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Convergence in nonisotropic regions of harmonic functions in $\mathbb{B}^n$

by

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To the memory of Joaquin M. Cascante

Abstract. We study the boundedness in $L^p(\mathbb{S}^n)$ of the projections onto spaces of functions with spectrum contained in horizontal strips. We obtain some results concerning convergence along nonisotropic regions of harmonic extensions of functions in $L^p(\mathbb{S}^n)$ with spectrum included in these horizontal strips.

1. Introduction. This work deals with some topics related to the expansion of functions in $L^2(\mathbb{S}^n)$, $\mathbb{S}^n$ the unit sphere in $\mathbb{C}^n$, in terms of harmonic homogeneous polynomials $H(r,s)$ of bidegree (r,s). The projections K_{rs} of $L^2(\mathbb{S}^n)$ onto $H(r,s)$ extend to $L^1(\mathbb{S}^n)$ and permit defining for every $f\in L^1(\mathbb{S}^n)$ the *spectrum* of f, $\operatorname{spec} f=\{(r,s)\in\mathbb{Z}_+\times\mathbb{Z}_+ : K_{rs}f\neq 0\}$. The orthogonal projection from $\bigoplus_{r,s}H(r,s)$ to $\bigoplus_r H(r,0)$ can be identified with the Cauchy–Szegő projection and it is well known that it can be continuously extended to $L^p(\mathbb{S}^n)$, $p>1$. What happens if we project to other $\bigoplus_{(r,s)\in\Omega}H(r,s)$? This is a very difficult problem whose answer is not known even for the Fourier expansions when $n=1$. The first object of this work is to study the boundedness in L^p when Ω is a horizontal strip $\Omega_{0k}=\{(r,s)\in\mathbb{Z}_+\times\mathbb{Z}_+ : 0\le s\le k\}$.

It is well known that the harmonic extensions of $L^p(\mathbb{S}^n)$ to $\mathbb{B}^n$ have a limit a.e. along nontangential regions and that if the function is in H^p, that is, its spectrum is in $\mathbb{Z}_+\times\{0\}$, then there is convergence along admissible regions that are tangential in some directions, if $n>1$. Is there any relation of this fact with the spectrum of the function? The second topic of this work is to study convergence along admissible and other tangential regions of harmonic extensions of functions with spectrum in Ω_{0k}.

The paper is organized as follows: in the second section we show that,

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as it happens with the Cauchy–Szegő projection, the orthogonal projection $C^*_{\Omega_{0k}}$ from $L^2(\mathbb{S}^n)$ onto $L^2_{\Omega_{0k}}$, the space of functions in $L^2(\mathbb{S}^n)$ with spectrum in Ω_{0k}, induces a bounded operator from $L^p(\mathbb{S}^n)$ to $L^p_{\Omega_{0k}}$, $p > 1$. This will be proved by obtaining an explicit formula for the projections, from which we deduce that they are operators of order 0, in the sense of [NaRoStWa].

In the third section we show that the space of harmonic transforms of functions in $L^p(\mathbb{S}^n)$ with spectrum in Ω_{0k} behaves, with respect to admissible convergence, very much like the space of holomorphic functions in $H^p(\mathbb{B}^n)$. In particular, for such spaces of harmonic functions there exists a strong L^p estimate of the admissible maximal function. We also prove that the theorem of Nagel, Rudin and Wainger ([NaRu] and [NaWa]), which shows that for any function in $H^\infty(\mathbb{B}^n)$, there exist radial limits at almost every point of a transverse curve, extends to bounded harmonic functions with spectrum in Ω_{0k}.

In the fourth section we study convergence in tangential approach regions of harmonic transforms of nonisotropic potentials of functions in $L^p_{\Omega_{0k}}$. The spaces of potentials in $L^p(\mathbb{S}^n)$, given by nonisotropic convolution with Riesz-type kernels, coincide, in the integer case, with nonisotropic Sobolev spaces on the unit sphere, and for the general case, when restricted to $L^p_{\Omega_{0k}}$, can be obtained by the complex interpolation method. A direct proof of this last fact is given in the appendix.

2. Boundedness in L^p. We begin the section with some definitions. Given $\Omega \subset \mathbb{Z}^2_+$, $\mathbb{Z}_+ = \{r \in \mathbb{Z} : r \geq 0\}$, let $\bigoplus_{(r,s)\in\Omega} H(r,s)$ be the algebraic sum of all spaces $H(r,s)$ with $(r,s) \in \Omega$. Here $H(r,s)$ is the space of harmonic homogeneous polynomials in $\mathbb{C}^n$ that have total degree r in the variables $z_1,\ldots,z_n$ and total degree s in $\overline{z}_1,\ldots,\overline{z}_n$. If $k \leq m$ we will write $\Omega_{km} = \{(r,s) \in \mathbb{Z}^2_+ : k \leq s \leq m\}$. If X is a Banach space of integrable functions on $\mathbb{S}^n$, containing the spaces $H(r,s)$, X_Ω will denote the closure of $\bigoplus_{(r,s)\in\Omega} H(r,s)$ in X. The space of harmonic extensions of functions in X_Ω will be denoted by $Q[X_\Omega]$. The Poisson kernel in the unit ball is given by

$$Q(z,\zeta) = \frac{1-|z|^2}{|z-\zeta|^{2n}}, \quad z \in \mathbb{B}^n,\ \zeta \in \mathbb{S}^n,$$

and if $f \in L^1(\mathbb{S}^n)$ and $z \in \mathbb{B}^n$, we denote the Poisson transform by

$$Q[f](z) = \int_{\mathbb{S}^n} Q(z,\zeta) f(\zeta)\, d\sigma(\zeta).$$

When $X = L^2(\mathbb{S}^n)$, the identification of the Hilbert space $L^2_\Omega(\mathbb{S}^n)$ with its harmonic extension gives for each $z \in \mathbb{B}^n$ a function $C_\Omega(z,\cdot) \in L^2_\Omega(\mathbb{S}^n)$

such that for any f in $Q[L^2_\Omega]$,

$$f(z) = \int_{\mathbb{S}^n} C_\Omega(z,\zeta) f(\zeta)\, d\sigma(\zeta).$$

If K_{rs} is the kernel associated with the orthogonal projection from $L^2(\mathbb{S}^n)$ onto $H(r,s)$, it is shown in [Do] that for $\zeta \in \mathbb{S}^n$ and $z \in \mathbb{B}^n$,

$$C_\Omega(z,\zeta) = \sum_{(r,s)\in\Omega} K_{rs}(z,\zeta). \tag{2.1}$$

The convergence is uniform in $\zeta \in \mathbb{S}^n$ and z in compact subsets of $\mathbb{B}^n$.

An explicit formula for the kernels K_{rs} can be found in [Al] (see also [Ru]). If $z \in \mathbb{B}^n$, $\zeta \in \mathbb{S}^n$, and $n \geq 2$, then

$$K_{rs}(z,\zeta) = D(r,s,n)(z\overline{\zeta})^r(\overline{z}\zeta)^s F\Big(-r,-s,n-1,1-\frac{|z|^2}{|z\overline{\zeta}|^2}\Big), \tag{2.2}$$

where

$$D(r,s,n) = \binom{r+n-2}{r}\binom{s+n-2}{s}\frac{r+s+n-1}{n-1} \tag{2.3}$$

is the dimension of $H(r,s)$, and

$$F(a,b,c,x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} x^k$$

is the hypergeometric function (here $(a)_k = a(a+1)\ldots(a+k-1)$).

It is also well known that if $z \in \mathbb{S}^n$, then

$$\|K_{rs}(z,\cdot)\|_2^2 = K_{rs}(z,z) = D(r,s,n). \tag{2.4}$$

For $r \geq s$, it is also shown in [Al] that

$$K_{rs}(z,\zeta) = D(r,s,n)(z\overline{\zeta})^{r-s}\mathcal{P}_s^{r-s,n-2}\Big(\frac{|z\overline{\zeta}|^2}{|z|^2}\Big)|z|^{2s}, \tag{2.5}$$

where

$$\mathcal{P}_s^{\alpha\beta}(x) = C\Big\{\sum_{m=0}^{s}\binom{s+\beta}{m}\binom{s+\alpha}{s-m}(x-1)^{s-m}x^m\Big\}$$

are the Jacobi polynomials, orthogonal in $L^2([0,1], x^\alpha(1-x)^\beta dx)$ to all polynomials of degree less than s, normalized by the condition $\mathcal{P}_s^{\alpha\beta}(1) = 1$. Observe that when $\Omega = \Omega_{00}$, $C_{\Omega_{00}}(z,\zeta)$ is the Cauchy–Szegő kernel and when $\Omega = \mathbb{Z}^2_+$, $C_\Omega(z,\zeta)$ is the harmonic Poisson kernel $Q(z,\zeta)$.

First we will calculate, for any $k \in \mathbb{Z}_+$, the kernel $C_{\Omega_{0k}}(z,\zeta)$. We want to obtain an expression to which we could apply the theory of operators of order 0 (in the sense of [NaRoStWa]).

For $s\in\mathbb{Z}_+$, $z\in\mathbb{B}^n$, and $\zeta\in\mathbb{S}^n$, we simply write $C_s(z,\zeta)$ instead of $C_{\Omega_{ss}}(z,\zeta)$ and C_{0k} instead of $C_{\Omega_{0k}}$.

THEOREM 2.1. *Assume $n\ge 2$. Then for any $z\in\mathbb{B}^n$ and $\zeta\in\mathbb{S}^n$, we have*

(i) $$C_s(z,\zeta)=\sum_{m=0}^{s}\frac{\binom{s+n-2}{m}}{(n-1)!(s-m)!}(|z\bar\zeta|^2-|z|^2)^{s-m}(\bar z\zeta)^m \times\left(\frac{(n-m+s-1)!}{(1-z\bar\zeta)^{n-m+s}}+\frac{s(n-m+s-2)!}{(1-z\bar\zeta)^{n-m+s-1}}\right)$$

(ii) $$C_{0k}(z,\zeta)=\binom{n+k-1}{k}\frac{(\bar z\zeta-|z|^2)^k}{(1-z\bar\zeta)^{n+k}} +\sum_{j=1}^{k}\binom{n+j-2}{j-1}\frac{(\bar z\zeta-|z|^2)^{j-1}(1-|z|^2)}{(1-z\bar\zeta)^{n+j-1}}.$$

REMARK. Since $|\bar z\zeta-|z|^2|<|1-z\bar\zeta|$, the limit as $k\to\infty$ of the first summand in (ii) is zero, whereas the second summand tends to
$$\frac{1-|z|^2}{(1-z\bar\zeta)^n}\left(1-\frac{\bar z\zeta-|z|^2}{1-z\bar\zeta}\right)^{-n}=\frac{1-|z|^2}{|\zeta-z|^{2n}},$$
the harmonic Poisson kernel in $\mathbb{B}^n$.

