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Averages of uniformly continuous retractions

by

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Abstract. Let X be an infinite-dimensional complex normed space, and let B and S be its closed unit ball and unit sphere, respectively. We prove that the identity map on B can be expressed as an average of three uniformly continuous retractions of B onto S. Moreover, for every $0 \le r < 1$, the three retractions are Lipschitz on rB. We also show that a stronger version where the retractions are required to be Lipschitz does not hold.

1. Introduction. Let Y be a strictly convex infinite-dimensional normed space, and let T be a topological space. Let $C = C(T,Y)$ be the normed space of continuous bounded functions from T into Y, with the usual uniform norm. Let B_Y and B_C be the closed unit balls of Y and C, respectively, and let S_Y be the unit sphere of Y. Note that f is an extreme point of B_C if and only if f maps into S_Y. Finally, for every metric space M denote the identity map on M by I_M.

Peck [8] proved that if T is a compact Hausdorff space, then B_C is the convex hull of its extreme points. In [2] it was proved that every $f \in B_C$ can be expressed as an average of four extreme points of B_C, a fact which implies that I_{B_Y} can be expressed as an average of four retractions of B_Y onto S_Y. Cantwell [3] conjectured that the number of retractions can be reduced. Indeed, the number of retractions was reduced in [6] to three, the lowest possible number, and in [4] it was proved that this result holds in every infinite-dimensional complex normed space.

In this paper we focus on two subspaces of $C(M,X)$, where M is a metric space and X is an infinite-dimensional complex normed space. Namely, we consider the subspace $U = U(M,X)$ of uniformly continuous functions, and its subspace $L = L(M,X)$ of Lipschitz functions.

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In Section 2 we prove the main theorem (2.1) of this paper: Let X be an infinite-dimensional complex normed space. Then I_{B_X} can be expressed as an average of three uniformly continuous retractions of B_X onto S_X. Moreover, for every $0 \le r < 1$, the three retractions are Lipschitz on $rB_X = \{rx \mid x \in B_X\}$. We do not know if this is true for every infinite-dimensional real normed space.

In Section 3 we show (Lemma 3.4) that if X is a Hilbert space, then I_{B_X} cannot be expressed as an average of any finite number of retractions of B_X onto S_X which are Lipschitz (or even Hölder with exponent $p > 1/2$). We do not know if this is true for every normed space.

We also show (Corollary 3.3) that if X is strictly convex then the convex hull of the extreme points of B_U (respectively, B_L) is equal to B_U (respectively, contains $B_L \setminus S_L$).

2. Main theorem. Let X be an infinite-dimensional complex normed space, and let B and S be its closed unit ball and unit sphere, respectively.

In this section we prove the following theorem:

THEOREM 2.1. *Let $\alpha_1, \alpha_2, \alpha_3 \in (0, 1/2)$ be such that $\sum_{i=1}^{3} \alpha_i = 1$. Then there are three uniformly continuous retractions $f_1, f_2, f_3 : B \to S$ such that $I_B \equiv \sum_{i=1}^{3} \alpha_i f_i$. Moreover, the restrictions $f_i|_{rB}$ are Lipschitz for every $0 \le r < 1$.*

We use the following theorem which was first proved by Nowak [7] for some Banach spaces, and later by Benyamini and Sternfeld [1] for arbitrary normed spaces (see also [5]).

THEOREM 2.2. *For every infinite-dimensional normed space Z, there exists a Lipschitz retraction from B_Z onto S_Z.*

We also need the following three lemmas. The first one (and its proof) also holds for real spaces.

LEMMA 2.3. *Let $\alpha \in [0, 1/2)$. Then there are two Lipschitz functions $g : B \to S$ and $h : B \to B$ such that $g|_S \equiv h|_S \equiv I_S$ and $I_B \equiv \alpha g + (1-\alpha)h$. Moreover, $h[rB] \subseteq rB$ for every $1/(2(1-\alpha)) \le r \le 1$.*

Proof. Let $\delta := (1-2\alpha)/2$ and note that $0 < \delta \le 1/2$. By Theorem 2.2 there is a Lipschitz retraction $f : \delta B \to \delta S$.

