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Normal Hilbert modules over the ball algebra $A(B)$

by

KUNYU GUO (Shanghai)

Abstract. The normal cohomology functor $\mathrm{Ext}_{\mathcal{N}}$ is introduced from the category of all normal Hilbert modules over the ball algebra to the category of $A(B)$-modules. From the calculation of $\mathrm{Ext}_{\mathcal{N}}$-groups, we show that every normal $C(\partial B)$-extension of a normal Hilbert module (viewed as a Hilbert module over $A(B)$) is normal projective and normal injective. It follows that there is a natural isomorphism between Hom of normal Shilov modules and that of their quotient modules, which is a new lifting theorem of normal Shilov modules. Finally, these results are applied to the discussion of rigidity and extensions of Hardy submodules over the ball algebra.

1. Introduction. During the past decade a systematic study of modules in the context of operator theory was undertaken [6]. A natural function algebra of holomorphic functions operating continuously on a module with a Hilbert space structure captures many of the features of single operator theory which provides a framework for investigating the multivariate case. However any attempt to apply standard homological algebra methods to the category of Hilbert modules immediately encounters some obstacles. What seems to make things most difficult is that the categories lack enough projective or injective objects. To avoid these obstacles, Douglas and Paulsen introduced in [6, Chapter 4] hypo-projective Hilbert modules and succeeded in characterizing the hypo-projectives and in using them to give a new proof of the lifting theorem.

In the present paper, we are especially interested in the recent work of Carlson and Clark [1, 2]. They began by introducing one of the central concepts from homological algebra. the Ext-functor, to the categories of Hilbert modules and using it to study problems from operator theory. But to calculate homology invariants of Hilbert modules is often very difficult. To overcome these difficulties, we begin by introducing the notion of normal Hilbert modules and trying to describe cohomology and extensions of normal Hilbert modules over the ball algebra $A(B)$, called normal cohomology and

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normal extensions. We expect that this kind of cohomology theory is a fruitful object of study and a useful tool in operator theory.

In Section 2, we introduce some basic homological constructions and notions which are necessary for the present paper. The proofs are similar to those in [1] which are minor variations on arguments in [7]. In Section 3, we consider a subclass, the normal Shilov modules over the ball algebra $A(B)$, which is a class of the most "natural" modules, and show that these modules have injective presentations in the category of normal Hilbert modules. Our principal result shows that there exists a natural isomorphism between Hom of normal Shilov modules and that of their quotient modules. Finally, these results are applied to the discussion of the rigidity and extensions of Hardy submodules over the ball algebra.

2. Homological preliminaries. Let B be the unit ball of $\mathbb{C}^n$ and $A(B)$ be the so-called *ball algebra*, i.e., the set of all functions continuous on the closure $\overline{B}$ of B and holomorphic in B. We say that a Hilbert space H is a *Hilbert module* over $A(B)$ if there exists a multiplication $(f,h)\mapsto fh$ from $A(B)\times H\to H$, making H into an $A(B)$-module and if, in addition, the action is jointly continuous in the sup-norm on $A(B)$ and the Hilbert space norm on H. A *Hilbert module map* between two Hilbert modules H_1 and H_2 is a bounded linear map $L: H_1\to H_2$ which commutes with the action of $A(B)$.

A Hilbert module H over $A(B)$ is called *normal* if for every $h\in H$, the map $f\mapsto fh$ is continuous from the weak* topology of $L^\infty(\partial B,d\sigma)$ restricted to $A(B)$ to the weak topology on H, where $d\sigma$ is the normalized sphere area measure on the boundary ∂B $(=\{\xi\in\mathbb{C}^n \mid \|\xi\|=1\})$ with $\sigma(\partial B)=1$. Some obvious facts are that the category $\mathcal{N}$ of all normal Hilbert modules over $A(B)$ is a proper subcategory of the category $\mathcal{H}$ of all Hilbert modules, and $\mathcal{N}$ is full in $\mathcal{H}$, which means that if $N_1,N_2\in\mathcal{N}$, then the set of all Hilbert maps from N_1 to N_2 in $\mathcal{N}$ is the same as in $\mathcal{H}$, that is, $\mathrm{Hom}_{\mathcal{N}}(N_1,N_2)=\mathrm{Hom}_{\mathcal{H}}(N_1,N_2)$.

In the present paper, we work in the category $\mathcal{N}$. For N_1,N_2 in $\mathcal{N}$, let $S(N_2,N_1)$ be the set of all short exact sequences in the category $\mathcal{N}$,

$$E: 0\to N_1\xrightarrow{\alpha} N\xrightarrow{\beta} N_2\to 0,$$

where α,β are Hilbert module maps. We call two such sequences E, E' *equivalent* if there exists a Hilbert module map θ such that the diagram

$$\begin{tikzcd}
E: & 0 \arrow[r] & N_1 \arrow[r, "\alpha"] \arrow[d, equal] & N \arrow[r, "\beta"] \arrow[d, "\theta"] & N_2 \arrow[r] \arrow[d, equal] & 0 \\
E': & 0 \arrow[r] & N_1 \arrow[r, "\alpha'"] & N' \arrow[r, "\beta'"] & N_2 \arrow[r] & 0
\end{tikzcd}$$

commutes. The set of equivalence classes of $S(N_2,N_1)$ under this relation is defined to be the *normal cohomology group*, $\mathrm{Ext}_{\mathcal{N}}(N_2,N_1)$. For the purpose of this paper, we omit the superscript "1" on $\mathrm{Ext}_{\mathcal{N}}$, and write $\mathrm{cls}(E)$ for the equivalence class of the short exact sequence E. The zero element of $\mathrm{Ext}_{\mathcal{N}}(N_2,N_1)$ is the split extension

$$0\to N_1\to N_1\oplus N_2\to N_2\to 0$$

where the middle term is the (orthogonal) direct sum of the two modules. We first prove that $\mathrm{Ext}_{\mathcal{N}}$ is a functor. Basically this amounts to showing the existence of pullbacks and pushouts in the category $\mathcal{N}$.

