STUDIA MATHEMATICA 135 (1) (1999)

Supercyclicity and weighted shifts

by

HÉCTOR N. SALAS (Mayagüez, Puerto Rico)

Abstract. An operator (linear and continuous) in a Fréchet space is hypercyclic if there exists a vector whose orbit under the operator is dense. If the scalar multiples of the elements in the orbit are dense, the operator is supercyclic. We give, for Fréchet space operators, a Supercyclicity Criterion reminiscent of the Hypercyclicity Criterion. We characterize the supercyclic bilateral weighted shifts in terms of their weight sequences. As a consequence, we show that a bilateral weighted shift is supercyclic if and only if it satisfies the Supercyclicity Criterion. We exhibit two supercyclic, irreducible Hilbert space operators which are C^*-isomorphic, but one is hypercyclic and the other is not. We prove that a Banach space operator which satisfies a version of the Supercyclicity Criterion, and has zero in its left essential spectrum, has an infinite-dimensional closed subspace whose nonzero vectors are supercyclic.

1. Introduction. Let T be a linear continuous mapping acting on a linear metric space E. We say that T is *cyclic* if there is a vector $x \in E$, called a *cyclic vector*, such that the linear span of the orbit $\{x, Tx, T^2x, \ldots\}$ is dense in E. We say that T is *hypercyclic* if the orbit of x itself is dense in E; x is then called a *hypercyclic vector*. An intermediate property that T can have is being *supercyclic*: there exists an $x \in E$, called a *supercyclic vector*, such that $\{\lambda x, \lambda Tx, \lambda T^2x, \ldots : \lambda \in \mathbb{C}\}$ (or $\{\lambda x, \lambda Tx, \lambda T^2x, \ldots : \lambda \in \mathbb{R}\}$ in a space over the real numbers) is dense in E.

Cyclic operators and cyclic vectors have been studied extensively for many years. A celebrated result of P. Enflo is the existence of an operator in a Banach space with only trivial invariant subspaces; i.e., every nonzero vector is cyclic. The corresponding problem for operators acting on Hilbert spaces is a well-known open problem.

Hypercyclicity is also a well established notion. Its origin goes back to a 1929 paper of G. D. Birkhoff [7] that shows that translation operators acting on the space of entire functions have hypercyclic vectors. The first examples

1991 *Mathematics Subject Classification*: Primary 47B99.

Key words and phrases: hypercyclic and supercyclic vectors, Hypercyclicity Criterion, Supercyclicity Criterion, weighted shifts, semi-Fredholm operators in Banach spaces, left essential spectrum, basic sequences.

of hypercyclic operators in Banach spaces were given by S. Rolewicz [30]. Good sources to learn about hypercyclicity are the paper by G. Godefroy and J. H. Shapiro [13], the third chapter in B. Beauzamy's book [4], and the seventh and eighth chapter in Shapiro's book [34].

C. J. Read constructed in [29] a remarkable Banach space operator in which every nonzero vector is hypercyclic. In [1], S. I. Ansari showed that every infinite-dimensional separable Banach space supports a hypercyclic operator, thus answering an old question of S. Rolewicz [30].

Supercyclicity is of a more recent vintage. It was invented by H. M. Hilden and L. J. Wallen [20], and it has been studied by several authors since then. B. Beauzamy [3], in his simplification of P. Enflo's example, obtained an operator such that each nonzero vector is supercyclic.

We now describe the content of this paper.

In the second section we present a sufficient condition for supercyclicity in Fréchet spaces along with some of its consequences. This condition, which we call the Supercyclicity Criterion, is reminiscent of the Hypercyclicity Criterion independently discovered by C. Kitai [21] and R. M. Gethner and J. H. Shapiro [12].

We also recover H. M. Hilden and L. J. Wallen's result that all unilateral backward shifts are supercyclic, but show that no vector is supercyclic for all unilateral backward weighted shifts simultaneously.

In the third section we characterize the supercyclic bilateral weighted shifts in terms of their weight sequence; H. N. Salas [33] did the same for hypercyclic bilateral weighted shifts. We also show that a bilateral weighted shift is supercyclic if and only if it satisfies the Supercyclicity Criterion.

In the (brief) fourth section, we show that hypercyclicity is not preserved by C^*-isomorphisms even when the operators in question are supercyclic.

The background for the fifth section is the following. D. A. Herrero [15] and P. S. Bourdon [8] showed, independently, that each hypercyclic operator has a dense linear manifold (with zero deleted) of hypercyclic vectors. In [5] L. Bernal-González and A. Montes-Rodríguez studied infinite-dimensional closed subspaces of (nonzero) hypercyclic vectors, and in [26] A. Montes-Rodríguez continued this study in the Banach space setting.

In the fifth section we start exploring under which conditions operators have infinite-dimensional subspaces (without zero) of supercyclic vectors. Theorem 5.1 states that if an operator satisfies the Supercyclicity Criterion and has zero in its left essential spectrum, then it has infinite-dimensional subspaces of supercyclic vectors. In [19] G. Herzog showed the existence of quasinilpotent supercyclic operators in any infinite-dimensional separable Banach space. As a corollary of Theorem 5.1 we deduce that the quasinilpotent supercyclic operator constructed by G. Herzog has infinite-dimensional subspaces of supercyclic vectors.

In the last section we present several questions. We also make some comments on a joint paper with A. Montes-Rodríguez [27], which is now in preparation.

2. Supercyclicity Criterion. All the Fréchet spaces under consideration will be complex, separable, in general infinite-dimensional, and the operators acting on them will be linear and continuous. $\mathcal{L}(E)$ denotes the algebra of operators acting on the Fréchet space E. A subspace of E is always a *closed* linear manifold.

We start out by recalling the *Hypercyclicity Criterion* (see Shapiro's book [34, p. 109]).

2.1. DEFINITION. Let E be a separable Fréchet space. $T\in\mathcal{L}(E)$ is said to satisfy the *Hypercyclicity Criterion* if:

(1) There exist dense sets X,Y and a right inverse S such that $S(Y)\subset Y$ and $TS=I_Y$.

(2) There exists a sequence $\{n_k\}$ such that $T^{n_k}x\to 0$ and $S^{n_k}y\to 0$ for all $x\in X$, $y\in Y$.

It is shown in [34] that operators that satisfy the Hypercyclicity Criterion are hypercyclic.

The following lemma consists of two particular cases of a theorem proved by Große-Erdmann [14, Theorem 1.2.2, p. 11]. It will be very useful in the sequel.

2.2. LEMMA. *Let E be a Fréchet space and $T\in\mathcal{L}(E)$.*

(1) *The operator T is supercyclic if and only if $\{(x,\alpha T^n x) : x\in E,\ \alpha\in\mathbb{Q}+i\mathbb{Q},\ n\in\mathbb{N}\}$ is dense in $E\times E$. Moreover, if T is supercyclic the set of supercyclic vectors is a dense G_δ subset of E.*

(2) *The operator T is hypercyclic if and only if $\{(x,T^n x) : x\in E,\ n\in\mathbb{N}\}$ is dense in $E\times E$. Moreover, if T is hypercyclic the set of supercyclic vectors is a dense G_δ subset of E.*

The proof of Lemma 2.2 is a direct application of Baire's Category Theorem. Since a countable intersection of dense G_δ sets is again a G_δ, it follows that

2.3. COROLLARY. *If $\{T_n : n\in\mathbb{N}\}$ is a countable set of supercyclic operators on E, then $\{x\in E : x$ is supercyclic for all $T_n\}$ is a dense G_δ set.*

The following corollary is known [2]. We give here a different proof, which is similar to the one given for hypercyclic invertible operators in [17].

2.4. COROLLARY. *Let E be a Fréchet space. If $T\in\mathcal{L}(E)$ is invertible and supercyclic, then T^{-1} is also supercyclic.*

Proof. This is also a direct consequence of Lemma 2.2. Let y, z be arbitrary vectors. Since T is supercyclic, for every open neighborhood V of 0 there exist a vector x and $n \in \mathbb{N}$, $\alpha \in \mathbb{C}$ such that $x - y \in V$ and $\alpha T^n x - z \in V$. We may assume that neither the vectors x, y, z nor the scalar α is 0. Set $\alpha T^n x = u$; then $u - z \in V$ and $(1/\alpha)T^{-n}u - y \in V$. ∎

The following definition is inspired by the Hypercyclicity Criterion, and we call it the *Supercyclicity Criterion.*

2.5. DEFINITION. Let E be a separable Fréchet space. $T \in \mathcal{L}(E)$ is said to satisfy the *Supercyclicity Criterion* if:

(1) There exist dense sets X, Y and a right inverse S such that $S(Y) \subset Y$ and $TS = I_Y$.

