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Boundedness of Marcinkiewicz functions

by

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Abstract. The L^p boundedness ($1 < p < \infty$) of Littlewood–Paley's g-function, Lusin's S function, Littlewood–Paley's g_λ^*-functions, and the Marcinkiewicz function is well known. In a sense, one can regard the Marcinkiewicz function as a variant of Littlewood–Paley's g-function. In this note, we treat counterparts μ_S^ϱ and $\mu_\lambda^{*,\varrho}$ to S and g_λ^*. The definition of $\mu_S^\varrho(f)$ is as follows:

$$\mu_S^\varrho(f)(x) = \Bigg(\int\limits_{|y-x|<t} \bigg|\frac{1}{t^\varrho}\int\limits_{|z|\le t} \frac{\Omega(z)}{|z|^{n-\varrho}} f(y-z)\,dz\bigg|^2 \frac{dy\,dt}{t^{n+1}}\Bigg)^{1/2},$$

where $\Omega(x)$ is a homogeneous function of degree 0 and Lipschitz continuous of order β ($0 < \beta \le 1$) on the unit sphere S^{n-1}, and $\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0$. We show that if $\sigma = \operatorname{Re}\varrho > 0$, then μ_S^ϱ is L^p bounded for $\max(1, 2n/(n+2\sigma)) < p < \infty$, and for $0 < \varrho \le n/2$ and $1 \le p \le 2n/(n+2\varrho)$, L^p boundedness does not hold in general, in contrast to the case of the S function. Similar results hold for $\mu_\lambda^{*,\varrho}$. Their boundedness in the Campanato space $\mathcal{E}^{\alpha,p}$ is also considered.

0. Introduction. Littlewood and Paley introduced the well known functions g and g_λ^* in their work on Fourier series, and Lusin introduced the area S function in his work on the boundary values of analytic functions. In this connection, Marcinkiewicz considered the function $\mu(f)$ given by

$$\mu(f)(x) = \bigg(\int\limits_0^{2\pi} \frac{|F(x+t)+F(x-t)-2F(x)|^2}{t^3}\,dt\bigg)^{1/2}, \quad x \in [0, 2\pi],$$

where $F(x) = \int_0^x f(t)\,dt$. The Marcinkiewicz function $\mu(f)$ was introduced in order to give an analogue of the Littlewood–Paley g function without going into the interior of the unit disk for its definition.

Stein [8] defined n-dimensional generalizations of these functions, and showed analogous results. Stein's generalization of the Marcinkiewicz func-

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tion is as follows. Let Ω be a function on $\mathbb{R}^n$ which satisfies the following two conditions:

(i) Ω is homogeneous of degree 0 and continuous on the unit sphere S^{n-1}, and satisfies a Lipschitz condition of order β $(0<\beta\le 1)$ there, i.e.

$$|\Omega(x')-\Omega(y')|\le C|x'-y'|^{\beta},\qquad x',y'\in S^{n-1}.$$

(ii) $\int_{S^{n-1}}\Omega(x')\,d\sigma(x')=0$, where $d\sigma$ is the surface Lebesgue measure on S^{n-1}.

Here and hereafter, C denotes a positive constant which may vary on each occasion.

For a locally integrable function f on $\mathbb{R}^n$ and $t>0$, let $F_t(f,x)=F_t(x)$ be given by

$$F_t(x)=\frac{1}{t}\int_{|y|\le t}\frac{\Omega(y)}{|y|^{n-1}}f(x-y)\,dy,\qquad x\in\mathbb{R}^n,$$

and define now $\mu(f)(x)$ by

$$\mu(f)(x)=\Big(\int_0^\infty\frac{|F_t(x)|^2}{t}\,dt\Big)^{1/2}.$$

It is known that for $1<p<\infty$,

$$\|\mu(f)\|_p\le C_p\|f\|_p,$$

and when $p=1$,

$$\lambda|\{\mu(f)>\lambda\}|\le C\|f\|_1\quad\text{ for all }\lambda>0$$

(Stein [8] and Hörmander [4]). In their work on Marcinkiewicz integral, A. Torchinsky and S. Wang [13] introduced the Marcinkiewicz functions $\mu_S(f)$ and $\mu_\lambda^*(f)$ corresponding to the S function and g_λ^* functions:

$$\mu_S(f)(x)=\Big(\int_{\Gamma(x)}\frac{|F_t(y)|^2}{t^{n+1}}\,dy\,dt\Big)^{1/2},$$

where $\Gamma(x)=\{(t,y)\in\mathbb{R}_+^{n+1}:|x-y|<t\}$, and

$$\mu_\lambda^*(f)(x)=\Big(\int_{\mathbb{R}_+^{n+1}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\frac{|F_t(y)|^2}{t^{n+1}}\,dy\,dt\Big)^{1/2},\qquad\lambda>1,$$

respectively. They proved that for $2\le p<\infty$,

$$\|\mu_S(f)\|_p\le C_p\|\mu_\lambda^*(f)\|_p\le C_p'\|f\|_p.$$

We shall see that these operators are L^p bounded for $p>\max(1,2n/(n+2))$. However, as is noted later, if $n\ge 3$ and $1\le p\le 2n/(n+2)$ then the L^p boundedness does not hold for $\Omega(x)=\operatorname{sign}x_1/|x|$.

On the other hand, in connection with $\mu(f)$ a parametrized Marcinkiewicz function $\mu^\varrho(f)$ was considered by L. Hörmander [4, p. 136]. Its definition is

$$\mu^\varrho(f)(x)=\Big(\int_0^\infty\Big|\frac{1}{t^\varrho}\int_{|x-z|\le t}\frac{\Omega(x-z)}{|x-z|^{n-\varrho}}f(z)\,dz\Big|^2\frac{dt}{t}\Big)^{1/2},$$

and it is proved to be L^p bounded for $1<p<\infty$ and $\varrho>0$.

So, it is natural to consider parametrized $\mu_S(f)$ and $\mu_\lambda^*(f)$ and to study their L^p boundedness, even in the non-parametrized case for $1<p<2$. They are defined by

$$\mu_S^\varrho(f)(x)=\Big(\int_{\Gamma(x)}\frac{1}{t^{n+1}}\Big|\frac{1}{t^\varrho}\int_{|z|\le t}\frac{\Omega(z)}{|z|^{n-\varrho}}f(y-z)\,dz\Big|^2dy\,dt\Big)^{1/2},$$

$$\mu_\lambda^{*,\varrho}(f)(x)=\Big(\int_{\mathbb{R}_+^{n+1}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\frac{1}{t^{n+1}}\Big|\frac{1}{t^\varrho}\int_{|z|\le t}\frac{\Omega(z)}{|z|^{n-\varrho}}f(y-z)\,dz\Big|^2dy\,dt\Big)^{1/2},$$

where $\varrho\in\mathbb{C}$ with $\operatorname{Re}\varrho>0$.

Besides L^p boundedness, we shall also consider boundedness of these operators in Campanato spaces; the boundedness of Littlewood–Paley's g, g_λ^*, Lusin's S function, and Marcinkiewicz $\mu(f)$ in Campanato spaces is already sufficiently discussed [16, 15, 3, 7, 14]. For $1\le p<\infty$ and $-n/p\le\alpha<1$, the *Campanato space* $\mathcal{E}^{\alpha,p}$ is defined to be the set of functions for which

$$\|f\|_{\alpha,p}=\sup_{x_0\in\mathbb{R}^n}\sup_{Q}\frac{1}{|Q|^{\alpha/n}}\Big(\frac{1}{|Q|}\int_Q|f(x)-f_Q|^p\,dx\Big)^{1/p}<\infty,$$

where Q runs over all cubes with sides parallel to the coordinate axes and centered at x_0, and f_Q is the average of f over Q, $(1/|Q|)\int_Q f(t)\,dt$. It is known that for $0<\alpha<1$, $\mathcal{E}^{\alpha,p}=\mathrm{Lip}_\alpha$, the Banach space of Lipschitz continuous functions of exponent α, and the norms are equivalent. For $\alpha=0$, $\mathcal{E}^{\alpha,p}$ coincides with BMO, the space of functions of bounded mean oscillation, and if $\alpha<0$, then $\mathcal{E}^{\alpha,p}$ is equivalent to the Morrey space $L^{p,n+p\alpha}$. It is also easily checked that

$$\|f\|_{\alpha,p}\le C\sup_Q\inf_{a\in\mathbb{C}}|Q|^{-\alpha/n}\Big(|Q|^{-1}\int_Q|f(x)-a|^p\,dx\Big)^{1/p}\qquad(-p/n\le\alpha<1),$$

and hence these norms are equivalent.

Now we state our results. In the sequel, $\varrho=\sigma+i\tau$ denotes a complex number and $\sigma=\operatorname{Re}\varrho$, $\tau=\operatorname{Im}\varrho$.

THEOREM 1. *Let $\sigma > 0$ and $1 < p < \infty$. Then there exists a positive constant C such that*

$$\|\mu^{\varrho}(f)\|_p \le C\,\frac{p^2(1+\sigma^4)(1+|\tau|^2)}{(p-1)\sigma^2}\|f\|_p.$$

THEOREM 2. *Let $\sigma > 0$ and $2 \le p < \infty$. Then there exists a positive constant C such that*

$$\|\mu_S^{\varrho}(f)\|_p \le C\,\frac{p(1+\sigma^2)(1+|\tau|)}{\sigma}\|f\|_p.$$

THEOREM 3. *Let $\sigma > n/2$ and $1 < p < 2$. Then there exists a positive constant C such that*

$$\|\mu_S^{\varrho}(f)\|_p \le \frac{C}{p-1}\left(\frac{\sigma(1+|\tau|)}{2\sigma-n} + (1+\sigma^4)(1+|\tau|^2)\right)\|f\|_p.$$

THEOREM 4. *Let $0 < \sigma \le n/2$ and $2n/(n+2\sigma) < p < 2$. Then there exists a positive constant C such that*

$$\|\mu_S^{\varrho}(f)\|_p \le C\frac{1+|\tau|^2}{\sigma\left(\frac{n+2\sigma}{2n}-\frac{1}{p}\right)^2}\|f\|_p.$$

Similar results hold for $\mu_\lambda^{*,\varrho}(f)$:

THEOREM 5. *Let $\sigma > 0$ and $2 \le p < \infty$. Then there exists a positive constant C such that*

$$\|\mu_\lambda^{*,\varrho}(f)\|_p \le C\,\frac{p(1+\sigma^2)(1+|\tau|)}{\sigma}\|f\|_p.$$

THEOREM 6. *Let $\sigma > n/2$, $1 < p < 2$, and $\lambda > 2/p$. Then there exists a positive constant C such that*

$$\|\mu_\lambda^{*,\varrho}(f)\|_p \le \frac{C}{p-1}\left(\frac{\sigma(1+|\tau|)}{2\sigma-n} + (1+\sigma^4)(1+|\tau|^2)\right)\|f\|_p.$$

THEOREM 7. *Let $0 < \sigma \le n/2$, $2n/(n+2\sigma) < p < 2$, and $\lambda > 2/p$. Then there exists a positive constant C such that*

$$\|\mu_\lambda^{*,\varrho}(f)\|_p \le C\frac{1+|\tau|^2}{\sigma\left(\frac{n+2\sigma}{2n}-\frac{1}{p}\right)^2}\|f\|_p.$$

In the case where $0 < \sigma \le n/2$ and $1 \le p \le 2n/(n+2\sigma)$, we have

THEOREM 8. *Let $0 < \varrho \le n/2$. Then, for $1 \le p \le 2n/(n+2\varrho)$, there exists a kernel $\Omega(z)$ satisfying* (i) *and* (ii) *such that $\mu_S^{\varrho}(f)$ and $\mu_\lambda^{*,\varrho}(f)$ are not bounded on L^p.*

As for the boundedness in the Campanato spaces, we have similar results to the case of Littlewood–Paley functions.

THEOREM 9. *Let $\sigma > 0$, $1 < p < \infty$, and $-n/p \le \alpha < \beta \le 1$. If $f \in \mathcal{E}^{\alpha,p}$, then either $\mu^{\varrho}(f)(x) = \infty$ a.e., or $\mu^{\varrho}(f)(x) < \infty$ a.e. on $\mathbb{R}^n$. In the latter case, there is a constant C independent of f such that*

$$\|\mu^{\varrho}(f)\|_{\alpha,p} \le C\|f\|_{\alpha,p}.$$

THEOREM 10. *Let $\sigma = \varrho > 0$, $\max(1, 2n/(n+2\sigma)) < p < \infty$, and $-n/p \le \alpha < 1/2$ or $1/2 \le \alpha < \min(\beta,\sigma)$. If $f \in \mathcal{E}^{\alpha,p}$, then either $\mu_S^{\varrho}(f)(x) = \infty$ a.e., or $\mu_S^{\varrho}(f)(x) < \infty$ a.e. on $\mathbb{R}^n$. In the latter case, there is a constant C independent of f such that*

$$\|\mu_S^{\varrho}(f)\|_{\alpha,p} \le C\|f\|_{\alpha,p}.$$

THEOREM 11. *Let $\sigma > 0$, $\max(1, 2n/(n+2\sigma)) < p < \infty$, $\lambda > \lambda_0$, and $-n/p \le \alpha < 1/2$ or $1/2 \le \alpha < \min(\beta,\sigma)$. If $f \in \mathcal{E}^{\alpha,p}$, then either $\mu_\lambda^{*,\varrho}(f)(x) = \infty$ a.e., or $\mu_\lambda^{*,\varrho}(f)(x) < \infty$ a.e. on $\mathbb{R}^n$. In the latter case, there is a constant C independent of f such that*

$$\|\mu_\lambda^{*,\varrho}(f)\|_{\alpha,p} \le C\|f\|_{\alpha,p}.$$

Here $\lambda_0 = \max(1, 2/p)$ for $-n/p \le \alpha < 0$, $\lambda_0 = 1$ for $0 \le \alpha < 1/2$, and $\lambda_0 = 1 + 2/n$ for $1/2 \le \alpha < 1$.

In the next section we prepare several facts to prove our theorems. In Section 2, we prove Theorems 1, 2, 5 and 9. In Section 3, Theorems 3, 4, 6, 7 and 8 are proved. We treat Theorem 10 in Section 4, and Theorem 11 in the final Section 5.

1. Preliminaries. In this section we prepare some lemmas. Let $Mf(x)$ be the Hardy–Littlewood maximal function, i.e.,

$$Mf(x) = \sup_Q \frac{1}{|Q|}\int_Q |f(y)|\,dy,$$

where Q runs over all cubes with sides parallel to the coordinate axes and centered at x. Then

LEMMA 1.1 (Stein [10]). *Let $1 < p < \infty$ and $f \in L^p$. Then $Mf \in L^p$ and there exists a positive constant C_p such that*

$$\|Mf\|_p \le C_p\|f\|_p.$$

LEMMA 1.2 (Qiu [7]). *Let $f \in \mathcal{E}^{\alpha,p}$ $(1 < p < \infty,\ -n/p \le \alpha < 1\ (\alpha \ne 0))$ and $d > \max(0,\alpha)$. Then there exists a positive constant C such that for any cube Q with center x_0 and side length r,*

$$\int_{\mathbb{R}^n} \frac{|f(x)-f_Q|}{y^{n+d}+|x-x_0|^{n+d}}\,dx \le Cy^{-d}(y^{\alpha}+r^{\alpha})\|f\|_{\alpha,p}.$$

LEMMA 1.3 (Kurtz [6]). *Let $f \in \mathcal{E}^{\alpha,p}$ $(1 < p < \infty,\ \alpha = 0)$ and $d > 0$. Then there exists a positive constant C such that for any cube Q with center*

x_0 and side length r,

$$\int_{\mathbb{R}^n} \frac{|f(x)-f_Q|}{y^{n+d}+|x-x_0|^{n+d}}\,dx \le Cy^{-d}\Big(1+\Big|\log\frac{y}{r}\Big|\Big)\|f\|_{\alpha,p}.$$

We next give an elementary inequality.

