ is residual in $\mathcal{C}(\mathbb{E})$, by Claim 2.

Let $X$ be in the set on the left side of (5.3). Let $u \in \mathbb{E}$ and $s > 0$. Take an $H \in \mathcal{F}_n$ which meets the open ball $U_{\mathbb{E}}(u, s)$. Since $X \in \mathcal{M}^H_{n+1}$, the set $M_{n+1}(X) \cap U_{\mathbb{E}}(u, s)$ is nonempty. It follows that $M_{n+1}(X)$ is dense in $\mathbb{E}$ and $X \in \mathcal{C}_{n+1}$, proving (5.3). Thus Claim 3 is satisfied, completing the proof of Theorem 5.1.

For $X \in \mathcal{C}(\mathbb{E})$ denote by $S(X)$ the single-valued locus of $q_X$, that is,

$$S(X) = \{e \in \mathbb{E} \mid \operatorname{card} q_X(e) = 1\}.$$

By results proved, in a more general setting, by Asplund [1] and Edelstein [10] (see Lau [15] and De Ville and Zizler [8] for generalizations), it follows that each $X \in \mathcal{C}(\mathbb{E})$ has a single-valued locus $S(X)$ which is residual in $\mathbb{E}$.

From Theorem 5.1 one has:

COROLLARY 5.1. *Let $\mathbb{E}$ be as in Theorem 5.1. A typical $X \in C(\mathbb{E})$ has the following property. For every $e \in \mathbb{E}$ there is a sequence $\{e_n\} \subset \mathbb{E}$, converging to $e$, such that*

$$\operatorname{card} q_X(e_n) = n + 1 \quad \textit{for every } n \in \mathbb{N}.$$

*If $e \in S(X)$, then $\lim_{n\to+\infty} h(q_X(e_n), q_X(e)) = 0$.*

Now we want to discuss some properties of the range of the metric antiprojection mapping $q_X$, with $X$ a typical compact convex subset of $\mathbb{E}$.

For $X \in \mathcal{K}(\mathbb{E})$ and $n \in \mathbb{N}$, $n \geq 2$, the sets

$$R_1(X) = \bigcup_{e \in S(X)} q_X(e), \qquad R_n(X) = \bigcup_{e \in M_n(X)} q_X(e)$$

are called, respectively, the *single-valued range of $q_X$* and *multivalued range of $q_X$ of cardinality $n$*.

THEOREM 5.2. *Let $\mathbb{E}$ be a real infinite-dimensional separable Hilbert space. Let $n \in \mathbb{N}$. Then the set*

$$\mathcal{R}_n = \{X \in \mathcal{C}(\mathbb{E}) \mid R_n(X) \textit{ is dense in } X\}$$

*is residual in $\mathcal{C}(\mathbb{E})$.*

Proof. Consider $n = 1$. For $k \in \mathbb{N}$, set

$$\mathcal{A}_k = \{X \in \mathcal{C}(\mathbb{E}) \mid h(R_1(X), X) < 1/k\}.$$

Here $h$ is the Hausdorff pseudometric in the space of nonempty bounded subsets of $\mathbb{E}$.

CLAIM. *The set* $\operatorname{int} \mathcal{A}_k$ *is dense in* $\mathcal{C}(\mathbb{E})$.

Let $A \in \mathcal{C}(\mathbb{E})$ and $\lambda > 0$. It suffices to prove that there exist $B \in \mathcal{C}(\mathbb{E})$ and $\sigma > 0$, with $U_{\mathcal{C}(\mathbb{E})}(B, \sigma) \subset U_{\mathcal{C}(\mathbb{E})}(A, \lambda)$, such that

$$h(R_1(X), X) < 1/k \tag{5.4}$$

for every $X \in U_{\mathcal{C}(\mathbb{E})}(B, \sigma)$.



To prove this, let $N = (a_1, \ldots, a_n)$ be a finite subset of $\mathbb{E}$ satisfying

$$h(N, A) < \theta \quad \text{where } 0 < \theta < 1/(4k). \tag{5.5}$$

Since $\dim \mathbb{E} = +\infty$, we can suppose that $N$ is also linearly independent. Set $B = \overline{\operatorname{co}}\, N$ and observe that $N = \exp B$, by the Krein–Milman theorem. It follows that for each $i = 1, \ldots, n$ there is $e_i \in \mathbb{E}$ such that

$$q_B(e_i) = a_i, \quad i = 1, \ldots, n.$$

Since the map $(e, X) \to q_X(e)$, from $\mathbb{E} \times \mathcal{C}(\mathbb{E})$ to $\mathcal{K}(\mathbb{E})$, is upper semicontinuous there is $0 < \sigma < \theta$ ($\sigma$ independent of $i$) such that, for each $i = 1, \ldots, n$,

$$e \in U_{\mathbb{E}}(e_i, \sigma) \;\&\; X \in U_{\mathcal{C}(\mathbb{E})}(B, \sigma) \Rightarrow q_X(e) \subset U_{\mathbb{E}}(a_i, \theta).$$