(2) There exists a sequence $\{n_k\}$ such that

$$\lim_{k\to\infty} p_V(T^{n_k}x)p_V(S^{n_k}y) = 0$$

for all $x \in X$, $y \in Y$ and for all convex, balanced neighborhoods V of zero, where p_V is the Minkowski functional associated with V.

2.6. LEMMA. *Let E be a separable Fréchet space. If $T \in \mathcal{L}(E)$ satisfies the Supercyclicity Criterion, then T is supercyclic.*

Proof. Let V be a convex, balanced neighborhood of zero, $V = \{z : p_V(z) < 1\}$. Let $e, f \in E$ and $x \in X$, $y \in Y$ be such that $e - x \in V/2$ and $f - y \in V/2$. We will choose $\widehat{x} = x + (1/c)S^{n_k}y$ for some suitable $k \in \mathbb{N}$, $c > 0$.

First choose $k \in \mathbb{N}$ such that $p_V(T^{n_k}x)p_V(S^{n_k}y) < 1/6$, and then choose, if $p_V(S^{n_k}y) \neq 0$, a number $c > 0$ such that $(1/c)p_V(S^{n_k}y) = 1/3$. Therefore

$$e - \widehat{x} = e - x - (1/c)S^{n_k}y \in V,$$

$$f - cT^{n_k}\widehat{x} = f - cT^{n_k}(x + (1/c)S^{n_k}y) = f - y - cT^{n_k}x \in V,$$

since $p_V(cT^{n_k}(x)) = cp_V(T^{n_k}(x)) = 3p_V(S^{n_k}y)p_V(T^{n_k}(x) < 1/2$. If $p_V(S^{n_k}y) = 0$, then choose c small enough so that $cT^{n_k}x \in V/2$. ∎

REMARKS. (1) The proof also works if the sequence n_k is allowed to depend on each pair of points $x \in X$, $y \in Y$.

(2) For a Banach space the condition is that $\lim_{k\to\infty} \|T^{n_k}x\| \cdot \|S^{n_k}y\| = 0$ for all $x \in X$, $y \in Y$.

2.7. EXAMPLES. Let T be a bilateral weighted shift acting on $l^p(\mathbb{Z})$, $1 \leq p < \infty$, or $c_0(\mathbb{Z})$. That is, $Te_n = w_n e_{n-1}$ and as usual, $e_n = (\ldots, 0, 1, 0, \ldots)$ where the 1 is in the nth position.

Let T_1 be the bilateral shift with weight sequence $w_n = 1$ if $n \leq 0$ and $w_n = (n+1)/n$ otherwise, and T_2 be the bilateral shift with weight sequence $w_n = n/(n-1)$ if $n < 0$ and $w_n = 1$ otherwise.

Let T_3 be the bilateral shift with weight sequence $w_n = (n-2)/(n-1)$ if $n \leq 0$ and $w_n = (n+1)^2/n^2$ otherwise; and T_4 be the bilateral shift with weight sequence $w_n = n^2/(n-1)^2$ if $n < 0$ and $w_n = n+1/(n+2)$ otherwise.

Let X be the linear manifold spanned by $\{e_n\}$, and observe that for all $x \in X$, $x \neq 0$, we have:

(a) $\lim_{n\to\infty} T_1^{-n}x = 0$ and $\lim_{n\to\infty} \|T_1^n x\| < \infty$.
(b) $\lim_{n\to\infty} T_2^n x = 0$ and $\lim_{n\to\infty} \|T_2^{-n}x\| < \infty$.
(c) $0 < \lim_{n\to\infty} \|T_3^n x\|/n < \infty$ and $0 < \lim_{n\to\infty} n^2\|T_3^{-n}x\| < \infty$.
(d) $0 < n^2 \lim_{n\to\infty} \|T_4^n x\| < \infty$ and $0 < \lim_{n\to\infty} \|T_4^{-n}x\|/n < \infty$.

The following sufficient condition for being supercyclic is in Kitai's thesis [21, Theorem 3.2] for the Banach space case, except that there the right inverse is bounded. It is a particular, but important case of Lemma 2.6. The proof consists in considering $X = Y = \bigcup_{n=1}^{\infty} \operatorname{Ker}(T^n)$.

2.8. COROLLARY. *Let E be a separable Fréchet space and $T \in \mathcal{L}(E)$. Then T is supercyclic if the following two conditions hold:*

(1) $\bigcup_{n=1}^{\infty} \operatorname{Ker}(T^n)$ is dense in E.

(2) There exists S, possibly unbounded, with domain $\bigcup_{n=1}^{\infty} \operatorname{Ker}(T^n)$ such that $TS = I|_{\bigcup_{n=1}^{\infty} \operatorname{Ker}(T^n)}$.

For the particular case of the unilateral backward shift in $H(\mathbb{C})$, the space of entire functions in the complex plane, we recover a result of Godefroy and Shapiro [13]. They also showed that no multiple of the backward shift is hypercyclic in $H(\mathbb{C})$.

Let T be a unilateral backward weighted shift acting on $l^p(\mathbb{Z}^+)$, $0 < p < \infty$, or $c_0(\mathbb{Z}^+)$ with positive weight sequence $\{w_n\}$; i.e., $Te_n = w_n e_{n-1}$ if $n > 0$ and $Te_0 = 0$. (As usual, $e_n = (0, \ldots, 1, 0, \ldots)$ where the 1 is in the nth position.) The following result was obtained by Hilden and Wallen [20].

2.9. COROLLARY. *If T is a unilateral backward weighted shift acting on $l^p(\mathbb{Z}^+)$ or $c_0(\mathbb{Z}^+)$, then T is supercyclic.*

2.10. PROPOSITION. *Let E be a Fréchet space. Let $E = F \oplus G$ and $T \in \mathcal{L}(E)$. Assume that G is an invariant subspace of T and P is the projection on F with $\operatorname{Ker}(T) = G$. If T is hypercyclic (supercyclic, cyclic), then so is $PTP|_F$.*

Proof. We prove the three assertions at the same time. The estimates given in [33, Proposition 2.7] work in this context too. Since $PT(I-P) = 0$, induction on n shows that $PT^nP = (PTP)^n$.

Let $x \in E$, $f \in F$ and $\sum_{j=0}^{m} a_j z^j$ be a polynomial. Let W be a neighborhood of 0. The continuity of P implies that there exists another open neighborhood V of 0 such that

$$\sum_{j=0}^{m} a_j(PTP)^j Px - f = \sum_{j=0}^{m} a_j(PTP)^j x - f = P\Big(\sum_{j=0}^{m} a_j T^j x - f\Big) \in W \cap F$$

whenever $\sum_{j=0}^{m} a_j T^j x - f \in V$. For hypercyclicity choose the polynomials z^n, while for supercyclicity choose $a_n z^n$. Thus if x is cyclic (hyper, super) for T, then Px is cyclic (hyper, super) for PTP. ■

We remark that the orthogonal sum of hypercyclic operators may even fail to be cyclic. See, for instance, [15, p. 189] or [32, p. 768].

The underlying spaces for hypercyclicity must be separable and infinite-dimensional [21, Theorem 1.2]. For supercyclicity the spaces must be either of dimension one or separable and infinite-dimensional [19, Theorem 1]. Therefore we have the following:

2.11. COROLLARY. *Hypercyclic operators have no invariant subspaces of finite codimension, and supercyclic operators have no invariant subspaces of finite codimension greater than one.*

The following result can be found in [34, p. 111] with "hypercyclic" in place of "supercyclic". We will use it in the proof of Theorem 3.1

2.12. THE SUPERCYCLIC COMPARISON PRINCIPLE. *Suppose E is a linear metric space, and F a dense subspace that is itself a linear metric space with a stronger topology. Suppose T is a linear transformation on E that also maps the smaller space F into itself, and is continuous in the topology of each space. If T is supercyclic on F, then it is also supercyclic on E, and has an E-supercyclic vector that belongs to F.*

It is also possible to have a *Cyclic Comparison Principle*.