LEMMA 1.4. *Let $0<a<1$ and $b\in\mathbb{R}$. Then for any $x,y\in\mathbb{R}^n$ we have*

$$||x|^a(|x|^{ib}-|x-y|^{ib})| \le (2\cdot 2^a + 2\log 2\cdot|b|)|y|^a. \tag{1.1}$$

Proof. If $|x|\le 2|y|$, we get

$$||x|^a(|x|^{ib}-|x-y|^{ib})| \le 2^a|y|^a\Big||x|^{ib}\Big(1-\frac{|x-y|^{ib}}{|x|^{ib}}\Big)\Big| \le 2\cdot 2^a|y|^a. \tag{1.2}$$

If $|x|>2|y|$, we have

$$1-|y|/|x| < |x-y|/|x| < 1+|y|/|x|.$$

Now, $0>\log(1-t)>-2\log 2\cdot t$ for $0<t<1/2$, and $0<\log(1+t)<t$ for $t>0$. Hence,

$$\Big|\log\frac{|x-y|}{|x|}\Big| < 2\log 2\frac{|y|}{|x|} \qquad (|x|>2|y|).$$

So, noting $|y|/|x|<1/2$ and $0<a<1$, we have

$$\begin{aligned} ||x|^{ib}-|x-y|^{ib}| &= |e^{ib\log|x|}-e^{ib\log|x-y|}| \le |1-e^{ib\log(|x-y|/|x|)}| \\ &\le \Big|b\log\frac{|x-y|}{|x|}\Big| < 2\log 2\cdot|b|\cdot\frac{|y|}{|x|} \\ &< 2\log 2\cdot|b|\Big(\frac{|y|}{|x|}\Big)^a. \end{aligned} \tag{1.3}$$

Therefore, in any case we have the desired inequality. ∎

LEMMA 1.5. *Let $a>0$, $0<\gamma\le 1$, and $b\in\mathbb{R}$. Then there exists a positive constant C such that for any $x,y\in\mathbb{R}^n$,*

$$\begin{aligned} \Big|\frac{1}{|x-y|^{a+ib}}-\frac{1}{|x|^{a+ib}}\Big| &\le C(1+|b|)\Big(\frac{|y|^\gamma}{|x|^a|x-y|^\gamma}+\frac{|y|^\gamma}{|x-y|^{a-[a]}|x|^{\gamma+[a]}}\Big) \\ &\quad + 2^{1-\gamma}\sum_{j=0}^{[a]-1}\Big\{\frac{|y|^\gamma}{|x-y|^{a-j}|x|^{j+\gamma}}+\frac{|y|^\gamma}{|x-y|^{a-j-1+\gamma}|x|^{j+1}}\Big\} \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} ||x-y|^{a+ib}-|x|^{a+ib}| &\le C(1+|b|)\Big(\frac{|x|^a|y|^\gamma}{|x-y|^\gamma}+\frac{|x-y|^{a-[a]}|x|^{[a]}|y|^\gamma}{|x|^\gamma}\Big) \\ &\quad + 2^{1-\gamma}\sum_{j=0}^{[a]-1}\Big\{\frac{|x-y|^{a-j}|x|^j|y|^\gamma}{|x|^\gamma}+\frac{|x-y|^{a-j-1}|x|^{j+1}|y|^\gamma}{|x-y|^\gamma}\Big\}. \end{aligned} \tag{1.4'}$$

Proof. We only prove (1.4); (1.4′) can be dealt with quite similarly and more easily. We have

$$\begin{aligned} \Big|\frac{1}{|x-y|^{a+ib}}-\frac{1}{|x|^{a+ib}}\Big| &\le \sum_{j=0}^{[a]-1}\Big|\frac{1}{|x-y|^{a+ib-j}|x|^j}-\frac{1}{|x-y|^{a-j-1+ib}|x|^{j+1}}\Big| \\ &\quad + \Big|\frac{1}{|x-y|^{a-[a]+ib}|x|^{[a]}}-\frac{1}{|x|^{a+ib}}\Big| \\ &\le \sum_{j=0}^{[a]-1}\frac{|y|}{|x-y|^{a-j}|x|^{j+1}}+\Big|\frac{|x|^{a-[a]+ib}-|x-y|^{a-[a]+ib}}{|x-y|^{a-[a]+ib}|x|^{a+ib}}\Big| \\ &\le \sum_{j=0}^{[a]-1}\frac{|y|}{|x-y|^{a-j}|x|^{j+1}}+\frac{|x|^{a-[a]}||x|^{ib}-|x-y|^{ib}|}{|x-y|^{a-[a]}|x|^a} \\ &\quad + \frac{||x|^{a-[a]}-|x-y|^{a-[a]}|}{|x-y|^{a-[a]}|x|^a} \\ &= I_1+I_2+I_3, \quad \text{say.} \end{aligned}$$

Now, if $|y|>2|x|$, we get $|y|/(2|y-x|)<1$, and hence, noting $0<\gamma<1$,

$$\frac{|y|}{2|y-x|} < \Big(\frac{|y|}{2|y-x|}\Big)^\gamma. \tag{1.5}$$

Similarly, if $|y|\le 2|x|$, we have

$$\frac{|y|}{2|x|} < \Big(\frac{|y|}{2|x|}\Big)^\gamma. \tag{1.6}$$

Hence

$$I_1 \le \sum_{j=0}^{a-1}\Big\{\frac{2^{1-\gamma}}{|x-y|^{a-j}|x|^j}\Big(\frac{|y|}{|x|}\Big)^\gamma+\frac{2^{1-\gamma}}{|x-y|^{a-j-1}|x|^{j+1}}\Big(\frac{|y|}{|x-y|}\Big)^\gamma\Big\}.$$

Using Lemma 1.1 we have

$$(1.7)\qquad I_2 \le (2^{\gamma+1} + 2\log 2\cdot|b|)\frac{|y|^\gamma}{|x-y|^{a-[a]}|x|^{\gamma+[a]}}.$$

As for I_3 we consider the following two cases.

(a) *The case* $a-[a]\le\gamma$. Using an elementary inequality $(1+t)^\alpha < 1+\alpha t^\gamma/\gamma$ for $0<\alpha<\gamma$ and $t>0$, we get

$$\begin{aligned}I_3 &\le \frac{1}{|x-y|^{a-[a]}|x|^a}\\ &\quad\times\left(\frac{a-[a]}{\gamma}|x|^{a-[a]}\left(\frac{|y|}{|x|}\right)^\gamma - \frac{a-[a]}{\gamma}|x-y|^{a-[a]}\left(\frac{|y|}{|x-y|}\right)^\gamma\right)\\ &\le \frac{a-[a]}{\gamma}\left(\frac{|y|^\gamma}{|x-y|^{a-[a]}|x|^{[a]+\gamma}} + \frac{|y|^\gamma}{|x-y|^\gamma|x|^a}\right).\end{aligned}$$

(b) *The case* $0<\gamma<a-[a]$. Similarly to the above case we have

$$I_3 \le \frac{|y|^{a-[a]}}{|x-y|^{a-[a]}|x|^a}.$$

Hence, as in the case of I_1 we get

$$I_3 \le 2^{a-[a]-\gamma}\left(\frac{|y|^\gamma}{|x-y|^{a-[a]}|x|^{[a]+\gamma}} + \frac{|y|^\gamma}{|x-y|^\gamma|x|^a}\right).\ \blacksquare$$

2. The case of μ^ϱ, and the L^p boundedness of μ_S^ϱ and $\mu_\lambda^{*,\varrho}$ $(p\ge 2)$. First, we discuss the L^2 boundedness of $\mu^\varrho(f)$.

LEMMA 2.1. *Let $1<p<\infty$, $n=1,2,\ldots$, $\varrho\in\mathbb{C}$, $\varrho=\sigma+i\tau$, $\sigma,\tau\in\mathbb{R}$, and $\sigma>0$. Then there exists a positive constant C such that*

$$(2.1)\qquad \|\mu^\varrho(f)\|_2 \le C\frac{(1+\sigma^2)(1+|\tau|)}{\sigma}\|f\|_2.$$

Proof. Using the Plancherel formula, we have

$$\begin{aligned}(2.2)\qquad \int\mu^\varrho(f)^2\,dx &= \int\int_0^\infty \frac{1}{t}\left|\frac{1}{t^{\sigma+i\tau}}\int_{|y|\le t}\frac{\Omega(y)}{|y|^{n-\sigma-i\tau}}f(x-y)\,dy\right|^2 dt\,dx\\ &= \int_0^\infty\frac{1}{t}\int\left|\frac{1}{t^{\sigma+i\tau}}\int_{|y|\le t}\frac{\Omega(y)}{|y|^{n-\sigma-i\tau}}f(x-y)\,dy\right|^2 dx\,dt\\ &= \int_0^\infty\frac{1}{t}\int|\widehat{f}(x)|^2\left|\frac{1}{t^{\sigma+i\tau}}\int_{|y|\le t}e^{-ix\cdot y}\frac{\Omega(y)}{|y|^{n-\sigma-i\tau}}\,dy\right|^2 dx\,dt\\ &= \int_0^\infty\frac{1}{t}\int|\widehat{f}(x)|^2|k_t^\varrho(x)|^2\,dx\,dt,\quad\text{say.}\end{aligned}$$

For $t\le 1/|x|$, using $\int_{|y|\le t}(\Omega(y)/|y|^{n-\sigma-i\tau})\,dy=0$, we get

$$\begin{aligned}(2.3)\qquad |k_t^\varrho(x)| &= \left|\frac{1}{t^{\sigma+i\tau}}\int_{|y|\le t}(e^{-ix\cdot y}-1)\frac{\Omega(y)}{|y|^{n-\sigma-i\tau}}\,dy\right|\\ &\le C\frac{|x|}{t^\sigma}\int_{|y|\le t}\frac{1}{|y|^{n-\sigma-1}}\,dy = C\frac{|x|}{t^\sigma}\int_0^t r^\sigma\,dr \le \frac{C}{\sigma+1}|x|t.\end{aligned}$$

For $t>1/|x|$, we put $\gamma=\frac12\min(1,1/\sigma)$. Then, putting $x=sx'$, $y=ry'$, we have

$$\begin{aligned}(2.4)\qquad k_t^\varrho(x) &= \int_{S^{n-1}}\frac{1}{t^{\sigma+i\tau}}\int_0^t e^{-is\cdot r\cos(x',y')}r^{\sigma-1+i\tau}\,dr\ \Omega(y')\,d\sigma(y')\\ &= \int_{S^{n-1}}P(y')\Omega(y')\,d\sigma(y'),\quad\text{say.}\end{aligned}$$

Then, setting $A=s\cos(x',y')$, we have

$$\begin{aligned}(2.5)\qquad P(y') &= \frac{1}{t^{\sigma+i\tau}}\int_0^1 e^{-iAtr}t^{\sigma-1+i\tau}r^{\sigma-1+i\tau}t\,dr = \int_0^1 e^{-iAtr}r^{\sigma-1+i\tau}\,dr\\ &= \int_0^1 e^{-iAtr}r^{\sigma-1}\cos(\tau\log r)\,dr + i\int_0^1 e^{-iAtr}r^{\sigma-1}\sin(\tau\log r)\,dr\\ &= P_1+P_2,\quad\text{say.}\end{aligned}$$

Let $b=1+3(\sigma+1)|\tau|$. Then by elementary calculation we see that the following four functions are non-increasing on the interval $(0,1]$.

$$\begin{gathered}(2(\sigma+1)r^{-1/2}+r^{\sigma-1})(b+\cos(\tau\log r)),\qquad 2(\sigma+1)r^{-1/2}(b+\cos(\tau\log r)),\\ b(2(\sigma+1)r^{-1/2}+r^{\sigma-1}),\qquad 2b(\sigma+1)r^{-1/2}.\end{gathered}$$

Hence we have

$$\begin{aligned}(2.6)\qquad |P_1| &= \Big|\int_0^1 e^{-iAtr}(2(\sigma+1)r^{-1/2}+r^{\sigma-1})(b+\cos(\tau\log r))\\ &\quad - 2(\sigma+1)r^{-1/2}(b+\cos(\tau\log r))\\ &\quad - b(2(\sigma+1)r^{-1/2}+r^{\sigma-1}) + 2b(\sigma+1)r^{-1/2})\,dr\Big|\\ &\le \left|\int_0^{\min(1,2\pi/(At))}(2(\sigma+1)r^{-1/2}+r^{\sigma-1})(b+\cos(\tau\log r))\,dr\right|\\ &\quad + \left|\int_0^{\min(1,2\pi/(At))}2(\sigma+1)r^{-1/2}(b+\cos(\tau\log r))\,dr\right|\end{aligned}$$

$$+\Big|\int_0^{\min(1,2\pi/(At))} b(2(\sigma+1)r^{-1/2}+r^{\sigma-1})\,dr\Big|$$
$$+\Big|\int_0^{\min(1,2\pi/(At))} 2b(\sigma+1)r^{-1/2}\,dr\Big|$$
$$\le \int_0^{(2\pi/|At|)^\gamma} (2(\sigma+1)r^{-1/2}+r^{\sigma-1})(1+b)\,dr$$
$$+\int_0^{(2\pi/|At|)^\gamma} 2(\sigma+1)r^{-1/2}(b+1)\,dr$$
$$+\int_0^{(2\pi/|At|)^\gamma} b(2(\sigma+1)r^{-1/2}+r^{\sigma-1})\,dr$$
$$+\int_0^{(2\pi/|At|)^\gamma} 2b(\sigma+1)r^{-1/2}\,dr$$
$$\le (1+b)\Big\{\frac{1}{\sigma}\Big(\frac{2\pi}{|A|t}\Big)^{\gamma\sigma}+4(\sigma+1)\Big(\frac{2\pi}{|A|t}\Big)^{\gamma/2}\Big\}$$
$$+4(\sigma+1)(1+b)\Big(\frac{2\pi}{|A|t}\Big)^{\gamma/2}$$
$$+b\Big\{\frac{1}{\sigma}\Big(\frac{2\pi}{|A|t}\Big)^{\gamma\sigma}+4(\sigma+1)\Big(\frac{2\pi}{|A|t}\Big)^{\gamma/2}\Big\}$$
$$+4b(\sigma+1)\Big(\frac{2\pi}{|A|t}\Big)^{\gamma/2}$$
$$\le C(1+b)\Big\{\frac{1}{\sigma}\Big(\frac{2\pi}{|A|t}\Big)^{\gamma\sigma}+4(\sigma+1)\Big(\frac{2\pi}{|A|t}\Big)^{\gamma/2}\Big\}.$$

Similarly we have

$$|P_2|\le C(1+b)\Big\{\frac{1}{\sigma}\Big(\frac{2\pi}{|A|t}\Big)^{\gamma\sigma}+4(\sigma+1)\Big(\frac{2\pi}{|A|t}\Big)^{\gamma/2}\Big\}. \tag{2.7}$$

Hence by (2.4)–(2.7) and the boundedness of $\Omega(x)$ we get

$$\begin{aligned}(2.8)\quad & |k_t^\varrho(x)|\\ &\le C\int_{S^{n-1}}(1+b)\Big\{\frac{1}{\sigma}\Big(\frac{2\pi}{|A|t}\Big)^{\gamma\sigma}+4(\sigma+1)\Big(\frac{2\pi}{|A|t}\Big)^{\gamma/2}\Big\}\,d\sigma(y')\\ &=C(1+b)\\ &\quad\times\int_{S^{n-1}}\Big(\frac{(2\pi)^{\gamma\sigma}}{\sigma|x|^{\gamma\sigma}|\cos(x',y')|^{\gamma\sigma}t^{\gamma\sigma}}+\frac{4(\sigma+1)(2\pi)^{\gamma/2}}{|x|^{\gamma/2}|\cos(x',y')|^{\gamma/2}t^{\gamma/2}}\Big)\,d\sigma(y')\\ &\le C\frac{1+b}{\sigma}\cdot\frac{(2\pi)^{\gamma\sigma}}{|x|^{\gamma\sigma}t^{\gamma\sigma}}+C(\sigma+1)(1+b)\frac{(2\pi)^{\gamma/2}}{|x|^{\gamma/2}t^{\gamma/2}}.\end{aligned}$$

Using (2.2), (2.3), (2.8) and noting $\gamma=\frac12\min(1,1/\sigma)$, we have

$$\begin{aligned}\int\mu^\varrho(f)^2\,dx&\le\int_0^{1/|x|}\frac1t\int|\widehat f(x)|^2\Big(\frac{C}{\sigma+1}|x|t\Big)^2\,dx\,dt\\ &\quad+\int_{1/|x|}^{\infty}\frac1t\int|\widehat f(x)|^2\\ &\quad\times\Big(C\frac{1+b}{\sigma}\cdot\frac{(2\pi)^{\gamma\sigma}}{|x|^{\gamma\sigma}t^{\gamma\sigma}}+C(1+b)\frac{(2\pi)^{\gamma/2}}{|x|^{\gamma/2}t^{\gamma/2}}\Big)^2\,dx\,dt\\ &\le\frac{C}{(\sigma+1)^2}\int|\widehat f(x)|^2|x|^2\int_0^{1/|x|}t\,dt\,dx\\ &\quad+C(2\pi)^{2\gamma\sigma}\frac{(1+b)^2}{\sigma^2}\int|\widehat f(x)|^2\frac{1}{|x|^{2\gamma\sigma}}\int_{1/|x|}^{\infty}\frac{1}{t^{2\gamma\sigma+1}}\,dt\,dx\\ &\quad+C(2\pi)^{\gamma}(1+b)^2\int|\widehat f(x)|^2\frac{1}{|x|^{\gamma}}\int_{1/|x|}^{\infty}\frac{1}{t^{\gamma+1}}\,dt\,dx\\ &\le C\frac{(1+\sigma^2)(1+(1+b)^2)}{\sigma^2}\|f\|_2^2\\ &\le C\frac{(1+\sigma^2)(1+(1+(\sigma+1)|\tau|)^2)}{\sigma^2}\|f\|_2^2.\end{aligned}$$

This implies (2.1). ∎

Now using the L^2 boundedness and reasoning as in Chanillo and Wheeden [2, pp. 280–281] or Torchinsky and Wang [13, pp. 241–242] we have

LEMMA 2.2. *Let $\lambda>1$. Then there exists a positive constant C such that*

$$\Big(\int_{\mathbb{R}^n}\mu_\lambda^{*,\varrho}(f)(x)^2\omega(x)\,dx\Big)^{1/2}\le C\frac{(1+\sigma^2)(1+|\tau|)}{\sigma}\Big(\int_{\mathbb{R}^n}|f(x)|^2M\omega(x)\,dx\Big)^{1/2}$$

for any non-negative and locally integrable function $\omega(x)$.