Proof (of Theorem 2.1). If we use (2.5), we obtain
$$\sum_{r\ge s}K_{rs}(z,\zeta)$$
$$=\sum_{r\ge s}\frac{\binom{r+n-2}{r}(r+s+n-1)}{n-1}(z\bar\zeta)^{r-s} \times\left\{\sum_{m=0}^{s}\binom{s+n-2}{m}\binom{r}{s-m}\left(\frac{|z\bar\zeta|^2}{|z|^2}-1\right)^{s-m}\left(\frac{|z\bar\zeta|^2}{|z|^2}\right)^m\right\}|z|^{2s}$$
$$=\sum_{r\ge s}\frac{\binom{r+n-2}{r}(r+s+n-1)}{n-1}(z\bar\zeta)^{r-s} \times\left\{\sum_{m=0}^{s}\binom{s+n-2}{m}\binom{r}{s-m}(|z\bar\zeta|^2-|z|^2)^{s-m}|z\bar\zeta|^{2m}\right\}$$
$$=\sum_{r\ge 0}\frac{\binom{r+s+n-2}{r+s}(r+2s+n-1)}{n-1}(z\bar\zeta)^{r} \times\left\{\sum_{m=0}^{s}\binom{s+n-2}{m}\binom{r+s}{s-m}(|z\bar\zeta|^2-|z|^2)^{s-m}|z\bar\zeta|^{2m}\right\}$$

$$=\sum_{m=0}^{s}\frac{\binom{s+n-2}{m}}{(n-1)!(s-m)!} \times\sum_{r\ge 0}\frac{(r+2s+n-1)(r+s+n-2)!}{(r+m)!}|z\bar\zeta|^{2m}(|z\bar\zeta|^2-|z|^2)^{s-m}(z\bar\zeta)^r$$
$$=\sum_{m=0}^{s}\frac{\binom{s+n-2}{m}}{(n-1)!(s-m)!}(|z\bar\zeta|^2-|z|^2)^{s-m}(\bar z\zeta)^m A_{sm}(z,\zeta),$$
where we have defined
$$A_{sm}(z,\zeta)=\sum_{r\ge 0}(r+2s+n-1)(r+s+n-2)(r+s+n-3)\ldots(r+m+1)(z\bar\zeta)^{r+m}.$$

Next
$$A_{sm}(z,\zeta)=\sum_{r\ge m}(r+2s-m+n-1)\frac{(r+s-m+n-2)!}{r!}(z\bar\zeta)^r =\frac{(n-m+s-1)!}{(1-z\bar\zeta)^{n-m+s}}+\frac{s(n-m+s-2)!}{(1-z\bar\zeta)^{n-m+s-1}} -\sum_{r\le m-1}\frac{(r+2s-m+n-1)(r+s-m+n-2)!}{r!}(z\bar\zeta)^r.$$

So we have obtained

(2.6) $$C_s(z,\zeta) =\sum K_{rs}(z,\zeta) =\sum_{r\le s-1}K_{rs}(z,\zeta)-\sum_{m=0}^{s}\frac{\binom{s+n-2}{m}}{(n-1)!(s-m)!}(|z\bar\zeta|^2-|z|^2)^{s-m}(z\bar\zeta)^m \times\left\{\sum_{r\le m-1}\frac{(r+2s-m+n-1)(r+s-m+n-2)!}{(n-1)!(s-m)!r!}(z\bar\zeta)^r\right\} +\sum_{m=0}^{s}\frac{\binom{s+n-2}{m}}{(n-1)!(s-m)!}(|z\bar\zeta|^2-|z|^2)^{s-m}(\bar z\zeta)^m \times\left\{\frac{(n-m+s-1)!}{(1-z\bar\zeta)^{n-m+s}}+\frac{s(n-m+s-2)!}{(1-z\bar\zeta)^{n}}\right\}.$$

Now, for each $r<s$,
$$K_{rs}(z,\zeta)=D(r,s,n)(z\bar\zeta)^r(\bar z\zeta)^sF\left(-r,-s,n-1,\frac{1-|z|^2}{|z\bar\zeta|^2}\right)$$

$$= \binom{r+n-2}{r}\binom{s+n-2}{s}\frac{r+s+n-1}{n-1} \times \sum_{m=0}^{r} \frac{r!s!}{m(m!)^2}(|z\bar{\zeta}|^2-|z|^2)^m(\bar{z}\zeta)^{s-r}|z\bar{\zeta}|^{2(r-m)}.$$

If we use this formula, a careful calculation of the sum of the first two sumands in (2.6) shows that its value is identically zero, and consequently we obtain part (i).

(ii) is a consequence of the following lemma.

LEMMA 2.2. *Assume $n \geq 2$ and let $s \geq 1$ be a nonnegative integer. Then for $z \in \mathbb{B}^n$ and $\zeta \in \mathbb{S}^n$, we have*

$$C_s(z,\zeta) + \binom{n+s-2}{s-1}\frac{(\bar{z}\zeta-|z|^2)^{s-1}}{(1-z\bar{\zeta})^{n+s-1}} = \binom{n+s-1}{s}\frac{(\bar{z}\zeta-|z|^2)^{s}}{(1-z\bar{\zeta})^{n+s}} + \binom{n+s-2}{s-1}\frac{(\bar{z}\zeta-|z|^2)^{s-1}}{(1-z\bar{\zeta})^{n+s-1}}(1-|z|^2).$$

Proof. In the formula obtained in Theorem 2.1(i) we write

$$(|z\bar{\zeta}|^2-|z|^2)^{s-m} = ((z\bar{\zeta}-1)\bar{z}\zeta+(\bar{z}\zeta-|z|^2))^{s-m},$$

develop the above sum, and group together the terms which have the same power of $1-z\bar{\zeta}$ in the denominator. We then obtain

$$\begin{aligned} C_s(z,\zeta) = \sum_{m=0}^{s}\frac{\binom{s+n-2}{m}}{(n-1)!(s-m)!}\Big\{&\frac{(n-m+s-1)!}{(1-z\bar{\zeta})^{n-m+s}}(\bar{z}\zeta)^m(\bar{z}\zeta-|z|^2)^{s-m} \\ &+\sum_{j=1}^{s-m}\frac{(-1)^j}{(1-z\bar{\zeta})^{n-m+s-j}} \\ &\times\Big((n-m+s-1)!\binom{s-m}{j}(\bar{z}\zeta)^{j+m}(\bar{z}\zeta-|z|^2)^{s-m-j} \\ &-s(n-m+s-2)!\binom{s-m}{j-1}(\bar{z}\zeta)^{j-1+m}(\bar{z}\zeta-|z|^2)^{s-m-j+1}\Big) \\ &+(-1)^{s-m}\frac{s(n-m+s-2)!}{(1-z\bar{\zeta})^{n-1}}(\bar{z}\zeta)^s\Big\}. \end{aligned}$$

Since

$$\sum_{m=0}^{s}\frac{\binom{s+n-2}{m}}{(s-m)!}(-1)^{s-m}(n-m+s-2)! = \frac{(s+n-2)!}{s!}\sum_{m=0}^{s}\binom{s}{m}(-1)^{s-m} = 0,$$

the right-hand side above can be rewritten as $A+B$, where

$$\begin{aligned} A = \sum_{m=0}^{s}\frac{\binom{s+n-2}{m}}{(n-1)!(s-m)!}\Big\{&\frac{(n-m+s-1)!}{(1-z\bar{\zeta})^{n-m+s}}(\bar{z}\zeta)^m(\bar{z}\zeta-|z|^2)^{s-m} \\ &+\sum_{j=1}^{s-m}(-1)^j\frac{(n-m+s-1)!\binom{s-m}{j}(\bar{z}\zeta)^{j+m}(\bar{z}\zeta-|z|^2)^{s-m-j}}{(1-z\bar{\zeta})^{n-m+s-j}}\Big\}, \end{aligned}$$

$$\begin{aligned} B = s\sum_{m=0}^{s}&\frac{\binom{s+n-2}{m}}{(n-1)!(s-m)!} \\ &\times\sum_{j=1}^{s-m}(-1)^j\frac{(n-m+s-2)!\binom{s-m}{j-1}(\bar{z}\zeta)^{j-1+m}(\bar{z}\zeta-|z|^2)^{s-m-j+1}}{(1-z\bar{\zeta})^{n-m+s-j}}. \end{aligned}$$

For each $0 \leq k \leq s-2$, we will check that the sum of the terms in A that have $(1-z\bar{\zeta})^{n+k}$ as common denominator is zero. Indeed this sum equals

$$\begin{aligned} (\bar{z}\zeta)^{s-k}(\bar{z}\zeta-|z|^2)^k\Big\{&\frac{\binom{s+n-2}{s-k}(n+k-1)!}{(n-1)!k!} \\ &+\sum_{m=0}^{s-k-1}\frac{\binom{s+n-2}{m}}{(n-1)!(s-m)!}(-1)^{s-m-k}(n-m+s-1)!\binom{s-m}{s-k-m}\Big\}. \end{aligned}$$

But

$$\begin{aligned} \frac{(s+n-2)!(n+k-1)}{(s-k)!(n-1)!k!} &+ \sum_{m=0}^{s-k-1}(-1)^{s-k-m}\frac{(n-m+s-1)(s+n-2)!}{(n-1)!m!(s-m-k)!k!} \\ &= \frac{(s+n-2)!}{(n-1)!k!}\Big\{\frac{n+k-1}{(s-k)!}+\sum_{m=0}^{s-k-1}(-1)^{s-k-m}\frac{n-m+s-1}{m!(s-m-k)!}\Big\} \\ &= \frac{(s+n-2)!}{(n-1)!k!}\Big\{\sum_{m=0}^{s-k}(-1)^{s-k-m}\frac{n-m+s-1}{m!(s-m-k)!}\Big\} \end{aligned}$$

and

$$\begin{aligned} \sum_{m=0}^{s-k}&(-1)^{s-k-m}\frac{n-m+s-1}{m!(s-m-k)!} \\ &= (-1)^{s-k-m}\Big\{\frac{n+s-1}{(s-k)!}\sum_{m=0}^{s-k}\binom{s-k}{m}-\frac{1}{(s-k)!}\sum_{m=0}^{s-k}m\binom{s-k}{m}\Big\}. \end{aligned}$$

The fact that

$$\frac{d}{dx}(x-1)^{q-k} = \sum_{m=1}^{s-k}\binom{s-k}{m}(-1)^m x^{m-1}m$$

easily implies that the above sum is zero, since $k \leq s-2$.

In consequence we obtain

$$A+\binom{n+s-2}{s-1}\frac{(\overline{z}\zeta-|z|^2)^{s-1}}{(1-z\overline{\zeta})^{n+s-1}}$$
$$=\left\{(n+s-2)\binom{n+s-2}{s-1}-\binom{n+s-1}{s}s+\binom{n+s-2}{s-1}\right\}\frac{(\overline{z}\zeta-|z|^2)^{s-1}}{(1-z\overline{\zeta})^{n+s-1}}$$
$$+\binom{n+s-1}{s}\frac{(\overline{z}\zeta-|z|^2)^{s}}{(1-z\overline{\zeta})^{n+s}}$$
$$=\binom{n+s-2}{s-1}\frac{(\overline{z}\zeta-|z|^2)^{s-1}}{(1-z\overline{\zeta})^{n+s-1}}(1-z\overline{\zeta})+\binom{n+s-1}{s}\frac{(\overline{z}\zeta-|z|^2)^{s}}{(1-z\overline{\zeta})^{n+s}}.$$

An analogous calculation shows that

$$B=\binom{n+s-2}{s-1}\frac{(\overline{z}\zeta-|z|^2)^{s}}{(1-z\overline{\zeta})^{n+s-1}}.$$

Adding both formulas we finally obtain the lemma. ■

Now formula (ii) of Theorem 2.1 is deduced from Lemma 2.2 and the expression of the Cauchy kernel. ■

We will next check that the limit as $|z|\to 1$ of the kernels $C_{\Omega_{0k}}$ is a singular operator of order 0 in the sense of [NaRoStWa]. We recall some definitions.

Let T_{ij}, $1\le i<j\le n$, be the complex tangential vector fields

$$T_{ij}=\overline{z}_i\frac{\partial}{\partial z_j}-\overline{z}_j\frac{\partial}{\partial z_i},\qquad \overline{T}_{ij}=z_i\frac{\partial}{\partial \overline{z}_j}-z_j\frac{\partial}{\partial \overline{z}_i}.$$

We denote by $X^I=X_1\ldots X_m$ a complex differential operator whose vector fields X_l, $1\le l\le m$, are of type T_{ij} or $\overline{T}_{ij}$ for some $i<j$. The weight of X^I is $\omega(X^I)=m/2$ if $X^I=X_1\ldots X_m$.

Let $Kf(\zeta)=\int_{\mathbb{S}^n}K(\zeta,\omega)f(\omega)\,d\sigma(\omega)$ for $f\in\mathcal{C}^\infty(\mathbb{S}^n)$, where $K(\zeta,\omega)$ is a distribution which is $\mathcal{C}^\infty$ outside the diagonal. The operator K is of order m if there exists a family of operators $K_\varepsilon[f](\zeta)=\int_{\mathbb{S}^n}K_\varepsilon(\zeta,\omega)f(\omega)\,d\sigma(\omega)$, such that

(a) $K_\varepsilon(\zeta,\omega)\in\mathcal{C}^\infty(\mathbb{S}^n\times\mathbb{S}^n)$.

(b) $K_\varepsilon f\to Kf$ in $\mathcal{C}^\infty(\mathbb{S}^n)$, for each f in $\mathcal{C}^\infty(\mathbb{S}^n)$.

(c) The following conditions hold uniformly in ε:

(c-1) For any X^I, X^J,

$$|X^I_\omega X^J_\zeta K_\varepsilon(\zeta,\omega)|\le C_{XY}|1-\zeta\overline{\omega}|^{m-\omega(X^I)-\omega(X^J)-n},$$

(c-2) For any $l\ge 0$ there exist N_l, $C_l>0$ so that for any smooth function φ supported in $B(\zeta_0,\delta)=\{\zeta\in\mathbb{S}^n:|1-\zeta_0\overline{\zeta}|<\delta\}$ and every X^I with $\omega(X^I)=l/2$,

$$X^IK_\varepsilon[\varphi](\zeta_0)|\le C_l\delta^{-l/2+m}\sup_\zeta\sum_{\omega(X^J)\le N_l}\delta^{\omega(X^J)}|X^J\varphi(\zeta)|.$$

The same estimates must hold for the adjoint operator K^* with associated kernel $\overline{K(\omega,\zeta)}$.