By using Corollary 2.3, we see that a countable set of unilateral backward weighted shifts has a common supercyclic vector; in fact, it has a dense G_δ of such vectors. On the other hand, the set of all unilateral *forward* weighted shifts has a common cyclic vector, namely e_0. It is then natural to ask if all backward shifts have a common supercyclic vector. The following proposition shows that the answer is negative.

2.13. PROPOSITION. *Given any vector in $l^1(\mathbb{Z}^+)$, there exists a backward shift for which the vector is not supercyclic.*

Proof. Let $x = \sum_{k=0}^{\infty} a_k e_k$; it is clear that the set $\{k : a_k \neq 0\}$ must be infinite. Otherwise for any backward shift T, $T^n x = 0$ for some n and therefore x cannot be supercyclic. Let $\{i_k : k \in \mathbb{N}\}$ with $i_1 > 0$ be the increasing sequence for which $a_{i_k} \neq 0$. Note that a_0 might be distinct from 0.

We will define inductively a decreasing weight sequence w_k for which $e_0 + e_1$ is not in the closure of $\{zT^n x : z \in \mathbb{C},\ n \in \mathbb{N}\}$, where T has weight sequence w_k. Set $w_j = 1$ for $1 \le j \le i_2 - 1$. Choose w_{i_2} such that $0 < w_{i_2} \le 1$ and

$$|a_{i_1}| \prod_{j=1}^{i_1} w_j = |a_{i_1}| \ge 2|a_{i_2}| \prod_{j=i_2-i_1+1}^{i_2} w_j.$$

Assume that we have already chosen w_j with $j \le i_n$. Set $w_j = w_{i_n}$ for $i_n < j < i_{n+1}$ and choose $w_{i_{n+1}}$ such that $0 < w_{i_{n+1}} \le w_{i_n}$ and

$$|a_{i_n}| \prod_{j=1}^{i_n} w_j \ge 2|a_{i_{n+1}}| \prod_{j=i_{n+1}-i_n+1}^{i_{n+1}} w_j.$$

It is clear that $\inf\{\|zx - e_0 - e_1\| : z \in \mathbb{C}\} = \min\{\|zx - e_0 - e_1\| : z \in \mathbb{C}\} > 0$.

Indeed, if there were a z for which the minimum is zero, then $z = 0$ since x has infinitely many nonzero coefficients. However, $\|0x - e_0 - e_1\| = 2$. Set $p = z\prod_{s=1}^{j} w_s a_j$ and $q = z\prod_{s=2}^{j+1} w_s a_{j+1}$. We now show $\|zT^j x - e_0 - e_1\| \ge |p-1| + |q-1| \ge 1/4$.

The left side above is greater than or equal to one unless $j = i_n$ and $j+1 = i_{n+1}$ for some n. If $|p-1| < 1/2$ then $|p| < 3/2$ and, from the way the weights have been chosen, $|q| < 3/4$. This completes the proof. ■

2.14. PROPOSITION. *If $x = \sum_{j=1}^{\infty} a_j e_j \in l^1(\mathbb{Z}^+)$, then there exists a hypercyclic unilateral backward weighted shift T with $\lim_{n\to\infty} T^n x = 0$.*

Proof. Assume that $\|x\| = 1$ and let $0 < \varepsilon < 1/2$. Let $\{n_k\}_{k=0}^{\infty}$, $n_0 = 0$, be an increasing sequence such that if $\sum_{n_k \le j < n_{k+1}} a_j e_j = y_k$, then $\|y_k\| < \varepsilon^k$, for all $k \ge 1$. [33, Theorem 2.8] states that the unilateral backward shift is hypercyclic if and only if $\limsup \prod_{i=1}^{n} w_i = \infty$. We choose the weights as follows: $w_i = 1$ if $0 = n_0 \le i < n_1$ and $w_i = 2^{1/(n_{k+1}-n_k)}$ if $n_k \le i < n_{k+1}$. Thus $\|T^n x\| \le \sum_{j=k}^{\infty} 2^j \|y_j\| \le \sum_{j=k}^{\infty} (2\varepsilon)^j$ for $n_k \le n$. The proof is now complete. ■

The proofs of Propositions 2.13 and 2.14 can be modified to work for $l^p(\mathbb{Z}^+)$ or $c_0(\mathbb{Z}^+)$.

3. Bilateral weighted shifts. In [33, Theorems 2.1 and 2.8] a characterization of hypercyclic weighted shifts was given in terms of the weight sequences. We now do something similar for supercyclic weighted shifts.

3.1. THEOREM. *Let T be a bilateral backward weighted shift, acting on $l^p(\mathbb{Z})$ with $p \ge 1$ or $c_0(\mathbb{Z})$ with positive weight sequence $\{w_n\}$; i.e., $Te_n = w_n e_{n-1}$ for $n \in \mathbb{Z}$. The operator T is supercyclic if and only if*

$$\liminf_{n\to\infty} \max\left\{\frac{\prod_{j+1-n\le k\le j} w_k}{\prod_{h+1\le k\le h+n} w_k} : |j|, |h| \le q\right\} = 0$$

for all $q \in \mathbb{N}$.

Proof. Let T be supercyclic. Let $z = y = \sum_{|j|\le q} e_j$ and $1/2 > \varepsilon > 0$. By Lemma 2.2, there is an arbitrarily large $n > 2q$, a vector $x = \sum x_k e_k$ and a complex number α such that $\|x - z\| < \varepsilon$ and $\|\alpha T^n x - y\| < \varepsilon$. Note that $\alpha \ne 0$. Thus $|x_j| > 1/2$ for $|j| \le q$ and

$$(1/2)|\alpha| \prod_{j+1-n\le k\le j} w_k \le |\alpha| \prod_{j+1-n\le k\le j} w_k |x_j| < \varepsilon.$$

On the other hand, for $|h| \le q$, we have $|x_{h+n}| < \varepsilon$ and

$$1/2 < 1 - \Big|\alpha \prod_{h+1\le k\le h+n} w_k x_{h+n} - 1\Big|$$
$$\le \Big|\alpha \prod_{h+1\le k\le h+n} w_k x_{h+n}\Big| \le |\alpha| \prod_{h+1\le k\le h+n} w_k \varepsilon.$$

Consequently, for all $|h|, |j| \le q$,

$$\frac{\prod_{j+1-n\le k\le j} w_k}{\prod_{h+1\le k\le h+n} w_k} \le 4\varepsilon^2.$$

For the other implication it suffices, by using the Supercyclity Comparison Principle, to prove it for the case $l^1(\mathbb{Z})$. Again we use Lemma 2.2. Let $z = \sum_{|j|\le q} z_j e_j$ and $y = \sum_{|j|\le q} y_j e_j$ be both different from 0. Given $\varepsilon > 0$, we want to find x and n such that $\|x - z\| \le \varepsilon$ and $\|\lambda T^n x - y\| \le \varepsilon$ for some λ.

Let $x = z + (1/\lambda)T^{-n}y$ with n to be determined but $\|(1/\lambda)T^{-n}y\| = \varepsilon$. Also let $T^{-n}y = \sum_{|h|\le q} u_h e_{h+n}$. Since $T^n x = T^n z + (1/\lambda)y$, it suffices to find n such that $\|\lambda T^n z\| < \varepsilon$. But

$$\|\lambda T^n x - y\| = \|\lambda T^n z\| = \|T^n z\| \cdot \|T^{-n} y\|/\varepsilon$$
$$\le \max\Big\{\prod_{j+1-n\le k\le j} w_k : |j| \le q\Big\}\|z\| \sum_{|h|\le q} |u_h|/\varepsilon$$
$$\le \|z\| \frac{\max\{\prod_{j+1-n\le k\le j} w_k : |j| \le q\}}{\min\{\prod_{h+1\le k\le h+n} w_k : |h| \le q\}} \sum_{|h|\le q}\Big(\prod_{h+1\le k\le h+n} w_k |u_h|/\varepsilon\Big)$$
$$\le \max \frac{\{\prod_{j+1-n\le k\le j} w_k : |j| \le q\}}{\{\prod_{h+1\le k\le h+n} w_k : |h| \le q\}} \|z\| \cdot \|y\|/\varepsilon$$

as $y = T^n \sum_{|h|\le q} u_h e_{h+n} = \sum_{|h|\le q}(\prod_{h+1\le k\le h+n} w_k) u_h e_h$. Now $\|z\| \cdot \|y\|/\varepsilon$ is a fixed quantity, so the hypothesis implies that there is an n large enough such that the maximum of the quotient in the last line can be made smaller than $\varepsilon^2/(\|z\| \cdot \|y\|)$. Thus $\|\lambda T^n x - y\| \le \varepsilon$. The proof is now complete. ∎

Note that an immediate consequence of Theorem 3.1 is that a supercyclic bilateral shift has no symmetric weight sequence, i.e., one with $w_n = w_{-n}$.