Proof of Theorem 5. Using Lemmas 2.2 and 1.1, and reasoning as in Stein [10, pp. 91–92] we have the desired result. ∎

Proof of Theorem 2. We can easily see that $\mu_S^\varrho(f)(x)\le 2^{\lambda n}\mu_\lambda^{*,\varrho}(f)(x)$, hence the conclusion of Theorem 2 follows from Theorem 5. ∎

Proof of Theorem 1. Let $k(t,x)=\Omega(x)|x|^{\varrho-n}t^{-\varrho}$ if $|x|\le t$, $=0$ if $|x|>t$, and consider the singular integral operator T defined by

$$Tf(x) = \int_{\mathbb{R}^n} k(t, x-y) f(y)\, dy$$

with values in $L^2((0,\infty), dt/t)$. Then $\mu^\varrho(f)(x) = (\int_0^\infty |Tf(x)|^2 dt/t)^{1/2}$. As is shown in Hörmander [4, p. 137],

$$\int_{|x|\ge 2|x-z|} \Big(\int_0^\infty |k(t,z) - k(t,x)|^2 \frac{dt}{t} \Big)^{1/2} dx \le C \frac{|\mathrm{Re}(\varrho - n)| + |\mathrm{Im}\, \varrho|}{\mathrm{Re}\, \varrho}.$$

Combining this with $L^2((0,\infty), dt/t)$ boundedness (Lemma 2.1) yields the desired conclusion (cf. the proof of Hörmander [4, Theorem 3.5] or Benedek, Calderón and Panzone [1, Theorem 2]). ■

Proof of Theorem 9. By the L^p boundedness of μ^ϱ (Theorem 1), the proof proceeds in quite a similar way to the case of Stein's Marcinkiewicz function $\mu(f) = \mu^1(f)$ (Han [3], Qiu [7] and Yabuta [14]). We omit the details. ■

3. L^p boundedness of μ_S^ϱ and $\mu_\lambda^{*,\varrho}$ $(1 \le p < 2)$. We begin with a lemma, concerning an estimate for the kernel $\Omega(x)$.

LEMMA 3.1. *Let $0 < \alpha \le 1$ and let $\Omega(x)$ be a homogeneous function of degree 0 on $\mathbb{R}^n$ with $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$. Suppose $\varrho \in \mathbb{C}$ with $\sigma = \mathrm{Re}\,\varrho > n/2$. Then for $\phi(x) = (\Omega(x)/|x|^{n-\varrho})\chi_{\{|x|\le 1\}}$ there exist $0 < \gamma < 1$ and $C > 0$ such that*

$$\int_{\mathbb{R}^n} |\phi(x-y) - \phi(x)|^2\, dx \le C \frac{\sigma(1+|\tau|)}{2\sigma - n} |y|^{2\gamma}, \quad y \in \mathbb{R}^n. \tag{3.1}$$

Proof. Since $\|\phi\|_2 = C/(2\sigma - n)$, we may assume $|y| < 1/2$. Further we set

$$\begin{aligned} I(y) &= \int_{\mathbb{R}^n} |\phi(x-y) - \phi(x)|^2\, dx \\ &= \int_{\{|x-y|\le 1,\, |x|>1\}} |\phi(x-y)|^2\, dx + \int_{\{|x-y|>1,\, |x|\le 1\}} |\phi(x)|^2\, dx \\ &\quad + \int_{\{|x-y|\le 1,\, |x|\le 1\}} |\phi(x-y) - \phi(x)|^2\, dx \\ &= I_1 + I_2 + I_3, \quad \text{say}. \end{aligned} \tag{3.2}$$

First, since $|y| < 1/2$, we have

$$\begin{aligned} I_1 &= \int_{\{1/2\le |x-y|\le 1,\, |x|>1\}} \frac{|\Omega(x-y)|^2}{|x-y|^{2(n-\sigma)}}\, dx \\ &\le C \int_{\{|x-y|\le 1,\, |x|>1\}} 2^{2(n-\sigma)}\, dx \le C|y|. \end{aligned} \tag{3.3}$$

Similarly we get

$$\begin{aligned} I_2 &= \int_{\{1/2\le |x|\le 1,\, |x-y|>1\}} \frac{|\Omega(x)|^2}{|x|^{2(n-\sigma)}}\, dx \\ &\le C \int_{\{1/2\le |x|\le 1,\, |x-y|>1\}} 2^{2(n-\sigma)}\, dx \le C|y|. \end{aligned} \tag{3.4}$$

We have

$$\begin{aligned} \Big| \frac{\Omega(x-y)}{|x-y|^{n-\varrho}} - \frac{\Omega(x)}{|x|^{n-\varrho}} \Big| &\le |\Omega(x-y)| \Big| \frac{1}{|x-y|^{n-\varrho}} - \frac{1}{|x|^{n-\varrho}} \Big| \\ &\quad + \frac{1}{|x|^{n-\sigma}} |\Omega(x-y) - \Omega(x)|. \end{aligned} \tag{3.5}$$

As for the second term in the above inequality, we have

$$\begin{aligned} |\Omega(x-y) - \Omega(x)| &\le C \Big| \frac{x-y}{|x-y|} - \frac{x}{|x|} \Big|^\alpha \le C 2^\alpha \min\Big(\frac{1}{|x-y|}, \frac{1}{|x|} \Big)^\alpha |y|^\alpha \\ &\le C|y|^\alpha \min\Big(\frac{1}{|x-y|^\alpha}, \frac{1}{|x|^\alpha} \Big). \end{aligned}$$

Now if $|y| > 2|x|$ and $0 < \gamma < \alpha$, we get $|y|/(2|y-x|) < 1$, and hence

$$\Big(\frac{|y|}{2|y-x|} \Big)^\alpha < \Big(\frac{|y|}{2|y-x|} \Big)^\gamma.$$

If $|y| \le 2|x|$ and $0 < \gamma < \alpha$, we get $|y|/(2|x|) < 1$, and hence

$$\Big(\frac{|y|}{2|x|} \Big)^\alpha < \Big(\frac{|y|}{2|x|} \Big)^\gamma.$$

Therefore

$$|\Omega(x-y) - \Omega(x)| \le c_\gamma \Big(\Big(\frac{|y|}{|x-y|} \Big)^\gamma + \Big(\frac{|y|}{|x|} \Big)^\gamma \Big). \tag{3.6}$$

We assume first $0 < \sigma < n$. We take $0 < \gamma < \min(\sigma - n/2, \alpha)$. Then, using (3.5), (3.6) and Lemma 1.5, we have

$$\begin{aligned} I_3 &\le C|y|^{2\gamma} \sum_{j=0}^{[n-\sigma]-1} \int_{\{|x-y|\le 1,\, |x|\le 1\}} \frac{1}{|x-y|^{2(n-\sigma-j)} |x|^{2(j+\gamma)}}\, dx \\ &\quad + C|y|^{2\gamma} \sum_{j=0}^{[n-\sigma]-1} \int_{\{|x-y|\le 1,\, |x|\le 1\}} \frac{1}{|x-y|^{2(n-\sigma-j-1+\gamma)} |x|^{2(j+1)}}\, dx \\ &\quad + C(1+|\tau|)|y|^{2\gamma} \int_{\{|x-y|\le 1,\, |x|\le 1\}} \frac{1}{|x|^{2(n-\sigma)} |x-y|^\gamma}\, dx \end{aligned}$$

$$+C(1+|\tau|)|y|^{2\gamma}\int_{\{|x-y|\le 1,\,|x|\le 1\}}\frac{1}{|x-y|^{2(n-\sigma-[n-\sigma])}|x|^{2([n-\sigma]+\gamma)}}\,dx$$

$$+C|y|^{2\gamma}\int_{\{|x-y|\le 1,\,|x|\le 1\}}\left\{\frac{1}{|x|^{n-\sigma}}\left(\frac{1}{|x-y|^{\gamma}}+\frac{1}{|x|^{\gamma}}\right)\right\}^2 dx$$

$$=C|y|^{2\gamma}(I_{3,1}+I_{3,2}+I_{3,3}+I_{3,4}+I_{3,5}).$$

Let $p_j=\frac{n-\sigma+\gamma}{n-\sigma-j}$ and $q_j=\frac{n-\sigma+\gamma}{j+\gamma}$. Then, since $\gamma<\min(\sigma-n/2,1)$, $p_j>1$ and $1/p_j+1/q_j=1$, by Hölder's inequality we have

$$I_{3,1}\le\sum_{j=0}^{[n-\sigma]-1}\left(\int_{\{|x-y|\le 1,\,|x|\le 1\}}\left(\frac{1}{|x-y|^{2(n-\sigma+\gamma)}}\right)^{1/p_j}\left(\frac{1}{|x|^{2(n-\sigma+\gamma)}}\right)^{1/q_j}dx\right)$$

$$\le\sum_{j=0}^{[n-\sigma]-1}\left(\int_0^1\frac{r^{n-1}}{r^{2(n-\sigma+\gamma)}}\,dr\right)^{1/p_j}\cdot\left(\int_0^1\frac{r^{n-1}}{r^{2(n-\sigma+\gamma)}}\,dr\right)^{1/q_j}$$

$$\le C\frac{|n-\sigma|}{-n+2(\sigma-\gamma)}\le C\frac{\sigma}{2\sigma-n}.$$

Similarly we have the same estimates for $I_{3,2}$, $I_{3,3}$, $I_{3,4}$ and $I_{3,5}$. Hence

$$I_3\le C\frac{\sigma(1+|\tau|)}{2\sigma-n}|y|^{2\gamma}. \tag{3.7}$$

Also in the case $\sigma\ge n$, we can obtain the same estimate for I_3 in quite a similar way. Summing up the estimates for I_1, I_2, I_3, we obtain the desired result. ∎

Proof of Theorem 3. Let γ be the constant given in Lemma 3.1, and $\phi(x)=(\Omega(x)/|x|^{n-\varrho})\chi_{\{|x|\le 1\}}$. Then by Lemma 3.1 we have

$$\int_{\mathbb{R}^n}|\phi(x-y)-\phi(x)|^2\,dx\le C\frac{\sigma(1+|\tau|)}{2\sigma-n}|y|^{2\gamma}. \tag{3.8}$$

By the assumption (ii) on Ω, we have

$$\int\phi(x)\,dx=\int\frac{\Omega(x)}{|x|^{n-\varrho}}\chi_{\{|x|\le 1\}}\,dx=0. \tag{3.9}$$

Since $\phi(x)=0$ $(|x|>1)$, we get

$$|\phi(x)|\le C/|x|^{n-\sigma}\le C/|x|^{n+\gamma}. \tag{3.10}$$

Using (3.8) and $\phi(x)=0$ $(|x|>1)$, we obtain

$$\int|\phi(x-y)-\phi(x)|\,dx\le C\frac{\sigma(1+|\tau|)}{2\sigma-n}|y|^{\gamma}. \tag{3.11}$$

Now $\mu_S^{\varrho}(f)$ can be rewritten as follows:

$$\mu_S^{\varrho}(f)(x)=\left(\int_{\Gamma(x)}\frac{1}{t^{n+1}}\left|\frac{1}{t^{\varrho}}\int_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\varrho}}f(z)\,dz\right|^2dy\,dt\right)^{1/2}$$

$$=\left(\int_0^{\infty}\int_{|y|<1}\frac{1}{t}\left|\frac{1}{t^n}\int\phi\left(\frac{x-z}{t}-y\right)f(z)\,dz\right|^2dy\,dt\right)^{1/2}.$$

We consider the Hilbert space $L^2((0,\infty)\times\{y\in\mathbb{R}^n:|y|\le 1\},dtdy/t)$ and the vector-valued singular integral with kernel $k(t,y,z)=t^{-n}\phi(z/t-y)$. We now check the following Hörmander condition (I):

$$\int_{|z|>2|x|}\left(\int_0^{\infty}\frac{1}{t^{2n+1}}\int_{|y|<1}\left|\phi\left(\frac{z-x}{t}-y\right)-\phi\left(\frac{z}{t}-y\right)\right|^2dy\,dt\right)^{1/2}dz\le C\frac{\sigma(1+|\tau|)}{2\sigma-n}. \tag{I}$$

We denote the left hand side of (I) by $I(x)$. Then, setting $\beta=\gamma/2$ and using Minkowski's inequality we have

(3.12) $I(x)$

$$\le\left(\int_{|z|>2|x|}|z|^{n+\beta}\int_0^{\infty}\frac{1}{t^{2n+1}}\int_{|y|<1}\left|\phi\left(\frac{z-x}{t}-y\right)-\phi\left(\frac{z}{t}-y\right)\right|^2dy\,dt\,dz\right)^{1/2}$$

$$\times\left(\int_{|z|>2|x|}|z|^{-n-\beta}\,dz\right)^{1/2}$$

$$=\left(\int_{|y|<1}\int_0^{\infty}\frac{1}{t^{2n+1}}\int_{|z|>2|x|}|z|^{n+\beta}\left|\phi\left(\frac{z-x}{t}-y\right)-\phi\left(\frac{z}{t}-y\right)\right|^2dz\,dt\,dy\right)^{1/2}$$

$$\times\left(\int_{2|x|}^{\infty}s^{-1-\beta}\,ds\right)^{1/2}$$

$$=J(x)^{1/2}\cdot C|x|^{-\beta/2}.$$

Now if $0<t<|x|/(2|y|)$, it follows that $|z|/t>2|x|/t>4|y|$, $|z-x|/t>(|z|-|x|)/t>|z|/(2t)>2|y|$, and hence we have

$$\left|\frac{z}{t}-y\right|>\frac{|z|}{t}-|y|>\frac{3|z|}{4t},\qquad\left|\frac{z-x}{t}-y\right|>\frac{|z|}{2t}-|y|>\frac{|z|}{4t}.$$

Thus, by (3.10) we get

$$\left|\phi\left(\frac{z}{t}-y\right)\right|\le C\left(\frac{t}{|z|}\right)^{n+\beta},\qquad\left|\phi\left(\frac{z-x}{t}-y\right)\right|\le C\left(\frac{t}{|z|}\right)^{n+\beta}. \tag{3.13}$$

Therefore

$$\text{(3.14)}\quad \int_0^{|x|/(2|y|)} \frac{1}{t^{2n+1}} \int_{|z|>2|x|} |z|^{n+\beta} \left| \phi\left(\frac{z-x}{t} - y\right) - \phi\left(\frac{z}{t} - y\right) \right|^2 dz\, dt$$

$$\le C \int_0^{|x|/(2|y|)} \frac{1}{t^{2n+1}} |z|^{n+\beta} \int_{|z|>2|x|} \left(\frac{t}{|z|}\right)^{n+\beta} \left| \phi\left(\frac{z-x}{t} - y\right) - \phi\left(\frac{z}{t} - y\right) \right| dz\, dt$$

$$\le C \int_0^{|x|/(2|y|)} t^{-n+\beta-1}$$

$$\times \left(\int_{|z|>2|x|} \left| \phi\left(\frac{z-x}{t} - y\right) \right| dz + \int_{|z|>2|x|} \left| \phi\left(\frac{z}{t} - y\right) \right| dz \right) dt$$

$$\le C \int_0^{|x|/(2|y|)} t^{-n+\beta-1} \cdot Ct^n \|\phi\|_1 \le C\|\phi\|_1 (|x|/|y|)^\beta.$$

Next we consider the case $t > |x|/(2|y|)$. If $|z| > 4t|y|$, we get

$$\left| \frac{z-x}{t} - y \right| > \frac{|x| - |z|}{t} - |y| > \frac{|z|}{4t}, \qquad \left| \frac{z}{t} - y \right| > \frac{|z|}{t} - |y| > \frac{|z|}{2t},$$

and so by (3.10), we obtain

$$\text{(3.15)}\quad \left| \phi\left(\frac{z}{t} - y\right) \right| \le C\left(\frac{t}{|z|}\right)^{n+\beta}, \qquad \left| \phi\left(\frac{z-x}{t} - y\right) \right| \le C\left(\frac{t}{|z|}\right)^{n+\beta}.$$