PROPOSITION 2.1. *Pullbacks and pushouts exist in the category $\mathcal{N}$.*

Proof. We refer to [7] for a more complete description of pullbacks and pushouts. For a pullback diagram

$$N_2\xrightarrow{\alpha_2} N\xleftarrow{\alpha_1} N_1$$

the pullback N' is obtained by setting

$$N'=\{(h_1,h_2)\in N_1\oplus N_2 \mid \alpha_1(h_1)=\alpha_2(h_2)\}$$

and defining β_1 from N' to N_1 by $\beta_1(h_1,h_2)=h_1$, and β_2 from N' to N_2 by $\beta_2(h_1,h_2)=h_2$. The module structure on N' is derived from that on the direct sum $N_1\oplus N_2$. The reader easily checks that N' is in $\mathcal{N}$, and $\{N',\beta_1,\beta_2\}$ forms the pullback for the given diagram.

Let N be in $\mathcal{N}$ and f be in $A(B)$. We write $T_f^{(N)}: N\to N$ for the linear map $T_f^{(N)}h=fh$. Then we may consider N to be a Hilbert module over $\overline{A(B)}$ (complex conjugates of elements in $A(B)$) by setting $\bar f\cdot h=T_f^{(N)*}h$, $f\in A(B)$, $h\in N$. For emphasis, we denote this $\overline{A(B)}$-module by N_*. It is easy to see that N_* is also a normal Hilbert module over $\overline{A(B)}$ $(\subset L^\infty(\partial B,d\sigma))$. If we use $\overline{\mathcal{N}}$ to denote the category of all normal Hilbert modules over $\overline{A(B)}$ $(\subset L^\infty(\partial B,d\sigma))$ then the opposite category of $\mathcal{N}$ is naturally identified with $\overline{\mathcal{N}}$ under the above defined relation. Similarly one can verify that pullbacks exist in the category $\overline{\mathcal{N}}$. By duality we see that pushouts exist in $\mathcal{N}$.

From Proposition 2.1, one can establish the functoriality of $\mathrm{Ext}_{\mathcal{N}}$. In fact, if $E: 0\to N_1\xrightarrow{\gamma} N\xrightarrow{\delta} N_2\to 0$ and if $\alpha: N_2'\to N_2$, then $\alpha^*(\mathrm{cls}(E))$ is defined to be the equivalence class of $E\alpha$ which is the upper row of the diagram

$$\begin{tikzcd}
E\alpha: & 0 \arrow[r] & N_1 \arrow[r, "\gamma'"] \arrow[d, equal] & N' \arrow[r, "\delta'"] \arrow[d, "\psi"] & N_2' \arrow[r] \arrow[d, "\alpha"] & 0 \\
E: & 0 \arrow[r] & N_1 \arrow[r, "\gamma"] & N \arrow[r, "\delta"] & N_2 \arrow[r] & 0
\end{tikzcd}$$

where $\{N', \delta', \psi\}$ is the pullback of the diagram in the lower right corner and γ' is given by $\gamma'(h) = (\gamma(h), 0)$ for $h \in N_1$. Likewise, if $\beta: N_1 \to N_1'$, then βE is obtained by taking the pushout of E along β. We have thus established the homomorphisms

$$\beta_* : \mathrm{Ext}_{\mathcal{N}}(N_2, N_1) \to \mathrm{Ext}_{\mathcal{N}}(N_2, N_1'),$$
$$\alpha^* : \mathrm{Ext}_{\mathcal{N}}(N_2, N_1) \to \mathrm{Ext}_{\mathcal{N}}(N_2', N_1),$$

where $\beta_*(\mathrm{cls}(E)) = \mathrm{cls}(\beta E)$, $\alpha^*(\mathrm{cls}(E)) = \mathrm{cls}(E\alpha)$.

Using the standard methods from homological algebra (see [7]), one can prove that the induced maps α^*, β_* satisfy $\alpha^*\beta_* = \beta_*\alpha^*$ and if $\alpha' : N_2'' \to N_2'$, $\beta' : N_1' \to N_1''$ are Hilbert module maps, then $(\alpha\alpha')^* = \alpha'^*\alpha^*$ and $(\beta'\beta)_* = \beta'_*\beta_*$. Furthermore, one may naturally make $\mathrm{Ext}_{\mathcal{N}}(N_2, N_1)$ into an $A(B)$-module. The module action of $A(B)$ on $\mathrm{Ext}_{\mathcal{N}}(N_2, N_1)$ is given by the diagram for $T_f^{(N_1)}E$, that is, $f \cdot \mathrm{cls}(E)$ is defined to be $\mathrm{cls}(T_f^{(N_1)}E)$. It is easy to check that $\mathrm{cls}(ET_f^{(N_2)}) = \mathrm{cls}(T_f^{(N_1)}E)$.

As we indicated above, we have shown the following:

PROPOSITION 2.2. *$\mathrm{Ext}_{\mathcal{N}}(-,-)$ is a bifunctor from $\mathcal{N}$ to the category of $A(B)$-modules.*

The next proposition is well known in the purely algebraic setting. In the context of Hilbert modules it is proved in [1]. Its proof in the present context is similar to that in [1].