This restriction no longer applies to cyclic bilateral shifts; for instance, $\{w_n = 1\}$ corresponding to $M_z \in L^2(\partial D, d\theta/(2\pi))$ is cyclic. The cyclic vectors of this unitary operator are known [35, p. 116].

It is shown in [23] that a bilateral weighted shift is hypercyclic if and only if it satisfies the Hypercyclicity Criterion. An analogous result is presented below.

3.2. Corollary. *A bilateral weighted shift is supercyclic if and only if it satisfies the Supercyclicity Criterion.*

Proof. Let T be a supercyclic bilateral backward weighted shift acting on $l^p(\mathbb{Z})$ or $c_0(\mathbb{Z})$ with positive weight sequence $\{w_n\}$. Let $X = Y =$ the manifold spanned by $\{e_n : n \in \mathbb{Z}\}$ and $x = \sum_{|j|\le q} a_j e_j$, $y = \sum_{|j|\le q} b_j e_j$. Then

$$\|T^n(x)\| \le \max\Big\{\prod_{j+1-n\le k\le j} w_k : |j| \le q\Big\} q\|x\|,$$
$$\|T^{-n}(y)\| \le \frac{1}{\min\{\prod_{h+1\le k\le h+n} w_k : |h| \le q\}} q\|y\|.$$

Thus

$$\|T^n(x)\| \cdot \|T^{-n}(y)\| \le \max\left\{\frac{\prod_{j+1-n\le k\le j} w_k}{\prod_{h+1\le k\le h+n} w_k} : |j|, |h| \le q\right\} q^2 \|x\| \cdot \|y\|.$$

By Theorem 3.1 there exists a sequence $\{n_r\}$ such that

$$\lim_{r\to\infty} \max\left\{\frac{\prod_{j+1-n_r\le k\le j} w_k}{\prod_{h+1\le k\le h+n_r} w_k} : |j|, |h| \le q\right\} q^2 \|x\| \cdot \|y\| = 0,$$

and consequently $\lim_{r\to\infty} \|T^{n_r}(x)\| \cdot \|T^{-n_r}(y)\| = 0$. Since the converse is always true, the proof is complete. ∎

3.3. Corollary. *Let T_i, $i = 1, \ldots, m$, be bilateral weighted shifts with weight sequences $\{w_{i,k} : i = 1, \ldots, m$ and $k \in \mathbb{N}\}$. The direct sum $\bigoplus_{i=1}^m T_i$ is supercyclic if and only if*

$$\liminf_{n\to\infty} \max\left\{\frac{\prod_{j+1-n\le k\le j} w_{i,k}}{\prod_{h+1\le k\le h+n} w_{i,k}} : |j|, |h| \le q,\ i = 1, \ldots, m\right\} = 0$$

for all $q \in \mathbb{N}$.

3.4. Examples. The following two bilateral backward shifts are supercyclic.

1. The shift T_1 with weight sequence $w_n = 1/n^2$ if $n < 0$, $w_0 = 1$ and $w_n = 1/n$ otherwise.

2. The shift T_2 with weight sequence w_n such that $w_0 = 1$ and

$$w_k = \begin{cases} \frac{k+1-\sum_{j=0}^{i} n_j}{k-\sum_{j=0}^{i} n_j} & \text{if } i \text{ is even and } \sum_{j=0}^{i} n_j < k \le \sum_{j=0}^{i+1} n_j, \\ 1 & \text{if } i \text{ is odd and } \sum_{j=0}^{i} n_j < k \le \sum_{j=0}^{i+1} n_j, \\ 1 & \text{if } i \text{ is even and } \sum_{j=0}^{i} n_j < -k \le \sum_{j=0}^{i+1} n_j, \\ \frac{|k|+1-\sum_{j=0}^{i} n_j}{|k|-\sum_{j=0}^{i} n_j} & \text{if } i \text{ is odd and } \sum_{j=0}^{i} n_j < -k \le \sum_{j=0}^{i+1} n_j, \end{cases}$$

where $\{n_i\}_{i=0}^{\infty}$ with $n_0 = 0$ is a sequence that increases very rapidly.

The first shift is quasinilpotent, and no multiple of it is hypercyclic. This is due to Kitai [21, Theorems 2.7 and 2.8] where she shows that any component of the spectrum of a hypercyclic operator intersects the unit circle. The spectrum of the second shift is the unit circle. No multiple of this operator is hypercyclic either [33, Theorem 2.1].

4. Hypercyclicity is not preserved by C^*-isomorphisms.

In this section we work with operators acting on separable Hilbert spaces. The basics of C^*-algebras are given, for instance, in J. B. Conway's book [11, Chapter 8]. Recall that operators S, T are said to be *similar* if there exists an invertible operator U such that $U^{-1}SU = T$. Operators S, T are said to be *quasisimilar* if there exist operators V, W that are one-to-one, have dense ranges and intertwine S, T, i.e., $SU = UT$ and $WS = TW$.

Similarity preserves hypercyclicity and so does quasisimilarity. On the other hand, cyclicity is not preserved by C^*-isomorphism, i.e., there exist T cyclic and S noncyclic such that there is a C^*-isomorphism π between $C^*(T)$ and $C^*(S)$ with $\pi(T) = S$ and $\pi(T^*) = S^*$. Thus one cannot expect hypercyclicity to be preserved by C^*-isomorphism either. This is the case even if both operators are supercyclic.

In [31] it was shown that there exists an injective bilateral weighted shift T with $\|T\| = 1$ such that for any other shift S with $\|S\| \le 1$ there is a C^*-homomorphism π with $\pi(T) = S$. Such a T is called a *universal bilateral weighted shift*, and almost all bilateral weighted shifts are universal. A necessary and sufficient condition for T to be universal is that its weight sequence $\{w_n\}_{n\in\mathbb{Z}}$ contains contiguous fragments that arbitrarily approximate any finite sequence sampled in $[0,1]$. To be precise: given $k \in \mathbb{N}$ and $(s_1, \ldots, s_k)$ with $0 \le s_j \le 1$ and $\varepsilon > 0$, there is $n \in \mathbb{Z}$ with $|s_j - w_{n+j}| < \varepsilon$ for $1 \le j \le k$ (see [28, Theorem 2.4.1] or [31, pp. 335–337]). We remark that [28, Theorem 2.4.1] still holds for injective shifts, even if they are not invertible. The method of proof given in [31, p. 343] of [28, Theorem 2.5.2] works also in this case. A universal bilateral shift is irreducible because the von Neumann algebra generated by it contains the compact operators.

4.1. Proposition. *There exist C^*-isomorphic irreducible operators $\widehat{S}$, $\widehat{T}$ such that*:

1. *$\widehat{S}$ is supercyclic but not hypercyclic.*
2. *$\widehat{T}$ is hypercyclic.*

Proof. S, T will be universal bilateral weighted shifts and therefore C^*-isomorphic. Let $\{(a_{i,j})_{i=1}^{m_j} : j \in \mathbb{N},\ a_{i,j} \in \mathbb{Q} \cap (0,1]\}$ be an enumeration of all finite positive rational sequences.

The weight sequence $\{w_n\}$ for S is the concatenation

$$(\ldots, w_{-3}, w_{-2}, w_{-1})(b_1)(a_{i,1})_{i=1}^{m_1}(b_2)(a_{i,2})_{i=1}^{m_2}(b_3)\ldots$$

In other words $w_0 = b_1, w_1 = a_{1,1}, \ldots, w_{m_1} = a_{m_1,1}, w_{m_1+1} = b_2, \ldots$ We require that $\{b_n\}$ goes to zero so rapidly that $\lim_{n\to\infty} |z|^n \prod_{k=1}^{n} w_k = 0$ for all $z > 1$.

We also require that for $n \in \mathbb{N}$, $w_{-n} = (1/n)w_n$. Thus the condition of Theorem 3.1 is satisfied and therefore S is supercyclic. At the same time, from the way $\{w_n : n \in \mathbb{N}\}$ was defined, [33, Theorem 2.1] shows that no multiple of S is hypercyclic.