Define the two required functions $g : B \to S$ and $h : B \to B$ by

$$g(x) = \begin{cases} \dfrac{x}{\|x\|} & \text{if } \|x\| \ge \delta, \\ \dfrac{f(x)}{\delta} & \text{if } \|x\| \le \delta, \end{cases} \qquad h(x) = \frac{x - \alpha g(x)}{1-\alpha}.$$

Clearly, g and h are Lipschitz, $I_B \equiv \alpha g + (1-\alpha)h$, and $g|_S \equiv h|_S \equiv I_S$. To see that h maps into B and the second property in the theorem holds, let $x \in B$ and let $1/(2(1-\alpha)) \le r \le 1$. Note that $2\alpha \le r$. Hence,

$$(*) \qquad |t - \alpha| \le r - \alpha$$

for every $0 \le t \le r$. Consider three cases.

1. If $0 \le \|x\| \le \delta$, then
$$\|h(x)\| \le \frac{\|x\| + \alpha\|g(x)\|}{1-\alpha} \le \frac{\delta+\alpha}{1-\alpha} = \frac{1}{2(1-\alpha)} \le r.$$
2. If $\delta \le \|x\| \le r$, then
$$\|h(x)\| = \frac{\|x - \alpha x/\|x\|\,\|}{1-\alpha} = \frac{|\,\|x\| - \alpha|}{1-\alpha} \le \frac{r-\alpha}{1-\alpha} \le r \quad \text{by } (*).$$
3. If $r \le \|x\| \le 1$, then
$$\|h(x)\| = \frac{|\,\|x\| - \alpha|}{1-\alpha} = \frac{\|x\| - \alpha}{1-\alpha} \le 1. \ \blacksquare$$

NOTATION. For every $0 \le r_1 \le r_2$, define $R(r_1, r_2) = \{x \in X \mid r_1 \le \|x\| \le r_2\}$.

The next lemma is the only place in the proof of the theorem where we use the fact that X is a complex space.

LEMMA 2.4. *Let $\alpha \in (0,1)$ and let $\beta \in [1/2 - \alpha/2, 1/2]$. Then there are two uniformly continuous retractions $\varphi_1, \varphi_2 : R(\alpha, 1) \to S$ such that $I_{R(\alpha,1)} \equiv \beta\varphi_1 + (1-\beta)\varphi_2$. Moreover, the restrictions $\varphi_i|_{R(\alpha,r)}$ are Lipschitz for every $\alpha \le r < 1$.*

Proof. Define a Lipschitz function $\Gamma : [\alpha, 1] \to [-1, 1]$ by

$$\Gamma(t) = \frac{t^2 + 2\beta - 1}{2\beta t}.$$

Define two functions $z_1, z_2 : [\alpha, 1] \to S_{\mathbb{C}}$ by

$$z_1(t) = \Gamma(t) + i\sqrt{1 - [\Gamma(t)]^2} \quad \text{and} \quad z_2(t) = \frac{t - \beta z_1(t)}{1-\beta}.$$

Then $z_1(1) = z_2(1) = 1$, $t = \beta z_1(t) + (1-\beta)z_2(t)$ for every $t \in [\alpha, 1]$, z_1 and z_2 are uniformly continuous, and $z_1|_{[\alpha,r]}$ and $z_2|_{[\alpha,r]}$ are Lipschitz for every $\alpha \le r < 1$.

Define the required retractions $\varphi_1, \varphi_2 : R(\alpha, 1) \to S$ by

$$\varphi_1(x) = z_1(\|x\|)\frac{x}{\|x\|} \quad \text{and} \quad \varphi_2(x) = z_2(\|x\|)\frac{x}{\|x\|}. \ \blacksquare$$

LEMMA 2.5. *Let $0 < d \le c \le b \le a \ne d$ be such that $a + d = b + c$. Let $g : B \to S$ and $h : B \to B$ be two uniformly continuous functions such* *that $g|_S \equiv h|_S \equiv I_S$. Then there are two uniformly continuous retractions $f_1, f_2 : B \to S$ such that:*

(1) $ag + dh \equiv cf_1 + bf_2$ *(as functions from B into X).*

(2) *For every $0 \le r < 1$, if $g|_{rB}$ and $h|_{rB}$ are Lipschitz and $h[rB] \subseteq rB$, then $f_i|_{rB}$ are Lipschitz.*

Proof. Let
$$\alpha := \frac{a-d}{a+d} \quad \text{and} \quad \beta := \frac{c}{a+d}.$$
Then $\alpha \in (0,1)$ and
$$\frac{1}{2} \ge \frac{c}{b+c} = \frac{c}{a+d} = \beta \ge \frac{d}{a+d} = \frac{1}{2} - \frac{\alpha}{2}.$$
Let φ_1 and φ_2 be the retractions from Lemma 2.4 with respect to α and β.