Thus, since $\beta = \gamma/2$, by (3.11) and (3.15) we have

$$\text{(3.16)}\quad \int_{|z|>4t|y|} |z|^{n+\beta} \left| \phi\left(\frac{z-x}{t} - y\right) - \phi\left(\frac{z}{t} - y\right) \right|^2 dz$$

$$\le C \int_{\mathbb{R}^n} |z|^{n+\beta} \left(\frac{t}{|z|}\right)^{n+\beta} \left| \phi\left(\frac{z-x}{t} - y\right) - \phi\left(\frac{z}{t} - y\right) \right| dz$$

$$= Ct^{n+\beta} \int_{\mathbb{R}^n} \left| \phi\left(\frac{z-x}{t} - y\right) - \phi\left(\frac{z}{t} - y\right) \right| dz$$

$$\le Ct^{2n+\beta} \cdot C\frac{\sigma(1+|\tau|)}{2\sigma - n} \left| \frac{x}{t} \right|^\gamma = C\frac{\sigma(1+|\tau|)}{2\sigma - n} t^{2n+\beta-\gamma} |x|^\gamma$$

$$= C\frac{\sigma(1+|\tau|)}{2\sigma - n} t^{2n-\beta} |x|^\gamma.$$

If $2|x| < |z| < 4t|y|$, then by (3.8) we have

$$\text{(3.17)}\quad \int_{2|x|<|z|<4t|y|} |z|^{n+\beta} \left| \phi\left(\frac{z-x}{t} - y\right) - \phi\left(\frac{z}{t} - y\right) \right|^2 dz$$

$$\le C(t|y|)^{n+\beta} \int_{\mathbb{R}^n} \left| \phi\left(\frac{z-x}{t} - y\right) - \phi\left(\frac{z}{t} - y\right) \right|^2 dz$$

$$\le C(t|y|)^{n+\beta} \int_{\mathbb{R}^n} t^n \left| \phi\left(z - \frac{x}{t} - y\right) - \phi(z - y) \right|^2 dz$$

$$\le Ct^{2n+\beta} |y|^{n+\beta} \frac{\sigma(1+|\tau|)}{2\sigma - n} \left| \frac{x}{t} \right|^{2\gamma}$$

$$= C\frac{\sigma(1+|\tau|)}{2\sigma - n} t^{2n-3\beta} |y|^{n+\beta} |x|^{2\gamma}.$$

Hence by (3.16) and (3.17) we have

$$\text{(3.18)}\quad \int_{|x|/(2|y|)}^\infty \frac{1}{t^{2n+1}} \int_{|z|>2|x|} |z|^{n+\beta} \left| \phi\left(\frac{z-x}{t} - y\right) - \phi\left(\frac{z}{t} - y\right) \right|^2 dz\, dt$$

$$\le \int_{|x|/(2|y|)}^\infty \frac{1}{t^{2n+1}} \cdot C\frac{\sigma(1+|\tau|)}{2\sigma - n} t^{2n-\beta} |x|^\gamma\, dt$$

$$+ \int_{|x|/(2|y|)}^\infty \frac{1}{t^{2n+1}} \cdot C\frac{\sigma(1+|\tau|)}{2\sigma - n} t^{2n-3\beta} |y|^{n+\beta} |x|^{2\gamma}\, dt$$

$$= C\frac{\sigma(1+|\tau|)}{2\sigma - n} |x|^\gamma \int_{|x|/(2|y|)}^\infty t^{-\beta-1}\, dt$$

$$+ C\frac{\sigma(1+|\tau|)}{2\sigma - n} |y|^{n+\beta} |x|^{2\gamma} \int_{|x|/(2|y|)}^\infty t^{-3\beta-1}\, dt$$

$$\le C\frac{\sigma(1+|\tau|)}{2\sigma - n} |x|^{\gamma/2} |y|^{\gamma/2} + C\frac{\sigma(1+|\tau|)}{2\sigma - n} |y|^{n+4\beta} |x|^\beta$$

$$= C\frac{\sigma(1+|\tau|)}{2\sigma - n} |x|^\beta |y|^\beta + C\frac{\sigma(1+|\tau|)}{2\sigma - n} |y|^{n+4\beta} |x|^\beta.$$

Therefore by (3.14) and (3.18) we get

$$\text{(3.19)}\quad J(x) \le C\frac{\sigma(1+|\tau|)}{2\sigma - n} \int_{|y|<1} \left(\frac{1}{|y|^\beta} + |y|^\beta + |y|^{n+4\beta} \right) |x|^\beta\, dy$$

$$\le C\frac{\sigma(1+|\tau|)}{2\sigma - n} |x|^\beta.$$

So, from (3.12) and (3.19) it follows that

$$I(x) \le J(x)^{1/2} \cdot C|x|^{-\beta/2} \le C\frac{\sigma(1+|\tau|)}{2\sigma - n}. \tag{3.20}$$

Thus we have shown the Hörmander condition (I).

Now combining this with $L^2((0,\infty)\times\{y\in\mathbb{R}^n : |y|\le 1\}, dt\,dy/t)$ boundedness (Theorem 2) yields the assertion of Theorem 3 (cf. Hörmander [4, Theorem 3.5]). ■

Proof of Theorem 4. Take $G(t,y)\in C_0^\infty(\Gamma(0))$ satisfying

$$\int\limits_{\Gamma(0)} |G(t,y)|^2 \frac{1}{t^{n+1}}\,dy\,dt \le 1.$$

For $z\in\mathbb{C}$ we set

$$\begin{aligned}(3.21)\quad & T_z(f)(x)\\ &= \int\limits_{\Gamma(0)} \frac{1}{t^{n(z+\eta)/2}} \int\limits_{|u|\le t} \frac{\Omega(u)}{|u|^{n-n(z+\eta)/2}} f(x-y-u)\,du\, G(t,y)\frac{1}{t^{n+1}}\,dy\,dt,\end{aligned}$$

where we take

$$\eta = 2\Big(1-\frac{\sigma}{n}\Big)\Big(\frac{2\sigma+n}{2n}-\frac{1}{p}\Big).$$

Note that $0<\eta<2$. We set $z_0+\eta = 2\varrho/n$. Then $0<z_0<1$. Further, we take $p_1>0$ so that $1/p=\frac{1}{2}(1-z_0)+z_0/p_1$. In this case we have $1<p_1<p$. Then we can check easily that

$$\frac{1}{p_1-1}\sim 1+\frac{2-p}{\frac{\sigma}{n}\big(\frac{2\sigma+n}{2n}-\frac{1}{p}\big)p}. \tag{3.22}$$

Now, for z with $0\le \operatorname{Re} z\le 1$, by Cauchy–Schwarz's inequality we have

$$\begin{aligned}(3.23)\quad & |T_z(f)(x)|\\ &\le \Bigg|\int\limits_{\Gamma(0)} \frac{1}{t^{n(z+\eta)/2}} \int\limits_{|u|\le t} \frac{\Omega(u)}{|u|^{n-n(z+\eta)/2}} f(x-y-u)\,du \frac{1}{t^{n+1}}\,dy\,dt\Bigg|\\ &\quad\times\Bigg|\int\limits_{\Gamma(0)} G(t,y)\frac{1}{t^{n+1}}\,dy\,dt\Bigg|\\ &\le \mu_S^{n(z+\eta)/2}(f)(x)\cdot \int\limits_{\Gamma(0)} |G(t,y)|\frac{1}{t^{n+1}}\,dy\,dt\\ &\le \mu_S^{n(z+\eta)/2}(f)(x)\cdot \Big(\int\limits_{\Gamma(0)} |G(t,y)|^2\frac{1}{t^{n+1}}\,dy\,dt\Big)^{1/2}\\ &= \mu_S^{n(z+\eta)/2}(f)(x)\|G(t,y)\|_{L^2(\Gamma(0),dy\,dt/t^{n+1})}.\end{aligned}$$

Since $0<n\eta/2\le \operatorname{Re}(n(z+\eta)/2)\le n(1+\eta)/2$ for $0\le\operatorname{Re} z\le 1$, it follows from Theorem 2 that

$$\|T_z(f)\|_{L^2(\mathbb{R}^n)} \le C\frac{1+|\operatorname{Im}(z)|}{\eta}\|f\|_{L^2(\mathbb{R}^n)}\|G\|_{L^2}. \tag{3.24}$$

And, since $0<\eta<2$, for z with $\operatorname{Re} z=1$ we get $n/2<\operatorname{Re}(n(z+\eta)/2) = n(1+\eta)/2<2n$. Thus, as $p_1>1$, by using Theorem 3, (3.22) and the definition of η we have

$$\begin{aligned}(3.25)\quad & \|T_{1+iy}(f)\|_{L^{p_1}(\mathbb{R}^n)}\\ &\le \frac{C}{p_1-1}\bigg\{\frac{1+|y|}{\eta}+\Big(1+\frac{n(1+\eta)}{2}\Big)^4(1+|y|^2)\bigg\}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|G\|_{L^2}\\ &\le C\frac{1+|y|^2}{\sigma\big(\frac{2\sigma+n}{2n}-\frac{1}{p}\big)^2}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|G\|_{L^2}.\end{aligned}$$

We can also see that $T_z(f)$ is an $L^2(\mathbb{R}^n)$-valued continuous function on $0\le\operatorname{Re} z\le 1$, and holomorphic in $0<\operatorname{Re} z<1$. Thus, using the complex interpolation theorem (cf. Stein and Weiss [11, Theorem 4.1]), we obtain

$$\|T_z(f)\|_{L^p} \le C\frac{1+|\operatorname{Im} z|^2}{\sigma\big(\frac{n+2\sigma}{2n}-\frac{1}{p}\big)^2}\|f\|_{L^p(\mathbb{R}^n)}\|G\|_{L^2}. \tag{3.26}$$

Hence we have

$$\|T_z(f)\|_{L^p} \le C\frac{1+|\operatorname{Im} z|^2}{\sigma\big(\frac{n+2\sigma}{2n}-\frac{1}{p}\big)^2}\|f\|_{L^p(\mathbb{R}^n)}. \tag{3.27}$$

Taking $z=z_0$, we get the assertion of Theorem 4. ■

Proofs of Theorems 6 and 7. Theorems 6 and 7 can be deduced from Theorems 3 and 4 respectively in the same way as in Torchinsky [13, pp. 314–318]. ■

Proof of Theorem 8. Set $\Omega(x)=x_1/|x|$ and $\gamma=(n+2\varrho)/2$. We first consider the case $p=2n/(n+2\varrho)$. Take $\varepsilon>0$ so that $1/p<\varepsilon<1/2+1/p$. Take

$$f(x)=\frac{1}{|x|^\gamma\log^\varepsilon\frac{1}{|x|}}\chi_{\{x_1<0,|x|<1\}}.$$

Then $f\in L^p(\mathbb{R}^n)$. We set

$$F_t(x)=\frac{1}{t^\varrho}\int\limits_{|z|\le t}\frac{\Omega(z)}{|z|^{n-\varrho}}f(x-z)\,dz.$$

Then for $0<4|x|<t<1/2$, $x_1>0$ we get

$$t^\varrho F_t^\varrho(x) = \int\limits_{\{|z|\le t,\ x_1<z_1,\ |x-z|<1\}} \frac{z_1}{|z|^{n-\varrho+1}|x-z|^\gamma\log^\varepsilon\frac{1}{|x-z|}}\,dz. \tag{3.28}$$

For $2|x|<|z|<t$, we get $|z|/2<|x-z|<|x|+|z|<\frac{3}{2}|z|<\frac{3}{2}t<3/4$, and hence

$$(3.29)\qquad t^{\varrho}F_t^{\varrho}(x) > C\int\limits_{\{2|x|<|z|<t,\,x_1<z_1\}}\frac{z_1}{|z|^{n-\varrho+1+\gamma}\log^{\varepsilon}\frac{2}{|z|}}\,dz$$

$$> C\int\limits_{2|x|<|z|<t}\frac{1}{|z|^{n-\varrho+\gamma}\log^{\varepsilon}\frac{2}{|z|}}\,dz$$

$$= C\int\limits_{2|x|}^{t}\frac{r^{n-1}}{r^{n-\varrho+\gamma}\log^{\varepsilon}\frac{2}{|r|}}\,dr > C\frac{1}{|x|^{\gamma-\varrho}\log^{\varepsilon}\frac{1}{|x|}}.$$

Suppose $x\in\mathbb{R}^n$ satisfies $x_1>0$, $|x|<1/8$. Then, if $4|x|<t<1/2$, for y with $y_1<x_1$, $|x-y|<t/4$ we have $4|x-y|<t$, $x_1-y_1>0$, $|y|<|y-x|+|x|<t/2$. Hence by (3.29) we get

$$(3.30)\qquad \int\limits_{|y|\le t}|F_t^{\varrho}(x-y)|^2\,dy$$

$$> C\frac{1}{t^{2\sigma}}\int\limits_{\{y_1<x_1,\,|x-y|<t/4\}}\frac{1}{|x-y|^{2(\gamma-\sigma)}\log^{2\varepsilon}\frac{1}{|x-y|}}\,dy$$

$$= \frac{C}{t^{2\sigma}}\int\limits_{\{y_1>0,\,|y|<t/4\}}\frac{1}{|y|^{n}\log^{2\varepsilon}\frac{1}{|y|}}\,dy$$

$$= \frac{C}{t^{2\sigma}}\int\limits_{0}^{t/4}\frac{1}{r\log^{2\varepsilon}\frac{1}{r}}\,dr = \frac{C}{t^{2\sigma}\log^{2\varepsilon-1}\frac{4}{t}}.$$

Therefore, for $x\in\mathbb{R}^n$ satisfying $x_1>0$, $|x|<1/8$, we have

$$(3.31)\qquad \mu_S^{\varrho}(f)(x) \ge C\biggl(\int\limits_{4|x|}^{1/2}\frac{1}{t^{n+2\sigma+1}\log^{2\varepsilon-1}\frac{1}{t}}\,dt\biggr)^{1/2}$$

$$\ge C\biggl(\frac{1}{|x|^{n+2\sigma}\log^{2\varepsilon-1}\frac{1}{|x|}}\biggr)^{1/2}$$

$$\ge \frac{C}{|x|^{(n+2\sigma)/2}\log^{\varepsilon-1/2}\frac{1}{|x|}}.$$

Since $\varepsilon<1/2+1/p$, we get $(\varepsilon-1/2)p<1$, and hence

$$(3.32)\qquad \int|\mu_S^{\varrho}(f)(x)|^p\,dx \ge C\int\limits_{\{x_1>0,\,|x|<1/8\}}\frac{1}{|x|^n\log^{(\varepsilon-1/2)p}\frac{1}{|x|}}\,dx=\infty.$$

This shows the statement in the case $p=2n/(n+2\varrho)$.

For $1\le p<2n/(n+2\varrho)$, taking Ω as above and $f(x)=|x|^{-\gamma}\chi_{\{x_1<0,|x|<1\}}$, we see in the same way that $f\in L^p$ and $\mu_S^{\varrho}(x)=\infty$ for x with $x_1>0$ and $|x|<1/4$.

Since $\mu_S^{\varrho}(x)<C\mu_\lambda^{*,\varrho}(f)(x)$, we have also shown the same results for $\mu_\lambda^{*,\varrho}(f)$. ■

4. Proof of Theorem 10. Let $f\in\mathcal{E}^{\alpha,p}$. Suppose $E=\{x\in\mathbb{R}^n:\mu_S^{\varrho}(f)(x)<\infty\}$ has positive measure. Let x_Q be a density point of E, and Q be a cube with center x_Q and side length r. Set

$$f=f_Q+(f-f_Q)\chi_Q+(f-f_Q)\chi_{Q^c}=f_1(x)+f_2(x)+f_3(x),\quad\text{say.}$$

Then $\mu_S^{\varrho}(f_1)(x)\equiv 0$, and by Theorems 2–4 we have $\|\mu_S^{\varrho}(f_2)\|_p\le C\|f_2\|_p=C(\int_Q|f-f_Q|^p\,dx)^{1/p}\le C\|f\|_{\alpha,p}|Q|^{1/p+\alpha/n}$. Hence, $\mu_S^{\varrho}(f_2)(x)<\infty$ a.e. on $\mathbb{R}^n$. So, $\mu_S^{\varrho}(f_2)(x)<\infty$ a.e. on E. Thus, there exists $x_0\in E\cap Q$ such that $\mu_S^{\varrho}(f_3)(x_0)\le\mu_S^{\varrho}(f)(x_0)+\mu_S^{\varrho}(f_2)(x_0)<\infty$. Therefore, as explained in Kurtz [6, pp. 660–661] or Qiu [7, p. 46], it suffices to show the following. For $\delta=1/(8\sqrt{n})$,

(i) if $\mu_S^{\varrho}(f)(x_0)<\infty$ for some $x_0\in\delta Q$, then $\mu_S^{\varrho}(f)(x)<\infty$ for all $x\in\delta Q$,

(ii) $|\mu_S^{\varrho}(f)(x)-\mu_S^{\varrho}(f)(x_0)|\le Cr^{\alpha}\|f\|_{\alpha,p}$ for all $x,x_0\in\delta Q$,

where δQ is the cube with the same center as Q and side length δr.