If l is a nonnegative integer, and $L^p_l(\mathbb{S}^n)$ is the nonisotropic Sobolev space of functions with tangential derivatives up to weight l in L^p, $p>1$, and K is an operator of order m, then (see [NaRoStWa]) K maps continuously $L^p_l(\mathbb{S}^n)$ in $L^p_{l+m}(\mathbb{S}^n)$.

It will be convenient for some computations to consider, for $1\le i,j\le n$, the generators of the tangent vector fields to $\mathbb{S}^n$

$$N_{ij}=z_i\frac{\partial}{\partial z_j}-\overline{z}_j\frac{\partial}{\partial \overline{z}_i},$$

which have the property (see [Ge]) that for any smooth functions $f:\mathbb{S}^n\to\mathbb{C}$ and $\Phi:\mathbb{C}\to\mathbb{C}$,

$$\int_{\mathbb{S}^n}N_{ij}\Phi(z\overline{\zeta})f(\zeta)\,d\sigma(\zeta)=\int_{\mathbb{S}^n}\Phi(z\overline{\zeta})N_{ij}f(\zeta)\,d\sigma(\zeta).$$

By induction it can be proved that if $X=X_1\ldots X_k$ is a differential operator with X_i, $i=1,\ldots,k$, tangent vector fields with coefficients in $\mathcal{C}^\infty(\overline{\mathbb{B}}^n)$, there exist differential operators $Y^\alpha=Y^\alpha_1\ldots Y^\alpha_{|\alpha|}$ with $|\alpha|\le k$, Y^α_i tangent, and smooth functions $\varphi_\alpha(z,\zeta)$ satisfying

$$|Z_z\varphi_\alpha(z,\zeta)|\le C|1-z\overline{\zeta}|^{\max(0,\omega(Y^\alpha)-\omega(X)-\omega(Z))}$$

such that

$$X_z\int_{\mathbb{S}^n}\Phi(z\overline{\zeta})f(\zeta)\,d\sigma(\zeta)=\sum_{|\alpha|\le k}\int_{\mathbb{S}^n}\Phi(z\overline{\zeta})\varphi_\alpha(z,\zeta)Y^\alpha f(\zeta)\,d\sigma(\zeta).\tag{2.7}$$

As already mentioned, the operators $C_s(z,\zeta)$ when $|z|\to 1$ can be realized as operators of order 0. It is easy to verify that if we define $C^r_s(\omega,\zeta)=C_s(r\omega,\zeta)$ for ζ,ω in $\mathbb{S}^n$ and $r<1$, then $\lim_{r\to1}C^r_s[f](\zeta)$ exists for any f in $\mathcal{C}^\infty(\mathbb{S}^n)$, and it defines a function in $\mathcal{C}^\infty(\mathbb{S}^n)$. We will write $C^*_s[f]$ for the value of this limit. We then have

PROPOSITION 2.3. *For each $s\ge 0$ the operator C^*_s is of order* 0.

Proof. Since the case $s=0$ is the Cauchy–Szegő projection and the result is well known, we assume that $s>0$. From the definition of C^*_s it is clear that conditions (a) and (b) are satisfied. We next check condition (c-1). Theorem 2.1(i) shows that for any $i<j$,

$$
\begin{aligned}
&T^{\omega}_{ij} C^r_s(\omega,\zeta)\\
&= \sum_{m=0}^{s} \frac{r^s\binom{s+n-2}{m}}{(n-1)!(s-m)!}\Big\{(s-m)(\overline{\omega}_i\overline{\zeta}_j-\overline{\omega}_j\overline{\zeta}_i)(\overline{\omega}\zeta)^{m+1}(|\omega\overline{\zeta}|^2-1)^{s-m-1}\\
&\quad\times\bigg(\frac{(n-m+s-1)!}{(1-r\omega\overline{\zeta})^{n-m+s}}+\frac{s(n-m+s-2)!}{(1-r\omega\overline{\zeta})^{n-m+s-1}}\bigg)+r(\overline{\omega}\zeta)^m(|\omega\overline{\zeta}|^2-1)^{s-m}\\
&\quad\times\bigg(\frac{(n-m+s-1)!(n-m+s)}{(1-r\omega\overline{\zeta})^{n-m+s+1}}(\overline{\omega}_i\overline{\zeta}_j-\overline{\omega}_j\overline{\zeta}_i)\\
&\quad+\frac{s(n-m+s-2)!(n-m+s-1)}{(1-r\omega\overline{\zeta})^{n-m+s}}(\overline{\omega}_i\overline{\zeta}_j-\overline{\omega}_j\overline{\zeta}_i)\bigg)\Big\}.
\end{aligned}
$$

Now the fact that $\sum_{i<j}|\omega_i\zeta_j-\omega_j\zeta_i|^2=1-|\omega\overline{\zeta}|^2$ gives

$$T^{\omega}_{ij} C^r_s(\omega,\zeta)=O\bigg(\frac{1}{|1-\omega\overline{\zeta}|^{n+1/2}}\bigg).$$

The same argument can be used to show a similar estimate for $\overline{T}^{\omega}_{ij} C^r_s(\omega,\zeta)$, and that for any composition of complex tangential vector fields X^I_ω, X^J_ζ,

$$|X^I_\omega X^J_\zeta C^r_s(\omega,\zeta)|=O\bigg(\frac{1}{|1-\omega\overline{\zeta}|^{n+\omega(X^I)+\omega(X^J)}}\bigg)$$

uniformly in r.

In order to finish we just need to check that the kernels C^r_s satisfy condition (c-2) uniformly in r. We will use the following

LEMMA 2.4. *Let k be a nonnegative integer. There exists $C>0$ so that for all $\varepsilon>0$ and $z\in\mathbb{B}^n$,*

$$\Big|\int\limits_{|1-z\overline{\zeta}|>\varepsilon} C_{0k}(z,\zeta)\,d\sigma(\zeta)\Big|\le C.$$

Proof. It is immediate to verify that all the summands in formula (ii) of Theorem 2.1 have the property that the integral over $\mathbb{S}^n$ of their modulus is bounded independently of z, except for

$$\binom{n+k-1}{k}\frac{(|z\overline{\zeta}|^2-|z|^2)^k}{(1-z\overline{\zeta})^{n+k}}.$$

So we are led to prove that for $z\in\mathbb{B}^n$,

$$\int\limits_{|1-z\overline{\zeta}|>\varepsilon}\frac{(|z\overline{\zeta}|^2-|z|^2)^k}{(1-z\overline{\zeta})^{n+k}}\,d\sigma(\zeta)=O(1). \tag{2.8}$$

Since for any $0<s\le k$,

$$\int\limits_{\mathbb{S}^n}\frac{(1-|z|^2)^s(1-|z\overline{\zeta}|^2)^{k-s}}{|1-z\overline{\zeta}|^{n+k}}\,d\sigma(\zeta)=O(1),$$

the estimates in (2.8) will hold once we prove that

$$\int\limits_{|1-z\overline{\zeta}|>\varepsilon}\frac{(1-|z\overline{\zeta}|^2)^k}{(1-z\overline{\zeta})^{n+k}}\,d\sigma(\zeta)=O(1)$$

uniformly in $1/2<|z|<1$, $\varepsilon>0$. Let $\lambda=|z|$. Then a unitary change of variables shows that the above is equivalent to

$$\int\limits_{|1-\lambda\overline{\zeta}_1|>\varepsilon}\frac{(1-\lambda^2|\zeta_1|^2)^k}{(1-\lambda\overline{\zeta}_1)^{n+k}}\,d\sigma(\zeta)=O(1).$$

But the above integral equals

$$
\begin{aligned}
&\frac{n-1}{\pi}\int\int\limits_{\{re^{i\theta}\in\mathbb{D}\,:\,\varepsilon<|1-\lambda re^{i\theta}|\}}\frac{(1-\lambda^2r^2)^{n-2+k}}{(1-\lambda re^{-i\theta})^{n+k}}r\,dr\,d\theta\\
&\qquad=\frac{n-1}{\pi\lambda^2}\int\int\limits_{\{\varrho e^{i\theta}\in\mathbb{D}\,:\,\varrho<\lambda,\ \varepsilon<|1-\varrho e^{i\theta}|\}}\frac{(1-\varrho^2)^{n-2+k}}{(1-\varrho e^{-i\theta})^{n+k}}\varrho\,d\varrho\,d\theta,
\end{aligned}
$$

and the last integral can be easily seen to be bounded uniformly in ε, which completes the proof. ■

Going back to the proof of condition (c-2), we have to prove that

$$|X^I C^r_s[\varphi](\zeta_0)|\preceq\delta^{-\omega(X^I)}\sup_{\omega(Y^J)\le N^I_X}\delta^{\omega(Y^J)}\|Y^J\varphi\|_\infty,$$

for any C^∞ function φ on $\mathbb{S}^n$ such that $\operatorname{supp}\varphi\subset B(\zeta_0,\delta)$ for $\zeta_0\in\mathbb{S}^n$, $\delta>0$, and any composition $X^I=X_1\ldots X_k$ of complex tangential vector fields. If $\delta\preceq1-r$, then

$$
\begin{aligned}
|X^I C^r_s[\varphi](\zeta_0)|&\preceq\int\limits_{B(\zeta_0,\delta)}|X^I C_s(r\zeta_0,\zeta)|\cdot|\varphi(\zeta)|\,d\sigma(\zeta)\\
&\preceq(1-r)^{-n-\omega(X^I)}\delta^n\|\varphi\|_\infty\preceq\delta^{-\omega(X^I)}\|\varphi\|_\infty.
\end{aligned}
$$

Thus it is enough to assume that $1-r\preceq\delta$.

Now, let $X=X_1\ldots X_k$ be a differential operator with X_i complex tangential vector fields. By (2.7),

$$
\begin{aligned}
&XC^r_s[\varphi](\zeta_0)\\
&\quad=\sum_{|\alpha|\le k}\int\limits_{\mathbb{S}^n}C_s(r\zeta_0,\zeta)\{\varphi_\alpha(r\zeta_0,\zeta)Y^\alpha\varphi(\zeta)-\varphi_\alpha(r\zeta_0,\zeta_0)Y^\alpha\varphi(\zeta_0)\}\,d\sigma(\zeta).
\end{aligned}
$$

Since φ is supported in $B(\zeta_0,\delta)$, the previous lemma shows that the part of the above integral over $\mathbb{S}^n \setminus B(\zeta_0,\delta)$ is bounded by

$$\sum_{|\alpha|\le k} \delta^{\max(0,\omega(Y^\alpha)-k/2)}\|Y^\alpha\varphi\|_\infty \le \sum_{|\alpha|\le k} \delta^{-k/2}\delta^{\omega(Y^\alpha)}\|Y^\alpha\varphi\|_\infty.$$

For the integral over $B(\zeta_0,\delta)$, the regularity of $Y^\alpha\varphi$ together with the properties of the functions φ_α show that (see [BrOr])

$$|Y^\alpha\varphi(\zeta)-Y^\alpha\varphi(\zeta_0)| \preceq \sum_{i<j}\|T_{ij}Y^\alpha\varphi\|_\infty|1-\zeta\overline{\zeta}_0|^{1/2},$$

$$|\varphi_\alpha(r\zeta_0,\zeta)-\varphi_\alpha(r\zeta_0,\zeta_0)| \preceq \delta^{\max(0,\omega(Y^\alpha)-\omega(X)-1/2)}|1-\zeta\overline{\zeta}_0|^{1/2},$$

which easily gives

$$\Big|\int_{B(\zeta_0,\delta)} C_s(r\zeta_0,\zeta)\{\varphi_\alpha(r\zeta_0,\zeta)Y^\alpha\varphi(\zeta)-\varphi_\alpha(r\zeta_0,\zeta_0)Y^\alpha\varphi(\zeta_0)\}\,d\sigma(\zeta)\Big|$$
$$\preceq \delta^{-k/2}\sum_Y \delta^{\omega(Y)}\|Y\varphi\|_\infty. \ \blacksquare$$

As a corollary we obtain

COROLLARY 2.5. *For any nonnegative integers k, l, and $1<p<\infty$, the singular operator $C^*_{\Omega_{0k}}$ maps continuously $L^p_l(\mathbb{S}^n)$ to itself.*

One consequence of the above corollary is that the functions in $L^p_{l,\Omega_{0k}}(\mathbb{S}^n)$ can be characterized in terms of their spectrum.