The weight sequence $\{v_n\}$ for T is the concatenation

$$(\ldots, w_{-3}, w_{-2}, w_{-1})(c_{i,1})_{i=1}^{n_1}(a_{i,1})_{i=1}^{m_1}(c_{i,2})_{i=1}^{n_2}(a_{i,2})_{i=1}^{m_2}(c_{i,3})_{i=1}^{n_3}\ldots$$

(that is, $v_n = w_n$ for $n < 0$) where $c_{i,j} = 1$ for $1 \le i \le n_j$. Thus $c_{1,1} = v_0$. Again $\{n_j\}$ is required to increase so rapidly that $\limsup_{n\to\infty} |z|^n \prod_{k=1}^{n} w_k = \infty$ for all $|z| > 1$. Since $\lim_{n\to\infty} |z|^n \prod_{k=1}^{n} w_{-k} = 0$, [33, Theorem 2.1] says that zT is hypercyclic for all $|z| > 1$. By letting $2T = \widehat{T}$ and $2S = \widehat{S}$ we conclude the proof. ∎

5. Infinite-dimensional spaces of supercyclic vectors.

In [1] Ansari proved that every space in a large class of topological vector spaces supports hypercyclic operators. In particular, each separable Banach spaces supports hypercyclic operators. León-Saavedra and Montes-Rodríguez [22] showed that some of the operators constructed by Ansari actually have infinite-dimensional subspaces of hypercyclic vectors. On the other hand, in [19] Herzog proved that every separable Banach space supports a quasinilpotent supercyclic operator. To complete this circle of ideas, we show that some supercyclic quasinilpotent operators, Herzog's included, have infinite-dimensional subspaces of supercyclic vectors.

We define $\mathcal{HC}(E) := \{T \in \mathcal{L}(E) : T \text{ is hypercyclic}\}$ and $\mathcal{SC}(E) := \{T \in \mathcal{L}(E) : T \text{ is supercyclic}\}$. So both classes are subsets of $\mathcal{L}(E)$. On the other hand, for $T \in \mathcal{L}(E)$, let $HC(T) := \{x : x \text{ is hypercyclic for } T\}$ and $SC(T) := \{x : x \text{ is supercyclic for } T\}$. Thus $HC(T)$ and $SC(T)$ are subsets of E. Following [23], we denote by $\mathcal{HC}_\infty(E)$ the set of operators which have an infinite-dimensional subspace (with zero deleted) of hypercyclic vectors. In a similar fashion we define $\mathcal{SC}_\infty(E)$.

It is clear that $\mathcal{HC}_\infty(E)$ and $\mathcal{SC}_\infty(E)$ are invariant under similarity since if $T=USU^{-1}$ and x is hypercyclic (supercyclic) for T then so is $U^{-1}x$ for S, and if M is an infinite-dimensional subspace of E then so is $U^{-1}M$.

Are the classes $\mathcal{HC}_\infty(E)$ and $\mathcal{SC}_\infty(E)$ invariant under quasisimilarity? We suspect that the answer is negative.

We should point out that the Supercyclicity Comparison Principle is not useful in the context of this section. Indeed, let $T \in \mathcal{SC}_\infty(F)$ and M be an infinite-dimensional subspace of F of supercyclic vectors. This means that M is a closed linear manifold in F, but in principle it is only a linear manifold in E.

Recall that $S \in \mathcal{L}(E)$ is said to be *semi-Fredholm* if the range of S is closed and either $\dim \mathrm{Ker}(S)$ is finite or $\mathrm{codim}\,\mathrm{Ran}(S)$ is finite. In the first case we write $S \in \phi_+(E)$ and in the second case $S \in \phi_-(E)$. The *left essential spectrum* of S is $\sigma_{le}(S) = \{\lambda : \lambda - S \notin \phi_+(E)\}$. See for instance J. Zemánek's paper [36].

The following theorem was inspired by [26, Theorem 2.2]. The condition there that there exists an infinite-dimensional $E_0 \subset E$ with $T^n e \to 0$ for all $e \in E_0$ is not useful for supercyclicity since we can always consider that $\|T\| < 1$. This condition is replaced by: There exists a normalized basic sequence $\{u_j\}$ such that Tu_j goes to zero. We will show in Proposition 5.4 that this happens if and only if $0 \in \sigma_{le}(T)$.

5.1. THEOREM. *Let E be a separable Banach space. Assume that:*

(1) *$T \in \mathcal{L}(E)$ satisfies the Supercyclicity Criterion with $X = Y$.*

(2) *There exists a normalized basic sequence $\{u_j\}$ such that Tu_j goes to zero; or equivalently, 0 is in the left essential spectrum of T.*

Then T has an infinite-dimensional subspace such that all its nonzero vectors are supercyclic.

Proof. Without loss of generality we may assume that T is a contraction and that $\{u_j\}$ is contained in X. As in [26, Theorem 2.2], we find it convenient to consider $i(m,n) = (m+n-1)(m+n)/2 - n + 1$, the natural numbers distributed in matrix form as

$$\begin{pmatrix} 1 & 3 & 6 & \ldots & i(m,1) & \ldots \\ 2 & 5 & 9 & \ldots & i(m,2) & \ldots \\ 4 & 8 & 13 & \ldots & i(m,3) & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \\ i(1,n) & i(2,n) & i(3,n) & \ldots & i(m,n) & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $\{y_k\}_{k=2}^\infty \subset X$ be a dense subset of the unit sphere of E. We will construct a sequence $\{z_n = \sum_{j=1}^\infty x_{j,n}\}$ of supercyclic vectors and a rapidly increasing subsequence $\{r_{i(m,n)}\}$ of $\{m_l\}$ (where $\{m_l\}$ and S are given by the Supercyclicity Criterion) such that:

1. $\{x_{1,n}\}$ is a subsequence of $\{u_k\}$.
2. For $j > 1$, $x_{j,n} = a_{j,n} S^{r_{i(j,n)}} y_j$ where $a_{j,n}$ are positive numbers.
3. $\sum_{n=1}^\infty \sum_{j=2}^\infty \|x_{j,n}\| < 1/(2K)$ where K is the basis constant corresponding to the basic sequence $\{u_j\}$.

The conditions above will be shown to imply that $\{z_n\}$ is also a basic sequence. The infinite-dimensional subspace we are looking for is the one generated by $\{z_n\}$.

We now build simultaneously the sequence $\{r_{i(m,n)}\}$ and the vectors $\{x_{m,n}\}$. Let $r_{i(1,1)} = m_1$, $r_{i(1,2)} = m_2$ and $x_{1,1} = u_1$, $x_{1,2} = u_2$. Choose $r_{i(2,1)}$ such that

$$\|T^{r_{i(2,1)}} u_v\| \cdot \|S^{r_{i(2,1)}} y_2\| < (K^{-1} 2^{-i(2,1)-2})^2/6$$

for $v = 1,2$. Choose $a_{2,1}$ such that

$$a_{2,1}\|S^{r_{i(2,1)}} y_2\| = K^{-1}2^{-i(2,1)-2} = K^{-1}2^{-5}.$$

Set $x_{2,1} = a_{2,1} S^{r_{i(2,1)}} y_2$. Assume that we have already chosen the vectors $x_{k,s}$ and the positive numbers $a_{k,s}$ for $i(k,s) < i(p,q)$. For convenience assume that $a_{1,s} = 1$. We need to consider two cases:

(i) If $p = 1$, choose u_l such that $u_l \neq x_{1,s}$ for $1 \le s \le q-1$ and

$$\|Tu_l\| < \min\{a_{k,s} : i(k,s) < i(1,q)\} K^{-1} 2^{-i(1,q)-2}. \tag{1}$$

Set $x_{1,q} = u_l$ and choose $r_{i(1,q)} > r_{i(q-1,1)}$.