Let $X \in U_{\mathcal{C}(\mathbb{E})}(B, \sigma)$. For each $i = 1, \ldots, n$ take a point $\widehat{e}_i \in U_{\mathbb{E}}(e_i, \sigma) \cap S(X)$, which certainly exists, for $S(X)$ is dense in $\mathbb{E}$. Thus

$$q_X(\widehat{e}_i) \in U_{\mathbb{E}}(a_i, \theta), \quad i = 1, \ldots, n. \tag{5.6}$$

As $\sigma < \theta$ and $h(B, A) < \theta$, we have $X \subset B + \theta U \subset A + 2\theta U$ and, by (5.5), $X \subset N + 3\theta U$. In view of (5.6), it follows that

$$X \subset R_1(X) + 4\theta U,$$

which yields (5.4), because $\theta < 1/(4k)$ and $R_1(X) \subset X$. Since $X \in U_{\mathcal{C}(\mathbb{E})}(B, \sigma)$ is arbitrary, the Claim is proved.

We are ready to complete the proof of the theorem. For $n = 1$ the statement of the theorem is satisfied since

$$\bigcap_{k=1}^{+\infty} \mathcal{A}_k \subset \mathcal{R}_1,$$

and the set on the left side is residual in $\mathcal{C}(\mathbb{E})$, by the Claim.

Consider $n \geq 2$. Let $\mathcal{C}_n$ be as in Theorem 5.1. It suffices to show that

$$\mathcal{C}_n \cap \mathcal{R}_1 \subset \mathcal{R}_n, \tag{5.7}$$

since $\mathcal{R}_1$ is residual in $\mathcal{C}(\mathbb{E})$ and so is $\mathcal{C}_n$, by Theorem 5.1. To prove (5.7), let $X \in \mathcal{C}_n \cap \mathcal{R}_1$. Let $x \in X$ and $\varepsilon > 0$. As $X \in \mathcal{R}_1$, there exists $e_0 \in S(X)$ such that $q_X(e_0) \in U_{\mathbb{E}}(x, \varepsilon)$. But $q_X$ is upper semicontinuous at $e_0$ and $X \in \mathcal{C}_n$, hence close to $e_0$ there is $\widehat{e} \in M_n(X)$ such that $q_X(\widehat{e}) \subset U_{\mathbb{E}}(x, \varepsilon)$. Therefore $\mathcal{R}_n(X)$ is dense in $X$, thus $X \in \mathcal{R}_n$, proving (5.7). This completes the proof.

From Theorem 5.2 one has:

COROLLARY 5.2. *Let $\mathbb{E}$ be as in Theorem 5.2. A typical $X \in \mathcal{C}(\mathbb{E})$ has the following property. For every $x \in X$ there is a sequence $\{e_n\} \subset \mathbb{E}$ such that*

$$\operatorname{card} q_X(e_n) = n + 1 \quad \textit{and} \quad \lim_{n\to+\infty} h(q_X(e_n), x) = 0.$$

Observe that for every $n \in \mathbb{N}$ we have $\mathcal{R}_n(X) \subset \exp X \subset \operatorname{ext} X$, where the inclusions can be strict. In view of that, Theorem 5.2 is a slightly stronger formulation (in a separable Hilbert space) of a well-known result of Klee [14] stating that, for a typical compact convex set $X$ in an infinite-dimensional Banach space, the set $\operatorname{ext} X$ is dense in $X$. Examples of sets $X$ with this property can be found in Bohnenblust and Karlin [3] and in Poulsen [8].

**6. Typical compact sets.** It is evident that Theorem 5.1 fails for the metric projection mapping $p_X$, with $X \in \mathcal{C}(\mathbb{E})$, since in this case $p_X$ is always single-valued. But if we let $X \in \mathcal{K}(\mathbb{E})$ then one can prove for $p_X$ a result similar to Theorem 5.1. The proof is as in the case of the metric antiprojection mapping $q_X$.

For $X \in \mathcal{K}(\mathbb{E})$ and $n \in \mathbb{N}$, let

$$M_{n+1}(X) = \{e \in \mathbb{E} \mid \operatorname{card} p_X(e) = n+1\},$$
$$\widehat{M}_{n+1}(X) = \{e \in \mathbb{E} \mid \operatorname{card} p_X(e) \geq n+1\}$$

be the multivalued locus of $p_X$ of cardinality $n+1$ and of cardinality at least $n+1$, respectively.

Lemmas 6.1 and 6.2 below are proved as the corresponding Lemmas 4.1 and 4.2.