COROLLARY 2.6. *Let $1<p<\infty$, and let k, l be nonnegative integers. Then*

$$L^p_{l,\Omega_{0k}}(\mathbb{S}^n)=\{f\in L^p_l(\mathbb{S}^n): f_{rs}=K_{rs}[f]=0 \text{ for all } s>k \text{ and any } r\}.$$

REMARK. Observe that for $k=0$ this corollary can be obtained directly. Using a Bochner–Riesz summation (see for instance [BoCl]), every function f in $L^p(\mathbb{S}^n)$ can be approximated by polynomials in $\bigoplus_{r+s\le k}H(r,s)$ whose spectra are included in the spectrum of f. In consequence, if $\Omega\subset\mathbb{Z}^2_+$, then the functions in $L^p(\mathbb{S}^n)$ with spectrum in Ω can be approximated in L^p by polynomials with spectrum in Ω.

Given a smooth function F on $\mathbb{B}^n$, we will say that $\overline{\partial}^kF=0$ if

$$\frac{\partial^k}{\partial\overline{z}_{i_1}\ldots\partial\overline{z}_{i_k}}F=0$$

for any $i_1,\ldots,i_k\in\{1,\ldots,n\}$. Then the following characterization of the harmonic extensions of L^p functions with spectrum in Ω_{0k} holds:

PROPOSITION 2.7 ([Do]). *If F is in $Q[L^p_l(\mathbb{S}^n)]$ and $\overline{\partial}^{k+1}F=0$, then $F\in Q[L^p_{l,\Omega_{0k}}(\mathbb{S}^n)]$. Conversely, any F in $Q[L^p_{l,\Omega_{0k}}(\mathbb{S}^n)]$ satisfies $\overline{\partial}^{k+1}F=0$.*

3. Admissible convergence of harmonic functions. In this section we will show that some assertions relating to admissible convergence of holomorphic functions in Hardy spaces still hold in the spaces of harmonic extensions of L^p functions with spectrum in Ω_{0k}. We recall some definitions.

Given $f:\mathbb{B}^n\to\mathbb{C}$, the *admissible maximal function* will be denoted by $M_{\rm adm}f(\zeta)=\sup_{z\in D_\alpha(\zeta)}|f(z)|$, where $D_\alpha(\zeta)$ is the *admissible region* given by $D_\alpha(\zeta)=\{z\in\mathbb{B}^n:|1-z\overline{\zeta}|<\alpha(1-|z|^2)\}$.

THEOREM 3.1. *Let $1<p<\infty$ and let k be a nonnegative integer. Then there exists $C>0$ so that for any f in $L^p(\mathbb{S}^n)$,*

$$\|M_{\rm adm}C_{\Omega_{0k}}[f]\|_p\le C\|f\|_p.$$

Proof. If $M_{\rm rad}$ is the radial maximal operator, it is well known that

$$\|M_{\rm rad}Q[f]\|_p\le C\|f\|_p$$

for any f in $L^p(\mathbb{S}^n)$. By Corollary 2.6, $C_{\Omega_{0k}}[f]=Q[C^*_{\Omega_{0k}}[f]]$, and

$$\|M_{\rm rad}Q[C^*_{\Omega_{0k}}[f]]\|_p\le C\|C^*_{\Omega_{0k}}[f]\|_p\le C\|f\|_p.$$

Next, by Proposition 2.7, any F in $Q[L^p_{\Omega_{0k}}]$ satisfies $\overline{\partial}^{k+1}F=0$, and [AhBr, Lemma 4.4] shows that in this case $\|M_{\rm adm}F\|_p\le C\|M_{\rm rad}F\|_p$. $\blacksquare$

COROLLARY 3.2. *Let $1<p<\infty$. Any harmonic function in h^p whose spectrum lies in Ω_{0k} has admissible limit at almost every $\zeta\in\mathbb{S}^n$.*

REMARK. Corollary 3.2 could have been obtained from the last remark, which together with Proposition 2.7 and Lemma 4.4 of [AhBr] shows that $\|M_{\rm adm}Q[f]\|_p\le C\|M_{\rm rad}Q[f]\|_p\le\|f\|_p$ for any $f\in L^p_{\Omega_{0k}}$.

In general, one cannot expect to prove for arbitrary Ω that the admissible maximal function $M_{\rm adm}C_\Omega$ is bounded in L^p, even in L^2. This would imply the existence, almost everywhere, of admissible limits of functions in L^2, which is well known (see [Zy]) not to be true. One could ask if for some other regions, different from the strips Ω_{0k}, there exist such L^p estimates. We will next see a simple result which shows that if we restrict ourselves to the case where $L^p_\Omega(\mathbb{S}^n)$ is a module over the ball algebra $A(\mathbb{S}^n)$, then the sets Ω_{0k} are the only ones for which the admissible maximal function $M_{\rm adm}C_\Omega$ is bounded in L^p, $p>1$. This module condition holds ([Do]) if and only if for any (r,s) in Ω, (k,m) is also in Ω provided $k\ge r$ and $m\le s$.

We show that if the set Ω has its second projection not bounded, then the admissible maximal function is not even weakly bounded in $L^2(\mathbb{S}^n)$.

PROPOSITION 3.3. *Assume $\Omega\subset\mathbb{Z}^2_+$ satisfies the module condition. If the set of $s\in\mathbb{Z}_+$ for which there exists $r\in\mathbb{Z}_+$ with $(r,s)\in\Omega$ is not bounded in $\mathbb{Z}_+$, then the admissible maximal function does not satisfy the weak L^2-type* *estimate, that is, it does not have the property that for some* $C > 0$,

$$\sigma(\{\zeta \in \mathbb{S}^n : M_{\mathrm{adm}} C_\Omega[f](\zeta) > \lambda\}) \le C\|f\|_2^2/\lambda^2$$

for all $f \in L^2(\mathbb{S}^n)$ *and* $\lambda > 0$.

Proof. If Ω satisfies the module condition, and its second projection is not bounded in $\mathbb{Z}_+$, one can easily construct a nondecreasing function $\varphi : \mathbb{Z}_+ \to \mathbb{Z}_+$ so that $\Omega = \{(r,s) \in \mathbb{Z}_+^2 : r \ge \varphi(s)\}$.

Assume now that the weak L^2-estimate holds, and take $\zeta_0 \in \mathbb{S}^n$ and $0 < \mu < 1$. Then for any $\zeta \in B(\zeta_0, \varepsilon(1-\mu))$ ($\varepsilon > 0$ small enough), $\mu\zeta_0 \in D_\alpha(\zeta)$. In particular, for any $f \in L_+^2(\mathbb{S}^n)$ with $\|f\|_2 = 1$, and $F = C_\Omega[f]$,

$$B(\zeta_0, \varepsilon(1-\mu)) \subset \{\zeta \in \mathbb{S}^n : M_{\mathrm{adm}}F(\zeta) \ge |F(\mu\zeta_0)|\}.$$

Consequently,

$$(1-\mu)^n \preceq \sigma(\{\zeta \in \mathbb{S}^n : M_{\mathrm{adm}}F(\zeta) \ge |F(\mu\zeta_0)|\}) \preceq \frac{\|f\|_2^2}{|F(\mu\zeta_0)|^2},$$

and $|F(\mu\zeta_0)| \preceq \|f\|_2/(1-\mu)^{n/2}$.

The above estimate shows that for any $f \in L^2(\mathbb{S}^n)$ with $\|f\|_2 = 1$,

$$\Big|\int_{\mathbb{S}^n} C_\Omega(\mu\zeta_0, \zeta) f(\zeta)\, d\sigma(\zeta)\Big| = |F(\mu\zeta_0)| \preceq \frac{1}{(1-\mu)^{n/2}},$$

which by duality gives

$$\|C_\Omega(\mu\zeta_0, \cdot)\|_2 \preceq \frac{1}{(1-\mu)^{n/2}}. \tag{3.1}$$

Next, formula (2.4) together with (2.5) leads to

$$\begin{aligned}\|C_\Omega(\mu\zeta_0,\cdot)\|_2^2 &= \sum_{(r,s)\in\Omega} \|K_{rs}(\mu\zeta_0,\cdot)\|_2^2 \\ &= \sum_{(r,s)\in\Omega} \binom{r+n-2}{r}\binom{s+n-2}{s}\frac{r+s+n-1}{n-1}\mu^{2(r+s)}.\end{aligned}$$

If $\varphi_1(s) = \max(\varphi(s), s)$, we then have

$$\begin{aligned}\|C_\Omega(\mu\zeta_0,\cdot)\|_2^2 &\succeq \sum_s \mu^{2s} \sum_{r \ge \varphi_1(s)} (r+n-2)\ldots(r+1)r\mu^{2r} \\ &\succeq \frac{1}{(1-\mu^2)^n}\sum_s \mu^{2(s+\varphi_1(s))}.\end{aligned}$$

Since $\varphi_1(s) \ge s$, the above gives

$$\frac{1}{(1-\mu^2)^n}\sum_{s\ge 0}\mu^{4\varphi_1(s)} \preceq \|C_\Omega(\mu\zeta_0,\cdot)\|_2^2.$$

If $g(\mu) = \sum_{s\ge 0}\mu^{\varphi_1(s)}$, let us check that $g(\mu) \to \infty$ as $\mu \to 1$. Let $N \in \mathbb{Z}_+$ and consider $\mu < 1$ such that $\mu^{\varphi_1(2N)} > 1/2$. Since φ_1 is nondecreasing, we have $\mu^{\varphi_1(i)} \ge \mu^{\varphi_1(2N)} > 1/2$ for $i \le 2N$. Hence $g(\mu) \ge \sum_{i=1}^{2N}\mu^{\varphi_1(i)} \ge N$, which together with the previous estimate contradicts (3.1). ■

In the last part of this section we study the convergence of bounded harmonic functions with spectrum in Ω_{0k}. Nagel, Rudin and Wainger (see [NaRu] and [NaWa]) showed that every function in $H^\infty(\mathbb{B}^n)$ has radial limits at almost every point of a transverse curve in $\mathbb{S}^n$ relative to its arc-length measure. Recall that a curve γ is *transverse* if for each t, $\gamma'(t)$ does not lie in the complex tangential space at $\gamma(t)$. We will prove that these theorems extend to bounded harmonic functions in $\mathbb{B}^n$ with spectrum in Ω_{0k}.

We will assume that $\gamma : I \to \mathbb{S}^n$ is a simple closed transverse curve of class $\mathcal{C}^1$. By making a convenient reparametrization, we may assume that $I = [-\pi, \pi]$, and that γ is 2π-periodic. For $\zeta \in \mathbb{S}^n$, a *ζ-curve* is a continuous map $\varphi : [0,1) \to \mathbb{B}^n$ so that $\lim_{t\to 1}\varphi(t) = \zeta$. It is *special* if

$$\lim_{t\to 1}\frac{|\varphi(t) - (\varphi(t)\overline{\zeta})\zeta|^2}{1 - |\varphi(t)\overline{\zeta}|^2} = 0,$$

and *restricted* if it also satisfies

$$\frac{|\varphi(t)\overline{\zeta} - 1|}{1 - |\varphi(t)\overline{\zeta}|} = O(1)$$

for $0 \le t < 1$.

A function $f : \mathbb{B}^n \to \mathbb{C}$ is said to have *restricted K-limit L at $\zeta \in \mathbb{S}^n$* if $\lim_{t\to 1} f(\varphi(t)) = L$ for any restricted ζ-curve φ.