PROPOSITION 2.3. *Suppose that N_1 and N_2 are normal Hilbert modules over $A(B)$. Then*

$$\mathrm{Ext}_{\mathcal{N}}(N_2, N_1) \cong \mathcal{U}_{\mathcal{N}}/\mathcal{B}_{\mathcal{N}}$$

where $\mathcal{U}_{\mathcal{N}} = \mathcal{U}_{\mathcal{N}}(N_2, N_1)$ is the set of all continuous bilinear functions $\sigma : A(B) \times N_2 \to N_1$ such that $f\sigma(g,h) + \sigma(f, gh) = \sigma(fg, h)$ for $f, g \in A(B)$ and $h \in N_2$, and, for every $h \in N_2$, the map $f \mapsto \sigma(f,h)$ is continuous from the weak topology of $L^\infty(\partial B, d\sigma)$ restricted to $A(B)$ to the weak topology on N_1; $\mathcal{B}_{\mathcal{N}} = \mathcal{B}_{\mathcal{N}}(N_2, N_1)$ is the subset of $\mathcal{U}_{\mathcal{N}}$ consisting of those functions σ that can be expressed in the form $\sigma(f,h) = fL(h) - L(fh)$ for $f \in A(B)$, $h \in N_2$ and for some bounded linear map $L : N_2 \to N_1$.*

We call the elements of $\mathcal{U}_{\mathcal{N}}$ *normal cocycles* and the elements of $\mathcal{B}_{\mathcal{N}}$ *normal coboundaries*. For $\sigma \in \mathcal{U}_{\mathcal{N}}$, we use $\widetilde{\sigma} = \sigma + \mathcal{B}_{\mathcal{N}}$ to denote the *normal cohomology class* of σ. Let $N_1 \widetilde{\oplus} N_2$ be the Hilbert space direct sum of N_1 and N_2 with the module structure defined by $f(h_1, h_2) = (fh_1 + \sigma(f, h_2), fh_2)$, $f \in A(B)$, $h_1 \in N_1$, $h_2 \in N_2$. Then the extension defined by σ is the exact sequence

$$E_\sigma : \quad 0 \to N_1 \xrightarrow{i} N_1 \widetilde{\oplus} N_2 \xrightarrow{\pi} N_2 \to 0$$

in the category $\mathcal{N}$, where $i(h_1) = (h_1, 0)$ and $\pi(h_1, h_2) = h_2$. The correspondence in Proposition 2.3 is given by $\widetilde{\sigma} \mapsto \mathrm{cls}(E_\sigma)$. Moreover, if $\alpha : N_1 \to N_1'$ and $\beta : N_2' \to N_2$ are Hilbert module maps, it can be seen that with regard to pullbacks and pushouts the extensions αE_σ and $E_\sigma \beta$ are $\sigma' \in \mathcal{U}_{\mathcal{N}}(N_2, N_1')$ and $\sigma'' \in \mathcal{U}_{\mathcal{N}}(N_2', N_1)$, respectively, where $\sigma'(f,h) = \alpha(\sigma(f,h))$ and $\sigma''(f,h') = \sigma(f, \beta(h'))$ for $f \in A(B)$, $h \in N_2$, $h' \in N_2'$.

Using these and Propositions 2.2 and 2.3 one can establish the following Hom-Ext sequences whose proof is very similar to that for modules (see [7]):

PROPOSITION 2.4. *If $E : 0 \to N_1 \xrightarrow{\alpha} N_2 \xrightarrow{\beta} N_3 \to 0$ is an exact sequence of normal Hilbert modules over $A(B)$ and N is normal, then we have the following Hom-Ext sequences:*

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{N}}(N, N_1) &\xrightarrow{\alpha_*} \mathrm{Hom}_{\mathcal{N}}(N, N_2) \xrightarrow{\beta_*} \mathrm{Hom}_{\mathcal{N}}(N, N_3) \\ &\xrightarrow{\delta} \mathrm{Ext}_{\mathcal{N}}(N, N_1) \xrightarrow{\alpha_*} \mathrm{Ext}_{\mathcal{N}}(N, N_2) \xrightarrow{\beta_*} \mathrm{Ext}_{\mathcal{N}}(N, N_3) \end{aligned}$$

where δ is the connecting homomorphism given by $\delta(\theta) = \mathrm{cls}(E\theta)$ for $\theta : N \to N_3$, and

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{N}}(N_3, N) &\xrightarrow{\beta^*} \mathrm{Hom}_{\mathcal{N}}(N_2, N) \xrightarrow{\alpha^*} \mathrm{Hom}_{\mathcal{N}}(N_1, N) \\ &\xrightarrow{\delta} \mathrm{Ext}_{\mathcal{N}}(N_3, N) \xrightarrow{\beta^*} \mathrm{Ext}_{\mathcal{N}}(N_2, N) \xrightarrow{\alpha^*} \mathrm{Ext}_{\mathcal{N}}(N_1, N) \end{aligned}$$

where $\delta(\theta) = \mathrm{cls}(\theta E)$ for $\theta : N_1 \to N$.

3. Normal Shilov modules over the ball algebra $A(B)$.

Let H be a normal Hilbert module over the algebra $C(\partial B)$ ($\subset L^\infty(\partial B, d\sigma)$) of all continuous functions on ∂B. By Kaplansky's density theorem and a simple continuity argument, H can be extended to a normal Hilbert module over $L^\infty(\partial B, d\sigma)$ without change of the module bound. Moreover, from [6] we know that H is similar to a normally contractive Hilbert module over $C(\partial B)$. Hence we concentrate on normally contractive Hilbert modules over $C(\partial B)$. Let N be a normally contractive Hilbert module over $C(\partial B)$. A closed subspace $M \subseteq N$ which is invariant for $A(B)$ is called a *normal Shilov module* for $A(B)$ and N is called a *normally contractive $C(\partial B)$-extension* of M. A normal Shilov module for $A(B)$ is *reductive* if it is invariant for $C(\partial B)$, and *pure* if no non-zero subspace of it is reductive.