(I) *The case* $-n/p\le\alpha<0$. Suppose $x\in\delta Q$. We set

$$\Gamma(x)^-=\{(t,y)\in\mathbb{R}_+^{n+1}:|x-y|<t,\ 0<t<\delta r\},$$
$$\Gamma(x)^+=\{(t,y)\in\mathbb{R}_+^{n+1}:|x-y|<t,\ t\ge\delta r\}.$$

For $z\in Q^c$ and $(t,y)\in\Gamma(x)^-$, we get $|x-z|>\frac{r}{2}(1-\delta)$, and hence $|y-z|\ge|x-z|-|x-y|>\frac{r}{2}(1-\delta)-\delta r>\delta r>t$. Hence we have

$$(4.1)\qquad \mu_S^{\varrho}(f_3)(x)=\biggl(\int\limits_{\Gamma(x)^+}\biggl|\frac{1}{t^{\varrho}}\int\limits_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\varrho}}f_3(z)\,dz\biggr|^2\frac{dy\,dt}{t^{n+1}}\biggr)^{1/2}$$

$$\le\biggl(\int\limits_{\Gamma(x_0)}\biggl|\frac{1}{t^{\varrho}}\int\limits_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\varrho}}f_3(z)\,dz\biggr|^2\frac{dy\,dt}{t^{n+1}}\biggr)^{1/2}$$

$$+\biggl(\int\limits_{\Gamma(x)^+\setminus\Gamma(x_0)}\biggl|\frac{1}{t^{\varrho}}\int\limits_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\varrho}}f_3(z)\,dz\biggr|^2\frac{dy\,dt}{t^{n+1}}\biggr)^{1/2}$$

$$=\mu_S^{\varrho}(f_3)(x_0)+A,\quad\text{say.}$$

We note first that if $x\in\delta Q$, $(t,y)\in\Gamma(x)^+$ and $|y-z|<t$, then

$$|x_Q-z|\le|x-x_Q|+|x-y|+|y-z|\le\frac{\sqrt{n}}{2}t+t+t<3\sqrt{n}t.$$

(Ia) We suppose $0<\sigma<n$. Then, since $p>2n/(n+2\sigma)$, we can choose p_0 and a so that $1/p_0+1/p_0'=1$, $n/(2(n-\sigma))>a>1-n/((n-\sigma)p_0')$, $\min(2,p)\ge p_0\ge\max(1,2n/(n+2\sigma))$. Then, because of the boundedness of Ω, by Hölder's inequality we obtain

$$A^2\le C\int\limits_{\Gamma(x)+\setminus\Gamma(x_0)}\frac{1}{t^{n+1+2\sigma}}\Big(\int\limits_{|y-z|\le t}\frac{1}{|y-z|^{(n-\sigma)(1-a)p_0'}}\,dz\Big)^{2/p_0'}\times\Big(\int\limits_{|y-z|\le t}\frac{\chi_{\{|z-x_Q|<3\sqrt{n}t\}}|f_3(z)|^{p_0}}{|y-z|^{(n-\sigma)ap_0}}\,dz\Big)^{2/p_0}\,dy\,dt.$$

As is easily calculated, we have

$$\Big(\int\limits_{|y-z|\le t}\frac{1}{|y-z|^{(n-\sigma)(1-a)p_0'}}\,dz\Big)^{2/p_0'}=\Big(\int_0^t\frac{s^{n-1}}{s^{(n-\sigma)(1-a)p_0'}}\,ds\Big)^{2/p_0'}=Ct^{(2/p_0')\{n-(n-\sigma)(1-a)p_0'\}}.$$

Therefore, since $p_0\le 2$, by Minkowski's inequality we get

$$A^2\le C\int_{\delta r}^{\infty}t^{(2/p_0')\{n-(n-\sigma)(1-a)p_0'\}-n-1-2\sigma}\times\Big\{\int\limits_{|z-x_Q|<3\sqrt{n}t}|f_3(z)|^{p_0}\Big(\int\limits_{\substack{|y-z|\le t\\|x-y|\le t\\|x_0-y|>t}}\frac{1}{|y-z|^{2(n-\sigma)a}}\,dy\Big)^{p_0/2}dz\Big\}^{2/p_0}dt.$$

By geometric considerations we have easily

$$\begin{aligned}\int\limits_{\substack{|y-z|\le t\\|x-y|\le t\\|x_0-y|>t}}\frac{1}{|y-z|^{2(n-\sigma)a}}\,dy&\le C\int_0^{\delta r}\frac{s^{n-1}}{s^{2(n-\sigma)a}}\,ds+C\delta r\int_{\delta r}^{t}\frac{s^{n-2}}{s^{2(n-\sigma)a}}\,ds\\&\le C(\delta r)^{n-2(n-\sigma)a}+(\delta r)\{t^{n-1-2(n-\sigma)a}+(\delta r)^{n-1-2(n-\sigma)a}\}\\&\le C(r^{n-2(n-\sigma)a}+rt^{n-1-2(n-\sigma)a}).\end{aligned}$$

Note that in the one-dimensional case, we do not need the second term in the second, third and the last expression. Thus we get

$$\begin{aligned}A^2&\le C\int_{\delta r}^{\infty}t^{(2/p_0')\{n-(n-\sigma)(1-a)p_0'\}-n-1-2\sigma}(r^{n-2(n-\sigma)a}+rt^{n-1-2(n-\sigma)a})\times\Big\{\int\limits_{\{|z-x_Q|<3\sqrt{n}t\}}|f_3(z)|^{p_0}\,dz\Big\}^{2/p_0}dt\\&=C\int_{\delta r}^{\infty}t^{(2/p_0')\{n-(n-\sigma)(1-a)p_0'\}-n-1-2\sigma}(r^{n-2(n-\sigma)a}+rt^{n-1-2(n-\sigma)a})\times\Big(\int\limits_{\substack{\{|z-x_Q|<3\sqrt{n}t\}\\z\in Q^c}}|f(z)-f_Q|^{p_0}\,dz\Big)^{2/p_0}dt.\end{aligned}$$

Now, since $p\ge p_0$, we get

$$\begin{aligned}\Big(\int\limits_{\substack{|z-x_Q|<3\sqrt{n}t\\z\in Q^c}}|f(z)-f_Q|^{p_0}\,dz\Big)^{1/p_0}&\le Ct^{n/p_0}\cdot t^{-n/p}\Big(\int\limits_{\substack{|z-x_Q|<3\sqrt{n}t\\z\in Q^c}}|f(z)-f_Q|^{p}\,dz\Big)^{1/p}\\&\le Ct^{n/p_0-n/p}\Big(\int\limits_{\substack{|z-x_Q|<3\sqrt{n}t\\z\in Q^c}}|f(z)-f_{\{|z-x_Q|<3\sqrt{n}t\}}|^{p}\,dz\Big)^{1/p}\\&\quad+Ct^{n/p_0-n/p}\Big(\int\limits_{\substack{|z-x_Q|<3\sqrt{n}t\\z\in Q^c}}|f_{\{|z-x_Q|<3\sqrt{n}t\}}-f_Q|^{p}\,dz\Big)^{1/p}\\&\le Ct^{n/p_0-n/p}\Big(\int\limits_{|z-x_Q|<3\sqrt{n}t}|f(z)-f_{\{|z-x_Q|<3\sqrt{n}t\}}|^{p}\,dz\Big)^{1/p}\\&\quad+Ct^{n/p_0-n/p}\Big(\int\limits_{|z-x_Q|<3\sqrt{n}t}|f_{\{|z-x_Q|<3\sqrt{n}t\}}-f_Q|^{p}dz\Big)^{1/p}\\&\le Ct^{n/p_0-n/p}t^{n(1/p+\alpha/n)}\|f\|_{\alpha,p}+Ct^{n/p_0-n/p}t^{n/p}|f_{\{|z-x_Q|<3\sqrt{n}t\}}-f_Q|.\end{aligned}$$

In a way similar to Qiu [7, pp. 43–45], we can see that

$$|f_{\{|z-x_Q|<3\sqrt{n}t\}}-f_Q|\le C(t^\alpha+r^\alpha)\|f\|_{\alpha,p}.\tag{4.2}$$

Hence we get

$$\begin{aligned}\Big(\int\limits_{\substack{|z-x_Q|<3\sqrt{n}t\\z\in Q^c}}|f(z)-f_Q|^{p_0}\,dz\Big)^{2/p_0}&\le\{Ct^{n/p_0-n/p}t^{n/p+\alpha}\|f\|_{\alpha,p}+Ct^{n/p_0-n/p}t^{n/p}(t^\alpha+r^\alpha)\|f\|_{\alpha,p}\}^2\\&\le Ct^{2n/p_0+2\alpha}\|f\|^2_{\alpha,p}+Ct^{2n/p_0}r^{2\alpha}\|f\|^2_{\alpha,p}.\end{aligned}$$

Thus, noting $1/p_0 + 1/p_0' = 1$, $\alpha < 0$, $n/2(n-\sigma) > a$, we have

$$\begin{aligned}
A^2 &\le C \int_{\delta r}^{\infty} t^{(2/p_0')\{n-(n-\sigma)(1-a)p_0'\}-n-1-2\sigma} (r^{n-2(n-\sigma)a} + rt^{n-1-2(n-\sigma)a}) \\
&\quad \times (t^{2n/p_0+2\alpha} + t^{2n/p_0} r^{2\alpha}) \|f\|_{\alpha,p}^2 \, dt \\
&\le C\|f\|_{\alpha,p}^2 \int_{\delta r}^{\infty} t^{2n/p_0'-2(n-\sigma)(1-a)-n-1-2\sigma} r^{n-2(n-\sigma)a} t^{2n/p_0+2\alpha} \, dt \\
&\quad + C\|f\|_{\alpha,p}^2 \int_{\delta r}^{\infty} t^{2n/p_0'-2(n-\sigma)(1-a)-n-1-2\sigma} r^{n-2(n-\sigma)a} t^{2n/p_0} r^{2\alpha} \, dt \\
&\quad + C\|f\|_{\alpha,p}^2 \int_{\delta r}^{\infty} t^{2n/p_0'-2(n-\sigma)(1-a)-n-1-2\sigma} r t^{n-1-2(n-\sigma)a} t^{2n/p_0+2\alpha} \, dt \\
&\quad + C\|f\|_{\alpha,p}^2 \int_{\delta r}^{\infty} t^{2n/p_0'-2(n-\sigma)(1-a)-n-1-2\sigma} r t^{n-1-2(n-\sigma)a} t^{2n/p_0} r^{2\alpha} \, dt \\
&\le C r^{n-2(n-\sigma)a} \|f\|_{\alpha,p}^2 \int_{\delta r}^{\infty} t^{2(n-\sigma)a-n-1+2\alpha} \, dt \\
&\quad + C r^{n-2(n-\sigma)a+2\alpha} \|f\|_{\alpha,p}^2 \int_{\delta r}^{\infty} t^{2(n-\sigma)a-n-1} \, dt \\
&\quad + Cr\|f\|_{\alpha,p} \int_{\delta r}^{\infty} t^{2\alpha-2} \, dt + Cr^{2\alpha+1} \|f\|_{\alpha,p}^2 \int_{\delta r}^{\infty} t^{-2} \, dt \\
&\le Cr^{2\alpha} \|f\|_{\alpha,p}^2.
\end{aligned}$$

Therefore by (4.1) we have

$$\mu_S^{\varrho}(f_3)(x) \le \mu_S^{\varrho}(f_3)(x_0) + Cr^{\alpha}\|f\|_{\alpha,p}.$$

Changing the role of x and x_0, we have

$$|\mu_S^{\varrho}(f_3)(x) - \mu_S^{\varrho}(f_3)(x_0)| \le Cr^{\alpha}\|f\|_{\alpha,p}.$$

Thus we have proved (i) and (ii) in this case.

(Ib) We next suppose $\sigma \ge n$. In this case, we take $p_0 = 1$, $p_0' = \infty$, and $a = 0$. Then the reasoning in step (Ia) still works.

(II) *The case* $0 \le \alpha < 1/2$. In this case the $\mathcal{E}^{\alpha,p_1}$ norm is equivalent to the $\mathcal{E}^{\alpha,p_2}$ norm for $1 < p_1 < p_2$. For $0 < \sigma < n$ we take $a = 0$ and $p = p_0 > n/\sigma$. Then $(n-\sigma)p_0' < n$, and using $\int_{|x-y|<t,\,|x_0-y|>t} dy \le Crt^{n-1}$ we can argue in the same way as in step (Ia), without using Minkowski's inequality. For $\sigma \ge n$, we only have to take $p_0 = 1$, $p_0' = \infty$, and $a = 0$. However, in the case $\alpha = 0$, we use the following variant of (4.2) and modify the proof somewhat:

$$|f_{\{|z-x_Q|<3\sqrt{n}t\}} - f_Q| \le C\Big(1 + \log\frac{t}{\delta r}\Big)\|f\|_{0,p}. \tag{4.2$'$}$$

(III) *The case* $1/2 \le \alpha < \min(\beta,\sigma)$. In this case, since $\mathrm{Lip}_{\beta_1}(S^{n-1}) \subset \mathrm{Lip}_{\beta_2}(S^{n-1})$ $(0 < \beta_1 < \beta_2)$, we may assume $\alpha < \beta < \min(1,\sigma)$. Now

$$\begin{aligned}
&|\mu_S^{\varrho}(f_3)(x) - \mu_S^{\varrho}(f_3)(x_0)| \\
&\le \bigg(\iint_{\Gamma} \bigg| \frac{1}{t^{\varrho}} \int_{|x-y-z|\le t} \frac{\Omega(x-y-z)}{|x-y-z|^{n-\varrho}} f_3(z)\,dz \\
&\qquad - \frac{1}{t^{\varrho}} \int_{|x_0-y-z|\le t} \frac{\Omega(x_0-y-z)}{|x_0-y-z|^{n-\varrho}} f_3(z)\,dz \bigg|^2 \frac{dy\,dt}{t^{n+1}} \bigg)^{1/2} \\
&\le \bigg(\iint_{\Gamma} \bigg| \frac{1}{t^{\varrho}} \int_{\substack{|x-y-z|<t\\ |x_0-y-z|>t}} \frac{\Omega(x-y-z)}{|x-y-z|^{n-\varrho}} f_3(z)\,dz \bigg|^2 \frac{dy\,dt}{t^{n+1}} \bigg)^{1/2} \\
&\qquad + \bigg(\iint_{\Gamma} \bigg| \frac{1}{t^{\varrho}} \int_{\substack{|x-y-z|>t\\ |x_0-y-z|<t}} \frac{\Omega(x_0-y-z)}{|x_0-y-z|^{n-\varrho}} f_3(z)\,dz \bigg|^2 \frac{dy\,dt}{t^{n+1}} \bigg)^{1/2} \\
&\qquad + \bigg(\iint_{\Gamma} \bigg[\frac{1}{t^{\sigma}} \int_{\substack{|x-y-z|<t\\ |x_0-y-z|<t}} \bigg| \frac{\Omega(x-y-z)}{|x-y-z|^{n-\varrho}} \\
&\qquad - \frac{\Omega(x_0-y-z)}{|x_0-y-z|^{n-\varrho}} \bigg| |f_3(z)|\,dz \bigg]^2 \frac{dy\,dt}{t^{n+1}} \bigg)^{1/2} \\
&= I_1 + I_2 + I_3, \quad \text{say.}
\end{aligned} \tag{4.3}$$

If $x \in \delta Q$, $z \in Q^c$, $(t,y) \in \Gamma$ and $|x-y-z| < t$, we get $t > |x-y-z| > |x-z| - |y| > |x-z| - t > \frac{r}{2}(1-\delta) - t$, and hence $t > \frac{1}{4}(1-\delta)r > \delta r$. So, we may assume $t > \delta r$.