THEOREM 3.4. *Let $\gamma : [-\pi, \pi] \to \mathbb{C}$ be a simple closed transverse curve of class $\mathcal{C}^1$, and let k be a nonnegative integer. If F is a bounded harmonic function with spectrum in Ω_{0k}, then F has restricted K-limit at $\gamma(t)$ for almost every $t \in [-\pi, \pi]$.*

Proof. The hypothesis on F implies that we may assume that $F = C_{\Omega_{0k}}[f]$ with $f \in L^\infty(\mathbb{S}^n)$. Next, Theorem 2.1(ii) shows that

$$\begin{aligned}F(z) &= \frac{(n+k-1)!}{(n-1)!k!}\int_{\mathbb{S}^n}\frac{(\overline{z}\zeta - |z|^2)^k}{(1-z\overline{\zeta})^{n+k}} f(\zeta)\, d\sigma(\zeta) \\ &\quad + \sum_{j=1}^k \frac{(n+j-2)!}{(n-1)!(j-1)!}(1-|z|^2)\int_{\mathbb{S}^n}\frac{(\overline{z}\zeta - |z|^2)^{j-1}}{(1-z\overline{\zeta})^{n+j-1}} f(\zeta)\, d\sigma(\zeta).\end{aligned}$$

Now, for each $1 \le j \le k$,

$$\Big|\int_{\mathbb{S}^n}\frac{(\overline{z}\zeta - |z|^2)^{j-1}}{(1-z\overline{\zeta})^{n+j-1}} f(\zeta)\, d\sigma(\zeta)\Big| \preceq \|f\|_\infty |\log(1-|z|^2)|$$

and

$$\lim_{|z|\to 1}(1-|z|^2)\int_{\mathbb{S}^n}\frac{(\bar z\zeta-|z|^2)^{j-1}}{(1-z\bar\zeta)^{n+j-1}}f(\zeta)\,d\sigma(\zeta)=0.$$

Thus in order to finish the proof we just need to deal with the function

$$F_1(z)=\int_{\mathbb{S}^n}\frac{(\bar z\zeta-|z|^2)^{k}}{(1-z\bar\zeta)^{n+k}}f(\zeta)\,d\sigma(\zeta).$$

A direct computation gives $\partial F_1/\partial\bar z_i = O((1-|z|^2)^{-1/2})$, $\partial F_1/\partial z_i = O((1-|z|^2)^{-1})$, $i=1,\dots,n$, and for any complex tangential operator X, $|XF_1(z)|=O((1-|z|^2)^{-1/2})$.

Now, the transversality of γ allows one to construct (see [Ru, p. 238]) a "quasianalytic" disc $\Phi=(\Phi_1,\dots,\Phi_n)$ from the closed unit disc $\overline{\mathbb{D}}$ in $\mathbb{C}$ into $\mathbb{B}^n$ with the following properties:

(i) $\Phi(e^{i\theta})=\gamma(\theta)$, $1-|\Phi(re^{i\theta})|\simeq 1-r$, and for any θ in $[-\pi,\pi]$, the curve $r\to\Phi(re^{i\theta})$ is a restricted $\gamma(\theta)$-curve.

(ii) $|\sum_{i=1}^n \partial\Phi_i/\partial\bar z|=O(1-|z|)$.

The above properties of Φ together with the estimates on the derivatives of F_1 show that the function $U=F_1\circ\Phi$ satisfies $|\partial U/\partial\bar z|=O((1-|z|)^{-1/2})$, and consequently (see for instance [BrCa]) there exists h in $\mathrm{Lip}_{1/2}(\overline{\mathbb{D}})$ so that $\partial U/\partial\bar z=\partial h/\partial\bar z$. Now, Fatou's theorem in one complex variable applied to the bounded analytic function $U-h$ implies that for almost every $t\in[-\pi,\pi]$, $\lim_{r\to1}(U(re^{it})-h(re^{it}))$ exists, and hence so does $\lim_{r\to1}(F\circ\Phi)(re^{it})$.

Since for each t the curve $\Phi(re^{it})$, $0\le r<1$, is a restricted $\gamma(t)$-curve, it remains to check that Chirka's theorem (see [Ch]), concerning sufficient conditions for the existence of restricted K-limits of bounded holomorphic functions, extends to our context. In the proof of this extension we follow closely Chirka's ideas, modifying some arguments due to the fact that our functions are not holomorphic.

PROPOSITION 3.5. *Let F be a bounded harmonic function with spectrum in Ω_{0k}, and let Ψ_0 be a special ζ-curve for some ζ in $\mathbb{S}^n$. If $\lim_{t\to1}F(\Psi_0(t))=L$, then F has restricted K-limit L at ζ.*

Proof. Let Ψ be a special ζ-curve, and $\psi=(\Psi\bar\zeta)\zeta$ be its orthogonal projection onto the complex line joining 0 and ζ. We first show that

$$\lim_{t\to1}(F(\Psi(t))-F(\psi(t)))=0. \tag{3.2}$$

We have $(\Psi-\psi)\perp\psi$, and if $t\in[0,1)$, then for any $|\lambda|<R=R(t)$ with $R^2=(1-|\psi|^2)/|\Psi-\psi|^2$, the point $(1-\lambda)\psi(t)+\lambda\Psi(t)$ is in $\mathbb{B}^n$. The fact that Ψ is a special ζ-curve implies that $R(t)\to\infty$ as $t\to1$.

Arguing as in Theorem 3.4 we just need to show that (3.2) holds with $F=C_{\Omega_{0k}}[f]$ replaced by F_1. Since $\partial F_1/\partial\bar z_i=O((1-|z|^2)^{-1/2})$, $i=1,\dots,n$, the function $g(\lambda)=F_1((1-\lambda)\psi(t)+\lambda\Psi(t))$ defined in $|\lambda|<R$ satisfies

$$\left|\frac{\partial g}{\partial\bar\lambda}(\lambda)\right|\preceq\frac{|\Psi-\psi|}{(1-|(1-\lambda)\psi+\lambda\Psi|^2)^{1/2}}.$$

Now, $|(1-\lambda)\psi+\lambda\Psi|^2=|\psi|^2+|\lambda|^2|\Psi-\psi|^2$, and the above estimate leads to

$$\left|\frac{\partial g}{\partial\bar\lambda}(\lambda)\right|\preceq\frac{1}{(R^2-|\lambda|^2)^{1/2}},\qquad |\lambda|<R.$$

In particular, the function $h(\lambda)=g(R\lambda)$ defined in the unit disc is in $C^1(\mathbb{D})$ and

$$\left|\frac{\partial h}{\partial\bar\lambda}(\lambda)\right|=O\bigg(\frac{1}{(1-|\lambda|^2)^{1/2}}\bigg).$$

Hence there exists h_1 in $\mathrm{Lip}_{1/2}(\overline{\mathbb{D}})$ with $\|h_1\|_{\mathrm{Lip}_{1/2}}\preceq\|f\|_\infty$ such that $h-h_1$ is a bounded holomorphic function in $\mathbb{D}$. Schwarz's lemma applied to $h-h_1$ gives

$$\begin{aligned}|F_1(\Psi(t))-F_1(\psi(t))|&=|g(1)-g(0)|=|h(1/R)-h(0)|\\&\le|(h-h_1)(1/R)-(h-h_1)(0)|+|h_1(1/R)-h_1(0)|\\&\preceq\frac{\|h-h_1\|_\infty}{R}+\frac{\|h_1\|_{\mathrm{Lip}_{1/2}}}{R^{1/2}}\preceq\frac{1}{R^{1/2}},\end{aligned}$$

which yields (3.2) since $R(t)\to\infty$.

If we apply (3.2) to the given special ζ-curve Ψ_0, the hypothesis on Ψ_0 gives

$$\lim_{t\to1}F_1(\psi_0(t))=L \tag{3.3}$$

with $\psi_0=(\Psi_0\bar\zeta)\zeta$. Next, we want to pass from this particular curve Ψ_0 to any restricted ζ-curve by applying Lindelöf's theorem, as in Chirka's theorem. Since the function $\lambda\to F_1(\lambda\zeta)$ defined in $\mathbb{D}$ is not holomorphic, we will correct it by solving a $\bar\partial$-equation. An argument like the previous one shows that there exists a function f_1 in $\mathrm{Lip}_{1/2}(\overline{\mathbb{D}})$ with $F_1(\cdot\,\zeta)-f_1$ in $H^\infty(\mathbb{D})$. And (3.3) implies that $\lim_{t\to1}(F_1(\psi_0(t))-f_1(\Psi_0(t)\bar\zeta))=L-f_1(1)$.

Finally, if Ψ is any restricted ζ-curve, its orthogonal projection ψ is nontangential, and Lindelöf's theorem (see [Li]) shows that

$$\lim_{t\to1}(F_1(\psi(t))-f_1(\Psi(t)\bar\zeta))=L-f_1(1)$$

and $\lim_{t\to1}F_1(\psi(t))=L$. Then (3.2) implies that $\lim_{t\to1}F_1(\Psi(t))=L$. ■

4. Tangential convergence. In this section we will show that for harmonic extensions of nonisotropic potentials with spectrum in Ω_{0k}, the limit along tangential approach regions exists at almost every point in $\mathbb{S}^n$. We first need some definitions. Let $\alpha \in \mathbb{C}$ with $0 < \operatorname{Re}\alpha < n$, $z \in \overline{\mathbb{B}}^n$ and $\zeta \in \mathbb{S}^n$. The nonisotropic kernel $I_\alpha(z,\zeta)$ is defined by

$$I_\alpha(z,\zeta) = \frac{C(n,\alpha)}{|1-z\overline{\zeta}|^{n-\alpha}},$$

where $C(n,\alpha) = (\Gamma(\frac{n+\alpha}{2}))^2/((n-1)!\Gamma(\alpha))$ is a normalization constant. If $1 \le p < \infty$, and $f \in L^p(\mathbb{S}^n)$, $z \in \overline{\mathbb{B}}^n$, the nonisotropic convolution with I_α is given by

$$I_\alpha * f(z) = \int_{\mathbb{S}^n} I_\alpha(z,\zeta) f(\zeta)\, d\sigma(\zeta).$$

The space of functions $I_\alpha * f$ with $f \in L^p(\mathbb{S}^n)$ will be denoted by $I_\alpha * L^p$. We recall that if $0 < \operatorname{Re}\alpha < n$, $u_{rs} \in H(r,s)$ and $z \in \overline{\mathbb{B}}^n$, then (see [AhCa])

$$(4.1)\qquad I_\alpha * u_{rs}(z) = \frac{\Gamma(\frac{n+\alpha}{2})^2\Gamma(\frac{n-\alpha}{2}+r)\Gamma(\frac{n-\alpha}{2}+s)}{\Gamma(\alpha)\Gamma(\frac{n-\alpha}{2})^2\Gamma(r+s+n)} \times F\left(\frac{n-\alpha}{2}+r, \frac{n-\alpha}{2}+s, r+s+n, |z|^2\right)u_{rs}(z),$$

where $F(a,b,c,x)$ is the hypergeometric function. The fact that if $n - \operatorname{Re} 2a > 0$, then

$$F(a+r, a+s, r+s+n, 1) = \frac{\Gamma(r+s+n)\Gamma(n-2a)}{\Gamma(n+r-a)\Gamma(n+s-a)},$$

implies in particular that $I_\alpha * 1_{|\mathbb{S}^n} \equiv 1$. If $T = \sum_{i<j} \overline{T}_{ij} T_{ij}$ and $\overline{T} = \sum_{i<j} T_{ij}\overline{T}_{ij}$, then (see [AhBr])

$$(4.2)\qquad Tu_{rs} = -r(s+n-1)u_{rs}, \quad \overline{T}u_{rs} = -s(r+n-1)u_{rs}.$$

In [Ge], it is proved that I_1 is the fundamental solution for the sublaplacian

$$(4.3)\qquad \mathcal{L} = \frac{1}{(\frac{n-1}{2})^2}\left(-\frac{1}{2}\sum_{i<j}(T_{ij}\overline{T}_{ij} + \overline{T}_{ij}T_{ij})\right).$$

This operator can be used to show that the spaces of potentials $I_m * L^p$, m a positive integer, coincide with the nonisotropic Sobolev spaces $L^p_m(\mathbb{S}^n)$. This is the result of the next proposition.

PROPOSITION 4.1. *Let $1 \le m \le n-1$ and $1 < p < \infty$. Then $I_m * L^p = L^p_m(\mathbb{S}^n)$.*

Proof. The operators I_m are of order m, and hence map L^p into $L^p_m(\mathbb{S}^n)$. The other inclusion is a consequence of the following facts:

(i) $I_1 * \overset{m}{\ldots} * I_1 * L^p = L^p_m$.
(ii) $I_1 * \overset{m}{\ldots} * I_1 * L^p \subset I_m * L^p$.

The nonisotropic convolution $I_1 * \overset{m}{\ldots} * I_1$ is an operator of order m (see [NaRoStWa]), and consequently maps L^p into L^p_m. Of course, $\mathcal{L}^m$ maps L^p_m in L^p. Since $\mathcal{L}^m(I_1 * \overset{m}{\ldots} * I_1)$ and $(I_1 * \overset{m}{\ldots} * I_1)\mathcal{L}^m$ are the identity on regular functions by Geller's result, we obtain (i).