The following proposition is basic for our analysis. It will help us to calculate $\mathrm{Ext}_{\mathcal{N}}$-groups of some normal Hilbert modules over $A(B)$.

PROPOSITION 3.1. *For any N in $\mathcal{N}$, the module action of $A(B)$ on N can be uniquely extended to $H^\infty(B)$ without change of the module bound of N, making N into a normal $H^\infty(B)$-Hilbert module, where $H^\infty(B)$ denotes the set of all bounded and holomorphic functions on B.*

Proof. By [5, Corollary 2.3], the unit ball of $A(B)$ is weak*-dense in the unit ball of $H^\infty(B)$. Also, since $H^\infty(B)$ is weak*-closed and the unit ball of $H^\infty(B)$ is weak*-compact and weak*-metrizable, a simple continuity argument implies the assertion.

According to Proposition 3.1, the category $\mathcal{N}$ is essentially the same as the category $\mathcal{N}^\infty$ of all normal Hilbert modules over $H^\infty(B)$. We thus conclude that Propositions 2.1–2.4 are valid in the category $\mathcal{N}^\infty$. For $N_1, N_2 \in \mathcal{N}$, since $\operatorname{Ext}_{\mathcal{N}}(N_2, N_1)$ is isomorphic to $\operatorname{Ext}_{\mathcal{N}^\infty}(N_2, N_1)$ as $A(B)$-modules, this implies that $\operatorname{Ext}_{\mathcal{N}}(N_2, N_1)$ can be extended to an $H^\infty(B)$-module.

From the preceding discussion and the theorem on the existence of inner functions in the unit ball B of $\mathbb{C}^n$ (see [12]), we are now in a position to give the main result of this section. First of all, the following notation is necessary. Let G be a semigroup. An *invariant mean* on G is a state μ on $l^\infty(G)$ such that $\mu(F) = \mu({}_gF)$, where ${}_gF(g') := F(gg')$ for all $g \in G$ and $F \in l^\infty(G)$. A basic fact is that every abelian semigroup has an invariant mean (see [11]).

THEOREM 3.2. *Let N be a normal $C(\partial B)$-Hilbert module (of course $N \in \mathcal{N}$). Then for every normal Hilbert module K over $A(B)$, we have*

$$\operatorname{Ext}_{\mathcal{N}}(K, N) = 0, \qquad \operatorname{Ext}_{\mathcal{N}}(N, K) = 0.$$

Proof. By the discussion above, we only need to prove

$$\operatorname{Ext}_{\mathcal{N}^\infty}(K, N) = 0, \qquad \operatorname{Ext}_{\mathcal{N}^\infty}(N, K) = 0$$

where N, K are regarded as normal Hilbert modules over $H^\infty(B)$. Moreover, N is also regarded as a normal Hilbert module over $L^\infty(\partial B, d\sigma)$. For every normal cocycle $\sigma : H^\infty(B) \times K \to N$, we therefore need to show that there exists a bounded linear operator $T : K \to N$ such that $\sigma(f, k) = \sigma_T(f, k) = Tfk - fTk$.

To do this, we write $B_1(N, K)$ for all trace class operators from N to K, $B(K, N)$ for all bounded linear operators from K to N, and identify $B(K, N)$ with $B_1^*(N, K)$ by setting

$$\langle T, C\rangle = \operatorname{tr}(TC), \qquad T \in B(K, N),\ C \in B_1(N, K).$$

Let μ be an invariant mean of the multiplication semigroup of all inner functions in $H^\infty(B)$. Define $T \in B(K, N) = B_1^*(N, K)$ by setting

$$\langle T, C\rangle = \mu_\eta(\langle T_{\overline{\eta}}^{(N)}\sigma(\eta, \cdot), C\rangle),$$

that is, $\langle T, C\rangle$ is the mean of the bounded complex function $\eta \mapsto \langle T_{\overline{\eta}}^{(N)}\sigma(\eta, \cdot), C\rangle$ on the multiplication semigroup of all inner functions in $H^\infty(B)$. For each inner function η', we have

$$\begin{aligned}
&\langle T_{\eta'}^{(N)}T - TT_{\eta'}^{(K)}, C\rangle \\
&\quad= \langle T, CT_{\eta'}^{(N)} - T_{\eta'}^{(K)}C\rangle \\
&\quad= \mu_\eta(\langle T_{\overline{\eta}}^{(N)}\sigma(\eta, \cdot), CT_{\eta'}^{(N)} - T_{\eta'}^{(K)}C\rangle) \\
&\quad= \mu_\eta(\langle T_{\overline{\eta}\eta'}^{(N)}\sigma(\eta, \cdot) - T_{\overline{\eta}}^{(N)}\sigma(\eta, \cdot)T_{\eta'}^{(K)}, C\rangle) \\
&\quad= \mu_\eta(\langle T_{\overline{\eta}\eta'}^{(N)}\sigma(\eta, \cdot) - T_{\overline{\eta}}^{(N)}(\sigma(\eta\eta', \cdot) - T_\eta^{(N)}\sigma(\eta', \cdot)), C\rangle) \\
&\quad= \mu_\eta(\langle \sigma(\eta', \cdot), C\rangle) + \mu_\eta(\langle T_{\overline{\eta}\eta'}^{(N)}\sigma(\eta, \cdot) - T_{\overline{\eta}}^{(N)}\sigma(\eta\eta', \cdot), C\rangle) \\
&\quad= \langle \sigma(\eta', \cdot), C\rangle + \mu_\eta(\langle T_{\overline{\eta}\eta'}^{(N)}\sigma(\eta, \cdot), C\rangle) - \mu_{\eta\eta'}(\langle T_{\overline{\eta\eta'}\eta'}^{(N)}\sigma(\eta\eta', \cdot), C\rangle) \\
&\quad= \langle \sigma(\eta', \cdot), C\rangle
\end{aligned}$$