Since Ω is bounded, we have

$$I_1 \le \|\Omega\|_\infty \bigg(\int_{\delta r}^{\infty} \int_{|y|<t} \bigg| \frac{1}{t^{\sigma}} \int_{\substack{|x-y-z|<t\\ |x_0-y-z|>t}} \frac{|f(z)-f_Q|\chi_{Q^c}}{|x-y-z|^{n-\sigma}}\,dz \bigg|^2 \frac{dy\,dt}{t^{n+1}} \bigg)^{1/2}.$$

Let $z \in Q^c$, $|z-x+y| < t$ and $|y| < t$. Then, since $f \in \mathcal{E}^{\alpha,p} = \mathrm{Lip}_\alpha$,

$$|f(z) - f_Q| \le \frac{1}{|Q|} \int_Q |f(z) - f(u)|\,du \le \frac{1}{|Q|} \int_Q C|z-u|^{\alpha}\|f\|_{\alpha,p}\,du.$$

Since $z\in Q^c, u\in Q, x\in \delta Q$, we have $|z-u|\le|z-x+y|+|x-y-u|\le t+|x-u|+|y|\le t+\sqrt{n}r+t\le Ct$. Hence

$$|f(z)-f_Q|\le \frac{1}{|Q|}\int_Q (Ct)^\alpha\|f\|_{\alpha,p}\,du\le Ct^\alpha\|f\|_{\alpha,p}. \tag{4.4}$$

Since $t>|x-y-z|=|x_0-y-z-x_0+x|>|x_0-y-z|-|x_0-x|>t-r/8$, we see that

$$\begin{aligned}\int\limits_{\substack{|x-y-z|<t\\|x_0-y-z|>t}}\frac{1}{|x-y-z|^{n-\sigma}}\,dz &\le \int\limits_{t-r/8<|x-y-z|<t}\frac{1}{|x-y-z|^{n-\sigma}}\,dz\\ &=\int\limits_{\max(0,t-r/8)}^{t}\frac{s^{n-1}}{s^{n-\sigma}}\,ds\le C(t^\sigma-\max(0,t-r/8)^\sigma)\\ &\le\begin{cases}Crt^{\sigma-1} & \text{if } t>r/8,\\ Ct^\sigma & \text{if } \delta r<t\le r/8.\end{cases}\end{aligned} \tag{4.5}$$

Therefore by noting $0<\alpha<1$ we have

$$\begin{aligned}I_1^2&\le C\|f\|_{\alpha,p}^2\Big(\int_{\delta r}^{r/8}\frac{t^{2\alpha}}{t^{n+1+2\sigma}}\int\limits_{|y|<t}t^{2\sigma}\,dy\,dt\\ &\quad+\int_{r/8}^{\infty}\frac{t^{2\alpha}}{t^{n+1+2\sigma}}\int\limits_{|y|<t}(rt^{\sigma-1})^2\,dy\,dt\Big)\\ &\le C\|f\|_{\alpha,p}^2\Big(\int_{\delta r}^{r/8}t^{2\alpha-n-1-2\sigma+2\sigma+n}\,dt+r^2\int_{r/8}^{\infty}t^{2\alpha-n-1-2\sigma+2\sigma-2+n}\,dt\Big)\\ &=C\|f\|_{\alpha,p}^2\Big(\int_{\delta r}^{r/8}t^{2\alpha-1}\,dt+r^2\int_{r/8}^{\infty}t^{2\alpha-3}\,dt\Big)\le Cr^{2\alpha}\|f\|_{\alpha,p}^2.\end{aligned} \tag{4.6}$$

Similarly we get

$$I_2\le Cr^\alpha\|f\|_{\alpha,p}. \tag{4.7}$$

As for I_3, since as before $|f(z)-f_Q|\le Ct^\alpha\|f\|_{\alpha,p}$ and

$$\begin{aligned}\Big|\frac{\Omega(x-y-z)}{|x-y-z|^{n-\varrho}}-\frac{\Omega(x_0-y-z)}{|x_0-y-z|^{n-\varrho}}\Big| &\le|\Omega(x-y-z)|\Big|\frac{1}{|x-y-z|^{n-\varrho}}-\frac{1}{|x_0-y-z|^{n-\varrho}}\Big|\\ &\quad+\frac{1}{|x_0-y-z|^{n-\sigma}}|\Omega(x-y-z)-\Omega(x_0-y-z)|,\end{aligned}$$

I_3 is bounded by

$$\begin{aligned}&C\|f\|_{\alpha,p}\|\Omega\|_\infty\\ &\quad\times\Big(\int_\Gamma\Big[\frac{t^\alpha}{t^\sigma}\int\limits_{\substack{|x-y-z|<t\\|x_0-y-z|<t}}\Big|\frac{1}{|x-y-z|^{n-\varrho}}-\frac{1}{|x_0-y-z|^{n-\varrho}}\Big|\,dz\Big]^2\frac{dy\,dt}{t^{n+1}}\Big)^{1/2}\\ &\quad+C\|f\|_{\alpha,p}\\ &\quad\times\Big(\int_\Gamma\Big[\frac{t^\alpha}{t^\sigma}\int\limits_{\substack{|x-y-z|<t\\|x_0-y-z|<t}}\frac{|\Omega(x-y-z)-\Omega(x_0-y-z)|}{|x_0-y-z|^{n-\sigma}}\,dz\Big]^2\frac{dy\,dt}{t^{n+1}}\Big)^{1/2}\\ &=I_{3,1}+I_{3,2},\quad\text{say.}\end{aligned} \tag{4.8}$$

Since $\Omega\in \mathrm{Lip}_\beta(S^{n-1})$, we get

$$\begin{aligned}&|\Omega(x-y-z)-\Omega(x_0-y-z)|\\ &\quad\le C\Big|\frac{x-y-z}{|x-y-z|}-\frac{x_0-y-z}{|x_0-y-z|}\Big|^\beta\\ &\quad\le C\min\Big(\frac{1}{|x-y-z|},\frac{1}{|x_0-y-z|}\Big)^\beta|x-x_0|^\beta\\ &\quad\le Cr^\beta\min\Big(\frac{1}{|x-y-z|^\beta},\frac{1}{|x_0-y-z|^\beta}\Big),\end{aligned}$$

and hence using $\beta<\sigma$,

$$\begin{aligned}\int\limits_{\substack{|x-y-z|<t\\|x_0-y-z|<t}}\frac{|\Omega(x-y-z)-\Omega(x_0-y-z)|}{|x_0-y-z|^{n-\sigma}}\,dz &\le C\int\limits_{|x_0-y-z|<t}\frac{r^\beta}{|x_0-y-z|^{n-\sigma+\beta}}\,dz=C\int_0^t r^\beta\frac{s^{n-1}}{s^{n-\sigma+\beta}}\,ds\\ &\le Cr^\beta t^{\sigma-\beta}.\end{aligned} \tag{4.9}$$

Thus noting $\alpha<\beta$ we see that

$$\begin{aligned}I_{3,2}&\le C\|f\|_{\alpha,p}\Big(\int_{\delta r}^{\infty}\int\limits_{|y|<t}\frac{t^{2\alpha}}{t^{n+1+2\sigma}}|r^\beta t^{\sigma-\beta}|^2\,dy\,dt\Big)^{1/2}\\ &\le Cr^\beta\|f\|_{\alpha,p}\Big(\int_{\delta r}^{\infty}t^{2\alpha-n-1-2\sigma+2\sigma-2\beta+n}\,dt\Big)^{1/2}\\ &=Cr^\beta\|f\|_{\alpha,p}\Big(\int_{\delta r}^{\infty}t^{2\alpha-2\beta-1}\,dt\Big)^{1/2}\le Cr^\beta\|f\|_{\alpha,p}r^{\alpha-\beta}\\ &=Cr^\alpha\|f\|_{\alpha,p}.\end{aligned} \tag{4.10}$$

As for $I_{3,1}$, we first consider the case $0<\sigma<n$. By using Lemma 1.5 we choose $0<\gamma<1$ so that $\alpha<\gamma<\sigma$ and

$$\left|\frac{1}{|x-y-z|^{n-\varrho}}-\frac{1}{|x_0-y-z|^{n-\varrho}}\right|$$
$$\le C\left(\frac{|x-x_0|^\gamma}{|x_0-y-z|^{n-\sigma}|x-y-z|^\gamma}+\frac{|x-x_0|^\gamma}{|x-y-z|^{n-\sigma-[n-\sigma]}|x|^{\gamma+[n-\sigma]}}\right)$$
$$+C\sum_{j=0}^{[n-\sigma]-1}\frac{|x-x_0|^\gamma}{|x-y-z|^{n-\sigma-j}|x_0-y-z|^{j+\gamma}}$$
$$+C\sum_{j=0}^{[n-\sigma]-1}\frac{|x-x_0|^\gamma}{|x-y-z|^{n-\sigma-j-1+\gamma}|x_0-y-z|^{j+1}}.$$

Hence

$$(4.11)\qquad \int\limits_{\substack{|x-y-z|<t\\|x_0-y-z|<t}}\left|\frac{1}{|x-y-z|^{n-\varrho}}-\frac{1}{|x_0-y-z|^{n-\varrho}}\right|dz$$
$$\le C(\delta r)^\gamma\int\limits_{\substack{|x-y-z|<t\\|x_0-y-z|<t}}\frac{1}{|x_0-y-z|^{n-\sigma}|x-y-z|^\gamma}\,dz$$
$$+C(\delta r)^\gamma\int\limits_{\substack{|x-y-z|<t\\|x_0-y-z|<t}}\frac{1}{|x-y-z|^{n-\sigma-[n-\sigma]}|x|^{\gamma+[n-\sigma]}}\,dz$$
$$+C(\delta r)^\gamma\sum_{j=0}^{[n-\sigma]-1}\int\limits_{\substack{|x-y-z|<t\\|x_0-y-z|<t}}\frac{1}{|x-y-z|^{n-\sigma-j}|x_0-y-z|^{j+\gamma}}\,dz$$
$$+C(\delta r)^\gamma\sum_{j=1}^{[n-\sigma]}\int\limits_{\substack{|x-y-z|<t\\|x_0-y-z|<t}}\frac{1}{|x-y-z|^{n-\sigma-j+\gamma}|x_0-y-z|^{j}}\,dz.$$

Now, since $\sigma>\gamma$, letting $p=\frac{n-\sigma+\gamma}{n-\sigma-j}$, $q=\frac{n-\sigma+\gamma}{j+\gamma}$ we have by Hölder's inequality

$$\int\limits_{\substack{|x-y-z|<t\\|x_0-y-z|<t}}\frac{1}{|x-y-z|^{n-\sigma-j}|x_0-y-z|^{j+\gamma}}\,dz$$
$$\le\Big(\int\limits_{|x-y-z|<t}\frac{1}{|x-y-z|^{n-\sigma+\gamma}}\,dz\Big)^{1/p}$$
$$\times\Big(\int\limits_{|x_0-y-z|<t}\frac{1}{|x_0-y-z|^{n-\sigma+\gamma}}\,dz\Big)^{1/q}$$

$$\le C\Big(\int_0^t\frac{s^{n-1}}{s^{n-\sigma+\gamma}}\,ds\Big)^{1/p}\Big(\int_0^t\frac{s^{n-1}}{s^{n-\sigma+\gamma}}\,ds\Big)^{1/q}\le Ct^{\sigma-\gamma}.$$

Similar estimates hold for the other integrals in (4.11). Hence using $\alpha<\gamma$ we obtain

$$(4.12)\qquad I_{3,1}\le C\|f\|_{\alpha,p}\Big(\int_{\delta r}^\infty t^{2\alpha-n-1-2\sigma}\int\limits_{|y|<t}(\delta r)^{2\gamma}t^{2\sigma-2\gamma}\,dy\,dt\Big)^{1/2}$$
$$\le C(\delta r)^\gamma\|f\|_{\alpha,p}\Big(\int_{\delta r}^\infty t^{2\alpha-n-1-2\gamma}\int\limits_{|y|<t}dy\,dt\Big)^{1/2}$$
$$\le C(\delta r)^\gamma\|f\|_{\alpha,p}\Big(\int_{\delta r}^\infty t^{2\alpha-n-1-2\gamma+n}\,dt\Big)^{1/2}$$
$$\le C(\delta r)^\gamma\|f\|_{\alpha,p}(\delta r)^{\alpha-\gamma}=Cr^\alpha\|f\|_{\alpha,p}.$$

In the case $\sigma\ge n$, we can similarly obtain the same estimates for $I_{3,1}$. Hence, by (4.8), (4.9) and (4.12) we have $I_3\le Cr^\alpha\|f\|_{\alpha,p}$, and finally combining this with (4.3), (4.6) and (4.7) we get

$$|\mu_S^\varrho(f_3)(x)-\mu_S^\varrho(f_3)(x_0)|\le Cr^\alpha\|f\|_{\alpha,p}.\ \blacksquare$$

5. Proof of Theorem 11. Let $f\in\mathcal{E}^{\alpha,p}$. Suppose $E=\{x\in\mathbb{R}^n:\mu_\lambda^{*,\varrho}(f)(x)<\infty\}$ has positive measure. Let E, x_Q, r, Q be as in the proof of Theorem 10, and let f_1, f_2, f_3 be the same decomposition of f. Then, as before, we only have to show that for $\delta=1/(8\sqrt{n})$,

(i) if $\mu_\lambda^{*,\varrho}(f)(x_0)<\infty$ for some $x_0\in\delta Q$, then $\mu_\lambda^{*,\varrho}(f)(x)<\infty$ for all $x\in\delta Q$,

(ii) $|\mu_\lambda^{*,\varrho}(f)(x)-\mu_\lambda^{*,\varrho}(f)(x_0)|\le Cr^\alpha\|f\|_{\alpha,p}$ for all $x,x_0\in\delta Q$.

(I) *The case* $-n/p\le\alpha<1/2$. We first consider the case $\alpha\ne0$. Suppose $x\in\delta Q$. We set

$$J(k)=\{(t,y)\in\mathbb{R}_+^{n+1}:|x_Q-y|<2^{k-2}r,\ 0<t<2^{k-2}r\}\qquad(k\ge0).$$

Then, if $|y-z|\le t$, $z\in Q^c$, we have $(t,y)\notin J(0)$. In fact, if $(t,y)\in J(0)$, then $t\ge|y-z|\ge|x_Q-z|-|x_Q-y|\ge\frac12r-\frac14r=\frac14r$, a contradiction. Thus,

$$(5.1)\qquad \mu_\lambda^{*,\varrho}(f_3)(x)$$
$$=\Big(\int\limits_{\mathbb{R}_+^{n+1}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\Big|\frac{1}{t^\varrho}\int\limits_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\varrho}}f_3(z)\,dz\Big|^2\frac{dy\,dt}{t^{n+1}}\Big)^{1/2}$$
$$=\Big(\int\limits_{\mathbb{R}_+^{n+1}\setminus J(0)}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\Big|\frac{1}{t^\varrho}\int\limits_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\varrho}}f_3(z)\,dz\Big|^2\frac{dy\,dt}{t^{n+1}}\Big)^{1/2}$$

$$\le \Big(\int_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}\Big|\frac{1}{t^\varrho}\int_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\varrho}}f_3(z)\,dz\Big|^2\frac{dy\,dt}{t^{n+1}}\Big)^{1/2}$$

$$+\Big(\int_{\mathbb{R}^{n+1}_+\setminus J(0)}\Big|\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}-\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}\Big|$$

$$\times\Big|\frac{1}{t^\varrho}\int_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\varrho}}f_3(z)\,dz\Big|^2\frac{dy\,dt}{t^{n+1}}\Big)^{1/2}$$

$$=\mu_\lambda^{*,\varrho}(f_3)(x_0)+A,\quad\text{say.}$$

Now, if $(t,y)\in J(k)\setminus J(k-1)$, we see easily that $t+|x-y|\sim t+|x_0-y|\sim t+|x_Q-y|\sim 2^kr$. By the mean-value theorem we get

$$\Big|\Big(\frac{1}{t+|x-y|}\Big)^{\lambda n}-\Big(\frac{1}{t+|x_0-y|}\Big)^{\lambda n}\Big|\le C\frac{|x-x_0|}{(t+|x_Q-y|)^{\lambda n+1}}\le Cr(2^kr)^{-\lambda n-1}.$$

Thus, since Ω is bounded,

$$(5.2)\quad A\le\Big(\sum_{k=1}^{\infty}\int_{J(k)\setminus J(k-1)}\Big|\Big(\frac{1}{t+|x-y|}\Big)^{\lambda n}-\Big(\frac{1}{t+|x_0-y|}\Big)^{\lambda n}\Big|$$

$$\times t^{\lambda n}\Big|\frac{1}{t^\varrho}\int_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\varrho}}f_3(z)\,dz\Big|^2\frac{dy\,dt}{t^{n+1}}\Big)^{1/2}$$

$$\le C\Big(\sum_{k=1}^{\infty}r(2^kr)^{-\lambda n-1}$$

$$\times\int_{J(k)\setminus J(k-1)}t^{\lambda n}\Big|\frac{1}{t^\varrho}\int_{|y-z|\le t}\frac{1}{|y-z|^{n-\varrho}}f_3(z)\,dz\Big|^2\frac{dy\,dt}{t^{n+1}}\Big)^{1/2}$$

$$\le C\Big(\sum_{k=1}^{\infty}r(2^kr)^{-\lambda n-1}$$

$$\times\int_{J(k)\setminus J(k-1)}t^{\lambda n}\Big|\frac{1}{t^\varrho}\int_{|y-z|\le t}\frac{(f(z)-f_Q)\chi_{Q^c}}{|y-z|^{n-\varrho}}\,dz\Big|^2\frac{dy\,dt}{t^{n+1}}\Big)^{1/2}$$

$$\le C\Big[\sum_{k=1}^{\infty}r(2^kr)^{-\lambda n-1}$$

$$\times\Big(\int_{J(k)\setminus J(k-1)}t^{\lambda n}\Big|\frac{1}{t^\varrho}\int_{Q^c_{k+1}}\frac{(f(z)-f_Q)\chi_{\{|y-z|\le t\}}}{|y-z|^{n-\varrho}}\,dz\Big|^2\frac{dy\,dt}{t^{n+1}}$$

$$+\int_{J(k)\setminus J(k-1)}t^{\lambda n}\Big|\frac{1}{t^\varrho}\int_{Q_{k+1}\setminus Q}\frac{(f(z)-f_Q)\chi_{\{|y-z|\le t\}}}{|y-z|^{n-\varrho}}\,dz\Big|^2\frac{dy\,dt}{t^{n+1}}\Big)\Big]^{1/2}$$

$$=C\Big(\sum_{k=1}^{\infty}r(2^kr)^{-\lambda n-1}(A_k+B_k)\Big)^{1/2},\ \text{say,}$$

where Q_{k+1} is the cube with center x_Q and side length $2^{k+1}r$.

We note here that from this expression we may assume $1<\lambda<2$.