For (ii) consider the differential operator given by

$$X^m = \prod_{i=0}^{m-1}(\alpha_i T + \beta_i \overline{T} + \gamma_i \mathrm{Id}),$$

where

$$\alpha_i = -\frac{1}{n-1}\left(\frac{n+m}{2} - 1 - i\right),$$
$$\beta_i = -\frac{1}{n-1}\left(\frac{n-m}{2} + i\right),$$
$$\gamma_i = \left(\frac{n-m}{2} + i\right)\left(\frac{n-m}{2} + m - 1 - i\right).$$

Applying (4.2) we easily deduce that if $u_{rs} \in H(r,s)$, then

$$I_1 * \overset{m}{\ldots} * I_1 * u_{rs} = I_m X^m I_1 * \overset{m}{\ldots} * I_1 * u_{rs}.$$

By density, the above equality also holds for L^p. The facts that $\omega(X^m) = m$, and that $I_1 * \overset{m}{\ldots} * I_1$ is a differential operator of order m yield finally that $g = X^m I_1 * \overset{m}{\ldots} * I_1 * f$ is in L^p, and consequently that $I_1 * \overset{m}{\ldots} * I_1 * f = I_m * g$ is in $I_m * L^p$. Observe that I_m gives a topological isomorphism from L^p to L^p_m. ■

REMARK. It is also easy to check that the operators I_m give a topological isomorphism from L^p_Ω to $L^p_{m,\Omega}$, which in particular shows that $I_m * L^p_{\Omega_{0k}} = L^p_{m,\Omega_{0k}}$. If α is real and noninteger, and k is a nonnegative integer, then the spaces $I_\alpha * L^p_{\Omega_{0k}}$ arise as complex method interpolation spaces of Sobolev spaces with spectrum in Ω_{0k}. A direct proof of this fact will be given in the appendix.

In order to state our results concerning the L^p-boundedness of tangential maximal operators, we recall some definitions. Let $1 < p < \infty$, $0 < \alpha$, $m = n - \alpha p \ge 0$, and let $\zeta \in \mathbb{S}^n$, $C > 0$. If $m > 0$ and $1 \le \tau \le n/m$, we consider the tangential approximation regions given by

$$\mathcal{D}_\tau(\zeta) = \{z \in \mathbb{B}^n : |1 - z\overline{\zeta}|^\tau < C(1-|z|)\}.$$

If $m = 0$ and $\mu \ge 1$, we also consider the regions

$$\mathcal{E}_\mu(\zeta) = \left\{z \in \mathbb{B}^n : |1 - z\overline{\zeta}| < \frac{C}{(\log\frac{1}{1-|z|})^{(p-1)\mu/n}}\right\}.$$

If $f:\mathbb{B}^n\to\mathbb{C}$, the corresponding maximal functions will be denoted by $M_\tau f(\zeta)=\sup_{z\in\mathcal{D}_\tau(\zeta)}|f(\zeta)|$ and $\mathcal{M}_\mu f(\zeta)=\sup_{z\in\mathcal{E}_\mu(\zeta)}|f(z)|$ respectively.

It is a well known fact that if $\alpha p>n$, then the space $I_\alpha * L^p$ consists of regular functions. If $\alpha p<n$, [NaRuSh] proved that in $\mathbb{R}^{n+1}_+$ there is a strong-type estimate for the analogous tangential maximal functions of Poisson transforms of Riesz potentials in $\mathbb{R}^n$, in the case $\tau=n/m$. The holomorphic version for the Hardy–Sobolev space H^p_α in the unit ball, in the case $\alpha p\le n$ and $\tau=n/m$, was proved in [Kr1] and [Kr2]. See also [Su] and [CaOr] for related results.

We will show that these strong-type estimates also hold for the space of harmonic transforms of functions in $I_\alpha * L^p$ with spectrum in Ω_{0k}.

THEOREM 4.2. *Let $1<p<\infty$, $\alpha>0$, $m=n-\alpha p\ge 0$, and let k be a nonnegative integer.*

(i) *If $m>0$, $1<\tau\le n/m$ and ν is a positive Borel measure on $\mathbb{S}^n$ such that for any $\zeta\in\mathbb{S}^n$ and $\delta>0$,*

$$\nu(B(\zeta,\delta))\preceq\delta^{\tau m},$$

then there exists $C>0$ such that for any $f\in L^p(\mathbb{S}^n)$,

$$\|M_\tau C_{\Omega_{0k}}[I_\alpha * f]\|_{L^p(d\nu)}\le C\|f\|_p.$$

(ii) *If $m=0$, $\mu\ge 1$ and ν is a finite positive Borel measure on $\mathbb{S}^n$ such that for any $\zeta\in\mathbb{S}^n$ and $\delta>0$,*

$$\nu(B(\zeta,\delta))\preceq\delta^{n/\mu},$$

then there exists $C>0$ so that for any $f\in L^p(\mathbb{S}^n)$,

$$\|\mathcal{M}_\mu C_{\Omega_{0k}}[I_\alpha * f]\|_{L^p(d\nu)}\le C\|f\|_p.$$

Proof. The proof is based on the following lemma.

LEMMA 4.3. *Let k be a nonnegative integer and $0<\alpha<n$. Then for any $z\in\mathbb{B}^n$ and $\omega\in\mathbb{S}^n$,*

$$\Big|\int_{\mathbb{S}^n} C_{\Omega_{0k}}(z,\zeta)I_\alpha(\zeta,\omega)\,d\sigma(\zeta)\Big|\preceq\frac{1}{|1-z\overline{\omega}|^{n-\alpha}}.$$

Proof. Let $z=rz_0$ where $z_0\in\mathbb{S}^n$. Assume first that $|1-z_0\overline{\omega}|<1-|z|$. Then $|1-z\overline{\omega}|\simeq 1-|z|$, and Theorem 2.1(ii) gives

$$\Big|\int_{\mathbb{S}^n} C_{\Omega_{0k}}(z,\zeta)I_\alpha(\zeta,\omega)\,d\sigma(\zeta)\Big|\le\int_{B(\omega,K(1-|z|))}\frac{d\sigma(\zeta)}{|1-z\overline{\zeta}|^n|1-\zeta\overline{\omega}|^{n-\alpha}}+\int_{B^c(\omega,K(1-|z|))}\frac{d\sigma(\zeta)}{|1-z\overline{\zeta}|^n|1-\zeta\overline{\omega}|^{n-\alpha}},$$

for some K large enough. The first integral on the right is bounded by

$$\frac{1}{(1-|z|)^n}\int_{B(\omega,K(1-|z|))}\frac{\sigma(\zeta)}{|1-\zeta\overline{\omega}|^{n-\alpha}}\preceq\frac{1}{(1-|z|)^{n-\alpha}}\simeq\frac{1}{|1-z\overline{\omega}|^{n-\alpha}}.$$

The second summand is bounded by

$$\int_{B^c(\omega,K(1-|z|))}\frac{d\sigma(\zeta)}{|1-\zeta\overline{\omega}|^{2n-\alpha}}\preceq\frac{1}{(1-|z|)^{n-\alpha}}\simeq\frac{1}{|1-z\overline{\omega}|^{n-\alpha}}.$$

If $1-|z|\le|1-z_0\overline{\omega}|$ then $|1-z\overline{\omega}|\simeq|1-z_0\overline{\omega}|$, and the estimate is deduced from the fact that the operators $C^*_{\Omega_{0k}} * I_\alpha$ are of order α, and hence satisfy

$$\Big|\int_{\mathbb{S}^n} C_{\Omega_{0k}}(z,\zeta)I_\alpha(\zeta,\omega)\,d\sigma(\zeta)\Big|\preceq\frac{1}{|1-z_0\overline{\omega}|^{n-\alpha}}$$

uniformly in $r=|z|$. ∎

Going back to the proof of the theorem, let $f\in L^p(\mathbb{S}^n)$ and consider $F=C_{\Omega_{0k}}[I_\alpha * f]$. Then the previous lemma gives

$$\begin{aligned}|F(z)|&=\Big|\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} C_{\Omega_{0k}}(z,\zeta)I_\alpha(\zeta,\omega)f(\omega)\,d\sigma(\omega)\,d\sigma(\zeta)\Big|\\&\le\int_{\mathbb{S}^n}\Big|\int_{\mathbb{S}^n} C_{\Omega_{0k}}(z,\zeta)I_\alpha(\zeta,\omega)\,d\sigma(\zeta)\Big||f(\omega)|\,d\sigma(\omega)\\&\preceq\int_{\mathbb{S}^n}\frac{|f(\omega)|}{|1-z\overline{\omega}|^{n-\alpha}}\,d\sigma(\omega).\end{aligned}$$

In particular, the above estimate together with [CaOr, Lemma 2.21] shows that $|F(z)|\le P[I_\alpha * |f|](z)$, with P the Poisson–Szegő kernel in $\mathbb{B}^n$. Now Theorem 4.2 is a consequence of [CaOr, Thms. 2.10 and 2.17] together with Remark 2.20 there (see also [Su]). ∎

A straightforward argument using Frostman's theorem gives the following

COROLLARY 4.4. *Let $1<p<\infty$, $\alpha>0$, $m=n-\alpha p\ge 0$, and let k be a nonnegative integer.*

(i) *If $m>0$, $1<\tau\le n/m$, and f is a harmonic extension of a function in $I_\alpha * L^p$ with spectrum in Ω_{0k}, then there exists a set $E\subset\mathbb{S}^n$ with $H^{\tau m}(E)=0$ so that for any $\zeta\notin E$ the limit of $f(z)$ exists as z approaches ζ within $\mathcal{D}_\tau(\zeta)$.*

(ii) *If $m=0$, $\mu\ge 1$, and f is a harmonic extension of a function in $I_\alpha * L^p$ with spectrum in Ω_{0k}, then there exists a set $E\subset\mathbb{S}^n$ with $H^{n/\mu}(E)=0$ so that for any $\zeta\notin E$, the limit of $f(z)$ exists as z approaches ζ within $\mathcal{E}_\mu(\zeta)$.*

5. Appendix. In this appendix we will give a constructive proof of the fact that the spaces of potentials $I_\alpha * L^p_{\Omega_{0k}}$ arise as interpolation spaces, by the complex method, of the Sobolev spaces $L^p_{j,\Omega_{0k}}$. We begin with the case $0<\alpha<1$, from which we deduce the general case.

THEOREM 5.1. *Let* $0<\alpha<1$ *and let* k *be a nonnegative integer. Then*

$$[L^p_{\Omega_{0k}}, L^p_{1,\Omega_{0k}}]_{[\alpha]} = I_\alpha * L^p_{\Omega_{0k}}.$$

Proof. We first prove the inclusion

$$I_\alpha * L^p_{\Omega_{0k}} \subset [L^p_{\Omega_{0k}}, L^p_{1,\Omega_{0k}}]_{[\alpha]}.$$

It is enough, given f in $L^p_{\Omega_{0k}}$, to construct a continuous vector-valued function φ from the closed strip $\overline{\mathcal S}=\{\lambda+it\in\mathbb C: 0\le\lambda\le 1\}$ to $L^p_{\Omega_{0k}}+L^p_{1,\Omega_{0k}}=L^p_{\Omega_{0k}}$, holomorphic on $\mathcal S$, with $\varphi(\alpha)=I_\alpha * f$, and such that for some $\gamma\in\mathbb R$,

$$\|\varphi(it)\|_{L^p}\le Ce^{\gamma|t|}\|f\|_p,\quad \|\varphi(1+it)\|_{L^p_1}\le Ce^{\gamma|t|}\|f\|_p. \tag{5.1}$$

If $0<\lambda\le 1$, define $\varphi(\lambda+it)=I_{\lambda+it}*f$. Let us first check that for $g\in L^p$, $\|I_{\lambda+it}*g\|_p\le Ce^{\gamma|t|}\|g\|_p$ for some $\gamma\in\mathbb R$.