Define a uniformly continuous function $f : B \to X$ by
$$f(x) = \frac{ag(x) + dh(x)}{a+d}.$$
Then f maps into $R(\alpha, 1)$ because for every $x \in B$,
$$1 \ge \|f(x)\| \ge \frac{|a\|g(x)\| - d\|h(x)\||}{a+d}$$
$$= \frac{|a-d\|h(x)\||}{a+d} = \frac{a-d\|h(x)\|}{a+d} \ge \frac{a-d}{a+d} = \alpha.$$
Also, $f|_S \equiv I_S$. Hence, we can define the two required uniformly continuous retractions by $f_i \equiv \varphi_i \circ f : B \to S$.

(1) holds because for every $x \in B$, by Lemma 2.4,
$$\frac{ag(x) + dh(x)}{a+d} = f(x) = \beta\varphi_1(f(x)) + (1-\beta)\varphi_2(f(x))$$
$$= \frac{c}{a+d}\varphi_1(f(x)) + \frac{b}{a+d}\varphi_2(f(x)) = \frac{c\varphi_1(f(x)) + b\varphi_2(f(x))}{a+d}.$$

(2) holds because if we let $0 \le r < 1$ and let g and h be as in (2), then $f|_{rB}$ is Lipschitz. Let
$$t := \frac{a+dr}{a+d}.$$
Then $\alpha \le t < 1$. By Lemma 2.4, $\varphi_i|_{R(\alpha,t)}$ are Lipschitz.

Since $h[rB] \subseteq rB$, for every $x \in rB$ we have
$$\alpha \le \|f(x)\| \le \frac{a\|g(x)\| + d\|h(x)\|}{a+d} \le \frac{a\|g(x)\| + dr}{a+d} = t.$$
Hence, $f|_{rB}$ maps into $R(\alpha, t)$. Therefore $f_i|_{rB} \equiv (\varphi_i \circ f)|_{rB} \equiv \varphi_i|_{R(\alpha,t)} \circ f|_{rB}$. Thus, $f_i|_{rB}$ are Lipschitz functions as compositions of two Lipschitz functions. ∎

Proof of Theorem 2.1. Let $\alpha_1, \alpha_2, \alpha_3 \in (0, 1/2)$ be such that $\sum_{i=1}^{3} \alpha_i = 1$. Assume that $\alpha_3 \le \alpha_2 \le \alpha_1$. Choose α_0 such that $\alpha_1 < \alpha_0 < 1/2$. Let $g : B \to S$ and $h : B \to B$ be the Lipschitz functions from Lemma 2.3 with respect to α_0. Then

(1) $$I_B \equiv \alpha_0 g + (1-\alpha_0)h \equiv \alpha_0 g + \alpha_2 h + (\alpha_1 + \alpha_3 - \alpha_0)h.$$

Applying Lemma 2.5 with respect to $0 < \alpha_2 \le \alpha_2 < \alpha_0 \le \alpha_0$, g, and h, we obtain two uniformly continuous retractions $f_2, f_0 : B \to S$ such that

(2) $$\alpha_0 g + \alpha_2 h \equiv \alpha_2 f_2 + \alpha_0 f_0.$$

Applying Lemma 2.5 again but with respect to $0 < \alpha_1 + \alpha_3 - \alpha_0 \le \alpha_3 \le \alpha_1 < \alpha_0$, h, and f_0, we obtain two uniformly continuous retractions $f_3, f_1 : B \to S$ such that

(3) $$\alpha_0 f_0 + (\alpha_1 + \alpha_3 - \alpha_0)h \equiv \alpha_3 f_3 + \alpha_1 f_1.$$

Combining (1), (2), and (3), we get $I_B \equiv \sum_{i=1}^{3} \alpha_i f_i$.

To prove the second property in the theorem, let $0 \le r < 1$. Choose r_0 such that $\max\{r, 1/(2(1-\alpha_0))\} \le r_0 < 1$. By Lemma 2.3, $h[r_0 B] \subseteq r_0 B$. Therefore, by Lemma 2.5, $f_2|_{r_0 B}$ and $f_0|_{r_0 B}$ are Lipschitz. Again by Lemma 2.5, $f_3|_{r_0 B}$ and $f_1|_{r_0 B}$ are Lipschitz. Thus, $f_i|_{rB}$ are Lipschitz since $r \le r_0$. ∎

3. Observations. First, we have two immediate corollaries of Theorem 2.1.

Corollary 3.1. *There are three uniformly continuous retractions $f_1, f_2, f_3 : B \to S$ such that $I_B \equiv \frac{1}{3}(f_1 + f_2 + f_3)$.*

Corollary 3.2. *Let $n \ge 3$ and let $0 < \alpha_1 \le \ldots \le \alpha_n < 1/2$ be such that $\sum_{i=1}^{n} \alpha_i = 1$. Then there are uniformly continuous retractions $f_1, \ldots, f_n : B \to S$ such that $I_B \equiv \sum_{i=1}^{n} \alpha_i f_i$.*

Remarks. 1. Each retraction $f_i : B \to S$ in Theorem 2.1 is a uniform limit of the Lipschitz functions $f_i^n : B \to S$ defined by $f_i^n(x) = f_i((1 - 1/n)x)$.