Now, we first treat A_k. For $(t,y)\in J(k)$, $z\notin Q_{k+1}$, we have $|y-z|\sim|x_Q-z|$. Also, $|x_Q-z|+2^kr\le|x_Q-z|+2^{k+1}r\le C|x_Q-z|$. Hence,

$$(5.3)\qquad |x_Q-z|+2^kr\sim|x_Q-z|\sim|y-z|.$$

Since $\lambda>1$, by (5.3) and Lemma 1.2 we have

$$(5.4)\quad A_k\le\int_{J(k)}t^{\lambda n}\Big|\frac{1}{t^\varrho}\int_{Q^c_{k+1}}\frac{(f(z)-f_Q)\chi_{\{|y-z|\le t\}}}{|y-z|^{n-\varrho}}\,dz\Big|^2\frac{dy\,dt}{t^{n+1}}$$

$$=C\int_{J(k)}t^{\lambda n+2}\Big|\int_{Q^c_{k+1}}\frac{(f(z)-f_Q)\chi_{\{|y-z|\le t\}}}{tt^\varrho|y-z|^{n-\varrho}}\,dz\Big|^2\frac{dy\,dt}{t^{n+1}}$$

$$\le C\int_{J(k)}t^{\lambda n+2}\Big|\int_{Q^c_{k+1}}\frac{(f(z)-f_Q)}{|y-z|^{n+1}}\,dz\Big|^2\frac{dy\,dt}{t^{n+1}}$$

$$\le C\int_{J(k)}t^{\lambda n+2}\Big|\int_{Q^c_{k+1}}\frac{(f(z)-f_Q)}{|x_Q-z|^{n+1}+(2^kr)^{n+1}}\,dz\Big|^2\frac{dy\,dt}{t^{n+1}}$$

$$\le C\int_0^{2^kr}\int_{|x_Q-y|<2^kr}t^{\lambda n+2}\Big(\frac{(2^kr)^\alpha+r^\alpha}{2^kr}\|f\|_{\alpha,p}\Big)^2\frac{dy\,dt}{t^{n+1}}$$

$$\le C(2^kr)^{2\alpha-2}\|f\|^2_{\alpha,p}\int_0^{2^kr}\int_{|x_Q-y|<2^kr}t^{\lambda n-n+1}\,dy\,dt$$

$$\le C(2^kr)^{2\alpha-2}\|f\|^2_{\alpha,p}(2^kr)^n\int_0^{2^kr}t^{\lambda n-n+1}\,dt$$

$$\le C(2^kr)^{2\alpha-2+n+\lambda n-n+2}\|f\|^2_{\alpha,p}\le C(2^kr)^{\lambda n+2\alpha}\|f\|^2_{\alpha,p}.$$

As for B_k, we consider the following two cases (Ia) and (Ib).

(Ia) *The case* $0<\sigma\le n/2$. By Minkowski's inequality we get

$$(5.5)\quad B_k\le\int_{J(k)}t^{\lambda n}\Big|\frac{1}{t^\varrho}\int_{Q_{k+1}\setminus Q}\frac{(f(z)-f_Q)\chi_{\{|y-z|\le t\}}}{|y-z|^{n-\varrho}}\,dz\Big|^2\frac{dy\,dt}{t^{n+1}}$$

$$\le\int_{\mathbb{R}^n}\Big[\int_{Q_{k+1}\setminus Q}\frac{|f(z)-f_Q|}{|y-z|^{n-\sigma}}$$

$$\times\Big(\int_0^{2^{k-2}r} t^{\lambda n-n-1}\chi_{\{|y-z|<t\}}t^{-2\sigma}\,dt\Big)^{1/2}dz\Big]^2 dy$$

$$= \int_{\mathbb{R}^n}\Big[\int_{Q_{k+1}\setminus Q}\frac{|f(z)-f_Q|}{|y-z|^{n-\sigma}}\Big(\int_{|y-z|}^{2^{k-2}r} t^{\lambda n-n-1-2\sigma}\,dt\Big)^{1/2}dz\Big]^2 dy$$

$$\le C\int_{\mathbb{R}^n}\Big(\int_{Q_{k+1}\setminus Q}\frac{|f(z)-f_Q|}{|y-z|^{n-\sigma}}$$

$$\times\big(|y-z|^{(\lambda n-n-2\sigma)/2}+(2^{k-2}r)^{(\lambda n-n-2\sigma)/2}\big)\,dz\Big)^2 dy$$

$$\le C(2^{k-2}r)^{\lambda n-n-2\sigma}\int_{\mathbb{R}^n}\Big(\int_{Q_{k+1}\setminus Q}\frac{|f(z)-f_Q|}{|y-z|^{n-\sigma}}\,dz\Big)^2 dy$$

$$+C\int_{\mathbb{R}^n}\Big(\int_{Q_{k+1}\setminus Q}|f(z)-f_Q|\cdot|y-z|^{-n+n(\lambda-1)/2}\,dz\Big)^2 dy$$

$$= C(2^{k-2}r)^{\lambda n-n-2\sigma}\cdot B_{k,1}+B_{k,2},\quad \text{say.}$$

If we set $\gamma=\sigma$, $p_0=2$, $1/q=1/2+\gamma/n=(n+2\sigma)/(2n)$, then we get $0\le\gamma<n, 1\le q\le p_0<\infty$. Hence using the Hardy–Littlewood–Sobolev Theorem for fractional integration, we have

$$B_{k,1}\le C\Big(\int_{Q_{k+1}\setminus Q}|f(z)-f_Q|^q\,dt\Big)^{2/q}.$$

Since

$$|f_{Q_j}-f_{Q_{j-1}}|\le\frac{1}{|Q_{j-1}|}\int_{Q_{j-1}}|f(x)-f_{Q_j}|\,dx\le C\frac{1}{|Q_j|}\int_{Q_j}|f(x)-f_{Q_j}|\,dx$$

$$\le C\frac{1}{|Q_j|}\Big(\int_{Q_j}|f(x)-f_{Q_j}|^p\,dx\Big)^{1/p}|Q_j|^{1-1/p}$$

$$\le C|Q_j|^{-1/p}|Q_j|^{(1+\alpha p/n)/p}\|f\|_{\alpha,p}$$

$$\le C|Q_j|^{\alpha/n}\|f\|_{\alpha,p}\le C(2^jr)^{\alpha}\|f\|_{\alpha,p},$$

we have

$$|f_{Q_{k+1}}-f_Q|\le\sum_{j=0}^{k+1}|f_{Q_j}-f_{Q_{j-1}}|\le C\|f\|_{\alpha,p}r^{\alpha}\sum_{j=0}^{k+1}2^{j\alpha}$$

$$\le C((2^kr)^{\alpha}+r^{\alpha})\|f\|_{\alpha,p}.$$

Hence, using $p>2n/(n+2\sigma)=q$, we have

$$\Big(\int_{Q_{k+1}\setminus Q}|f(z)-f_Q|^q\,dt\Big)^{1/q}=\Big(\int_{Q_{k+1}\setminus Q}|f(z)-f_{Q_{k+1}}+f_{Q_{k+1}}-f_Q|^q\,dt\Big)^{1/q}$$

$$\le\Big(\int_{Q_{k+1}\setminus Q}|f(z)-f_{Q_{k+1}}|^q\,dt\Big)^{1/q}+\Big(\int_{Q_{k+1}\setminus Q}|f_{Q_{k+1}}-f_Q|^q\,dt\Big)^{1/q}$$

$$\le\Big(\int_{Q_{k+1}}|f(z)-f_{Q_{k+1}}|^q\,dt\Big)^{1/q}+\Big(\int_{Q_{k+1}}|f_{Q_{k+1}}-f_Q|^q\,dt\Big)^{1/q}$$

$$\le|Q_{k+1}|^{1/q-1/p}\Big(\int_{Q_{k+1}}|f(z)-f_{Q_{k+1}}|^p\,dt\Big)^{1/p}+|Q_{k+1}|^{1/q}|f_{Q_{k+1}}-f_Q|$$

$$\le C(2^kr)^{n(1/q-1/p)}(2^kr)^{n(1+\alpha p/n)/p}\|f\|_{\alpha,p}+C(2^kr)^{n/q}((2^kr)^{\alpha}+r^{\alpha})\|f\|_{\alpha,p}$$

$$\le C(2^kr)^{n/q}r^{\alpha}((2^k)^{\alpha}+1)\|f\|_{\alpha,p}$$

$$=C(2^kr)^{(n+2\sigma)/2}r^{\alpha}((2^k)^{\alpha}+1)\|f\|_{\alpha,p}.$$

Thus, we get

$$B_{k,1}\le C(2^kr)^{n+2\sigma}r^{2\alpha}((2^k)^{2\alpha}+1)\|f\|^2_{\alpha,p}. \tag{5.6}$$

Next, we treat $B_{k,2}$. We set $\gamma=n(\lambda-1)/2$, $p_0=2$, $1/q=1/2+(\lambda-1)/2=\lambda/2$. Then, since $1<\lambda<2$, we get $0<\gamma<n$, $1\le q\le p_0<\infty$, and again by the Hardy–Littlewood–Sobolev Theorem for fractional integration we have

$$B_{k,2}\le C\Big(\int_{Q_{k+1}\setminus Q}|f(z)-f_Q|^q\,dt\Big)^{2/q}.$$

As above, because $p>2/\lambda=q$ we obtain

$$\Big(\int_{Q_{k+1}\setminus Q}|f(z)-f_Q|^q\,dt\Big)^{1/q}\le C(2^kr)^{n/q}r^{\alpha}((2^k)^{\alpha}+1)\|f\|_{\alpha,p}$$

$$=C(2^kr)^{\lambda n/2}r^{\alpha}((2^k)^{\alpha}+1)\|f\|_{\alpha,p}.$$

Hence

$$B_{k,2}\le C(2^kr)^{\lambda n}r^{2\alpha}((2^k)^{2\alpha}+1)\|f\|^2_{\alpha,p}. \tag{5.7}$$

Therefore by (5.5)–(5.7) we have

$$(5.8)\qquad \begin{aligned} B_k &\le C(2^k r)^{\lambda n-n-2\sigma}\cdot C(2^k r)^{n+2\sigma} r^{2\alpha}((2^k)^{2\alpha}+1)\|f\|^2_{\alpha,p}\\ &\quad + C(2^k r)^{\lambda n} r^{2\alpha}((2^k)^{2\alpha}+1)\|f\|^2_{\alpha,p}\\ &\le C(2^k r)^{\lambda n} r^{2\alpha}((2^k r)^{2\alpha}+1)\|f\|^2_{\alpha,p}. \end{aligned}$$

(Ib) *The case* $\sigma > n/2$. Since $1 < \lambda < 2 < 1 + 2\sigma/n$, using Minkowski's inequality we get

$$\begin{aligned} B_k &\le \int\limits_{J(k)} t^{\lambda n}\bigg|\frac{1}{t^\varrho}\int\limits_{Q_{k+1}\setminus Q}\frac{(f(z)-f_Q)\chi_{\{|y-z|\le t\}}}{|y-z|^{n-\varrho}}\,dz\bigg|^2\frac{dy\,dt}{t^{n+1}}\\ &\le \int\limits_{\mathbb{R}^n}\bigg[\int\limits_{Q_{k+1}\setminus Q}\frac{|f(z)-f_Q|}{|y-z|^{n-\sigma}}\Big(\int\limits_0^\infty t^{\lambda n-n-1}\chi_{\{|y-z|<t\}}t^{-2\sigma}\,dt\Big)^{1/2}dz\bigg]^2 dy\\ &\le \int\limits_{\mathbb{R}^n}\bigg[\int\limits_{Q_{k+1}\setminus Q}\frac{|f(z)-f_Q|}{|y-z|^{n-\sigma}}\Big(\int\limits_{|y-z|}^\infty t^{\lambda n-n-1-2\sigma}\,dt\Big)^{1/2}dz\bigg]^2 dy\\ &\le C\int\limits_{\mathbb{R}^n}\bigg[\int\limits_{Q_{k+1}\setminus Q}\frac{|f(z)-f_Q|}{|y-z|^{n-\sigma}}|y-z|^{(\lambda n-n-2\sigma)/2}\,dz\bigg]^2 dy\\ &= C\int\limits_{\mathbb{R}^n}\Big(\int\limits_{Q_{k+1}\setminus Q}|f(z)-f_Q|\cdot|y-z|^{-n+n(\lambda-1)/2}\,dz\Big)^2 dy. \end{aligned}$$

Hence, as in case (Ia) for $B_{k,2}$, we get

$$(5.9)\qquad B_k \le C(2^k r)^{\lambda n} r^{2\alpha}((2^k)^{2\alpha}+1)\|f\|^2_{\alpha,p}.$$

In any case, by (5.4), (5.8) and (5.9) we have

$$(5.10)\qquad \begin{aligned} A_k + B_k &\le C(2^k r)^{\lambda n+2\alpha}\|f\|^2_{\alpha,p}\\ &\quad + C(2^k r)^{\lambda n} r^{2\alpha}((2^k)^{2\alpha}+1)\|f\|^2_{\alpha,p}\\ &\le C(2^k r)^{\lambda n+2\alpha}(1+r^{2\alpha})\|f\|^2_{\alpha,p}. \end{aligned}$$

Thus, since $-n/p \le \alpha < 1/2$, by (5.2) and (5.10) we have

$$\begin{aligned} A &\le C\Big[\sum_{k=1}^\infty r(2^k r)^{-\lambda n-1}((2^k r)^{\lambda n+2\alpha}\|f\|^2_{\alpha,p} + (2^k r)^{\lambda n} r^{2\alpha}\|f\|^2_{\alpha,p})\Big]^{1/2}\\ &= C\Big[\sum_{k=1}^\infty r(2^k r)^{2\alpha-1}\|f\|^2_{\alpha,p} + \sum_{k=1}^\infty r(2^k r)^{-1} r^{2\alpha}\|f\|^2_{\alpha,p}\Big]^{1/2}\\ &= C\Big[\sum_{k=1}^\infty r^{2\alpha}2^{k(2\alpha-1)}\|f\|^2_{\alpha,p} + \sum_{k=1}^\infty 2^{-k} r^{2\alpha}\|f\|^2_{\alpha,p}\Big]^{1/2} \le Cr^\alpha\|f\|_{\alpha,p}, \end{aligned}$$

and hence

$$\mu_\lambda^{*,\varrho}(f_3)(x) \le \mu_\lambda^{*,\varrho}(f_3)(x_0) + Cr^\alpha\|f\|_{\alpha,p}.$$

Changing the roles of x and x_0, we have

$$|\mu_\lambda^{*,\varrho}(f_3)(x) - \mu_\lambda^{*,\varrho}(f_3)(x_0)| \le Cr^\alpha\|f\|_{\alpha,p}.$$

In the case of $\alpha = 0$, using Lemma 1.3 in place of Lemma 1.2 and modifying the above proof somewhat, we can obtain the same conclusion. Thus we have shown (i) and (ii) for $-p/n \le \alpha < 1/2$.