Formula (4.1) shows that if $u_{rs}\in H(r,s)$, then

$$I_{\lambda+it}*u_{rs}(\zeta)=C_{\lambda+it}(r,s)u_{rs}(\zeta) \tag{5.2}$$

with

$$C_{\lambda+it}(r,s)=\frac{\Gamma\big(\frac{n+\lambda+it}{2}\big)^2}{\Gamma\big(\frac{n-\lambda-it}{2}\big)^2}\frac{\Gamma\big(r+\frac{n-\lambda-it}{2}\big)\Gamma\big(s+\frac{n-\lambda-it}{2}\big)}{\Gamma\big(r+\frac{n+\lambda+it}{2}\big)\Gamma\big(s+\frac{n+\lambda+it}{2}\big)}$$
$$=\frac{r-1+\frac{n-\lambda-it}{2}}{r-1+\frac{n+\lambda+it}{2}}\cdots\frac{\frac{n-\lambda-it}{2}}{\frac{n+\lambda+it}{2}}\cdot\frac{s-1+\frac{n-\lambda-it}{2}}{s-1+\frac{n+\lambda+it}{2}}\cdots\frac{\frac{n-\lambda-it}{2}}{\frac{n+\lambda+it}{2}}$$

for $r,s\ge 1$. Since for each $x\ge 1$ and $0<\lambda\le 1$,

$$\left|\frac{x-\frac{\lambda+it}{2}}{x+\frac{\lambda+it}{2}}\right|\le 1,$$

we have $|C_{\lambda+it}(r,s)|\le 1$. The case of r or s equal to zero is treated in a similar way. Hence we deduce that $\|I_{\lambda+it}g\|_2\le\|g\|_2$ for $g\in L^2$.

Next if $z=|z|z_0\in\mathbb B^n$ and $\zeta\in\mathbb S^n$, then

$$|\nabla I_{\lambda+it}(z,\zeta)|\le C(n+|t|)A_{\lambda+it}\frac{1}{|1-z_0\overline{\zeta}|^{n-\lambda+1}},$$

where C is a constant independent of λ and t, and

$$A_{\lambda+it}=\left|\frac{\Gamma(\frac{n+\lambda+it}{2})^2}{\Gamma(\lambda+it)}\right|.$$

Stirling's formula implies easily that $A_{\lambda+it}=O(e^{\gamma|t|})$ for some $\gamma\in\mathbb R$. Since the estimate on the derivatives implies that the kernels $I_{\lambda+it}$ satisfy Hörmander's condition [GaRu] with bounds $Ce^{\gamma|t|}$, we finally get $\|I_{\lambda+it}*g\|_p\le Ce^{\gamma|t|}\|g\|_p$ for $g\in L^p$.

Next observe that formula (5.2) shows that for any $t_0\in\mathbb R$ and $u_{rs}\in H(r,s)$, the limit $\lim_{\lambda+it\to it_0}I_{\lambda+it}*u_{rs}$ exists, and equals u_{rs} if $t_0=0$. Since $\bigoplus H(r,s)$ is dense in L^p, this fact, together with the uniform boundedness of $\|I_{\lambda+it}\|_{p,p}$, for t bounded, shows that for any $g\in L^p$ and $t_0\in\mathbb R$, the limit in L^p of $I_{\lambda+it}*g$ exists as $\lambda+it$ tends to it_0. If we denote it by $I_{it_0}*g$, we also have $\|I_{it}\|_{p,p}\le Ce^{\gamma|t|}$ and $I_0=\mathrm{Id}$. Thus we have continuously extended φ to all $\mathcal S$, and proved that $\|\varphi(it)\|_p\le Ce^{\gamma|t|}\|f\|_p$. Since $\varphi(s+it)$ is in $L^p_{\Omega_{0k}}$ so is $\varphi(it)$.

We now show that $\|\varphi(1+it)\|_{L^p_1}\le Ce^{\gamma|t|}\|f\|_p$. Given $0<\mu<1$, we set $I^\mu_{1+it}(\zeta,\omega)=I_{1+it}(\mu\zeta,\omega)$ for $\zeta,\omega\in\mathbb S^n$. Since the derivatives of order two of $I^\mu_{1+it}(\zeta,\omega)$ are pointwise bounded uniformly in μ by $Ce^{\gamma|t|}/|1-\zeta\overline{\omega}|^{n+1}$, the classical theory of singular integral operators shows that the desired estimate in (5.1) for $\|\varphi(1+it)\|_{L^p_1}$ will follow once we show that

$$\|\mathcal L I_{1+it}*u\|_2\le Ce^{\gamma|t|}\|u\|_2 \tag{5.3}$$

for every $u\in L^2(\mathbb S^n)$.

Now, if $u_{rs}\in H(r,s)$, then $I_{1+it}*u_{rs}(\zeta)=\psi_{1+it}(r,s)u_{rs}(\zeta)$, where

$$\psi_{1+it}(r,s)=\frac{\Gamma\big(\frac{n+1+it}{2}\big)^2}{\Gamma\big(\frac{n-1-it}{2}\big)^2}\frac{\Gamma\big(r+\frac{n-1-it}{2}\big)\Gamma\big(s+\frac{n-1-it}{2}\big)}{\Gamma\big(r+\frac{n+1+it}{2}\big)\Gamma\big(s+\frac{n+1+it}{2}\big)}.$$

Since

$$|\psi_{1+it}(r,s)|\preceq\frac{|n-1+it|^2}{\big|r+\frac{n-1+it}{2}\big|\big|s+\frac{n-1+it}{2}\big|},$$

and $\mathcal L u_{rs}=\big(r+\frac{n-1}{2}\big)\big(s+\frac{n-1}{2}\big)u_{rs}$, we obtain (5.3).

REMARK. The above shows, in particular, that for $t\in\mathbb R$ the operator $S_t=\mathcal L I_{1-it}$ satisfies $\|S_t[f]\|_p\le Ce^{\gamma|t|}\|f\|_p$ for any $f\in L^p$. Since for every $u_{rs}\in H(r,s)$, $S_t(I_1*u_{rs})=I_{1-it}*u_{rs}$, and $S_t\circ I_1$, I_{1-it} are bounded in L^p, we deduce that $\|I_{1-it}*h\|_p\le Ce^{\gamma|t|}\|I_1*h\|_p$ for any $h\in L^p$.

In $\mathbb R^n$, the Riesz kernels are additive with respect to convolution. The next lemma, which will be used to finish the proof the theorem, gives some results concerning the nonisotropic convolution of the kernels I_α.

LEMMA 5.2. *Let* $0\le j\le n-1$, $0<\alpha<1$, *and let* k *be a nonnegative integer. Then:*

(i) *If* $f\in L^p_{\Omega_{0k}}$ *and* $\mathcal L I_{1-\alpha}*f\in L^p_{\Omega_{0k}}$, *then there exists* $g\in L^p_{\Omega_{0k}}$ *such that* $\mathcal L^j I_{\alpha+j}*g=f$.

(ii) $I_\alpha * L^p_{\Omega_{0k}}=\{f\in L^p: I_{1-\alpha}*f\in L^p_{1,\Omega_{0k}}\}$.

(iii) $I_j * (I_\alpha * L^p_{\Omega_{0k}}) = I_{\alpha+j} * L^p_{\Omega_{0k}}$.

(iv) *If* $f \in L^p$, *then* $I_1 * f$ *admits an expression* $I_1 * f = I_\alpha * g$ *with* $\|g\|_p \preceq \|I_{1-\alpha} * f\|_p$.

Proof. Observe that (i) would follow if we could find a bounded operator S in L^p such that $\mathcal{L}^j I_{\alpha+j} S \mathcal{L} I_{1-\alpha} = \mathrm{Id}_{L^p}$. We will show that, except for some terms with good behavior, this is what happens. The proof is based on the action of these operators on the spaces $H(r,s)$ and asymptotic developments of gamma functions.

Assume $0 \le j \le n-1$, $0 < \alpha < 1$ and k is a nonnegative integer. It suffices to show the lemma for each $L^p_{\Omega_{ss}}$, $0 \le s \le k$. Applying (4.1), we find that for any u_{rs} in $H(r,s)$,

$$I_{j+\alpha} * I_{1-\alpha} * u_{rs}(\zeta) = C_s \frac{\Gamma\left(\frac{n-j-\alpha}{2}+r\right)\Gamma\left(\frac{n-1+\alpha}{2}+r\right)}{\Gamma\left(\frac{n+j+\alpha}{2}+r\right)\Gamma\left(\frac{n+1-\alpha}{2}+r\right)} u_{rs}(\zeta),$$

where C_s is a constant depending on n, α and s.

The asymptotic development in [TrEr] together with Stirling's formula implies that there exist $\lambda_i(\alpha,n)$, $i \in \mathbb{Z}_+$, with $\lambda_0 \ne 0$ so that for each $l > 0$, the numbers

$$b_{rl} = 1 - \frac{\Gamma\left(\frac{n-\alpha}{2}+r\right)}{\Gamma\left(\frac{n+\alpha}{2}+r\right)}\left(\lambda_0\left(r+\frac{n-1}{2}\right)^{\alpha} + \ldots + \lambda_{l-1}\left(r+\frac{n-1}{2}\right)^{\alpha-l+1}\right)$$

satisfy $|b_{rl}| \le (2|\lambda_l|+1)/(r+1)^l$ if r is large enough.

In particular, there exist $\lambda_i(\alpha,n)$, $\mu_i(\alpha,n)$ such that

$$\begin{aligned}
&\frac{\Gamma\left(\frac{n-j-\alpha}{2}+r\right)\Gamma\left(\frac{n-1+\alpha}{2}+r\right)}{\Gamma\left(\frac{n+j+\alpha}{2}+r\right)\Gamma\left(\frac{n+1-\alpha}{2}+r\right)} \\
&\quad\times\left(\lambda_0\left(r+\frac{n-1}{2}\right)^{j+\alpha} + \ldots + \lambda_{l-1}\left(r+\frac{n-1}{2}\right)^{j+\alpha-l+1}\right) \\
&\quad\times\left(\mu_0\left(r+\frac{n-1}{2}\right)^{1-\alpha} + \ldots + \mu_{l-1}\left(r+\frac{n-1}{2}\right)^{1-\alpha-l+1}\right) = 1 - c_{rl},
\end{aligned}$$

with $|c_{rl}| \preceq d_l/(r+1)^l$. Next

$$\begin{aligned}
&\left(\lambda_0\left(r+\frac{n-1}{2}\right)^{j+\alpha} + \ldots + \lambda_{l-1}\left(r+\frac{n-1}{2}\right)^{j+\alpha-l+1}\right) \\
&\qquad\times\left(\mu_0\left(r+\frac{n-1}{2}\right)^{1-\alpha} + \ldots + \mu_{l-1}\left(r+\frac{n-1}{2}\right)^{1-\alpha-l+1}\right) \\
&\quad= \sum_{k=3-2l}^{1} \gamma_k \left(r+\frac{n-1}{2}\right)^{j+k}.
\end{aligned}$$

Since $\mathcal{L}u_{rs} = \left(r+\frac{n-1}{2}\right)\left(s+\frac{n-1}{2}\right)u_{rs}$ for $u_{rs} \in H(r,s)$, and

$$I_1 * u_{rs} = \frac{n-1}{2}\,\frac{1}{\left(r+\frac{n-1}{2}\right)\left(s+\frac{n-1}{2}\right)} u_{rs},$$

we can write

$$\sum_{k=3-2l}^{1} \gamma_k \left(r+\frac{n-1}{2}\right)^{j+k} u_{rs} = \widetilde{\gamma}\mathcal{L}^{j+1}u_{rs} + \sum_{k=0}^{2l-3} \widetilde{\gamma}_k \mathcal{L}^j (I_1 * \overset{k}{\ldots} * I_1) * u_{rs}$$

with $\widetilde{\gamma} \ne 0$.

We define an operator in $\bigoplus_r H(r,s)$ by

$$T_{rs}u_{rs} = \begin{cases} u_{rs} & \text{if } r \le r_0, \\ \left(\widetilde{\gamma}\mathcal{L}^{j+1} + \sum_{k=0}^{2l-3} \widetilde{\gamma}_k \mathcal{L}^j (I_1 * \overset{k}{\ldots} * I_1)\right) * I_{\alpha+j} * I_{1-\alpha} * u_{rs}/C_s & \text{if } r > r_0, \end{cases}$$

where r_0 is to be chosen. Then

$$(\mathrm{Id} - T_{rs})u_{rs} = \begin{cases} 0 & \text{if } r \le r_0, \\ c_{rl}u_{rs} & \text{if } r > r_0. \end{cases}$$

We will check that there exists $\varepsilon < 1$ such that $\|f - Tf\|_p \le \varepsilon\|f\|_p$ for any $f \in L^p_{\Omega_{ss}}$.