2. Corollary 3.2 fails if $\alpha_n > 1/2$ since otherwise it follows from $0_X = \sum_{i=1}^{n} \alpha_i f_i(0_X)$ that
$$1 = \|-f_n(0_X)\| = \left\| \sum_{i=1}^{n-1} \frac{\alpha_i}{\alpha_n} f_i(0_X) \right\| \le \sum_{i=1}^{n-1} \frac{\alpha_i}{\alpha_n} \|f_i(0_X)\| = \frac{1-\alpha_n}{\alpha_n} < 1.$$
This argument also shows that for strictly convex spaces the corollary fails if $\alpha_n = 1/2$.

Let U, L, B_U and B_L be as in the Introduction, and let E_U and E_L be the extreme points of B_U and B_L, respectively.

Corollary 3.3. *Assume that X is strictly convex. Let $\alpha_1, \alpha_2, \alpha_3 \in (0, 1/2)$ be such that $\sum_{i=1}^3 \alpha_i = 1$. Then there exists a uniformly continuous function $\mathcal{F} : B_U \to E_U \times E_U \times E_U$ such that for every $g \in B_U$, $\sum_{i=1}^3 \alpha_i(\mathcal{F}(g))_i \equiv g$. Moreover, for every $0 \le r < 1$, $\mathcal{F}|_{rB_L}$ is a Lipschitz function into $E_L \times E_L \times E_L$. Hence, $B_U = \frac{1}{3}(E_U + E_U + E_U)$ and $B_L \setminus S_L \subseteq \frac{1}{3}(E_L + E_L + E_L)$.*

Proof. Let f_i be the retractions from Theorem 2.1. We leave it to the reader to check that $\mathcal{F}(g) := (f_1 \circ g, f_2 \circ g, f_3 \circ g)$ is as required. ■

Lemma 3.4. *Let H be an infinite-dimensional real Hilbert space. Let $n \ge 3$ and let $\alpha_1, \ldots, \alpha_n \in (0, 1/2)$ be such that $\sum_{i=1}^n \alpha_i = 1$. Let $f_1, \ldots, f_n$ be retractions of B_H into S_H such that $I_{B_H} \equiv \sum_{i=1}^n \alpha_i f_i$. Then there is $1 \le j \le n$ such that f_j is not a locally p-Hölder function for any $p > 1/2$; in particular, f_j is not a locally Lipschitz function.*

Proof. Let $x \in S_H$. Assume, for contradiction, that for every $1 \le i \le n$ there is a neighborhood N of x and there are $1/2 < p_i \le 1$ such that $f_i|_N$ is a p_i-Hölder function with constant $k_i \ge 0$. Let $p := \min\{p_1, \ldots, p_n\}$ and $k := \max\{k_1, \ldots, k_n\}$.

Let $0 < t < 1$ be large enough so that

$$\frac{\sqrt{2(1-t)}}{(1-t)^p} > k \quad \text{and} \quad tx \in N.$$

Then there is $1 \le j \le n$ such that $\langle x, f_j(tx)\rangle \le t$, since otherwise

$$t = \langle x, tx\rangle = \Big\langle x, \sum_{i=1}^n \alpha_i f_i(tx)\Big\rangle = \sum_{i=1}^n \alpha_i \langle x, f_i(tx)\rangle > \sum_{i=1}^n \alpha_i t = t.$$

Therefore,

$$\begin{aligned}\|f_j(x) - f_j(tx)\| &= \sqrt{\|f_j(x)\|^2 + \|f_j(tx)\|^2 - 2\langle f_j(x), f_j(tx)\rangle} \\ &= \sqrt{1 + 1 - 2\langle x, f_j(tx)\rangle} = \sqrt{2(1 - \langle x, f_j(tx)\rangle)} \\ &\ge \sqrt{2(1-t)} > k(1-t)^p = k\|x - tx\|^p \ge k_j\|x - tx\|^{p_j},\end{aligned}$$

contradicting the fact that $f_j|_N$ is a p_j-Hölder function with constant k_j. ■

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