(ii) If $p > 1$, choose $r_{i(p,q)}$ such that for each $i(k,s) < i(p,q)$ we have $r_{i(k,s)} < r_{i(p,q)}$ and

$$\|T^{r_{i(p,q)}} x_{k,s}\| \cdot \|S^{r_{i(p,q)}} y_p\| < (\min\{a_{k,s} : i(k,s) < i(p,q)\} K^{-1} 2^{-i(p,q)-2})^2/(2i(p,q)). \tag{2}$$

We still have to choose $a_{p,q}$; we do it in such a way that

$$a_{p,q}\|S^{r_{i(p,q)}} y_p\| = \min\{a_{k,s} : i(k,s) < i(p,q)\} K^{-1} 2^{-i(p,q)-2}. \tag{3}$$

Define $x_{p,q} = a_{p,q} S^{r_{i(p,q)}} y_p$. The last inequality and the fact that $a_{1,s} = 1$ imply that

$$\|x_{p,q}\| = a_{p,q}\|S^{r_{i(p,q)}} y_p\| \le K^{-1} 2^{-i(p,q)-2}.$$

Therefore if $\{z_n = \sum_{j=1}^\infty x_{j,n}\}$, then $\{z_n\}$ is a basic sequence since

$$\sum_{n=1}^\infty \|z_n - x_{1,n}\| \le \sum_{n=1}^\infty \sum_{j=2}^\infty \|x_{j,n}\| \le 1/(2K).$$

ASSERTION. *If* $z=\sum_{n=1}^{\infty} b_n z_n \neq 0$, *then* z *is supercyclic.*

Since for $j>1$ we have

$$T^{r_{i(j,q)}} x_{j,q} = T^{r_{i(j,q)}} S^{r_{i(j,q)}} a_{j,q} y_j = a_{j,q} y_j$$

it follows that

$$\frac{1}{b_q a_{j,q}} T^{r_{i(j,q)}} z - y_j = \sum_{i(k,s)<i(j,q)} \frac{b_s}{b_q a_{j,q}} T^{r_{i(j,q)}} x_{k,s} + \sum_{i(j,q)<i(k,s);k\neq 1} \frac{b_s a_{k,s}}{b_q a_{j,q}} T^{r_{i(j,q)}} S^{r_{i(k,s)}} y_k + \sum_{i(j,q)<i(1,s)} \frac{b_s}{b_q a_{j,q}} T^{r_{i(j,q)}} x_{1,s}.$$

Let q be such that

$$|b_q| \geq |b_n| \tag{4}$$

for all n. For $i(k,s)<i(j,q)$, the first inequality below is due to (4); the equality is due to (3); and the last inequality is due to (2).

$$\left\|\frac{b_s}{b_q a_{j,q}} T^{r_{i(j,q)}} x_{k,s}\right\| \leq \left\|\frac{1}{a_{j,q}} T^{r_{i(j,q)}} x_{k,s}\right\| = \|T^{r_{i(j,q)}} x_{k,s}\| \cdot \frac{\|S^{r_{i(j,q)}} y_j\|}{\min\{a_{k,s} : i(k,s)<i(j,q)\} K^{-1} 2^{-i(j,q)-2}} \leq \frac{\min\{a_{k,s} : i(k,s)<i(j,q)\} K^{-1} 2^{-i(j,q)-2}}{2i(j,q)}.$$

Therefore

$$\left\|\sum_{i(k,s)<i(j,q)} \frac{b_s}{b_q a_{j,q}} T^{r_{i(j,q)}} x_{k,s}\right\| \leq K^{-1} 2^{-i(j,q)-2}/2. \tag{5}$$

For $k\neq 1$ and $i(j,q)<i(k,s)$, (3) and (4) and $\|T\|\leq 1$ imply that

$$\left\|\frac{b_s a_{k,s}}{b_q a_{j,q}} T^{r_{i(j,q)}} S^{r_{i(k,s)}} y_k\right\| \leq \left\|\frac{a_{k,s}}{a_{j,q}} S^{r_{i(k,s)}} y_k\right\| \leq K^{-1} 2^{-i(k,s)-2}.$$

Consequently,

$$\left\|\sum_{i(j,q)<i(k,s);k\neq 1} \frac{b_s a_{k,s}}{b_q a_{j,q}} T^{r_{i(j,q)}} S^{r_{i(k,s)}} y_k\right\| \leq \sum_{i(j,q)<i(k,s);k\neq 1} K^{-1} 2^{-i(k,s)-2}. \tag{6}$$

For $i(j,q)<i(1,s)$, inequalities (1), (4) and $\|T\|\leq 1$ imply that

$$\left\|\frac{b_s}{b_q a_{j,q}} T^{r_{i(j,q)}} x_{1,s}\right\| \leq \left\|\frac{1}{a_{j,q}} T x_{1,s}\right\| \leq K^{-1} 2^{-i(1,s)-2},$$

which implies that

$$\left\|\sum_{i(j,q)<i(1,s)} \frac{b_s}{b_q a_{j,q}} T^{r_{i(j,q)}} x_{1,s}\right\| \leq \sum_{i(j,q)<i(1,s)} K^{-1} 2^{-i(1,s)-2}. \tag{7}$$

By using (5), (6), (7) we have

$$\left\|\frac{1}{b_q a_{j,q}} T^{r_{i(j,q)}} z - y_j\right\| \leq \sum_{i(j,q)\leq i(k,s)} K^{-1} 2^{-i(k,s)-1}.$$

Since the right hand side in the last inequality goes to zero, the vector z is indeed supercyclic. The proof is complete. ∎

The next two corollaries are immediate consequences of the above theorem.

5.2. COROLLARY. *Let E be a separable Banach space. Let $T\in\mathcal{L}(E)$ and S be a map such that:*

(1) *The linear manifold $\bigcup_{n=1}^{\infty} \operatorname{Ker}(T^n)$ is dense in E.*
(2) $S(\bigcup_{n=1}^{\infty} \operatorname{Ker}(T^n)) \subset \bigcup_{n=1}^{\infty} \operatorname{Ker}(T^n)$ *and* $TS = I|_{\bigcup_{n=1}^{\infty} \operatorname{Ker}(T^n)}$.
(3) *There is a normalized basic sequence $\{u_j\}$ such that Tu_j goes to zero.*

Then $T\in SC_\infty(E)$.

5.3. COROLLARY. (1) *If $T\in\mathcal{L}(l^2(\mathbb{Z}))$ is a supercyclic bilateral weighted shift such that a subsequence of its weight sequence goes to zero, then $T\in SC_\infty(l^2(\mathbb{Z}))$.*

(2) *If $T\in\mathcal{L}(l^2(\mathbb{Z}^+))$ is a unilateral backward weighted shift such that a subsequence of its weight sequence goes to zero, then $T\in SC_\infty(l^2(\mathbb{Z}^+))$.*

The following proposition is probably known to experts, but we have been unable to find a reference.

5.4. PROPOSITION. *Let E be a Banach space and $T\in\mathcal{L}(E)$. There exists a normalized basic sequence $\{u_j\}$ such that Tu_j goes to zero if and only if $0\in\sigma_{le}(T)$.*

Proof. Equivalently, we have to show that there exists a normalized basic sequence $\{u_j\}$ such that Tu_j goes to zero if and only if $\operatorname{Ker}(T)$ is infinite-dimensional or $E=\operatorname{Ker}(T)\oplus E_1$ and $T|_{E_1}$ is one-to-one but not bounded below. Since every infinite-dimensional Banach space contains a basic sequence, we need only consider the case in which $0\leq \dim\operatorname{Ker}(T) < \infty$.

Suppose $0 < \dim \operatorname{Ker}(T) < \infty$ and there exists a normalized basic sequence $\{u_j\}$ such that Tu_j goes to zero and $T|_{E_1}$ is bounded below. We can write $u_j = k_j + e_j$, with $k_j \in \operatorname{Ker}(T)$ and $e_j \in E_1$. Let $\inf\{\|Tx\| : \|x\| = 1, x \in E_1\} = c > 0$. Then $\|e_j\| \le (1/c)\|Te_j\| = (1/c)\|Tu_j\| \to 0$.

Since $\|k_j\| \to 1$, the sequence $\{k_j\}$ in the finite-dimensional space $\operatorname{Ker}(T)$ has a subsequence that is convergent to a nonzero vector k.

Consequently, $k \in \bigcap_{n=1}^{\infty} \overline{\operatorname{span}\{u_j : n \le j\}}$. However, this contradicts the fact that u_j is a basic sequence.

Lastly, we have to consider the case when $T|_{E_1}$ is one-to-one but not bounded below. As in the proof of [24, Lemma 1.a.6], assume that we have found unit vectors x_i, $i = 1, \ldots, n$, such that

$$\|Tx_k\| < 1/k \quad \text{and} \quad \|y\| \le \prod_{j=1}^{k-1} (1 + 2^{-j}) \|y + zx_k\|$$

for any $y \in \operatorname{span}\{x_j : 1 \le j \le k-1\}$ for any $k \le n$ and scalar z.