Assume first that $p \le 2$, and take $f \in L^2(\mathbb{S}^n)$. In [Al, p. 118], it is shown that if $\omega, \zeta \in \mathbb{S}^n$, then $|K_{rs}(\omega,\zeta)| \le D(r,s,n)$. Since

$$D(r,s,n) = \binom{r+n-2}{r}\binom{s+n-2}{s}\frac{r+s+n-1}{n-1}$$

we deduce that $|K_{rs}(\omega,\zeta)| \preceq (r+1)^{n-1}$, with constant depending on s and n. Hence

$$|f_{rs}(\omega)| = \left|\int_{\mathbb{S}^n} K_{rs}(z,\zeta) f(\zeta)\,d\sigma(\zeta)\right| \preceq (r+1)^{n-1}\|f\|_1.$$

This implies that if $f \in L^2_{\Omega_{ss}}$, then

$$\begin{aligned}
\|f - Tf\|_2^2 &\preceq \sum_{r \ge r_0} c_{rl}^2 \|f_{rs}\|_2^2 \preceq \sum_{r > r_0} \frac{1}{(r+1)^{2l}} \int_{\mathbb{S}^n} |f_{rs}(\omega)|^2\,d\sigma(\omega) \\
&\preceq \sum_{r > r_0} \frac{(r+1)^{2(n-1)}}{(r+1)^{2l}} \|f\|_1^2 \le \varepsilon\|f\|_1^2,
\end{aligned}$$

provided r_0 is large enough and l satisfies $2l - 2(n-1) > 1$.

Finally, since $1 < p \le 2$,

$$\|f - Tf\|_p^2 \le \|f - Tf\|_2^2 \preceq \varepsilon\|f\|_1^2 \preceq \varepsilon\|f\|_p^2.$$

In particular, the operator T is invertible in $L^p_{\Omega_{ss}}$ and there exists an operator $S : L^p_{\Omega_{ss}} \to L^p_{\Omega_{ss}}$ with $ST = TS = \mathrm{Id}_{L^p_{\Omega_{ss}}}$.

The case $p \ge 2$ can be deduced from the previous one by duality, since the operator T is selfadjoint.

The definition of T shows that we can write

$$T = T_1 + R_1 + R_2, \tag{5.4}$$

where

$$T_1 = \frac{\widetilde{\gamma}}{C_s}\mathcal{L}^{j+1} I_{\alpha+j} * I_{1-\alpha}$$

is an operator of order zero,

$$R_1 = \frac{1}{C_s}\sum_{k=0}^{2l-3} \widetilde{\gamma}_k \mathcal{L}^j (I_1 * \overset{k}{\ldots} * I_1) * I_{\alpha+j} * I_{1-\alpha}$$

is an operator of order greater than or equal to 1, and where

$$R_2 = \sum_{r\le r_0}\Big(\mathrm{Id} - \frac{1}{C_s}\Big(\widetilde{\gamma}\mathcal{L}^{j+1} + \sum_{k=0}^{2l-3}\widetilde{\gamma}_k \mathcal{L}^j (I_1 * \overset{k}{\ldots} * I_1)\Big) * I_{\alpha+j} * I_{1-\alpha}\Big) K_{rs}$$

is an operator of finite rank, and consequently, bounded in $L^p_{\Omega_{ss}}$. If we apply S to (5.4), the boundedness in L^p of the operators S, T_1, and R_j, $j = 1, 2$, together with the fact that they commute in L^2, shows that if we take $f \in L^p_{\Omega_{ss}}$ such that $\mathcal{L} I_{1-\alpha} * f$ is also in $L^p_{\Omega_{ss}}$, then

$$f = \frac{\widetilde{\gamma}}{C_s}\mathcal{L}^j I_{\alpha+j} * S\mathcal{L} I_{1-\alpha} * f + \frac{1}{C_s}\mathcal{L}^j I_{\alpha+j} * K_1 f + \frac{1}{C_s}\mathcal{L}^j I_{\alpha+j} * K_2 f,$$

where

$$K_1 f = \frac{1}{C_s}\sum_{k=0}^{2l-3}\widetilde{\gamma}_k S(I_1 * \overset{k}{\ldots} * I_1) * I_{1-\alpha} * f$$

is in L^p, and K_2 is of finite rank. Thus we have shown that if $f \in L^p_{\Omega_{ss}}$ and $\mathcal{L} I_{1-\alpha} * f$ is also in $L^p_{\Omega_{ss}}$, then $f = \mathcal{L}^j I_{j+\alpha} * g$ with g in L^p. Thus (i) is proved.

Next, part (i) with $j = 0$ gives

$$\{f \in L^p_{\Omega_{ss}} : I_{1-\alpha} * f \in L^p_{1,\Omega_{ss}}\} \subset I_\alpha * L^p_{\Omega_{ss}}.$$

The other inclusion follows from the fact that $I_{1-\alpha} * I_\alpha$ is an operator of order 1 and, consequently, maps L^p to L^p_1.

Since $\mathcal{L}^j I^j = \mathrm{Id}_{L^p}$ and $I^j \mathcal{L}^j = \mathrm{Id}_{L^p_j}$, (iii) will follow once we show that

$$I_\alpha * L^p_{\Omega_{ss}} = \mathcal{L}^j I_{\alpha+j} * L^p_{\Omega_{ss}}.$$

Now, $I_{1-\alpha}\mathcal{L}^j I_{\alpha+j}$ is an operator of order 1, and this gives

$$\mathcal{L}^j I_{\alpha+j} * L^p_{\Omega_{ss}} \subset \{f \in L^p_{\Omega_{ss}} : I_{1-\alpha} * f \in L^p_{1,\Omega_{ss}}\}.$$

Part (i) shows that the inclusion in the other direction also holds, and finally (ii) implies (iii).

If we take $j = 0$ in (i), and apply S followed by I_1 in (5.4), it is then easy to check from the above calculations that (iv) is satisfied. ∎

Let us now finish the proof of Theorem 5.1. We must show that

$$[L^p_{\Omega_{0k}}, L^p_{1,\Omega_{0k}}]_{[\alpha]} \subset I_\alpha * L^p_{\Omega_{0k}}.$$

By a lemma of Stafney (see [St]) it is enough to show that if φ_k are holomorphic on $\mathcal{S}$ and continuous on its closure $\overline{\mathcal{S}}$, and $h_k \in L^p_{\Omega_{0k}}$, then

$$\Big\|\sum_k \varphi_k(\alpha) I_1 * h_k\Big\|_{I_\alpha * L^p}$$
$$\le C\max\Big(\sup_t e^{\gamma|t|}\Big\|\sum_k \varphi_k(it) I_1 * h_k\Big\|_p, \sup_t e^{\gamma|t|}\Big\|\sum_k \varphi_k(1-it) I_1 * h_k\Big\|_{I_1 * L^p}\Big).$$

By Lemma 5.2(iv),

$$\Big\|\sum_k \varphi_k(\alpha) I_1 * h_k\Big\|_{I_\alpha * L^p} \preceq \Big\|I_{1-\alpha} * \Big(\sum_k \varphi_k(\cdot) h_k\Big)\Big\|_p$$
$$= \sup_{\|g\|_{p'}\le 1}\Big|\int_{\mathbb{S}^n}\sum_k \varphi_k(\alpha) I_{1-\alpha} * h_k(\zeta) g(\zeta)\, d\sigma(\zeta)\Big|.$$

Now the map defined on $\overline{\mathcal{S}}$ by

$$\omega \to \int_{\mathbb{S}^n}\sum_k \varphi_k(\omega) I_{1-\omega} h_k(\zeta) g(\zeta)\, d\sigma(\zeta)$$

is holomorphic in $\mathcal{S}$ and continuous on $\overline{\mathcal{S}}$. By Lemma 4.3.2 of [BeLo], this implies that

$$\Big|\int_{\mathbb{S}^n} (\varphi_k(\alpha) I_{1-\alpha} h_k(\zeta)) g(\zeta)\, d\sigma(\zeta)\Big|$$
$$\le \Big(\frac{1}{1-\alpha}\int_{-\infty}^{\infty}\Big|\int_{\mathbb{S}^n}(\varphi_k(it) I_{1-it} h_k(\zeta)) g(\zeta)\, d\sigma(\zeta)\Big|\mathcal{P}_0(\alpha, t)\, dt\Big)^{1-\alpha}$$
$$\times\Big(\frac{1}{\alpha}\int_{-\infty}^{\infty}\Big|\int_{\mathbb{S}^n}(\varphi_k(1+it) I_{-it} h_k(\zeta)) g(\zeta)\, d\sigma(\zeta)\Big|\mathcal{P}_1(\alpha, t)\, dt\Big)^{\alpha},$$

where for $m = 0, 1$,

$$\mathcal{P}_m(s + it, \tau) = \frac{e^{-\pi(\tau - t)}\sin \pi s}{\sin^2 \pi s + (\cos \pi s - e^{im\pi - \pi(\tau - t)})^2}.$$

Taking supremum over g in the previous estimate we get

$$\Big\|\sum_k \varphi_k(\alpha) I_1 * h_k\Big\|_{I_\alpha * L^p_{\Omega_{0k}}} \le \Big(\frac{1}{1-\alpha}\int_{-\infty}^{\infty}\Big\|\sum_k \varphi_k(it) I_{1-it} * h_k\Big\|_p \mathcal{P}_0(\alpha,t)\,dt\Big)^{1-\alpha} \times \Big(\frac{1}{\alpha}\int_{-\infty}^{\infty}\Big\|\sum_k \varphi_k(1+it) I_{-it} * h_k\Big\|_p \mathcal{P}_1(\alpha,t)\,dt\Big)^{\alpha}.$$

But we have seen in the previous remark that $\|I_{1-it} * h\|_p \le e^{\gamma|t|}\|I_1 * h\|_p$ and the definition of I_{-it} shows that $\|I_{-it}(h)\|_p \le e^{\gamma|t|}\|h\|_p$. Thus the above can be estimated by

$$\Big(\frac{1}{1-\alpha}\int_{-\infty}^{\infty} e^{\gamma|t|}\Big\|\sum_k \varphi_k(it) I_1 * h_k\Big\|_p \mathcal{P}_0(\alpha,t)\,dt\Big)^{1-\alpha} \times \Big(\frac{1}{\alpha}\int_{-\infty}^{\infty} e^{\gamma|t|}\Big\|\sum_k \varphi_k(1+it) h_k\Big\|_p \mathcal{P}_1(\alpha,t)\,dt\Big)^{\alpha}$$

$$\le \sup_t\Big(e^{\gamma|t|}\Big\|\sum_k \varphi_k(it) I_1 * h_k\Big\|_p\Big)^{1-\alpha} \cdot \sup_t\Big(e^{\gamma|t|}\Big\|\sum_k \varphi_k(1+it) h_k\Big\|_p\Big)^{\alpha}$$

$$\le \max\Big(\sup_t e^{\gamma|t|}\Big\|\sum_k \varphi_k(it) I_1 * h_k\Big\|_p, \sup_t e^{\gamma|t|}\Big\|\sum_k \varphi_k(1+it) h_k\Big\|_p\Big),$$

where we have used the fact that

$$\frac{1}{1-\alpha}\int_{-\infty}^{\infty}\mathcal{P}_0(\alpha,t)\,dt = \frac{1}{\alpha}\int_{-\infty}^{\infty}\mathcal{P}_1(\alpha,t)\,dt = 1. \ \blacksquare$$

THEOREM 5.3. *Let $j \in \mathbb{Z}_+$, $0 \le j \le n-1$, $0 < \alpha < 1$, and let k be a nonnegative integer. Then*

$$[L^p_{j,\Omega_{0k}}, L^p_{j+1,\Omega_{0k}}]_{[\alpha]} = I_{\alpha+j} * L^p_{\Omega_{0k}}.$$

Proof. If $j \le 1$, then $I_j : L^p_{\Omega_{ss}} \to L^p_{j,\Omega_{ss}}$ and $I_j : L^p_{1,\Omega_{ss}} \to L^p_{j+1,\Omega_{ss}}$ are topological isomorphisms and consequently

$$I_j : [L^p_{\Omega_{ss}}, L^p_{1,\Omega_{ss}}]_{[\alpha]} \to [L^p_{j,\Omega_{ss}}, L^p_{j+1,\Omega_{ss}}]_{[\alpha]}$$

is also a topological isomorphism. But Theorem 5.1 gives

$$[L^p_{\Omega_{ss}}, L^p_{1,\Omega_{ss}}]_{[\alpha]} = I_\alpha * L^p_{\Omega_{ss}}.$$

Hence $I_j * (I_\alpha * L^p_{\Omega_{ss}}) = [L^p_{j,\Omega_{ss}}, L^p_{j+1,\Omega_{ss}}]_{[\alpha]}$. Now, Lemma 5.2(iii) finishes the proof. ■

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