for all $C \in B_1(N, K)$, so that $\sigma(\eta', \cdot) = T_{\eta'}^{(N)}T - TT_{\eta'}^{(K)}$. Since all inner functions generate $H^\infty(B)$ in the weak* topology (see [12]), we see that $\sigma = \sigma_T$. This is just what is needed. Thus $\operatorname{Ext}_{\mathcal{N}}(K, N) = 0$.

The proof that $\operatorname{Ext}_{\mathcal{N}}(N, K)$ is zero comes from the following fact. Since $\operatorname{Ext}_{\mathcal{N}}(N, K)$ is isomorphic to $\operatorname{Ext}_{\overline{\mathcal{N}}}(K_*, N_*)$ as groups by duality, and N_* is also a normal Hilbert module over $C(\partial B)$, one can deduce that $\operatorname{Ext}_{\overline{\mathcal{N}}}(K_*, N_*) = 0$. Thus $\operatorname{Ext}_{\mathcal{N}}(N, K) = 0$. This completes the proof of Theorem 3.2.

For $P \in \mathcal{N}$, we say that P is *normally projective* if for each pair $N_1, N_2 \in \mathcal{N}$, and every pair of Hilbert module maps $\psi : P \to N_2$ and $\phi : N_1 \to N_2$ with ϕ onto, there exists a Hilbert module map $\widetilde{\psi} : P \to N_1$ such that $\psi = \phi\widetilde{\psi}$. Also, for $I \in \mathcal{N}$, I is called *normally injective* if for every pair $N_1, N_2 \in \mathcal{N}$ and every pair of Hilbert module maps $\psi : N_1 \to I$ and $\phi : N_1 \to N_2$ with ϕ one-to-one and having closed range, there exists a Hilbert module map $\widetilde{\psi} : N_2 \to I$ such that $\psi = \widetilde{\psi}\phi$. Using Proposition 2.4 and Theorem 3.2 we have

COROLLARY 3.3. *Let N be a normal Hilbert module over $C(\partial B)$. Then N (viewed as an $A(B)$-Hilbert module) is normally projective and normally injective.*

REMARK 3.4. (1) In their book [6], Douglas and Paulsen asked whether there is any function algebra, other than $C(X)$, with a (non-zero) projective module (see Problem 4.6). In [2], they proved that every unitary $C(\partial D)$-Hilbert module (viewed as a Hilbert module over the disk algebra $A(D)$) is projective and injective. Xiaoman Chen and Kunyu Guo pointed out in [4] that there exist non-zero projective modules over every unit modulus algebra. Although we do not know if there is any non-zero projective module over the ball algebra $A(B)$ $(n>1)$, Corollary 3.3 guarantees that there exist normally projective modules and normally injective modules.

(2) In the purely algebraic setting, one knows from [7] that there is no non-zero module which is projective and injective over any principal ideal domain (other than a field). Hence, Corollary 3.3 shows a very different character of normal Hilbert modules.

COROLLARY 3.5. *Let N_0 be a normal Shilov module over $A(B)$ and N be a normally contractive $C(\partial B)$-extension of N_0. Then the following statements are equivalent:*

(1) *N_0 is normally injective;*

(2) *$N\ominus N_0$ is normally projective;*

(3) *N_0 is reductive;*

(4) *the short exact sequence $E_{N_0}: 0\to N_0 \xrightarrow{i} N \xrightarrow{\pi} N\ominus N_0\to 0$ is split, where i is the inclusion map and π the quotient map, that is, π is the orthogonal projection $P_{N\ominus N_0}$ from N onto $N\ominus N_0$. As usual, the action of $A(B)$ on $N\ominus N_0$ is given by the formula $f\cdot h = P_{N\ominus N_0}T_f^{(N)}h$ for $f\in A(B)$ and $h\in N\ominus N_0$.*

Proof. It is easy to see that N being normally projective and normally injective implies that (1), (2) and (4) are equivalent. From Corollary 3.3, one can check that (3) leads to (1). To show that (4) implies (3), we may regard N_0, N and $N\ominus N_0$ in the category $\mathcal{N}^\infty$. If E_{N_0} is split, then there is a splitting map $\sigma: N\ominus N_0\to N$ such that $\pi\sigma = 1_{N\ominus N_0}$. Taking any $\xi\in N_0$ and any inner function η, we write $T_{\overline{\eta}}^{(N)}\xi = \xi_1+\xi_2$, $\xi_1\in N_0$, $\xi_2\in N\ominus N_0$. Hence

$$\xi = T_\eta^{(N)}\xi_1 + T_\eta^{(N)}\xi_2.$$

So

$$\pi(T_\eta^{(N)}\xi_2) = T_\eta^{(N\ominus N_0)}\xi_2 = 0.$$

This leads to

$$\sigma(T_\eta^{(N\ominus N_0)}\xi_2) = T_\eta^{(N)}\sigma(\xi_2) = 0,$$

i.e., $\sigma(\xi_2)=0$. Since σ is an injective Hilbert module map, we thus conclude that $\xi_2 = 0$. So N_0 is reductive.