As in the above-mentioned proof, let $\{y_i\}_{i=1}^m$ be vectors of norm 1 in $\operatorname{span}\{x_j : 1 \le j \le n\}$ such that for every $y \in \operatorname{span}\{x_j : 1 \le j \le n\}$ with $\|y\| = 1$ there is an i for which $\|y - y_i\| < 2^{-n-2}$. Let $\{y_i^*\}_{i=1}^m$ be functionals of norm 1 such that $y_i^* y_i = 1$. Note that we are assuming that $y_i^* \in E_1^*$. Thus we need $x \in \bigcap_{i=1}^m \operatorname{Ker}(y_i^*)$ with $\|x\| = 1$ and $\|Tx\| < 1/(n+1)$. Such an x would satisfy $\|y\| \le \prod_{j=1}^n (1+2^{-j})\|y + zx\|$ for any $y \in \operatorname{span}\{x_j : 1 \le j \le n\}$ and scalar z. Now $E_1 = E_0 \oplus \bigcap_{i=1}^m \operatorname{Ker}(y_i^*)$, where $\dim E_0 < \infty$. Since $T|_{E_1}$ is one-to-one, $T|_{E_0}$ is bounded below. Therefore $T|_{\bigcap_{i=1}^m \operatorname{Ker}(y_i^*)}$ is not bounded below.

Indeed, if it were, then $T(\bigcap_{i=1}^m \operatorname{Ker}(y_i^*))$ would be a closed subspace. Let $u_j = k_j + e_j$, with $k_j \in E_0$ and $e_j \in \bigcap_{i=1}^m \operatorname{Ker}(y_i^*)$, be such that $\|u_j\| = 1$ and $Tu_j \to 0$. Again, we may assume that k_j is convergent to k and therefore $Tk_j \to Tk$. But then $Te_j \to -Tk$, and there would exist an $e \in \bigcap_{i=1}^m \operatorname{Ker}(y_i^*)$ with $Te = -Tk$. Since $T|_{E_1}$ is one-to-one, this means that $k = 0$. Therefore $Te_j \to -Tk = 0$, and $T|_{\bigcap_{i=1}^m \operatorname{Ker}(y_i^*)}$ is not bounded below, a contradiction.

Thus we can choose $x \in \bigcap_{i=1}^m \operatorname{Ker}(y_i^*)$ with $\|x\| = 1$ such that $\|Tx\| < 1/(n+1)$; set $x = x_{n+1}$. As in [24, Theorem 1.a5], $\{x_n\}_{n=1}^{\infty}$ is a basic sequence with basis constant $\prod_{j=1}^{\infty}(1 + 2^{-j})$.

We have proved that there exists a basic sequence $\{x_n\}$ such that $\|Tx_n\| < 1/n$. ∎

Recall that $\lambda \in \sigma(T)$ is a *normal eigenvalue* if it is isolated and the corresponding Riesz spectral invariant subspace of T is finite-dimensional. It is known that if $T \in \mathcal{L}(E)$ and $\lambda \in \partial\sigma(T)$ and λ is not a normal eigenvalue, then $\lambda \in \sigma_{le}(T)$. In the case where E is a Hilbert space this fact can be found in [11, p. 366].

5.5. COROLLARY. *Let E be an infinite-dimensional Banach space and $T \in \mathcal{L}(E)$ be quasinilpotent. Then there exists a normalized basic sequence $\{u_j\}$ such that Tu_j goes to zero.*

Proof. 0 is not a normal eigenvalue of T. Thus $0 \in \sigma_{le}(T)$. ∎

6. Concluding remarks and questions. 1) Does there exist a "nice" characterization of cyclic bilateral weighted shifts in terms of their weight sequence? An answer to this should also answer:

a) Herrero's question [35, question 27]: must T or T^* have a cyclic vector?

b) Shapiro's query: There are examples coming from function theory of T cyclic but T^{-1} not cyclic. Are there "easier" examples, for instance, invertible bilateral weighted shifts, of this behavior? [2, Proposition 4.2] does not exclude in principle this possibility since 0 belongs to the bounded component of the resolvent of such operators.

2) In [18] Herrero and Wang showed that a compact perturbation of the identity is hypercyclic. This compact perturbation can be in any Schatten class but cannot be of finite rank, as shown by K. C. Chan and J. H. Shapiro [10, Theorem 3.2 and Corollary 4.2]. On the other hand, in [9] Bourdon showed that hyponormal operators are not supercyclic.

If T is hyponormal plus finite rank, must T be nonsupercyclic? A consequence would be that the Volterra operator $Vf(x) = \int_0^x f(t)\,dt$ is not supercyclic since V can be represented as a sum of a normal operator and an operator of finite rank. This representation can be seen by considering the matrix of T with respect to the basis $\{\exp(i2n\pi x) : n \in \mathbb{Z}\}$ of $L^2([0,1], dx)$.

3) In [16] Herrero asked if $T \oplus T$ is hypercyclic whenever T is. Although the question remains open, J. P. Bès has reformulated the problem in a tantalizing way. In [6, Theorem 1] he shows that the following three conditions are equivalent (there are two more equivalences):

(1) $T \oplus T$ is hypercyclic.
(2) T satisfies the Hypercyclicity Criterion.
(3) T is hereditarily hypercyclic.

His Hypercyclicity Criterion is a little more general than the usual one; namely, the "inverse" mapping is actually an asymptotic inverse.

We can ask then the equivalent of Herrero's question: Is $T \oplus T$ supercyclic whenever T is supercyclic and T^* does not have an eigenvector? The last condition is necessary by [15 , Lemma 3.1].

4) Herrero's Conjecture 1 in [16] is that multi-hypercyclicity implies hypercyclicity. In other words: If $T \in \mathcal{L}(E)$ is such that there is a finite set $F \subset E$ with $\{T^n x : n \in \mathbb{N},\ x \in F\}$ dense in E, then T is hypercyclic. He also conjectured that multi-supercyclicity implies supercyclicity. The problem remains open but V. G. Miller made progress in [25]. By using argu-

ments similar to the ones used in the third section, we can show that a multi-supercyclic bilateral weighted shift is supercyclic.

5) Let B be the unilateral backward shift whose weights are all 1. What are $\bigcap_{|z|>1} HC(zB)$ and $\bigcup_{|z|>1} HC(zB)$?

Corollary 2.3 says that $\bigcap_{z\in A} HC(zB)$ is a dense G_δ whenever A is a denumerable subset of $\{|z|>1\}$. What can we say about $\bigcap_{|z|>1} HC(zB)$?

Going in the other direction, since there is a supercyclic vector x such that given $|z|>1$, the sequence $\|z^nB^nx\|$ is bounded for all n, it follows that $\bigcup_{|z|>1} HC(zB)$ is properly contained in $SC(B)$.

Consider the unilateral backward shift T with weight sequence $\{w_n = (n+1)/n : n \in \mathbb{N}\}$. Since its essential spectrum $\sigma_e(T)$ is the unit circle, [23, Theorem 3.3] says that there exists an infinite-dimensional subspace of hypercyclic vectors of T. Is $\bigcap_{|z|=1} HC(zT) = HC(T)$?

On the other hand, if $|z|>1$, then $\sigma_e(zT) = \{|\lambda| = |z|\}$. [23, Theorem 3.4] says that zT does not have an infinite-dimensional subspace of hypercyclic vectors. But zT still has an infinite-dimensional subspace of supercyclic vectors; namely, any infinite-dimensional subspace of hypercyclic vectors of T.

6) In [26, Theorem 3.4] and [23, Theorem 3.4] the proof that certain hypercyclic operators do not have an infinite-dimensional subspace of hypercyclic vectors consists in exhibiting a vector whose orbit goes (in norm) to infinity. Nonetheless, the possibility remains that the orbit of such a vector could be "tamed" by appropriate multiples and thus be dense in H. In a joint paper with Montes-Rodríguez [27], which is now in preparation, it is proved that this is not always the case. For instance, the closed subspaces (without zero) of supercyclic vectors of the unilateral backward shift $B = M_z^*$ acting on the Hardy space $H^2(D)$ are all finite-dimensional.