(II) *The case* $1/2 \le \alpha < 1$. Since $\alpha < \min(\beta,\sigma)$, we may assume $\alpha < \beta < \sigma$ and $1 + 2\beta/n < \lambda$. In this case, we set

$$J(k) = \{(t,y)\in\mathbb{R}^{n+1}_+ : |y| < 2^{k-2}r,\ 0 < t < 2^{k-2}r\}\qquad (k \ge 0).$$

Then as in case (I) we see that $(t,y)\notin J(0)$ for $z \in Q^c$ with $|x-y-z| \le t$. Thus

$$(5.11)\qquad \mu_\lambda^{*,\varrho}(f_3)(x)$$

$$\begin{aligned} &= \bigg(\int\limits_{\mathbb{R}^{n+1}_+\setminus J(0)}\Big(\frac{t}{t+|y|}\Big)^{\lambda n}\bigg|\frac{1}{t^\varrho}\int\limits_{|x-y-z|\le t}\frac{\Omega(x-y-z)}{|x-y-z|^{n-\varrho}}f_3(z)\,dz\bigg|^2\frac{dy\,dt}{t^{n+1}}\bigg)^{1/2}\\ &\le \bigg(\int\limits_{\mathbb{R}^{n+1}_+\setminus J(0)}\Big(\frac{t}{t+|y|}\Big)^{\lambda n}\bigg|\frac{1}{t^\varrho}\int\limits_{|x_0-y-z|\le t}\frac{\Omega(x_0-y-z)}{|x_0-y-z|^{n-\varrho}}f_3(z)\,dz\bigg|^2\frac{dy\,dt}{t^{n+1}}\bigg)^{1/2}\\ &\quad + \bigg(\int\limits_{\mathbb{R}^{n+1}_+\setminus J(0)}\Big(\frac{t}{t+|y|}\Big)^{\lambda n}\bigg|\frac{1}{t^\varrho}\int\limits_{|x-y-z|\le t}\frac{\Omega(x-y-z)}{|x-y-z|^{n-\varrho}}f_3(z)\,dz\\ &\qquad - \frac{1}{t^\varrho}\int\limits_{|x_0-y-z|\le t}\frac{\Omega(x_0-y-z)}{|x_0-y-z|^{n-\varrho}}f_3(z)\,dz\bigg|^2\frac{dy\,dt}{t^{n+1}}\bigg)^{1/2}\\ &\le \mu_\lambda^{*,\varrho}(f_3)(x_0)\\ &\quad + \bigg(\int\limits_{\mathbb{R}^{n+1}_+\setminus J(0)}\Big(\frac{t}{t+|y|}\Big)^{\lambda n}\bigg|\frac{1}{t^\varrho}\int\limits_{\substack{|x-y-z|<t\\|x_0-y-z|>t}}\frac{\Omega(x-y-z)}{|x-y-z|^{n-\varrho}}f_3(z)\,dz\bigg|^2\frac{dy\,dt}{t^{n+1}}\bigg)^{1/2}\\ &\quad + \bigg(\int\limits_{\mathbb{R}^{n+1}_+\setminus J(0)}\Big(\frac{t}{t+|y|}\Big)^{\lambda n}\bigg|\frac{1}{t^\varrho}\int\limits_{\substack{|x-y-z|>t\\|x_0-y-z|<t}}\frac{\Omega(x_0-y-z)}{|x_0-y-z|^{n-\varrho}}f_3(z)\,dz\bigg|^2\frac{dy\,dt}{t^{n+1}}\bigg)^{1/2}\end{aligned}$$

$$+\Big(\int\limits_{\mathbb{R}^{n+1}_+\setminus J(0)}\Big(\frac{t}{t+|y|}\Big)^{\lambda n}$$

$$\times\Big[\frac{1}{t^{\varrho}}\int\limits_{\substack{|x-y-z|<t\\|x_0-y-z|<t}}\Big|\frac{\Omega(x-y-z)}{|x-y-z|^{n-\varrho}}-\frac{\Omega(x_0-y-z)}{|x_0-y-z|^{n-\varrho}}\Big|\,|f_3(z)|\,dz\Big]^2\frac{dy\,dt}{t^{n+1}}\Big)^{1/2}$$

$$=\mu_\lambda^{*,\varrho}(f_3)(x_0)+I_1+I_2+I_3,\quad\text{say.}$$

Here,

$$|f(z)-f_Q|\le\frac{1}{|Q|}\int\limits_Q|f(z)-f(u)|\,du\le\frac{1}{|Q|}\int\limits_Q C|z-u|^{\alpha}\|f\|_{\alpha,p}\,du.$$

For $z\in Q^{c}$, $u\in Q$ and $(t,y)\in J_k$ with $|x-y-z|<t$ we have

$$\begin{aligned}|z-u|&\le|z-x+y|+|x-y-u|\le t+|x-u|+|y|\\&\le 2^{k-2}r+\sqrt{n}r+2^{k-2}r\le c2^{k}r,\end{aligned}$$

and hence $|f(z)-f_Q|\le C2^{\alpha k}r^{\alpha}\|f\|_{\alpha,p}$. For $(t,y)\in J(k)\setminus J(k-1)$, we have

$$\Big(\frac{1}{t+|y|}\Big)^{\lambda n}\sim(2^kr)^{-\lambda n}.$$

Therefore, since Ω is bounded,

$$I_1^2\le\sum_{k=1}^{\infty}\int\limits_{J(k)\setminus J(k-1)}\Big(\frac{t}{t+|y|}\Big)^{\lambda n}$$

$$\times\Big|\frac{1}{t^{\varrho}}\int\limits_{\substack{|x-y-z|<t\\|x_0-y-z|>t}}\frac{\Omega(x-y-z)}{|x-y-z|^{n-\varrho}}f_3(z)\,dz\Big|^2\frac{dy\,dt}{t^{n+1}}$$

$$\le Cr^{2\alpha}\|f\|_{\alpha,p}^2\sum_{k=1}^{\infty}2^{2\alpha k}(2^kr)^{-\lambda n}$$

$$\times\int\limits_{J(k)\setminus J(k-1)}t^{\lambda n}\Big|\frac{1}{t^{\varrho}}\int\limits_{\substack{|x-y-z|<t\\|x_0-y-z|>t}}\frac{1}{|x-y-z|^{n-\varrho}}\,dz\Big|^2\frac{dy\,dt}{t^{n+1}}$$

$$\le Cr^{2\alpha}\|f\|_{\alpha,p}^2\sum_{k=1}^{\infty}2^{2\alpha k}(2^kr)^{-\lambda n}$$

$$\times\int\limits_{J(k)}t^{\lambda n}\Big|\frac{1}{t^{\sigma}}\int\limits_{\substack{|x-y-z|<t\\|x_0-y-z|>t}}\frac{1}{|x-y-z|^{n-\sigma}}\,dz\Big|^2\frac{dy\,dt}{t^{n+1}}.$$

Using $t>|x-y-z|=|x_0-y-z-x_0+x|>|x_0-y-z|-|x_0-x|>t-r/8$, we get the inequality (4.5) in §4,

$$\int\limits_{\substack{|x-y-z|<t\\|x_0-y-z|>t}}\frac{1}{|x-y-z|^{n-\sigma}}\,dz\le\begin{cases}Crt^{\sigma-1} & \text{if } t>r/8,\\ Ct^{\sigma} & \text{if } \delta r<t\le r/8.\end{cases}$$

Thus, since $1/2\le\alpha<1$ and $\lambda>1+2\alpha/n$, we obtain

$$I_1^2\le Cr^{2\alpha}\|f\|_{\alpha,p}^2\sum_{k=1}^{\infty}2^{2\alpha k}(2^kr)^{-\lambda n}\tag{5.12}$$

$$\times\Big(\int\limits_0^{r/8}\int\limits_{|y|<2^{k-2}r}t^{\lambda n}\frac{dy\,dt}{t^{n+1}}+\int\limits_{r/8}^{2^{k-2}r}\int\limits_{|y|<2^{k-2}r}r^2t^{\lambda n-2}\frac{dy\,dt}{t^{n+1}}\Big)$$

$$\le Cr^{2\alpha}\|f\|_{\alpha,p}^2\sum_{k=1}^{\infty}2^{2\alpha k}(2^kr)^{-\lambda n}$$

$$\times\Big(\int\limits_0^{r/8}t^{\lambda n-n-1}(2^kr)^n\,dt+\int\limits_{r/8}^{2^{k-2}r}r^2t^{\lambda n-n-3}(2^kr)^n\,dt\Big)$$

$$\le Cr^{2\alpha}\|f\|_{\alpha,p}^2$$

$$\times\sum_{k=1}^{\infty}2^{2\alpha k}(2^kr)^{-\lambda n}(r^2(2^kr)^{\lambda n-n-2+n}+(2^kr)^n(r/8)^{\lambda n-n})$$

$$\le Cr^{2\alpha}\|f\|_{\alpha,p}^2\Big(\sum_{k=1}^{\infty}2^{k(2\alpha-2)}+\sum_{k=1}^{\infty}2^{k(2\alpha-\lambda n+n)}\Big)\le Cr^{2\alpha}\|f\|_{\alpha,p}^2.$$

Similarly we have

$$I_2\le Cr^{\alpha}\|f\|_{\alpha,p}.\tag{5.13}$$

As for I_3, since $|f(z)-f_Q|\le C2^{\alpha k}r^{\alpha}\|f\|_{\alpha,p}$ for $z\in Q^{c}$ and $(t,y)\in J(k)\setminus J(k-1)$ with $|x-y-z|<t$, we have

$$I_3\le\Big(\sum_{k=1}^{\infty}\int\limits_{J(k)\setminus J(k-1)}\Big(\frac{t}{t+|y|}\Big)^{\lambda n}\tag{5.14}$$

$$\times\Big[\frac{1}{t^{\sigma}}\int\limits_{\substack{|x-y-z|<t\\|x_0-y-z|<t}}\Big|\frac{\Omega(x-y-z)}{|x-y-z|^{n-\varrho}}-\frac{\Omega(x_0-y-z)}{|x_0-y-z|^{n-\varrho}}\Big|\,|f_3(z)|\,dz\Big]^2\frac{dy\,dt}{t^{n+1}}\Big)^{1/2}$$

$$\le Cr^{\alpha}\|f\|_{\alpha,p}\Big(\sum_{k=1}^{\infty}2^{2\alpha k}(2^kr)^{-\lambda n}\int\limits_{J(k)\setminus J(k-1)}t^{\lambda n}$$

$$\times\Big[\frac{1}{t^{\sigma}}\int\limits_{\substack{|x-y-z|<t\\|x_0-y-z|<t}}\Big|\frac{\Omega(x-y-z)}{|x-y-z|^{n-\varrho}}-\frac{\Omega(x_0-y-z)}{|x_0-y-z|^{n-\varrho}}\Big|\,dz\Big]^2\frac{dy\,dt}{t^{n+1}}\Big)^{1/2}$$

$$\leq Cr^\alpha \|f\|_{\alpha,p}\Big(\sum_{k=1}^\infty 2^{2\alpha k}(2^k r)^{-\lambda n}\int_{J(k)} t^{\lambda n}$$
$$\times\Big[\frac{1}{t^\sigma}\int_{\substack{|x-y-z|<t\\|x_0-y-z|<t}}\frac{|\Omega(x-y-z)-\Omega(x_0-y-z)|}{|x_0-y-z|^{n-\sigma}}\,dz\Big]^2\frac{dy\,dt}{t^{n+1}}\Big)^{1/2}$$
$$+Cr^\alpha\|f\|_{\alpha,p}\Big(\sum_{k=1}^\infty 2^{2\alpha k}(2^k r)^{-\lambda n}\int_{J(k)} t^{\lambda n}\Big[\frac{1}{t^\sigma}\int_{\substack{|x-y-z|<t\\|x_0-y-z|<t}}|\Omega(x-y-z)|$$
$$\times\Big|\frac{1}{|x-y-z|^{n-\varrho}}-\frac{1}{|x_0-y-z|^{n-\varrho}}\Big|\,dz\Big]^2\frac{dy\,dt}{t^{n+1}}\Big)^{1/2}$$
$$=I_{3,1}+I_{3,2},\quad \text{say.}$$

Since $\Omega\in \mathrm{Lip}_\beta(S^{n-1})$, the inequality (4.9) also holds in this case. Hence noting $\alpha<\beta\leq 1$, $\lambda>1+2\beta/n$, we have

$$I_{3,1}\leq Cr^\alpha\|f\|_{\alpha,p} \tag{5.15}$$
$$\times\Big(\sum_{k=1}^\infty 2^{2\alpha k}(2^k r)^{-\lambda n}\int_{J(k)} t^{\lambda n}\Big[\frac{1}{t^\sigma}Cr^\beta t^{\sigma-\beta}\Big]^2\frac{dy\,dt}{t^{n+1}}\Big)^{1/2}$$
$$\leq Cr^{\alpha+\beta}\|f\|_{\alpha,p}$$
$$\times\Big(\sum_{k=1}^\infty 2^{2\alpha k}(2^k r)^{-\lambda n}\int_0^{2^{k-2}r}\int_{|y|<2^{k-2}r} t^{\lambda n-2\beta}\frac{dy\,dt}{t^{n+1}}\Big)^{1/2}$$
$$\leq Cr^{\alpha+\beta}\|f\|_{\alpha,p}\Big(\sum_{k=1}^\infty 2^{2\alpha k}(2^k r)^{-\lambda n+n}\int_0^{2^{k-2}r} t^{\lambda n-n-1-2\beta}\,dt\Big)^{1/2}$$
$$=Cr^{\alpha+\beta}\|f\|_{\alpha,p}\Big(\sum_{k=1}^\infty 2^{2\alpha k}(2^k r)^{-\lambda n+n+\lambda n-n-2\beta}\Big)^{1/2}$$
$$\leq Cr^{\alpha+\beta}\|f\|_{\alpha,p}\Big(\sum_{k=1}^\infty 2^{k(2\alpha-2\beta)}r^{-2\beta}\Big)^{1/2}\leq Cr^\alpha\|f\|_{\alpha,p}.$$

As for $I_{3,2}$, since Ω is bounded,

$$I_{3,2}\leq Cr^\alpha\|f\|_{\alpha,p}\Big(\sum_{k=1}^\infty 2^{2\alpha k}(2^k r)^{-\lambda n}\int_{J(k)} t^{\lambda n}$$
$$\times\Big[\frac{1}{t^\sigma}\int_{\substack{|x-y-z|<t\\|x_0-y-z|<t}}\Big|\frac{1}{|x-y-z|^{n-\varrho}}-\frac{1}{|x_0-y-z|^{n-\varrho}}\Big|\,dz\Big]^2\frac{dy\,dt}{t^{n+1}}\Big)^{1/2}.$$

Take $0<\gamma<1$ so that $\alpha<\gamma<\sigma$, $1+2\gamma/n<\lambda$ and the estimate in Lemma

1.5 holds. Then, as in the proof of Theorem 10,

$$\int_{\substack{|x-y-z|<t\\|x_0-y-z|<t}}\Big|\frac{1}{|x-y-z|^{n-\varrho}}-\frac{1}{|x_0-y-z|^{n-\varrho}}\Big|\,dz\leq C(\delta r)^\gamma t^{\sigma-\gamma}.$$

Since $\alpha<\gamma<1$ and $\lambda>1+2\gamma/n$, we have

$$I_{3,2}\leq Cr^\alpha\|f\|_{\alpha,p}\Big(\sum_{k=1}^\infty 2^{2\alpha k}(2^k r)^{-\lambda n}\int_{J(k)} t^{\lambda n-2\sigma}(\delta r)^{2\gamma}t^{2\sigma-2\gamma}\frac{dy\,dt}{t^{n+1}}\Big)^{1/2}$$
$$\leq Cr^\alpha\|f\|_{\alpha,p}(\delta r)^\gamma\Big(\sum_{k=1}^\infty 2^{2\alpha k}(2^k r)^{-\lambda n}\int_0^{2^{k-2}r}\int_{|y|<2^{k-2}r} t^{\lambda n-2\gamma}\frac{dy\,dt}{t^{n+1}}\Big)^{1/2}$$
$$\leq C(\delta r)^\gamma r^\alpha\|f\|_{\alpha,p}\Big(\sum_{k=1}^\infty 2^{2\alpha k}(2^k r)^{-\lambda n+n}\int_0^{2^{k-2}r} t^{\lambda n-1-2\gamma}\,dt\Big)^{1/2}$$
$$\leq Cr^\alpha\|f\|_{\alpha,p}\Big(\sum_{k=1}^\infty 2^{2k(\alpha-\gamma)}\Big)^{1/2}\leq Cr^\alpha\|f\|_{\alpha,p}.$$

Therefore, combining this with (5.14) and (5.15) we get

$$I_3\leq Cr^\alpha\|f\|_{\alpha,p}, \tag{5.16}$$

and hence by (5.11)–(5.13) and (5.16),

$$\mu_\lambda^{*,\varrho}(f_3)(x)\leq \mu_\lambda^{*,\varrho}(f_3)(x_0)+Cr^\alpha\|f\|_{\alpha,p}.$$

Changing the roles of x and x_0, we get

$$|\mu_\lambda^{*,\varrho}(f_3)(x)-\mu_\lambda^{*,\varrho}(f_3)(x_0)|\leq Cr^\alpha\|f\|_{\alpha,p}. \blacksquare$$

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Some geometric properties of typical compact convex sets in Hilbert spaces

by

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Abstract. An investigation is carried out of the compact convex sets X in an infinite-dimensional separable Hilbert space $\mathbb{E}$, for which the metric antiprojection $q_X(e)$ from e to X has fixed cardinality $n+1$ ($n \in \mathbb{N}$ arbitrary) for every e in a dense subset of $\mathbb{E}$. A similar study is performed in the case of the metric projection $p_X(e)$ from e to X where X is a compact subset of $\mathbb{E}$.

1. Introduction. One of the methods used to study the geometry of convex sets is based, as is well known, on the Baire category. This method, which goes back to the fundamental contributions by Klee [14] and, independently, by Gruber [11], has made it possible to discover several elusive and even unexpected properties of convex sets (see Gruber [12], Schneider [19], Schneider and Wieacker [20], Wieacker [23], Zamfirescu [24]). We refer to Gruber [13] and Zamfirescu [26] for a survey about this area of research and for additional bibliography.

In the present paper the Baire category will be used to investigate some geometric properties of typical compact convex sets. Let $\mathbb{E}$ be a real infinite-dimensional separable Hilbert space. In [6] it has been recently shown that, for a typical compact convex set $X \subset \mathbb{E}$ and any $n \in \mathbb{N}$, the metric antiprojection $q_X(e)$ from e to X (that is, the set of all points of X which are farthest from e) is such that $\operatorname{card} q_X(e) \geq n+1$ for every e in a dense subset of $\mathbb{E}$. The aim of the present paper is to establish a stronger version of this result. In fact, it is proved (Theorem 5.1) that, for a typical compact convex set $X \subset \mathbb{E}$ and any $n \in \mathbb{N}$, one actually has

$$\operatorname{card} q_X(e) = n+1$$

for every e in a dense subset of $\mathbb{E}$. Similarly, it is shown (Theorem 6.1) that for a typical compact set $X \subset \mathbb{E}$ and any $n \in \mathbb{N}$, the metric projection $p_X(e)$

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