For a normal Hilbert module M, let N be any normally contractive $C(\partial B)$-extension of M. We say that N is *minimal* if $C(\partial B)\cdot M$ is dense in N. From [6, Corollary 2.14], one can show that the minimal normally contractive $C(\partial B)$-extension of M is essentially unique.

LEMMA 3.6. *Let M_i be normal Shilov modules over $A(B)$ and N_i be normally contractive $C(\partial B)$-extensions of M_i, $i=1,2$. Then each $\theta\in$*

$\mathrm{Hom}_{\mathcal{N}}(M_1,M_2)$ lifts to a $C(\partial B)$-Hilbert module map $\theta': N_1\to N_2$. Furthermore, if N_1 is minimal, then the lifting is unique.

Proof. By Proposition 3.1, we may regard M_i as normally contractive Hilbert modules over $H^\infty(B)$ and N_i as normally contractive Hilbert modules over $L^\infty(\partial B, d\sigma)$ for $i=1,2$. Putting $D=\{\overline{\eta}h \mid h\in M_1,\ \eta$ are inner functions$\}$, it is easy to see that D is a linear subspace of N_1. Because all inner functions and their adjoints generate $L^\infty(\partial B,d\sigma)$ in the weak* topology (see [12]), it follows that the closure $\overline{D}$ of D is an $L^\infty(\partial B,d\sigma)$-Hilbert submodule of N_1. If we set $\theta''(\overline{\eta}h) = \overline{\eta}\theta(h)$ for inner functions η and $h\in M_1$, then it is easy to check that θ'' is well defined and can be continuously extended to an $L^\infty(\partial B,d\sigma)$-Hilbert module map from $\overline{D}$ to N_2. Hence if we use θ' to denote the $L^\infty(\partial B,d\sigma)$-Hilbert module map from N_1 to N_2 defined by setting $\theta'(h)=\theta''(h)$ for $h\in\overline{D}$, and $\theta'(h)=0$ for $h\in N_1\ominus\overline{D}$, then θ' is a $C(\partial B)$-lifting of θ. In particular, if N_1 is minimal, then the lifting is unique.

THEOREM 3.7. *Let M_1, M_2 be normal Shilov modules over $A(B)$, and N_1 be a minimal normally contractive $C(\partial B)$-extension of M_1. If N_2 is a normally contractive $C(\partial B)$-extension of M_2, and M_2 is pure, then*

$$\mathrm{Hom}_{\mathcal{N}}(M_1,M_2)\cong \mathrm{Hom}_{\mathcal{N}}(N_1\ominus M_1, N_2\ominus M_2)$$

as $A(B)$-modules. The isomorphism is given by $\beta(\theta) = P_{N_2\ominus M_2}\theta'|_{N_1\ominus M_1}$ for $\theta\in\mathrm{Hom}_{\mathcal{N}}(M_1,M_2)$, where θ' is uniquely determined from θ by Lemma 3.6, and $P_{N_2\ominus M_2}$ is the orthogonal projection from N_2 onto $N_2\ominus M_2$.

Proof. The statement can be expressed as the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "i_1"] \arrow[d, "\theta"] & N_1 \arrow[r, "\pi_1"] \arrow[d, "\theta'"] & N_1\ominus M_1 \arrow[r] \arrow[d, "\beta(\theta)"] & 0 \\
0 \arrow[r] & M_2 \arrow[r, "i_2"] & N_2 \arrow[r, "\pi_2"] & N_2\ominus M_2 \arrow[r] & 0
\end{tikzcd}$$

where i_1, i_2 are the inclusion maps and π_1,π_2 the quotient maps. By Lemma 3.6, it is easy to see that

$$\beta: \mathrm{Hom}_{\mathcal{N}}(M_1,M_2)\to \mathrm{Hom}_{\mathcal{N}}(N_1\ominus M_1, N_2\ominus M_2)$$

is an $A(B)$-module homomorphism, where the module structure of $\mathrm{Hom}_{\mathcal{N}}(M_1,M_2)$ is given by $(f\cdot\theta)(h) = \theta(f\cdot h)$ for $f\in A(B)$, $h\in M_1$, and the definition of the module structure of $\mathrm{Hom}_{\mathcal{N}}(N_1\ominus M_1, N_2\ominus M_2)$ is similar. Since M_2 is pure, Lemma 3.6 also implies that β is injective. Since N_1 is normally projective, β is surjective. This completes the proof of Theorem 3.7.

For Hardy submodules, Theorem 3.7 has a natural form. To state it, let Γ be a subset of $L^2(\partial B, d\sigma)$. A Borel set $E\subseteq\partial B$ is said to be the *support* of Γ (denoted by $S(\Gamma)$) if each function from Γ vanishes on $\partial B - E$, and for any Borel subset E' of E with $\sigma(E') > 0$, there exists a function $f \in \Gamma$ such that $f|_{E'} \neq 0$. For a Hilbert submodule M of $L^2(\partial B, d\sigma)$ over $A(B)$, it is not difficult to prove that $\chi_{S(M)} L^2(\partial B, d\sigma)$ is its minimal $C(\partial B)$-extension, where $\chi_{S(M)}$ is the characteristic function of $S(M)$. We also note that a Hilbert submodule M' of $L^2(\partial B, d\sigma)$ is pure if and only if $\sigma(S(M'^{\perp})) = 1$. Thus Theorem 3.7 implies the following:

COROLLARY 3.8. *Let M_1 and M_2 be Hilbert submodules of $L^2(\partial B, d\sigma)$ over $A(B)$, and $\sigma(S(M_1)) = \sigma(S(M_2^{\perp})) = 1$. Then*

$$\mathrm{Hom}_{\mathcal{N}}(M_1, M_2) \cong \mathrm{Hom}_{\mathcal{N}}(L^2(\partial B, d\sigma) \ominus M_1, L^2(\partial B, d\sigma) \ominus M_2).$$

The isomorphism is given by $\varphi \mapsto H_\varphi^{[M_2]}|_{L^2(\partial B, d\sigma)\ominus M_1}$, where $H_\varphi^{[M_2]}$ is defined by $H_\varphi^{[M_2]} f = P_{L^2(\partial B, d\sigma)\ominus M_2}(\varphi f)$ for all $f \in L^2(\partial B, d\sigma)$.

EXAMPLE 3.9. Let $H^2(\partial B)$ be the usual Hardy module over $A(B)$ and $H^2(\partial B)^{\perp}$ $(= L^2(\partial B, d\sigma) \ominus H^2(\partial B))$ be the corresponding quotient module. By Corollary 3.8, one can obtain

$$\mathrm{Hom}_{\mathcal{N}}(H^2(\partial B)^{\perp}, H^2(\partial B)^{\perp}) \cong H^\infty(B).$$

If we define a Hankel-type operator A_f for f in $L^\infty(\partial B, d\sigma)$ by

$$A_f(h) = P_{L^2(\partial B, d\sigma)\ominus H^2(\partial B)}(fh), \qquad h \in L^2(\partial B, d\sigma) \ominus H^2(\partial B),$$

then the commutant of $\{A_{z_1}, \ldots, A_{z_n}\}$ is equal to $\{A_f \mid f \in H^\infty(B)\}$.

4. Applications to rigidity and extensions of Hardy submodules. Let N_0 be a normally Shilov module and N be any normally contractive $C(\partial B)$-extension of N_0. It follows that we have a normally injective presentation of N_0:

$$E_{N_0}: \quad 0 \to N_0 \xrightarrow{i} N \xrightarrow{\pi} N \ominus N_0 \to 0.$$

From Propositions 2.3, 2.4 and Theorem 3.2, we have

PROPOSITION 4.1.

(1) $\quad \mathrm{Ext}_{\mathcal{N}}(M, N_0) \cong \mathrm{coker}(\pi_* : \mathrm{Hom}_{\mathcal{N}}(M, N) \to \mathrm{Hom}_{\mathcal{N}}(M, N \ominus N_0)).$

The correspondence is given by $\delta(\widetilde{\theta}) = \widetilde{r}_\theta$ for $\theta \in \mathrm{Hom}_{\mathcal{N}}(M, N \ominus N_0)$, where $\widetilde{r}_\theta$ is the normal cohomology class of r_θ defined by $r_\theta(f, h) = P_{N_0} T_f^{(N)} \theta(h)$, P_{N_0} is the orthogonal projection from N to N_0, and $f \in A(B)$, $h \in M$.

(2) $\quad \mathrm{Ext}_{\mathcal{N}}(N \ominus N_0, M) \cong \mathrm{coker}(i^* : \mathrm{Hom}_{\mathcal{N}}(N, M) \to \mathrm{Hom}_{\mathcal{N}}(N_0, M)).$

The correspondence is given by $\delta(\widetilde{\theta}) = \widetilde{\sigma}_\theta$ for $\theta \in \mathrm{Hom}_{\mathcal{N}}(N_0, M)$, where σ_θ is defined by $\sigma_\theta(f, h) = \theta(P_{N_0} T_f^{(N)} h)$, $f \in A(B)$, $h \in N \ominus N_0$.

Proposition 4.1 provides a very useful method for calculating normal cohomology groups of Hilbert modules in $\mathcal{N}$. In particular, if M (or N_0) is cyclic, then the characterizations of $\mathrm{Ext}_{\mathcal{N}}(M, N_0)$ $(\mathrm{Ext}_{\mathcal{N}}(N \ominus N_0, M))$ may be summed up as the action of module maps on cyclic vectors.

We now return to the calculation of $\mathrm{Ext}_{\mathcal{N}}$-groups of Hardy submodules over the ball algebra. For an $A(B)$-Hilbert submodule N of $L^2(\partial B, d\sigma)$, we define a function space $B(N)$ as follows. A function $\varphi \in L^2(\partial B, d\sigma)$ is said to be in $B(N)$ if the densely defined Hankel operator $H_\varphi^{(N)} : H^2(\partial B) \to L^2(\partial B, d\sigma) \ominus N$ can be continuously extended to $H^2(\partial B)$, where $H_\varphi^{(N)} f := P_{L^2(\partial B, d\sigma)\ominus N}(\varphi f)$, $f \in A(B)$. It is easy to check that for every $\varphi \in B(N)$, $H_\varphi^{(N)}$ is a Hilbert module map from $H^2(\partial B)$ to $L^2(\partial B, d\sigma) \ominus N$, and each Hilbert module map β from $H^2(\partial B)$ to $L^2(\partial B, d\sigma) \ominus N$ has such a form, that is, there exists a $\varphi \in B(N)$ such that $\beta = H_\varphi^{(N)}$. Furthermore, for a nonzero Hardy submodule N_0 $(\subseteq H^2(\partial B))$, another function space $B(N_0, N)$ is defined by $\varphi \in B(N_0, N)$ if $\varphi \in B(N)$ and $\ker H_\varphi^{(N)} \supseteq N_0$. From Proposition 4.1 and Lemma 3.6, the following are immediate:

PROPOSITION 4.2.