LEMMA 6.1. *The statement of Lemma* 4.1 *remains valid with* $\mathcal{C}(\mathbb{E})$ *replaced by* $\mathcal{K}(\mathbb{E})$ *and with* $\widehat{M}_{n+1}(X)$ *defined above.*

Define $\mathcal{N}_{e,r,p}^{H_n}$ as in Section 4, replacing $\mathcal{C}(\mathbb{E})$ by $\mathcal{K}(\mathbb{E})$ and $q_X$ by $p_X$.

LEMMA 6.2. $\mathcal{N}_{e,r,p}^{H_n}$ *is nowhere dense in* $\mathcal{K}(\mathbb{E})$.

By virtue of Lemmas 6.1 and 6.2, using the argument of Theorem 5.1 one can prove the following

THEOREM 6.1. *Let* $\mathbb{E}$ *be a real infinite-dimensional separable Hilbert space. Let* $n \in \mathbb{N}$. *Then the set*

$$\mathcal{K}_{n+1} = \{X \in \mathcal{K}(\mathbb{E}) \mid M_{n+1}(X) \text{ is dense in } \mathbb{E}\}$$

*is residual in* $\mathcal{K}(\mathbb{E})$.

For $X \in \mathcal{K}(\mathbb{E})$ let $S(X)$ be the single-valued locus of $p_X$, that is,

$$S(X) = \{e \in \mathbb{E} \mid \operatorname{card} p_X(e) = 1\}.$$

By a classical theorem proved (in a more general setting) by Stechkin [22] it follows that each $X \in \mathcal{K}(\mathbb{E})$ has a single-valued locus $S(X)$ which is residual in $\mathbb{E}$.

From Theorem 6.1 one has:

COROLLARY 6.1. *Let* $\mathbb{E}$ *be as in Theorem* 6.1. *A typical* $X \in \mathcal{K}(\mathbb{E})$ *has the following property. For every* $e \in \mathbb{E}$ *there is a sequence* $\{e_n\} \subset \mathbb{E}$, *converging to* $e$, *such that*

$$\operatorname{card} p_X(e_n) = n+1 \quad \text{for every } n \in \mathbb{N}.$$

*If* $e \in S(X)$ *then* $\lim_{n\to+\infty} h(p_X(e_n), p_X(e)) = 0$.



An account of the properties of single-valued loci for metric projection mappings and optimization problems can be found in Singer [21], Dontchev and Zolezzi [9] and Borwein and Fitzpatrick [4]. The first result concerning dense multivalued loci of metric projection mappings on compact subsets of $\mathbb{R}^d$, $d \geq 2$, is due to Zamfirescu [25]. Infinite-dimensional generalizations have recently been obtained by Zhivkov [27], [28] who also investigates dense multivalued loci with two-valued projections. Further topological properties of multivalued loci of metric projection mappings can be found in Bartke and Berens [2].

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*Received November 8, 1996*
*Revised version March 18, 1998* (3773)


STUDIA MATHEMATICA 135 (2) (1999)

# Spectrum for a solvable Lie algebra of operators


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**Abstract.** A new concept of spectrum for a solvable Lie algebra of operators is introduced, extending the Taylor spectrum for commuting tuples. This spectrum has the projection property on any Lie subalgebra and, for algebras of compact operators, it may be computed by means of a variant of the classical Ringrose theorem.


**0. Introduction.** A first non-commutative version of the Taylor spectrum of commuting tuples of operators ([17]) was studied in [7] for families of operators generating nilpotent Lie algebras. Independently, a spectral theory for solvable Lie algebras of operators was introduced in [3], [2]. Some complements to this theory are made in [10], [11] using the (Taylor type) spectrum $\sigma(\varrho)$, where $\varrho : E \to \mathcal{B}(\mathcal{X})$ is a representation of a Lie algebra $E$ on a Banach space $\mathcal{X}$.

One of the main results of [3] is the projection property of the spectrum on Lie ideals. As remarked in [3], this projection property does not hold on any subalgebra of the given solvable Lie algebra. However, [7] shows that in the case of nilpotent Lie algebras we have:

0.1. Proposition. *If $\varrho : E \to \mathcal{B}(\mathcal{X})$ is a representation of a nilpotent Lie algebra $E$ and $F$ is a Lie subalgebra of $E$, then*

$$\sigma(\varrho|_F) = \sigma(\varrho)|_F.$$

0.2. Corollary. *In the situation of Proposition* 0.1, *if we take an arbitrary element $e$ of $E$ and set $T = \varrho(e)$ then*

$$\sigma(T) = \{\lambda(e) \mid \lambda \in \sigma(\varrho)\}.$$

In the present paper we introduce a new concept of spectrum for a solvable Lie algebra of operators, which agrees in the case of a nilpotent Lie algebra with the spectrum from [7], [3]. Our spectrum has the projection property on *any* Lie subalgebra, exactly as in the nilpotent case (see Proposition 0.1 above), not only on Lie ideals. As a consequence, the spectrum of

1991 *Mathematics Subject Classification*: Primary 47A13; Secondary 17B30.