A result in [27] states that certain bilateral shifts, which are compact perturbations of a unitary operator, have infinite-dimensional subspaces of supercyclic vectors. The spectra and the essential spectra of these operators are all the same, namely, the unit circle. On the other hand, a related result states that for the inverses of the above-mentioned shifts all their subspaces of supercyclic vectors are finite-dimensional. These results complement the recent work of A. Montes-Rodríguez and F. León-Saavedra for hypercyclic operators [26], [22] and [23].

Acknowledgments. This work was done while I was on a sabbatical leave at Michigan State University, and I thank Professor Joel H. Shapiro for his hospitality. I also thank the Informal Analysis Seminar group at Kent State University, especially the organizer Professor Richard M. Aron, for inviting me to speak on the topics of this paper during the November 1997 Meeting and the March 1998 Conference on Hypercyclicity. I also

thank Juan P. Bès, Nathan S. Feldman, Eva Gallardo-Gutiérrez, Alfonso Montes-Rodríguez and Joel H. Shapiro for their helpful comments during the preparation of this paper. I also thank the referees for their valuable suggestions.

References

[1] S. I. Ansari, *Existence of hypercyclic operators on topological vector spaces*, J. Funct. Anal. 148 (1997), 384–390.

[2] S. I. Ansari and P. S. Bourdon, *Some properties of cyclic operators*, Acta Sci. Math. (Szeged) 63 (1997), 195–207.

[3] B. Beauzamy, *Un opérateur sans sous-espace invariant: Simplification de l'exemple de P. Enflo*, J. Integral Equations Operator Theory 8 (1985), 314–384.

[4] —, *Introduction to Operator Theory and Invariant Subspaces*, North-Holland Math. Library 42, North-Holland, 1988.

[5] L. Bernal-González and A. Montes-Rodríguez, *Non-finite dimensional closed vector spaces of universal functions for composition operators*, J. Approx. Theory 82 (1995), 375–391.

[6] J. P. Bès, *Hereditary hypercyclic operators and the hypercyclicity criterion*, preprint.

[7] G. D. Birkhoff, *Démonstration d'un théorème élémentaire sur les fonctions entières*, C. R. Acad. Sci. Paris 189 (1929), 473–475.

[8] P. S. Bourdon, *Invariant manifolds of hypercyclic operators*, Proc. Amer. Math. Soc. 118 (1993), 1577–1581.

[9] —, *Orbits of hyponormal operators*, Michigan Math. J. 44 (1997), 345–353.

[10] K. C. Chan and J. H. Shapiro, *The cyclic behavior of translation operators on Hilbert spaces of entire functions*, Indiana Univ. Math. J. 40 (1991), 1421–1449.

[11] J. B. Conway, *A Course in Functional Analysis*, 2nd ed., Springer, New York, 1990.

[12] R. M. Gethner and J. H. Shapiro, *Universal vectors for operators on spaces of holomorphic functions*, Proc. Amer. Math. Soc. 100 (1987), 281–288.

[13] G. Godefroy and J. H. Shapiro, *Operators with dense, invariant, cyclic vector manifolds*, J. Funct. Anal. 98 (1991), 239–269.

[14] K.-G. Große-Erdmann, *Holomorphe Monster und universelle Funktionen*, Mitt. Math. Sem. Giessen 176 (1987), 1–84.

[15] D. A. Herrero, *Limits of hypercyclic and supercyclic operators*, J. Funct. Anal. 99 (1991), 179–190.

[16] —, *Hypercyclic operators and chaos*, J. Operator Theory 28 (1992), 93–103.

[17] D. A. Herrero and C. Kitai, *On invertible hypercyclic operators*, Proc. Amer. Math. Soc. 116 (1992), 873–875.

[18] D. A. Herroro and Z. Y. Wang, *Compact perturbations of hypercyclic and supercyclic operators*, Indiana Univ. Math. J. 39 (1990), 819–829.

[19] G. Herzog, *On linear operators having supercyclic vectors*, Studia Math. 103 (1992), 295–298.

[20] H. M. Hilden and L. J. Wallen, *Some cyclic and non-cyclic vectors of certain operators*, Indiana Univ. Math. J. 24 (1974), 557–565.

[21] C. Kitai, *Invariant closed sets for linear operators*, thesis, Univ. of Toronto, 1982.

[22] F. León-Saavedra and A. Montes-Rodríguez, *Linear structure of hypercyclic vectors*, J. Funct. Anal. 148 (1997), 524–545.

[23] F. León-Saavedra and A. Montes-Rodríguez, *Spectral theory and hypercyclic subspaces*, Trans. Amer. Math. Soc., to appear.
[24] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer, Berlin, 1977.
[25] V. G. Miller, *Remarks on finitely hypercyclic and finitely supercyclic operators*, Integral Equations Operator Theory 29 (1997), 110–115.
[26] A. Montes-Rodríguez, *Banach spaces of hypercyclic vectors*, Michigan Math. J. 43 (1996), 419–436.
[27] A. Montes-Rodríguez and H. N. Salas, *Supercyclic operators*, in preparation.
[28] D. P. O'Donovan, *Weighted shifts and covariance algebras*, Trans. Amer. Math. Soc. 208 (1975), 1–25.
[29] C. J. Read, *The invariant subspace problem for a class of Banach spaces, 2: Hypercyclic operators*, Israel J. Math. 63 (1988), 1–40.
[30] S. Rolewicz, *On orbits of elements*, Studia Math. 32 (1969), 17–22.
[31] H. N. Salas, *Semigroup of isometries with commuting range projections*, J. Operator Theory 14 (1985), 331–346.
[32] —, *A hypercyclic operator whose adjoint is also hypercyclic*, Proc. Amer. Math. Soc. 112 (1991), 765–770.
[33] —, *Hypercyclic weighted shifts*, Trans. Amer. Math. Soc. 347 (1995), 993–1004.
[34] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Springer, New York, 1993.
[35] A. L. Shields, *Weighted shift operators and analytic function theory*, in: Topics in Operator Theory, Math. Surveys 13, Amer. Math. Soc., Providence, R.I., 2nd printing, 1979, 49–128.
[36] J. Zemánek, *The semi-Fredholm radius of a linear operator*, Bull. Polish Acad. Sci. Math. 32 (1984), 67–76.

Departamento de Matemáticas
Universidad de Puerto Rico
Mayagüez, Puerto Rico 00681
E-mail: salas@math.upr.clu.edu

Received April 21, 1998
Revised version October 23, 1998
(4090)

STUDIA MATHEMATICA 135 (1) (1999)

Averages of uniformly continuous retractions

by

A. JIMÉNEZ-VARGAS (Almería), J. F. MENA-JURADO (Granada), R. NAHUM (Haifa) and J. C. NAVARRO-PASCUAL (Almería)

Abstract. Let X be an infinite-dimensional complex normed space, and let B and S be its closed unit ball and unit sphere, respectively. We prove that the identity map on B can be expressed as an average of three uniformly continuous retractions of B onto S. Moreover, for every $0 \le r < 1$, the three retractions are Lipschitz on rB. We also show that a stronger version where the retractions are required to be Lipschitz does not hold.

1. Introduction. Let Y be a strictly convex infinite-dimensional normed space, and let T be a topological space. Let $C = C(T,Y)$ be the normed space of continuous bounded functions from T into Y, with the usual uniform norm. Let B_Y and B_C be the closed unit balls of Y and C, respectively, and let S_Y be the unit sphere of Y. Note that f is an extreme point of B_C if and only if f maps into S_Y. Finally, for every metric space M denote the identity map on M by I_M.

Peck [8] proved that if T is a compact Hausdorff space, then B_C is the convex hull of its extreme points. In [2] it was proved that every $f \in B_C$ can be expressed as an average of four extreme points of B_C, a fact which implies that I_{B_Y} can be expressed as an average of four retractions of B_Y onto S_Y. Cantwell [3] conjectured that the number of retractions can be reduced. Indeed, the number of retractions was reduced in [6] to three, the lowest possible number, and in [4] it was proved that this result holds in every infinite-dimensional complex normed space.

In this paper we focus on two subspaces of $C(M,X)$, where M is a metric space and X is an infinite-dimensional complex normed space. Namely, we consider the subspace $U = U(M,X)$ of uniformly continuous functions, and its subspace $L = L(M,X)$ of Lipschitz functions.

1991 *Mathematics Subject Classification*: Primary 46B20.
Key words and phrases: uniformly continuous retraction, Lipschitz retraction, extreme point.
The second author was partially supported by DGICYT PB 95-1146 and DGES PB 96-1406.