(1) $\quad \mathrm{Ext}_{\mathcal{N}}(H^2(\partial B), N) \cong B(N)/(L^\infty(\partial B, d\sigma) + N),$

(2) $\quad \mathrm{Ext}_{\mathcal{N}}(H^2(\partial B) \ominus N_0, N) \cong B(N_0, N)/N.$

REMARK 4.3. If B is the unit ball in $\mathbb{C}^n$ with $n > 1$, then one can check that $\mathrm{Ext}_{\mathcal{N}}(H^2(\partial B), H^2(\partial B)) \neq 0$. This says that $H^2(\partial B)$ is never normally projective. In the case $n = 1$, we have $\mathrm{Ext}_{\mathcal{N}}(H^2(\partial D), H^2(\partial D)) = 0$ from the Nehari theorem [1, 4]. However, we do not know if $H^2(\partial D)$ is normally projective.

Let two Hardy submodules N_1, N_2 satisfy $0 \neq N_1 \subseteq N_2 \subsetneq H^2(\partial B)$. Hence 1 is in $B(N_1, N_2)$ and it follows that $\mathrm{Ext}_{\mathcal{N}}(H^2(\partial B) \ominus N_1, N_2)$ is not zero. This indicates that for Hardy submodules N_1, N_2 and $N_1 \neq 0$, if $\mathrm{Ext}_{\mathcal{N}}(H^2(\partial B) \ominus N_1, N_2) = 0$, then there is no proper Hardy submodule N_3 such that $N_3 \supseteq N_1$ and N_3 is similar to N_2. The next proposition gives us some information on the rigidity of Hardy submodules.

PROPOSITION 4.4. *Let B^2 be the unit ball of $\mathbb{C}^2$ and let N be a Hardy submodule of finite codimension in $H^2(\partial B^2)$. Then*

$$\mathrm{Ext}_{\mathcal{N}}(H^2(\partial B^2) \ominus N, H^2(\partial B^2)) = 0.$$

Proof. By [8], $R \cap N$ is dense in N, and the set of common zeros (in $\mathbb{C}^2$) of the members of $R \cap N$ is finite and lies in B^2, where R is the ring of all polynomials on $\mathbb{C}^2$. For $\phi \in B(N, H^2(\partial B^2))$, we have

$$\phi(R \cap N) \subseteq H^2(\partial B^2).$$

Using the harmonic extension of ϕ and the removable singularities theorem (see [9]), one easily checks that ϕ is in $H^2(\partial B^2)$. Proposition 4.2 thus shows that $\mathrm{Ext}_{\mathcal{N}}(H^2(\partial B^2)\ominus N, H^2(\partial B^2))$ is zero. The proof is complete.

For a proper Hardy submodule N of finite codimension in $H^2(\partial B^2)$, since $\mathrm{Ext}_{\mathcal{N}}(H^2(\partial B^2)\ominus N, N)$ is never zero, it follows that N is never similar to $H^2(\partial B^2)$ by Proposition 4.4. We refer the reader to [3] for a further consideration of the rigidity of Hardy submodules over the ball algebra.

REMARK 4.5. The main results of the present paper are also valid for strongly pseudoconvex domains with smooth boundary by [5, 10].

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Department of Mathematics
Fudan University
Shanghai 200433, P.R. China
E-mail: kyguo@moboss.fudan.edu.cn

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Canonical functional extensions on von Neumann algebras

by

CARLO CECCHINI (Udine)

Abstract. The topology and the structure of the set of the canonical extensions of positive, weakly continuous functionals from a von Neumann subalgebra M_0 to a von Neumann algebra M are described.

1. Introduction. The aim of this paper is to give some results about the structure of the set $R(M, M_0)$ of the canonical extensions of positive, weakly continuous functionals (called canonical functional extensions, c.f.e.) from a von Neumann subalgebra M_0 to a von Neumann algebra M (cf. [3]–[5]). After the necessary preliminaries in Section 2, Section 3 is devoted to the introduction of a set $V(\omega_0)$ of vectors in the Hilbert space of the standard representation for M, canonically associated with $R(M, M_0)$ in the framework of the modular theory of von Neumann algebras. Section 4 contains some topological results on $R(M, M_0)$ and $V(\omega_0)$. In Section 5 structural properties for different c.f.e. are compared and the possibility of defining Radon–Nikodym derivatives for c.f.e. in the spirit of Connes' type cocycles for conditional expectations on von Neumann algebras (cf. [6]) is considered. In Section 6 we consider the special situation in which a c.f.e. is dominated (i.e. majorized by some multiple of another) in order to obtain some further comparison results, and we conclude by giving a sufficient condition for a given c.f.e. to dominate no other c.f.e.

2. Preliminaries and notations. Let M be a von Neumann algebra acting on a Hilbert space H. We denote by $S(M)$ (resp. $S_{\mathrm{f}}(M)$) the set of normal (resp. normal faithful) states on M. For ξ in H and a in M we set $\omega_\xi(a) = \langle \xi, a\xi\rangle$. Let ϕ and ω be in $(M_*)^+$. We say that ϕ is *dominated* by ω (and denote by $m(\omega)$ the set of such functionals) if it is majorized by some positive multiple of ω. If M_0 is a von Neumann subalgebra of M we set $\omega_0 = \omega|M_0$ for all ω in M_*. For ω in $S(M)$ we denote by $[\varepsilon(\omega)]$ the ω conditional expectation from M to M_0 introduced in [1] (see also [4], [5]).

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